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## Logarithmic density of a sequence of integers and density of its ratio set

par LADISLAV MIŠÍK et JÁNOS T. TÓTH

RÉSUMÉ. Nous donnons des conditions suffisantes pour que l'ensemble  $R(A)$  des fractions d'un ensemble d'entiers  $A$  soit dense dans  $\mathbb{R}^+$ , en termes des densités logarithmiques de  $A$ . Ces conditions diffèrent sensiblement de celles précédemment obtenues en termes des densités asymptotiques.

ABSTRACT. In the paper sufficient conditions for the  $(R)$ -density of a set of positive integers in terms of logarithmic densities are given. They differ substantially from those derived previously in terms of asymptotic densities.

### 1. Preliminaries

Denote by  $\mathbb{N}$  and  $\mathbb{R}^+$  the set of all positive integers and positive real numbers, respectively. For  $A \subset \mathbb{N}$  and  $x \in \mathbb{R}^+$  let  $A(x) = \{a \in A; a \leq x\}$ . Denote by  $R(A) = \{\frac{a}{b}; a \in A, b \in A\}$  the *ratio set of  $A$*  and say that a set  $A$  is  $(R)$ -dense if  $R(A)$  is (topologically) dense in the set  $\mathbb{R}^+$  (see [3]). Let us notice that the  $(R)$ -density of a set  $A$  is equivalent to the density of  $R(A)$  in the set  $(1, \infty)$ .

Define

$$\underline{d}(A) = \liminf_{x \rightarrow \infty} \frac{\#A(x)}{x}, \quad \bar{d}(A) = \limsup_{x \rightarrow \infty} \frac{\#A(x)}{x}, \quad d(A) = \lim_{x \rightarrow \infty} \frac{\#A(x)}{x}.$$

the *lower asymptotic density*, *upper asymptotic density*, and *asymptotic density* (if defined), respectively.

Similarly, define

$$\underline{\delta}(A) = \liminf_{x \rightarrow \infty} \frac{\sum_{a \in A(x)} \frac{1}{a}}{\ln x}, \quad \bar{\delta}(A) = \limsup_{x \rightarrow \infty} \frac{\sum_{a \in A(x)} \frac{1}{a}}{\ln x}, \quad \delta(A) = \lim_{x \rightarrow \infty} \frac{\sum_{a \in A(x)} \frac{1}{a}}{\ln x},$$

the *lower logarithmic density*, *upper logarithmic density*, and *logarithmic density* (if defined), respectively.

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The following relations between asymptotic density and  $(R)$ -density are known

(S1) If  $d(A) > 0$  then  $A$  is  $(R)$ -dense (see [3], [4]).

(S2) If  $\underline{d}(A) \geq \frac{1}{2}$  then  $A$  is  $(R)$ -dense and for all  $b \in (0, \frac{1}{2})$  there is a set  $B$  such that  $\underline{d}(B) = b$  and  $B$  is not  $(R)$ -dense (see [2], [5]).

(S3) If  $\bar{d}(A) = 1$  then  $A$  is  $(R)$ -dense and for all  $b \in (0, 1)$  there is a set  $B$  such that  $\bar{d}(B) = b$  and  $B$  is not  $(R)$ -dense (see [3], [4]).

Notice that the results (S1), (S2) and (S3) can be formulated in a common way as results about maximal sets (with respect to the corresponding density) which are not  $(R)$ -dense as follows. Denote by  $\mathbb{D} = \{A \subset \mathbb{N}; A \text{ is not } (R)\text{-dense}\}$ . Then we have

(S1)  $\{d(A); A \in \mathbb{D}\} = \{0\}$ .

(S2)  $\{\underline{d}(A); A \in \mathbb{D}\} = (0, \frac{1}{2})$ .

(S3)  $\{\bar{d}(A); A \in \mathbb{D}\} = (0, 1)$ .

The aim of this paper is to prove corresponding relations for logarithmic density. It appears (see Theorem 2 and Corollary 1) that they differ substantially from the above ones for asymptotic density.

## 2. Logarithmic density and $(R)$ -density

First, let us introduce a useful technique for calculation densities. It can be easily seen that in practical calculation of densities of a set  $A$ , the following method can be used.

Write the set  $A$  as  $A = \bigcup_{n=1}^{\infty} (p_n + 1, q_n) \cap \mathbb{N}$ , where  $0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \dots$  are integers. Then

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n (q_i - p_i)}{p_{n+1}}, \quad \bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n (q_i - p_i)}{q_n},$$

and

$$\underline{\delta}(A) = \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n \ln \frac{q_i}{p_i}}{\ln p_{n+1}}, \quad \bar{\delta}(A) = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n \ln \frac{q_i}{p_i}}{\ln q_n}.$$

In practice the bounds  $p_n$ ,  $q_n$  of intervals determining the set  $A$  are often real numbers instead of integers. Then it may be convenient to use the following lemma. In fact, we will use it in later calculations.

**Lemma 1.** *Let  $0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \dots$  be given real numbers such that  $\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty$  and let  $|a_n| \leq d$ ,  $|b_n| \leq d$  for each  $n \in \mathbb{N}$  and some*

fixed  $d \in \mathbb{N}$ . Let  $A = \bigcup_{n=1}^{\infty} \langle p_n + a_n + 1, q_n + b_n \rangle \cap \mathbb{N}$ . Then

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n (q_i - p_i)}{p_{n+1}}, \quad \bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n (q_i - p_i)}{q_n}.$$

$$\underline{\delta}(A) = \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n \ln \frac{q_i}{p_i}}{\ln p_{n+1}}, \quad \bar{\delta}(A) = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n \ln \frac{q_i}{p_i}}{\ln q_n}.$$

*Proof.* For all  $n \in \mathbb{N}$  let  $a_n, b_n$  be such that

1.  $|a_n| \leq d, |b_n| \leq d,$
2. both  $p_n + a_n$  and  $q_n + b_n$  are integers,
3.  $A = \bigcup_{n=1}^{\infty} \langle p_n + a_n + 1, q_n + b_n \rangle \cap \mathbb{N}.$

First, let us notice that it is known that if for an increasing sequence of positive integers  $\{p_n\}$  the series  $\sum_{n=1}^{\infty} \frac{1}{p_n}$  converges then  $\lim_{n \rightarrow \infty} \frac{n}{p_n} = 0$  ([1], 80.

Theorem, p.124) and trivially also  $\lim_{n \rightarrow \infty} \frac{n}{p_n+r} = 0$  for any fixed  $r \in \mathbb{R}$ .

Now a simple analysis shows that for each  $n \in \mathbb{N}$

$$\frac{\sum_{i=1}^n (q_i - p_i - 2d)}{p_{n+1} + d} \leq \frac{\sum_{i=1}^n ((q_i + b_i) - (p_i + a_i))}{p_{n+1} + a_{n+1}} \leq \frac{\sum_{i=1}^n (q_i - p_i + 2d)}{p_{n+1} - d}$$

and, using Lemma 1,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n (q_i - p_i - 2d)}{p_{n+1} + d} &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n ((q_i + b_i) - (p_i + a_i))}{p_{n+1} + a_{n+1}} = \underline{d}(A) \\ &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^n (q_i - p_i + 2d)}{p_{n+1} - d} \end{aligned}$$

or, rewritten,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left( \frac{\sum_{i=1}^n (q_i - p_i)}{p_{n+1} + d} - \frac{2dn}{p_{n+1} + d} \right) &\leq \underline{d}(A) \\ &\leq \liminf_{n \rightarrow \infty} \left( \frac{\sum_{i=1}^n (q_i - p_i)}{p_{n+1} - d} + \frac{2dn}{p_{n+1} - d} \right). \end{aligned}$$

As both  $\lim_{n \rightarrow \infty} \frac{n}{p_{n+1}+d}$  and  $\lim_{n \rightarrow \infty} \frac{n}{p_{n+1}-d}$  equal 0 an application of the Sandwich Theorem completes the proof for  $\underline{d}(A)$ . In a very similar way one can prove the corresponding statement for  $\overline{d}(A)$ .

Let  $s$  be the first positive integer  $i$  such that  $p_i - d > 0$ . The following inequalities hold for every  $i = s, s + 1, \dots$

$$(I) \quad \ln \frac{q_i - d}{p_i + d} = \ln \frac{q_i}{p_i} - \int_{p_i}^{p_i+d} \frac{1}{x} dx - \int_{q_i-d}^{q_i} \frac{1}{x} dx \geq \ln \frac{q_i}{p_i} - \frac{2d}{p_i - d}$$

and

$$(II) \quad \ln \frac{q_i + d}{p_i - d} = \ln \frac{q_i}{p_i} + \int_{p_i-d}^{p_i} \frac{1}{x} dx + \int_{q_i}^{q_i+d} \frac{1}{x} dx \leq \ln \frac{q_i}{p_i} - \frac{2d}{p_i - d}.$$

Again, a simple analysis shows that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\sum_{i=s}^n \ln \frac{q_i-d}{p_i+d}}{\ln(p_{n+1} + d)} &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{i=s}^n \ln \frac{q_i+b_i}{p_i+a_i}}{\ln(p_{n+1} + a_{n+1})} = \underline{\delta}(A) \\ &\leq \liminf_{n \rightarrow \infty} \frac{\sum_{s=1}^n \ln \frac{q_i+d}{p_i-d}}{\ln(p_{n+1} - d)} \end{aligned}$$

and, using (I) and (II),

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left( \frac{\sum_{i=s}^n \ln \frac{q_i}{p_i}}{\ln(p_{n+1} + d)} - 2d \frac{\sum_{i=s}^n \frac{1}{p_i-d}}{\ln(p_{n+1} + d)} \right) &\leq \underline{\delta}(A) \\ &\leq \liminf_{n \rightarrow \infty} \left( \frac{\sum_{i=s}^n \ln \frac{q_i}{p_i}}{\ln(p_{n+1} - d)} + 2d \frac{\sum_{i=s}^n \frac{1}{p_i-d}}{\ln(p_{n+1} - d)} \right). \end{aligned}$$

As the series  $\sum_{i=1}^n \frac{1}{p_i-d}$  is convergent and  $\lim_{n \rightarrow \infty} \frac{\ln(p_{n+1}+d)}{\ln(p_{n+1}-d)} = 1$  an application of the Sandwich Theorem completes the proof for  $\underline{\delta}(A)$ . In a very similar way one can prove the corresponding statement for  $\overline{\delta}(A)$ .  $\square$

The following simple lemma will be used in later calculations.

**Lemma 2.** *Let  $(a_i)_{1 \leq i \leq n}$  and  $(b_i)_{1 \leq i \leq n}$  such that  $1 \leq a_1$ ,  $a_i < b_i$ ,  $i = 1, 2, \dots, n$ ,  $b_i \leq a_{i+1}$ ,  $i = 1, 2, \dots, n-1$ , and  $b_n \leq w$ . Then  $\ln w \geq \sum_{i=1}^n \ln \frac{b_i}{a_i}$ .*

*Proof.* The statement of the Lemma is a straightforward consequence of the following relations

$$w \geq b_n = \frac{b_n}{a_n} \frac{a_n}{b_{n-1}} \frac{b_{n-1}}{a_{n-1}} \dots \frac{b_2}{a_2} \frac{a_2}{b_1} \frac{b_1}{a_1} a_1 = a_1 \left( \prod_{i=1}^n \frac{b_i}{a_i} \right) \left( \prod_{i=1}^{n-1} \frac{a_{i+1}}{b_i} \right) \geq \prod_{i=1}^n \frac{b_i}{a_i}.$$

□

The following class of sets plays an important role in our consideration.

$$\mathbb{A} = \left\{ A(a, b) = \bigcup_{n=s}^{\infty} \langle a^n b^n + 1, a^{n+1} b^n \rangle \cap \mathbb{N}; 1 < a < b \right\}$$

where  $s = \min\{n \in \mathbb{N}; a^n b^n + 1 \leq a^{n+1} b^n\}$ .

**Theorem 1.** *Let  $1 < a < b$  and  $A = A(a, b) \in \mathbb{A}$ . Then*

(i) 
$$R(A) \cap \langle a, b \rangle = \emptyset,$$

(ii) 
$$\underline{d}(A) = \frac{a-1}{ab-1}, \quad \bar{d}(A) = \frac{b(a-1)}{ab-1},$$

(iii) 
$$\delta(A) = \frac{\ln a}{\ln a + \ln b}.$$

*Proof.* (i) Let  $x \in A$ ,  $y \in A$  and  $x < y$ . First, let there be a  $n \in \mathbb{N}$  such that both  $x$  and  $y$  belong to the block  $\langle a^n b^n + 1, a^{n+1} b^n \rangle$ . Then

$$\frac{y}{x} \leq \frac{a^{n+1} b^n}{a^n b^n + 1} < \frac{a^{n+1} b^n}{a^n b^n} = a.$$

On the other hand, let  $x \in \langle a^n b^n + 1, a^{n+1} b^n \rangle$  and  $y \in \langle a^m b^m + 1, a^{m+1} b^m \rangle$  for  $m > n$ . Then

$$\frac{y}{x} \geq \frac{a^m b^m + 1}{a^{n+1} b^n} > \frac{a^m b^m}{a^{n+1} b^n} \geq b.$$

In both cases  $\frac{y}{x}$  does not belong to  $\langle a, b \rangle$ .

(ii) Calculate, using Lemma 1,

$$\begin{aligned} \underline{d}(A) &= \liminf_{n \rightarrow \infty} \frac{\sum_{i=s}^n (a^{i+1} b^i - a^i b^i)}{a^{n+1} b^{n+1}} = (a-1) a^s b^s \liminf_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-s} a^i b^i}{a^{n+1} b^{n+1}} \\ &= \frac{a-1}{ab-1} \lim_{n \rightarrow \infty} \frac{a^{n+1} b^{n+1} - a^s b^s}{a^{n+1} b^{n+1}} = \frac{a-1}{ab-1}. \end{aligned}$$

The corresponding value of  $\bar{d}(A)$  can be calculated in a very similar way.

(iii) Again, using Lemma 1, we have

$$\underline{\delta}(A) = \liminf_{n \rightarrow \infty} \frac{\sum_{i=s}^n \ln \frac{a^{i+1} b^i}{a^i b^i}}{\ln(a^{n+1} b^{n+1})} = \lim_{n \rightarrow \infty} \frac{(n-s+1) \ln a}{(n+1)(\ln a + \ln b)} = \frac{\ln a}{\ln a + \ln b}.$$

□

**Remark 1.** A simple analysis of equalities (ii) in Theorem 1 in comparison to the results (S1), (S2), (S3) shows

$$(A1) \quad \{d(A); A \in \mathbb{D}\} = \{0\} = \{d(A); A \in \mathbb{A}\}.$$

$$(A2) \quad \{\underline{d}(A); A \in \mathbb{D}\} = \langle 0, \frac{1}{2} \rangle = \{\underline{d}(A); A \in \mathbb{A}\}.$$

$$(A3) \quad \{\bar{d}(A); A \in \mathbb{D}\} = \langle 0, 1 \rangle = \{\bar{d}(A); A \in \mathbb{A}\}.$$

A similar analysis of equality (iii) in Theorem 1 leads to the following.

**Conjecture.** *The following equalities hold*

$$\begin{aligned} & \{\underline{\delta}(A); A \in \mathbb{D}\} = \{\underline{\delta}(A); A \in \mathbb{A}\} = \{\bar{\delta}(A); A \in \mathbb{D}\} = \\ & = \{\bar{\delta}(A); A \in \mathbb{A}\} = \{\delta(A); A \in \mathbb{D}\} = \{\delta(A); A \in \mathbb{A}\} = \left\langle 0, \frac{1}{2} \right\rangle. \end{aligned}$$

The purpose of the rest of this paper is to prove this conjecture. All the corresponding results will be corollaries to the following.

**Theorem 2.** *Let  $1 < a < b$  and  $A = A(a, b) \in \mathbb{A}$ . Then the set  $A$  is maximal element in the set  $\{X \subset \mathbb{N}; R(X) \cap \langle a, b \rangle = \emptyset\}$  with respect to the partial order induced by any of  $\underline{\delta}$ ,  $\bar{\delta}$ ,  $\delta$ .*

*Proof.* Let  $X \subset \mathbb{N}$  be an infinite set such that  $R(X) \cap \langle a, b \rangle = \emptyset$ . Then  $X$  can be written in the form

$$X = \bigcup_{n=1}^{\infty} \langle p_n + 1, q_n \rangle \cap \mathbb{N}; \quad 0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \dots \text{ are integers.}$$

For the proof it is sufficient to show (taking into account Theorem 1 (iii))

$$\bar{\delta}(X) \leq \bar{\delta}(A(a, b)) = \delta(A(a, b)) = \frac{\ln a}{\ln a + \ln b}.$$

Thus we can also suppose

$$(0) \quad \bar{\delta}(X) > 0.$$

The proof will be carried in several steps.

*Step 1.* In this step we will prove

$$(1) \quad \text{if } p_n \geq \frac{1}{b-a} \text{ then } q_n < a(p_n + 1).$$

*Proof of (1).* Suppose  $q_n \geq a(p_n + 1)$ . Then  $\frac{q_n}{p_n+1} \geq a$  and also  $\frac{p_n+1}{p_n+1} < a$  and, as  $R(X) \cap \langle a, b \rangle = \emptyset$ , there exists  $m \in \langle p_n + 1, q_n \rangle \cap \mathbb{N}$  such that  $\frac{m}{p_n+1} < a$  and  $\frac{m+1}{p_n+1} > b$ . Consequently  $\frac{m+1}{p_n+1} - \frac{m}{p_n+1} > b - a$  which implies  $p_n < \frac{1}{b-a}$ , a contradiction.

*Step 2.* In this step we will prove

$$(2) \quad \text{if } p_n \geq \frac{ab}{b-a} \text{ then } X \cap \langle a(p_n + 1), bq_n \rangle = \emptyset.$$

*Proof of (2).* Let  $p_n \geq \frac{ab}{b-a}$ . Then also  $p_n \geq \frac{1}{b-a}$  and, by the previous step,  $q_n < a(p_n + 1)$ . Suppose on the contrary that there exists  $x \in X \cap \langle a(p_n + 1), bq_n \rangle$ . Then  $\frac{x}{p_n+1} \geq \frac{a(p_n+1)}{p_n+1} = a$ ,  $\frac{x}{q_n} \leq \frac{bq_n}{q_n} = b$  and, as  $R(X) \cap \langle a, b \rangle = \emptyset$ , there exists  $m \in \langle p_n + 1, q_n \rangle \cap \mathbb{N}$  such that  $\frac{x}{m} > b$  and  $\frac{x}{m+1} < a$ . Consequently

$$b-a < \frac{x}{m} - \frac{x}{m+1} = \frac{x}{m(m+1)} < \frac{x}{m^2} < \frac{bq_n}{(p_n+1)^2} < \frac{ba(p_n+1)}{(p_n+1)^2} = \frac{ab}{p_n+1}$$

which implies  $p_n < \frac{ab}{b-a}$ , a contradiction.

*Step 3.* In this step we will introduce some useful notation. Denote by  $k_0$  the smallest integer  $k$  such that  $p_k \geq \frac{ab}{b-a}$  and let  $K_0 = \{k_0, k_0+1, k_0+2, \dots\}$ . From (2) we have for every  $k \in K_0$

$$(3) \quad \langle a(p_k + 1), bq_k \rangle \cap \mathbb{N} \subset \mathbb{N} - X$$

and so we can define a function  $\varphi : K_0 \rightarrow K_0$  by

$$(4) \quad \langle a(p_k + 1), bq_k \rangle \subset (q_{\varphi(k)}, p_{\varphi(k)+1} + 1).$$

The range of this function  $\varphi(K_0) = \{l_1 < l_2 < \dots\}$  is infinite, denote  $K_n = \varphi^{-1}(l_n)$  for each  $n \in \mathbb{N}$ . Evidently  $K_0 = \bigcup_{n=1}^{\infty} K_n$  and for every  $m < n$ ,  $x \in K_m$  and  $y \in K_n$  it is  $x < y$ . Let us call a **big gap in X** any interval of the form  $(q_{\varphi(k)}, p_{\varphi(k)+1} + 1)$  where  $k \in K_n$  for  $n \in \mathbb{N}$ . Finally, let us introduce two sequences  $\{u_n\}_{n=1}^{\infty}$ ,  $\{v_n\}_{n=1}^{\infty}$  by

$$u_n = p_{\min K_n}, \quad v_n = q_{\max K_n} \quad \text{for } n = 1, 2, 3, \dots$$

The above definitions imply that both  $a(u_n + 1)$  and  $bv_n$  belong to the same big gap in  $X$  and, consequently,

$$(5) \quad v_n < a(u_n + 1) \quad \text{for } n = 1, 2, 3, \dots$$

*Step 4.* In this step we will present and prove (if necessary) some simple relations and statement which will be used in the final calculation.



An easy analysis proves

$$(6) \quad \sum_{a=p+1}^q \frac{1}{a} \leq \ln \frac{q}{p} \quad \text{for all positive integers } p < q.$$

From Lemma 1 and (0) it can be seen that

$$(7) \quad \text{the series } \sum_{i=1}^{\infty} \ln \frac{q_i}{p_i} \text{ is divergent.}$$

As  $a(u_{i+1} + 1)$  belongs to the big gap next to the big gap in which  $b(u_i + 1)$  lies we have  $u_{i+1} + 1 \geq \frac{b}{a}(u_i + 1)$  for every  $i \in \mathbb{N}$ . Therefore the series  $\sum_{i=1}^{\infty} \frac{1}{u_i}$  is convergent and, consequently,

$$(8) \quad \sum_{i=1}^{\infty} \ln \frac{u_i + 1}{u_i} < \infty, \quad \lim_{i \rightarrow \infty} \ln \left(1 + \frac{1}{u_i}\right) = 0.$$

*Step 5.* For the rest of proof suppose that  $m$  is a sufficiently large fixed positive integer and denote by  $n$  the greatest (fixed from this moment) positive integer  $k$  for which  $bv_k \leq m$ . Thus, using (5), we have

$$(9) \quad bv_n \leq m < bv_{n+1} < ab(u_{n+1} + 1).$$

The considerations in Step 3 imply that the intervals  $\langle p_i + 1, q_i \rangle$  for  $i = 1, 2, \dots, \max K_n$  and  $\langle a(u_j + 1), bv_j \rangle$  for  $j = 1, 2, \dots, n$  are mutually disjoint and, by definition of numbers  $m$  and  $n$ , they are all contained in the interval  $(1, m)$ . Thus, by Lemma 2, we have

$$(10) \quad \ln m \geq \sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \sum_{j=1}^n \ln \frac{b}{a} \frac{v_j}{u_j + 1}.$$

For similar reason we have

$$(11) \quad \sum_{j=1}^{\max K_n} \ln \frac{q_j}{p_j} \leq \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j} + \sum_{i=1}^n \ln \frac{v_i}{u_i}.$$

The last inequality together with (5) imply

$$(12) \quad \sum_{j=1}^{\max K_n} \ln \frac{q_j}{p_j} \leq \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j} + n \ln a + \sum_{i=1}^n \ln \left(1 + \frac{1}{u_i}\right).$$

Step 6. Denote  $c = \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j}$ . Now we are able to estimate

$$\begin{aligned}
 s(m) &= \frac{\sum_{a \in X, a \leq m} \frac{1}{a}}{\ln m} = \frac{\sum_{a \in X, a \leq v_n} \frac{1}{a} + \sum_{a \in X, u_{n+1} < a \leq m} \frac{1}{a}}{\ln m} \\
 &\leq \frac{\sum_{i=1}^{\max K_n} \sum_{a=p_i+1}^{q_i} \frac{1}{a} + \sum_{a=u_{n+1}+1}^m \frac{1}{a}}{\ln m} \leq \frac{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \ln \frac{m}{u_{n+1}}}{\ln m} \quad \text{by (6)} \\
 &\leq \frac{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \ln \frac{ab(u_{n+1}+1)}{u_{n+1}}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + n \ln \frac{b}{a} + \sum_{i=1}^n \ln \frac{v_i}{u_i} + \sum_{i=1}^n \ln \frac{u_i}{u_i+1}} \quad \text{by (9), (10)} \\
 &= \frac{1 + \frac{\ln ab + \ln(1 + \frac{1}{u_{n+1}})}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}}{1 + \frac{n \ln \frac{b}{a}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}} + \frac{\sum_{i=1}^n \ln \frac{v_i}{u_i}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}} + \frac{\sum_{i=1}^n \ln \frac{u_i}{u_i+1}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}} \\
 &\leq \frac{1 + \frac{\ln ab + \ln(1 + \frac{1}{u_{n+1}})}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}}{1 + \frac{n \ln \frac{b}{a}}{c + n \ln a + \sum_{i=1}^n \ln(1 + \frac{1}{u_i})} + \frac{(\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}) - c}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}} + \frac{\sum_{i=1}^n \ln \frac{u_i}{u_i+1}}{\sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i}}} \quad \text{by (11), (12)} \\
 &= S(m).
 \end{aligned}$$

Step 7. In this step we will complete the proof by limit process. Let  $m \rightarrow \infty$  (and consequently  $n \rightarrow \infty$ ). Then, using (7) and (8), we have

$$\begin{aligned}
 \bar{\delta}(X) &= \limsup_{m \rightarrow \infty} s(m) \leq \limsup_{m \rightarrow \infty} S(m) = \lim_{m \rightarrow \infty} \frac{1 + 0}{1 + \frac{\ln b - \ln a}{\ln a} + 1 + 0} \\
 &= \frac{\ln a}{\ln a + \ln b}.
 \end{aligned}$$

□

The following corollary is a direct consequence of the previous theorem and it shows that the relations between  $(R)$ -density and logarithmic densities are completely different from those between  $(R)$ -density and asymptotic densities.

**Corollary.** *The following relations hold*

$$\{\underline{\delta}(A); A \in \mathbb{D}\} = \{\overline{\delta}(A); A \in \mathbb{D}\} = \{\delta(A); A \in \mathbb{D}\} = \left\langle 0, \frac{1}{2} \right\rangle.$$

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