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The Complex Sum of Digits Function and Primes

par JÖRG M. THUSWALDNER

RÉSUMÉ. La notion de développement q-adique d'un entier, pour une base q donnée, se généralise dans l'anneau des entiers de Gauss $\mathbb{Z}[i]$ au développement d'un entier de Gauss suivant une certaine base $b \in \mathbb{Z}[i]$, ce développement étant unique. Dans cet article, on s'intéresse à la fonction ν_b, désignant la somme de chiffres dans le développement suivant la base b. On montre un résultat sur la fonction somme de chiffres pour les nombres non multiples d'une puissance f-ième d'un nombre premier. On établit aussi pour ν_b un théorème du type Erdős-Kac. Dans ces résultats, l'équidistribution de ν_b joue un rôle essentiel. Partant de cela, les démonstrations font alors appel à des méthodes de crible, ainsi qu'à une version du modèle de Kubilius.

ABSTRACT. Canonical number systems in the ring of Gaussian integers $\mathbb{Z}[i]$ are the natural generalization of ordinary q-adic number systems to $\mathbb{Z}[i]$. It turns out, that each Gaussian integer has a unique representation with respect to the powers of a certain base number b. In this paper we investigate the sum of digits function ν_b of such number systems. First we prove a theorem on the sum of digits of numbers, that are not divisible by the f-th power of a prime. Furthermore, we establish an Erdős-Kac type theorem for ν_b. In all proofs the equidistribution of ν_b in residue classes plays a crucial rôle. Starting from this fact we use sieve methods and a version of the model of Kubilius to prove our results.

1. INTRODUCTION

Let $\nu_q(n)$ denote the sum of digits of the q-adic representation of a positive integer n. Gelfond [5] proved, that $\nu_q(n)$ is equidistributed in residue classes modulo an integer. In particular, he established the following result. For $r, m \in \mathbb{Z}$ with

$$(m, q-1) = 1 \tag{1}$$

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one has

$$(2) \qquad \left|\{n < N \mid n \equiv a\,(s), \nu_q(n) \equiv r\,(m)\}\right| = \frac{N}{ms} + \mathcal{O}(N^{\lambda_1}).$$

The exponent $\lambda_1 < 1$ in the error term can be computed explicitly and does not depend on a, s, r and N. This result forms the basis for the application of sieve methods, in order to get results on the sum of digits of prime numbers. The first result of this type, also contained in Gelfond's paper [5], is the following. Let $S_{r,m}(N) = \{n < N \mid \nu_q(n) \equiv r\,(m)\}$. Then the number of elements of $S_{r,m}(N)$, being not divisible by the f-th power of a prime is given by

$$(3) \qquad \frac{N}{m\zeta(f)} + \mathcal{O}(N^{\lambda_2})$$

with a constant $\lambda_2 < 1$. Here ζ denotes the Riemann zeta function. Recently, Mauduit and Sárközy [15] proved an Erdős-Kac type theorem for the elements of the set $S_{r,m}(N)$: Let $m, r \in \mathbb{Z}$ such that (1) holds. Denote the normal law by $\psi(x)$ and let $\omega(n)$ count the distinct prime divisors of n. Then

(4)

$$\left|\frac{1}{|S_{r,m}(N)|}\left|\{n \mid n \in S_{r,m}(N),\, \omega(n) - \log\log N \le x\sqrt{\log\log N}\}\right| - \psi(x)\right| = \mathcal{O}\left(\frac{\log\log\log N}{\sqrt{\log\log N}}\right).$$

The aim of this paper is to extend these results to the sum of digits function of canonical number systems in the ring of Gaussian integers $\mathbb{Z}[i]$. First we give a definition of these number systems. Let $b \in \mathbb{Z}[i]$ and $\mathcal{N} = \{0, 1, \dots, N(b) - 1\}$, where $N(b)$ denotes the norm of b over $\mathbb{Q}$. If any number $\gamma \in \mathbb{Z}[i]$ admits a representation of the form

$$\gamma = c_0 + c_1 b + \cdots + c_h b^h$$

for $c_j \in \{0, 1, \dots, N(b) - 1\}$ $(0 \le j \le h)$ and $c_h \ne 0$ for $h \ne 0$, then $(b, \mathcal{N})$ is called a canonical number system in $\mathbb{Z}[i]$. b is called the base of this number system. In Kátai, Szabó [11] it is shown that b can serve as a base of a canonical number system in $\mathbb{Z}[i]$ if and only if

$$b = -n \pm i \quad (n \in \mathbb{N}).$$

The sum of digits function of the number system $(b, \mathcal{N})$ is defined by

$$\nu_b(\gamma) = \nu_b(c_0 + c_1 b + \cdots + c_h b^h) = c_0 + c_1 + \cdots + c_h.$$

Some properties of this function have been investigated in recent papers. An asymptotic formula of the summatory function of $\nu_b(\gamma)$ in large circles was computed by Grabner, Kirschenhofer and Prodinger [7]. Moreover, Gittenberger and Thuswaldner [6] established asymptotic formulas for the moments of $\nu_b(\gamma)$.

The notion of canonical number systems can be extended to arbitrary number fields, provided that they have a power integral basis (cf. Kovács [12]). The bases of these number systems are characterized in Kovács, Pethő [13]. This characterization is not explicit and depends on the shape of the integral basis of the number field. Some results on the sum of digits function can be extended to the general case of canonical number systems in number fields. The results (3) and (4) can be generalized to number fields, whose ring of integers is a unique factorization domain. For further generalizations a new notion of the sum of digits function with ideals as arguments seems to be necessary. Since we also need the finiteness of the group of unity of the ring of integers under consideration, we confine ourselves to the Gaussian case.

Again our results are derived from a result of the type (2). The number field version of (2) is established in Thuswaldner [18]. We only need the "Gaussian" case of this result: Let $b = -n \pm i$ be the base of a canonical number system in $\mathbb{Z}[i]$ with minimal polynomial $p_b(x)$. We define the sets

$$U_{r,m}(N) := \{z \in \mathbb{Z}[i] \,|\, |z|^2 < N,\ \nu_b(z) \equiv r\,(m)\}.$$

Then, for $m, r \in \mathbb{Z}$, $a \in \mathbb{Z}[i]$ and an ideal $\mathfrak{s}$ of $\mathbb{Z}[i]$, satisfying

$$(p_b(1), m) = 1 \tag{5}$$

we have the estimate

$$\big|\{z \,|\, z \in U_{r,m}(N), z \equiv a\,(\mathfrak{s})\}\big| = \frac{\pi N}{mN(\mathfrak{s})} + \mathcal{O}(N^\lambda). \tag{6}$$

$\lambda < 1$ is an effectively computable constant independent of r, a, $\mathfrak{s}$, and N.

In Section 2 we will provide the necessary tools needed in the proofs of our theorems, Section 3 is devoted to the generalization of (3) and in Section 4 the Erdős-Kac type result (4) is established for canonical number systems in $\mathbb{Z}[i]$.

2. Auxiliary Results

In this section we want to list various results from algebraic number theory, that will be needed further on. First we give some estimates for sums over prime ideals of $\mathbb{Z}[i]$ (in fact, these estimates remain valid for rings of integers of arbitrary number fields). In Narkiewicz [17, Lemmas 7.3–7.6] it is proved, that

$$\sum_{N(\mathfrak{p}) \le x} \log N(\mathfrak{p}) = x + o(x). \tag{7}$$

The sum is extended over all prime ideals $\mathfrak{p}$ whose norm is less than x. Recall Abel's identity (cf. Hardy, Wright [8, Theorem 421])

$$\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt$$

for $A(t) = \sum_{n\leq t} a_n$ and a continuously differentiable function $f(t)$. Applying this to (7) yields the estimate

$$\sum_{N(\mathfrak{p})\leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \log x + o(\log x). \tag{8}$$

Furthermore, we have (cf. Narkiewicz [17, Chapter 7, Exercise 7])

$$\sum_{N(\mathfrak{p})\leq x} \frac{1}{N(\mathfrak{p})} = \log\log x + B + o(1). \tag{9}$$

We will also need the estimate

$$\prod_{N(\mathfrak{p})\leq x} \left(1 - \frac{1}{N(\mathfrak{p})}\right) = \mathcal{O}\left(\frac{1}{\log x}\right) \tag{10}$$

which can be proved in a similar way as Theorem 429 in [8].

Next we recall a result due to Hua [9] on exponential sums over number fields. Let $\operatorname{tr}(z)$ denote the trace of z over $\mathbb{Q}$ and let $\mathfrak{d} = 2\mathbb{Z}[i]$ be the different of $\mathbb{Q}(i)$. Then, writing $e(x) = \exp(2\pi i x)$,

$$\sum_{\xi} e(\operatorname{tr}(\xi a)) = \begin{cases} N(k) & \text{if } k \mid a \\ 0 & \text{if } k \nmid a \end{cases} \tag{11}$$

where ξ runs over a complete residue system in $(k\mathfrak{d})^{-1}$ modulo $\mathfrak{d}^{-1}$.

In order to prove our results we will need a version of Selberg's sieve method and a generalized Kubilius model in number fields. These objects are discussed in Kubilius [14, Chapter X] (cf. also Danilov [2] for related results). In a more general setting they are studied in Juskys [10] and Zhang [20]. Moreover, Thuswaldner [19] gives a survey on Selberg's sieve and the Kubilius model in number fields. We start with the statement of a quantitative version of Selberg's sieve in $\mathbb{Q}(i)$ (cf. [19]).

Lemma 2.1. *Let $a_n \in \mathbb{Z}[i]$ $(1 \leq n \leq N)$ and $\rho > 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be prime ideals of $\mathbb{Z}[i]$ with $N(\mathfrak{p}_1) \leq \dots \leq N(\mathfrak{p}_s) \leq \rho$ and set $\Omega = \mathfrak{p}_1 \cdots \mathfrak{p}_s$. If $\mathfrak{d}$ is an ideal that divides Ω $(\mathfrak{d}|\Omega)$, then we assume that there exists a representation*

$$\sum_{\substack{n=1\\ a_n\equiv 0(\mathfrak{d})}}^{N} 1 = \eta(\mathfrak{d})X + R(N,\mathfrak{d}), \tag{12}$$

where X and R are real numbers, $X \geq 0$, and $\eta(\mathfrak{d}_1\mathfrak{d}_2) = \eta(\mathfrak{d}_1)\eta(\mathfrak{d}_2)$ for divisors of $\mathfrak{Q}$ with $(\mathfrak{d}_1, \mathfrak{d}_2) = \mathfrak{o}$ ($\mathfrak{o}$ denotes the unit ideal). Furthermore for each prime ideal $\mathfrak{p}$ we assume that $0 \leq \eta(\mathfrak{p}) < 1$, holds.

Set

$$I(N, \mathfrak{Q}) = \sum_{\substack{n=1 \\ (a_n, \mathfrak{Q})=1}}^{N} 1.$$

Then the estimate

$$I(N, \mathfrak{Q}) = (1 + 2\theta_1 H) X \prod_{\mathfrak{p}|\mathfrak{Q}} (1 - \eta(\mathfrak{p})) + 2\theta_2 \sum_{\substack{\mathfrak{d}|\mathfrak{Q} \\ N(\mathfrak{d}) \leq \tau^3}} 3^{\omega(\mathfrak{d})} |R(N, \mathfrak{d})|$$

holds uniformly for $\rho \geq 2$, $\max(\log \rho, S) \leq \frac{1}{8} \log \tau$, where $|\theta_1| \leq 1$, $|\theta_2| \leq 1$,

$$H = \exp\left(-\frac{\log \tau}{\log \rho} \left\{ \log \frac{\log \tau}{S} - \log \log \frac{\log \tau}{S} - \frac{2S}{\log \tau} \right\} \right),$$

$\omega(\mathfrak{d})$ is the number of distinct prime ideal factors of $\mathfrak{d}$, and

$$S = \sum_{\mathfrak{p}|\mathfrak{Q}} \frac{\eta(\mathfrak{p})}{1 - \eta(\mathfrak{p})} \log N(\mathfrak{p}).$$

Now we sketch the construction of the generalized Kubilius model for $\mathbb{Z}[i]$. Again we refer to [19] for details. Let the same notation as in Lemma 2.1 be in force. Set

$$E(\mathfrak{p}) = \{a_n \mid n = 1, \ldots, N,\ a_n \equiv 0(\mathfrak{p})\}.$$
$$\bar{E}(\mathfrak{p}) = \{a_n \mid n = 1, \ldots, N,\ a_n \not\equiv 0(\mathfrak{p})\}.$$

and, for $\mathfrak{k}|\mathfrak{Q}$,

$$E_{\mathfrak{k}} = \bigcap_{\mathfrak{p}|\mathfrak{k}} E(\mathfrak{p}) \bigcap_{\mathfrak{p}|\frac{\mathfrak{Q}}{\mathfrak{k}}} \bar{E}(\mathfrak{p}).$$

Let $\mathcal{A}$ be the set algebra generated by the sets $E_{\mathfrak{k}}$. In order to make $\mathcal{A}$ to a probability space, we define the measure

$$\nu E_{\mathfrak{k}} = \frac{|E_{\mathfrak{k}}|}{N}.$$

If all conditions for the application of Lemma 2.1 are fulfilled, it can be applied to $\nu E_{\mathfrak{k}}$ to get the estimate

$$\begin{aligned} \nu E_{\mathfrak{k}} &= (1 + 2\theta_1 H)\eta(\mathfrak{k}) \prod_{\mathfrak{p}|\frac{\mathfrak{Q}}{\mathfrak{k}}} (1 - \eta(\mathfrak{p})) \\ &\quad + 2\theta_2 X^{-1} \sum_{\substack{\mathfrak{d}|\frac{\mathfrak{Q}}{\mathfrak{k}} \\ N(\mathfrak{d}) \leq \tau^3}} 3^{\omega(\mathfrak{d})} |R(N, [\mathfrak{k}, \mathfrak{d}])| \end{aligned} \tag{13}$$

with H as in Lemma 2.1, $|\theta_1| \le 1$ and $|\theta_2| \le 1$. Here $[\mathfrak{k}, \mathfrak{d}]$ denotes the least common multiple of $\mathfrak{k}$ and $\mathfrak{d}$.

Now we set

$$\mu_{\mathfrak{k}} := \eta(\mathfrak{k}) \prod_{\mathfrak{p} | \frac{\Omega}{\mathfrak{k}}} (1 - \eta(\mathfrak{p}))$$

for each $\mathfrak{k}|\Omega$. Unfortunately, $\mu E_{\mathfrak{k}} := \mu_{\mathfrak{k}}$ in general does not define a probability measure on $\mathcal{A}$, because there may exist empty sets $E_{\mathfrak{k}}$ with positive $\mu E_{\mathfrak{k}}$. Therefore we have to use a trick due to Zhang [20] in order to construct a sequence of independent random variables such that the distribution of their sum has density $\mu_{\mathfrak{k}}$.

Remark 2.1. The fact that $\mu E_{\mathfrak{k}}$ in general does not define a measure on $\mathcal{A}$ was observed by Zhang [20]. Unfortunately, many versions of Kubilius models proposed so far, treat $\mu E_{\mathfrak{k}}$ as a measure (cf. for instance [14, 3, 19]). This lacuna can be mended in many cases with help of a trick used in Zhang [20] (cf. also the proof of Lemma 2.3 of the present paper). We want to mention explicitely, that the Erdős-Kac type theorems in Mauduit-Sárközy [15, 16] are not affected by this lacuna. Their proofs can be adapted by reasoning in the same way as in the present paper.

Lemma 2.2. *Having the same assumptions as in Lemma 2.1, suppose, that* $R(N, \mathfrak{k}) = \mathcal{O}(N^{\lambda'})$ *and* $\tau = \mathcal{O}(N^{\delta})$*, such that* $\lambda' + 7\delta < 1$*. Then* $E_{\mathfrak{k}} \ne \emptyset$ *for* $X\eta(\mathfrak{k}) \ge N^{\lambda'+7\delta}$ *and* N *large enough.*

Proof. Using (10) we easily obtain

$$(1 + 2\theta_1 H) X\eta(\mathfrak{k}) \prod_{\mathfrak{p} | \frac{\Omega}{\mathfrak{k}}} = (1 - \eta(\mathfrak{p})) \ge c_1 N^{\lambda'+7\delta} \frac{1}{\log N}.$$

On the other hand, by the assumptions in the statement of the present lemma we have

$$\sum_{\substack{\mathfrak{d} | \frac{\Omega}{\mathfrak{k}} \\ N(\mathfrak{d}) \le \tau^3}} 3^{\omega(\mathfrak{d})} R(N, [\mathfrak{k}, \mathfrak{d}]) \le c_2 N^{\lambda'+6\delta}.$$

Both constants c_1, c_2 are absolute. By the estimate (13) the result follows. □

After these preparations we establish the following result (cf. also Elliot [3, Lemma 3.5]).

Lemma 2.3. *Having the same assumptions as in Lemma 2.1, suppose that* $\rho \ge 2$ *and* $\max(\log \rho, S) \le \frac{1}{8} \log \tau$ *hold. Furthermore, assume that* $E_{\mathfrak{k}} \ne \emptyset$

for $N(\mathfrak{k}) \le \tau^{\beta}$ for some $\beta > 0$. Define the strongly additive function

$$g(z) = \sum_{\substack{\mathfrak{p}|z \\ \mathfrak{p}|\mathfrak{Q}}} l(\mathfrak{p}),$$

where $l(\mathfrak{p})$ is a function from $\mathbb{Z}[i]$ to $\mathbb{R}$. Define the independent random variables $W_{\mathfrak{p}}$ for each prime divisor $\mathfrak{p}$ of $\mathfrak{Q}$ by

$$W_{\mathfrak{p}} = \begin{cases} l(\mathfrak{p}) & \text{with probability } \eta(\mathfrak{p}) \\ 0 & \text{with probability } 1-\eta(\mathfrak{p}) \end{cases}.$$

Then

$$\nu(g(a_n) \in F) \quad \text{and} \quad P\Big(\sum_{\mathfrak{p}|\mathfrak{Q}} W_{\mathfrak{p}} \in F\Big)$$

do not differ by more than

$$\mathcal{O}\left(\exp\left(-\beta' \frac{\log \tau}{\log r} \log\left(\frac{\log \tau}{S}\right)\right)\right) + \mathcal{O}\left(X^{-1} \sum_{\substack{N(\mathfrak{m}) \le \tau^4 \\ \mathfrak{m}|\mathfrak{Q}}} 4^{\omega(\mathfrak{m})} |R(N,\mathfrak{m})|\right),$$

where $\beta' = \min(\frac{1}{8}, \beta)$. This holds uniformly for all sets F.

Proof. Consider the variables

$$W_{\mathfrak{p}}(n) := \begin{cases} l(\mathfrak{p}) & \text{if } a_n \equiv 0\,(\mathfrak{p}) \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to see, that $P(\sum W_{\mathfrak{p}} \in F) = \mu(n : g(a_n) \in F)$. Thus it remains to show that μ and ν approximate each other.

Let $\mathcal{E}$ be the set of all $\mathfrak{d}|\mathfrak{Q}$, for which $E_{\mathfrak{d}} \neq \emptyset$. Then ν is again a probability measure on the set algebra generated by the elements of $\mathcal{E}$.

Following Thuswaldner [19, p. 496] we get the estimate

$$\text{(14)} \quad \Big|\sum_{\substack{N(\mathfrak{k}) \le \tau \\ \mathfrak{k} \in \mathcal{E}}} \nu E_{\mathfrak{k}} - \sum_{\substack{N(\mathfrak{k}) \le \tau \\ \mathfrak{k} \in \mathcal{E}}} \mu_{\mathfrak{k}}\Big| \le 10 \exp\left(-\frac{\log \tau}{8 \log \rho} \log\left(\frac{\log \tau}{S}\right)\right) + 12 X^{-1} \sum_{\substack{N(\mathfrak{m}) \le \tau^4 \\ \mathfrak{m}|\mathfrak{Q}}} 4^{\omega(\mathfrak{m})} |R(N,\mathfrak{m})|.$$

It remains to treat the sets $E_{\mathfrak{k}}$ with $N(\mathfrak{k}) > \tau$. We get from [19, Lemma 1] (cf. also [3, Lemma 2.3])

$$\text{(15)} \qquad \sum_{N(\mathfrak{k}) > \tau} \mu_{\mathfrak{k}} = \mathcal{O}\left(\exp\left(-\frac{\log \tau}{\log \rho} \log \frac{\log \tau}{S}(1 + o(1))\right)\right).$$

Since $\sum_{\mathfrak{k}\in\mathcal{E}} \nu(E_\mathfrak{k}) = 1$ and $\sum_{\mathfrak{k}\in\mathcal{E}} \mu_\mathfrak{k} = 1 - \sum_{\mathfrak{k}\notin\mathcal{E}} \mu_\mathfrak{k}$, (14) implies

$$(16)\quad \Bigl|\sum_{\substack{N(\mathfrak{k})>\tau\\ \mathfrak{k}\in\mathcal{E}}} \nu E_\mathfrak{k} - \sum_{\substack{N(\mathfrak{k})>\tau\\ \mathfrak{k}\in\mathcal{E}}} \mu_\mathfrak{k}\Bigr| \le 10\exp\left(-\frac{\log\tau}{8\log\rho}\log\left(\frac{\log\tau}{S}\right)\right) + 12X^{-1}\sum_{\substack{N(\mathfrak{m})\le\tau^4\\ \mathfrak{m}|\mathfrak{Q}}} 4^{\omega(\mathfrak{m})}|R(N,\mathfrak{m})| + \sum_{\mathfrak{k}\notin\mathcal{E}} \mu_\mathfrak{k}.$$

By our assumption we have

$$\sum_{\mathfrak{k}\notin\mathcal{E}} \mu_\mathfrak{k} \le \sum_{N(\mathfrak{k})>\tau^\beta} \mu_\mathfrak{k}.$$

Applying (15) to the sums containing $\mu_\mathfrak{k}$ in (16), we arrive at

$$(17)\qquad \sum_{N(\mathfrak{k})>\tau} \nu E_\mathfrak{k} = \mathcal{O}\left(\exp\left(-\beta\frac{\log\tau}{\log\rho}\log\frac{\log\tau}{S}(1+o(1))\right)\right).$$

(14) (15) and (17) now yield the result. □

3. Prime Powers

This section is devoted to the generalization of Gelfond's result (3). Instead of the ordinary zeta function in our theorem the Dedekind zeta function $\zeta_{\mathbb{Q}(i)}$ of the number field $\mathbb{Q}(i)$ occurs.

Theorem 3.1. *Suppose $m, r \in \mathbb{Z}$ fulfill (5) for a base b of a canonical number system in $\mathbb{Z}[i]$. Let $T_{b,f}(N)$ be the number of elements of $U_{r,m}(N)$ that are not divisible by the f-th power of a prime ($f \ge 2$). Then there is a constant $\mu < 1$ independent of N, such that the estimate*

$$T_{b,f}(N) = \frac{\pi N}{m\zeta_{\mathbb{Q}(i)}(f)} + \mathcal{O}(N^\mu)$$

holds.

Proof. Define the function

$$\phi(z) = \begin{cases} 1 & \text{if } z \in U_{r,m}(N) \\ 0 & \text{otherwise} \end{cases}.$$

Then $T_{b,f}(N)$ has the representation

$$\begin{aligned} T_{b,f}(N) &= \sum_{|z|^2<N} \phi(z) \sum_{z\equiv 0\,(\mathfrak{s}^f)} \mu(\mathfrak{s}) \\ &= \sum_{N(\mathfrak{s}^f)<N} \mu(\mathfrak{s}) \sum_{\substack{|z|^2<N\\ z\equiv 0\,(\mathfrak{s}^f)}} \phi(z), \end{aligned}$$

where the $\mathfrak{s}$–sums run over all ideals of $\mathbb{Z}[i]$ in the indicated range. Setting $N_1 = N^{(1-\lambda)/2}$, where λ is as in (6), we split the last double sum in two parts:

$$\begin{aligned} T_{b,f}(N) &= \sum_{N(\mathfrak{s}^f)<N_1} \mu(\mathfrak{s}) \sum_{\substack{|z|^2<N \\ z\equiv 0\,(\mathfrak{s}^f)}} \phi(z) + \sum_{N_1\le N(\mathfrak{s}^f)<N} \mu(\mathfrak{s}) \sum_{\substack{|z|^2<N \\ z\equiv 0\,(\mathfrak{s}^f)}} \phi(z) \\ &= R_1 + R_2. \end{aligned}$$

For $|R_2|$ we obtain the estimate

$$|R_2| \le \sum_{N_1\le N(\mathfrak{s}^f)<N} \left(\frac{\pi N}{N(\mathfrak{s}^f)} + \mathcal{O}\left(\sqrt{\frac{N}{N(\mathfrak{s}^f)}}\right)\right) = \mathcal{O}\Bigg(\sum_{N_1\le N(\mathfrak{s}^f)<N} \frac{N}{N(\mathfrak{s}^f)}\Bigg).$$

Since there are $\mathcal{O}(r^\varepsilon)$ ($\varepsilon > 0$, arbitrary) elements k in $\mathbb{Z}[i]$ with $N(k) = r$ (cf. [17, Lemma 4.2]), we get

$$\sum_{N_1\le N(\mathfrak{s}^f)<N} \frac{N}{N(\mathfrak{s}^f)} < \sum_{n=N_1^{1/f}}^{N^{1/f}} \frac{N^{1+\varepsilon}}{n^f} \le N^{1+\varepsilon}\frac{1}{f-1}N_1^{1/f-1}$$

and, hence

$$|R_2| = \mathcal{O}\big(N^{1+\varepsilon}N_1^{1/f-1}\big).$$

Keeping in mind, that $N_1 = N^{(1-\lambda)/2}$, we get for ε small enough

$$|R_2| = \mathcal{O}(N^{\alpha_1}) \quad (\alpha_1 < 1).$$

In order to extract the main term, we apply (6) to R_1 to derive

$$R_1 = \sum_{N(\mathfrak{s})<N_1^{1/f}} \mu(\mathfrak{s})\Big(\frac{\pi N}{mN(\mathfrak{s}^f)} + \mathcal{O}(N^\lambda)\Big).$$

Since

$$\begin{aligned} \sum_{N(\mathfrak{s})<N_1^{1/f}} \mu(\mathfrak{s})\frac{\pi N}{mN(\mathfrak{s}^f)} &= \frac{\pi N}{m\zeta_{\mathbb{Q}(i)}(f)} + \mathcal{O}\left(\frac{N}{N_1^{1-1/f-\varepsilon}}\right) \\ &= \frac{\pi N}{m\zeta_{\mathbb{Q}(i)}(f)} + \mathcal{O}(N^{\alpha_2}) \quad (\alpha_2 < 1) \end{aligned}$$

and

$$\mathcal{O}\Bigg(\sum_{N(\mathfrak{s})<N_1^{1/f}} N^\lambda\Bigg) = \mathcal{O}(N^{(1+\lambda)/2}) = \mathcal{O}(N^{\alpha_3}) \quad (\alpha_3 < 1)$$

we have

$$R_1 = \frac{\pi N}{m\zeta_{\mathbb{Q}(i)}(f)} + \mathcal{O}(N^{\max(\alpha_2,\alpha_3)}).$$

Setting $\mu = \max(\alpha_1, \alpha_2, \alpha_3) < 1$ yields the result. □

4. An Erdős-Kac Type Theorem

Now we prove an Erdős-Kac type theorem for the complex sum of digits function. This extends (4) to number systems in $\mathbb{Z}[i]$. Throughout this section $\psi(x)$ denotes the normal law. Furthermore we use the notation $f(x) \ll g(x)$ for $f(x) = \mathcal{O}(g(x))$.

Theorem 4.1. *Suppose $m, r \in \mathbb{Z}$ fulfill (5) for a base b of a canonical number system in $\mathbb{Q}(i)$. Define the frequency*

$$Q_N(X) = \frac{1}{|U_{r,m}(N)|}\left|\{z \mid z \in U_{r,m}(N),\ \omega(z) - \log\log N \le X\sqrt{\log\log N}\}\right|,$$

where $\omega(z)$ counts the distinct prime factors of $z \in \mathbb{Z}[i]$. Then

$$\left|Q_N(X) - \psi(X)\right| \ll \frac{\log\log\log N}{\sqrt{\log\log N}}$$

holds uniformly for $X \in \mathbb{R}$ and $N \ge 8$.

Proof. We will construct a measure, with respect to that the additive function $\omega(z)$ behaves like a sum of independent random variables. To this matter we will apply the general Kubilius model defined in Section 2. In a second step we will use the Berry-Esseen Theorem (cf. [1, 4]) to show the convergence to the normal law.

Set $t = \exp\left(\frac{\log N}{\log\log\log N}\right)$ and let $\omega_1(z)$ count the distinct prime factors p of z, whose norms satisfy $N(b) \le N(p) \le t$. $L_N(X)$ shall denote the frequency emerging from $Q_N(X)$ if one replaces $\omega(z)$ by $\omega_1(z)$ (We will also use these functions with ideals as arguments. In this case they count the distinct prime ideal factors of their argument. Since $\mathbb{Z}[i]$ is a principal ideal domain we have $\omega(z) = \omega(z\mathbb{Z}[i])$ and $\omega_1(z) = \omega_1(z\mathbb{Z}[i])$ for any $z \in \mathbb{Z}[i]$). Now we want to apply the Kubilius model: assign the elements of $U_{r,m}(N)$ to the numbers a_j of the model and set $X = |U_{r,m}(N)|$ and $\eta(\mathfrak{q}) = N(\mathfrak{q})^{-1}$. For the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of the model we take all prime ideals $\mathfrak{p}$ with $N(b) \le N(\mathfrak{p}) \le t$. The function S of the model turns out to be

$$S = \sum_{N(b) \le N(\mathfrak{p}) \le t} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p}) - 1}.$$

Using (8) we get the estimate $S = \log t(1 + o(1))$. Hence, the conditions of the model allow τ to be any power of N. Our first goal is the application of Lemma 2.3. To this matter we have to show, that the sets $E_{\mathfrak{k}}$ are non-empty up to a certain value of $N(\mathfrak{k})$. We do this by using Lemma 2.2. Hence, we have to estimate the remainder terms $R(N,\mathfrak{k})$. It is easy to see, that in our case the remainder terms $R(N,\mathfrak{k})$ are given by

$$R(N,\mathfrak{k}) = \sum_{\substack{z\in U_{r,m}(N)\\ z\in\mathfrak{k}}} 1 - \frac{|U_{r,m}(N)|}{N(\mathfrak{k})}.$$

In order to estimate the $R(N,\mathfrak{k})$ we use Hua's result (11). Let $\mathfrak{d}$ be the different of $\mathbb{Z}[i]$. Then

$$R(N,\mathfrak{k}) = \frac{1}{N(\mathfrak{k})} \sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}} \sum_{z\in U_{r,m}} e(\operatorname{tr}(\xi z)).$$

The sum $\sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}}$ runs over a complete residue system in $(\mathfrak{d}\mathfrak{k})^{-1}$ modulo $\mathfrak{d}^{-1}$ not containing the element $\equiv 0\,(\mathfrak{d}^{-1})$. We will use this notation in the rest of this paper. Setting

$$H_l(\alpha,N) = \sum_{|z|^2<N} e\left(\operatorname{tr}(\alpha z) + \frac{l}{m}\nu_b(z)\right)$$

for $\alpha\in\mathbb{Q}(i)$ yields

$$\sum_{l=0}^{m-1} e\left(-\frac{lr}{m}\right) H_l(\alpha,N) = m \sum_{z\in U_{r,m}(N)} e(\operatorname{tr}(\alpha z)).$$

In the next part of our proof we need the following estimates (cf. [18, Lemmas 3.2 and 3.1]):

For $l=0$ we have

$$\sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}} H_0(\xi,N) = \sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}} \sum_{|z|^2<N} e(\operatorname{tr}(\xi z)) = \mathcal{O}(N(\mathfrak{k})\sqrt{N}). \tag{18}$$

Moreover, for any $\xi\in\mathbb{Q}(i)$ we have, for $1\le l\le m-1$, with $(m,p_b(1))=1$,

$$H_l(\xi,N) = \mathcal{O}(N^\lambda). \tag{19}$$

Now we get, following [15],

$$\begin{aligned} |R(N,\mathfrak{k})| &= \frac{1}{N(\mathfrak{k})}\left| \sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}} \frac{1}{m}\sum_{l=0}^{m-1} e\left(-\frac{lr}{m}\right) H_l(\xi,N)\right| \\ &\ll \frac{1}{N(\mathfrak{k})m}\left(\sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}} \left|H_0(\xi,N)\right| + \sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\backslash\mathfrak{d}^{-1}} \sum_{l=1}^{m-1} \left|H_l(\xi,N)\right| \right). \end{aligned}$$

Using the estimates (18) and (19) yields

$$\begin{aligned}|R(N,\mathfrak{k})| &\ll \frac{1}{N(\mathfrak{k})m}\Big(\gamma_1 N(\mathfrak{k})\sqrt{N} + \sum_{\xi\in(\mathfrak{d}\mathfrak{k})^{-1}\setminus\mathfrak{d}^{-1}} \gamma_2 N^{\lambda}\Big)\\ &\ll \Big(\gamma_1\sqrt{N}+\gamma_2 N^{\lambda}\Big) \ll N^{\max(\frac{1}{2},\lambda)}.\end{aligned}$$

Hence, with $\lambda' = \max(\frac{1}{2},\lambda)$ we have

$$R(N,\mathfrak{k}) = \mathcal{O}(N^{\lambda'}). \tag{20}$$

Set now $\delta = \frac{1-\lambda'}{10}$ and

$$\tau = N^{\delta}. \tag{21}$$

By (20) and (21) the conditions for the application of Lemma 2.2 are fulfilled. Since $X = |U_{r,m}(N)| = \mathcal{O}(N)$ by (6), we conclude, that $E_{\mathfrak{k}} \neq \emptyset$ for $N(\mathfrak{k}) \leq N^{1-\lambda'-7\delta}$. With that we have established the conditions for the application of Lemma 2.3. Define a collection of independent random variables by

$$W_{\mathfrak{p}} = \begin{cases} 1 & \text{with probability } N(\mathfrak{p})^{-1} \\ 0 & \text{with probability } 1 - N(\mathfrak{p})^{-1} \end{cases}.$$

Lemma 2.3 now yields

$$\begin{aligned}\left| L_N(X) - P\Bigg(\sum_{N(\mathfrak{b})\leq N(\mathfrak{p})\leq t} W_{\mathfrak{p}} \leq X\sqrt{\log\log N} + \log\log N\Bigg)\right| \\ \leq \mathcal{O}\left(\exp\left(-\beta'\frac{\log\tau}{\log t}\log\left(\frac{\log\tau}{S}\right)\right)\right) \\ + \mathcal{O}\left(\frac{1}{|U_{r,m}(N)|}\sideset{}{'}\sum_{N(\mathfrak{k})\leq\tau^4} 4^{\omega(\mathfrak{k})}|R(N,\mathfrak{k})|\right),\end{aligned} \tag{22}$$

where $\sum'$ indicates, that the sum runs over all square-free ideals $\mathfrak{k}$, whose prime factors $\mathfrak{p}$ have $N(\mathfrak{b}) \leq N(\mathfrak{p}) \leq t$.

The estimate (22) holds for all $t \geq 2$ and for $\max(\log t, S) \leq \frac{\log\tau}{8}$. Some calculations yield that the first error term in (22) is $\mathcal{O}((\log\log N)^{-d})$ for each $d \in \mathbb{N}$.

Now we estimate the second error term. Recall that there are $\mathcal{O}(r^{\varepsilon})$ ($\varepsilon > 0$, arbitrary) ideals in $\mathbb{Z}[i]$ whose norm is equal to r. Since $|R(N,\mathfrak{k})| = \mathcal{O}(N^{\lambda'})$ we get for $\varepsilon < \delta$

$$\sideset{}{'}\sum_{N(\mathfrak{k})<\tau^4} 4^{\omega(\mathfrak{k})}N^{\lambda'} \ll N^{9\delta+\lambda'} = o\left(\frac{|U_{r,m}(N)|}{\log\log N}\right).$$

The last step is a consequence of (6).

This implies, together with (22), that

$$(23)\quad \left|L_N(X) - P\Big(\sum_{N(\mathfrak{b})\le N(\mathfrak{p})\le t} W_{\mathfrak{p}} \le X\sqrt{\log\log N} + \log\log N\Big)\right| = o\left((\log\log N)^{-1}\right).$$

Now we turn to the second step of our proof. In order to apply the Berry-Esseen Theorem on the convergence of a sum of independent random variables to the normal law (cf. [1, 4]), we define the random variables $Z_{\mathfrak{p}}$ by

$$Z_{\mathfrak{p}} := (W_{\mathfrak{p}} - N(\mathfrak{p})^{-1}).$$

Let $\sigma^2 = \sum_{N(\mathfrak{b})\le N(\mathfrak{p})\le t} N(\mathfrak{p})^{-1}(1 - N(\mathfrak{p})^{-1})$. Then, by (9) we have

$$\sigma^2 = \log\log N + \mathcal{O}(\log\log\log N).$$

The Berry-Esseen Theorem now yields, again using (9),

$$\left|\psi(X) - P\Big(\sigma^{-1}\sum_{N(\mathfrak{b})\le N(\mathfrak{p})\le t} Z_{\mathfrak{p}} \le X\Big)\right| \ll \sigma^{-3}\sum_{N(\mathfrak{b})\le N(\mathfrak{p})\le t} E(|Z_{\mathfrak{p}}|^3) \ll (\log\log N)^{-\frac{1}{2}}.$$

For the random variables $W_{\mathfrak{p}}$ we derive

$$P\Big(\sum_{N(\mathfrak{b})\le N(\mathfrak{p})\le t} W_{\mathfrak{p}} \le X\sqrt{\log\log N} + \log\log N\Big) = \psi\left(\frac{1}{\sigma}\Big(X\sqrt{\log\log N} + \log\log N - \sum_{N(\mathfrak{b})\le N(\mathfrak{p})\le t} N(\mathfrak{p})^{-1}\Big)\right) + \mathcal{O}\left(\frac{1}{\sqrt{\log\log N}}\right).$$

Together with (23) this yields

$$\left|L_N(X) - \psi(X)\right| \ll \frac{1}{\sqrt{\log\log N}}.$$

Since a number z with $|z|^2 < N$ has at most $\mathcal{O}(\log\log\log N)$ prime factors p with norms not contained in the interval $[N(\mathfrak{b}), t]$, we can replace $L_N(X)$ by $Q_N(X)$ and derive $\left|Q_N(X) - \psi(X)\right| \ll \frac{\log\log\log N}{\sqrt{\log\log N}}$. □