## PaUla B. Cohen

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Journal de Théorie des Nombres de Bordeaux, tome 11, n 1 (1999), p. 15-30<br>[http://www.numdam.org/item?id=JTNB_1999_11_1_15_0](http://www.numdam.org/item?id=JTNB_1999_11_1_15_0)

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# A $C^{*}$-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking 

par Paula B. COHEN


#### Abstract

Résumé. Dans cet article nous étendons une construction de Bost-Connes, au cas d'un corps de nombres quelconque, d'un $C^{*}$ système dynamique à brisure spontanée de symétrie et fonction de partition la fonction zeta de Riemann.


Abstract. In this paper we extend to arbitrary number fields a construction of Bost-Connes of a $C^{*}$-dynamical sytem with spontaneous symmetry breaking and partition function the Riemann zeta function.

## 1. Introduction

In [BC], J-B. Bost and A. Connes, motivated most notably by work of B. Julia (see for example [J]), develop the idea that by displaying the Riemann zeta function as the partition function of a dynamical system with spontaneous symmetry breaking at the pole of the zeta function, one can gain insight into the statistics of the primes of the field of rational numbers using the tools of quantum statistical mechanics. Their construction of such a dynamical system as a 1-parameter automorphism group on an appropriate Hecke algebra has done a lot to enrich the dictionary between concepts from number theory and concepts from quantum statistical mechanics. Moreover, it has been a motivation and guide for the considerations of the proposed approach to the Riemann Hypothesis in [C].

A generalization of the work of $[\mathrm{BC}]$ to the case of arbitrary global fields was proposed in [HaLe]. In the number field case, a Hecke algebra construction using semi-group crossed products was proposed in [ALR], see also [LR1] and [LR2]. A more general study developed in [L] applies in particular to the dynamics on this algebra for the class number 1 case. In the present article, which owes a great deal to both approaches, we construct a different generalization for number fields of the dynamical system of [BC] having the full Dedekind zeta function of the number field as partition function. In [HaLe] and [L] this is only achieved when the number
field has class number 1. For class number greater than 1 , the construction of [HaLe] is not canonical and the partition function recovered is the Dedekind zeta function with a finite number of Euler factors removed. The advantage of our treatment comes from viewing, by contrast to these other approaches, the ideals rather than just the principal ideals as playing the same role as the positive integers do in [BC].

The dynamical system we construct has a natural symmetry group which displays the phenomenon of spontaneous symmetry breaking at the pole of the Dedekind zeta function. In mechanical terms, this means that for inverse temperature $\beta$ less than 1 the temperature is high enough to create disorder in the system, so that the equilibrium state is unique and invariant under the action of the symmetry group. At the critical temperature $\beta=1$ a phase transition occurs, so that for $\beta>1$, when the temperature is low enough, the particles of the system start to align and the symmetry is broken. The equilibrium states are then no longer unique and the symmetry group acts on the extremal points of the compact convex space of equilibrium states. These equilibrium states are, in the $C^{*}$-algebraic formulation, the $\mathrm{KMS}_{\beta}$ states. In $[\mathrm{BC}], \S 1$, an overview of the $C^{*}$-algebraic approach to quantum statistical mechanics is given, including the definition of $\mathrm{KMS}_{\beta}$ states.

I would like to thank M. Laca and I. Raeburn of Newcastle University, Australia, for their hospitality in August 1996 and August 1997 and for introducing me to their work with J. Arledge and to the developments subsequently pursued by M. Laca. I also thank D. Harari and E. Leichtnam for their interest in my ideas.

## 2. Statement of the Theorem

We begin by introducing some notations and conventions. Let $K$ be a number field of degree $d$ over $\mathbb{Q}$ and with ring of integers $\mathcal{O}$. Let $M_{K}$ be the set of places of $K$ and $M_{K}^{o}$ the subset of finite places. For $v \in M_{K}$ we choose a valuation $\left|\left.\right|_{v}\right.$ normalised as follows: let $K_{v}$ be the completion of $K$ at $v$ and $d_{v}=\left[K_{v}: \mathbb{Q}_{v}\right]$ be the local degree. Then for $x \in K_{v}$ we set

$$
|x|_{v}=\|x\|_{v}^{\frac{d_{v}}{d}}
$$

where $\left\|\|_{v}\right.$ is the unique valuation on $K_{v}$ extending the usual $p$-adic or archimedean valuation on $\mathbb{Q}_{v}$. In particular, for this normalisation we have for all $x \in K, x \neq 0$ the product formula

$$
\prod_{v \in M_{K}}|x|_{v}=1
$$

For $v \in M_{K}^{o}$, let

$$
\mathcal{O}_{v}=\left\{x \in K_{v}:|x|_{v} \leq 1\right\}
$$

be the ring of integers of $K_{v}$ with unit subgroup

$$
\mathcal{O}_{v}^{*}=\left\{x \in K_{v}:|x|_{v}=1\right\}
$$

For most of this article we shall work with the finite adèles $A$, that is the restricted product of the $K_{v}$ with respect to the $\mathcal{O}_{v}, v \in M_{K}^{o}$. An element $a \in A$ is therefore an infinite vector $a=\left(a_{v}\right)_{v \in M_{K}^{o}}$ indexed by the set $M_{K}^{o}$ with $a_{v} \in K_{v}$ and $a_{v} \in \mathcal{O}_{v}$ for all but finitely many $v \in$ $M_{K}^{o}$. The finite adèles form a ring with respect to component-wise addition and multiplication. We have a natural embedding $K \hookrightarrow A$, whose image is called the principal adèles, which is just the diagonal embedding $x \mapsto$ $(x)_{v \in M_{K}^{o}}$ induced by the embeddings of $K$ into $K_{v}$ for $v \in M_{K}^{o}$. The image under this embedding of an element $a \in \mathcal{O}$ is an element of $R=\prod_{v \in M_{K}^{o}} \mathcal{O}_{v}$, the maximal compact subring of $A$. Let $W=\prod_{v \in M_{K}^{o}} \mathcal{O}_{v}^{*}$ be the group of units of $R$. Let $J$ be the group of finite idèles, that is the group of invertible elements of $A$ consisting of the restricted product of $K_{v}^{*}=K_{v} \backslash\{0\}$ with respect to $\mathcal{O}_{v}^{*}, v \in M_{K}^{o}$. An element $j \in J$ is therefore an infinite vector $j=\left(j_{v}\right)_{v \in M_{K}^{\circ}}$ indexed by the set $M_{K}^{o}$ with $j_{v} \in K_{v}^{*}$ and $j_{v} \in \mathcal{O}_{v}^{*}$ for all but finitely many $v \in M_{K}^{o}$. The module on $J$ is defined by

$$
\begin{gathered}
\left|\mid: J \rightarrow \mathbb{R}_{+}^{*}\right. \\
\left|\left|: j=\left(j_{v}\right)_{v} \mapsto \prod_{v \in M_{K}^{o}}\right| j_{v}\right|_{v}
\end{gathered}
$$

The image under the diagonal embedding of $K^{*}$ into $J$ is called the principal idèle group. The semigroup $I=J \cap R$ satisfies $I^{-1} I=J$. For $v \in M_{K}^{o}$, let $\mathfrak{P}_{v}$ be the prime ideal associated with $v$,

$$
\mathfrak{P}_{v}=\left\{a \in \mathcal{O} ;|a|_{v}<1\right\} .
$$

The quotient $\mathcal{O} / \mathfrak{P}_{v}$ has finite cardinality $N\left(\mathfrak{P}_{v}\right)$ and for any $a_{v} \in \mathcal{O}_{v}$, there is a unique integer $\operatorname{ord}_{v} a_{v}$ such that $\operatorname{Card}\left(\mathcal{O}_{v} / a_{v} \mathcal{O}_{v}\right)=N\left(\mathfrak{P}_{v}\right)^{\operatorname{ord}_{v} a_{v}}$. For $a \in I$ we denote by $\mathfrak{A}$ the ideal

$$
\mathfrak{A}=\prod_{v \in M_{K}^{o}} \mathfrak{P}_{v}^{\operatorname{ord}_{v} a_{v}}
$$

and call it the ideal associated to $a$. It is well-defined as $\operatorname{ord}_{v} a_{v}=0$ for almost all $v \in M_{K}^{o}$, and

$$
N(\mathfrak{A})=\operatorname{Card}(\mathcal{O} / \mathfrak{A})=\prod_{v \in M_{K}^{o}} N\left(\mathfrak{P}_{v}\right)^{\operatorname{ord}_{v} a_{v}}
$$

Moreover, we have in this way a short exact sequence, with arrows semigroup homomorphisms,

$$
1 \rightarrow W \rightarrow I \rightarrow \mathcal{I} \rightarrow 1
$$

where $\mathcal{I}$ denotes the semigroup of integral ideals of $\mathcal{O}$. By the Strong Approximation Theorem, there are additive isomorphisms between $A / R \simeq$ $\oplus_{v \in M_{K}^{o}} K_{v} / \mathcal{O}_{v}$ and $K / \mathcal{O}$. The semigroup $I$ acts on $A$ by multiplication and preserves $R$, so that multiplication by $a \in I$ induces an endomorphism of $A / R$

$$
x \mapsto a \cdot x, \quad x \in A / R .
$$

If $a \in I$ and $y \in A / R$ the equation $a \cdot x=y$ has $N(\mathfrak{A})$ solutions in $A / R$ where $\mathfrak{A}$ is the ideal associated to $a$. We denote the set of these solutions by $[x: a \cdot x=y]$. Let $\mathbb{C}(A / R)=: \operatorname{span}\left\{\delta_{x}: x \in A / R\right\}$. The formula

$$
\alpha_{a}\left(\delta_{y}\right)=\frac{1}{N(\mathfrak{A})} \sum_{[x: a \cdot x=y]} \delta_{x}
$$

where $a \in I$ and $\mathfrak{A} \in \mathcal{I}$ is the associated ideal, defines an action of $I$ by endomorphisms of the associated $C^{*}$-algebra $C^{*}(A / R)$.

Let $+: \mathcal{I} \rightarrow I$ with $+: j \mapsto j^{+}$denote any splitting semigroup homomorphism of the above exact sequence such that $\mathcal{O}^{+}=(1,1, \ldots)$, the identity in $I$, and such that for any principal prime ideal $\pi \mathcal{O}$ with generator $\pi$ (so that we could replace $\pi$ by $u \pi$ for any unit $u$ of $\mathcal{O}$ ) we have $(\pi \mathcal{O})^{+}=(\pi, \pi, \ldots)$, the image of the natural embedding of $\pi$ as a principal idèle. This condition is essential to ensure that there is sufficient interaction between the different primes to exhibit the phenomenon of spontaneous symmetry breaking. We call such a splitting of the short exact sequence an interactive splitting. Let $I_{+}$be the sub-semigroup of $I$ given by the image of $\mathcal{I}$ under a fixed map + . We take as our basic algebra the crossed product associated to the triple $\left(C^{*}(A / R), I_{+}, \alpha\right)$ in the sense of [ALR], see also $\S 4$ of the present paper. This is the universal object for covariant representations of this triple, namely pairs $(\pi, V)$ where $\pi$ is a unital representation of $C^{*}(A / R)$ on a Hilbert space $\mathcal{H}$ and $V$ is an isometric representation of $I$ on $\mathcal{H}$ satisfying

$$
\pi\left(\alpha_{a}(f)\right)=V_{a} \pi(f) V_{a}^{*}, \quad a \in I, f \in C^{*}(A / R)
$$

We denote this semigroup crossed product by $C_{K}=C^{*}(A / R) \rtimes_{\alpha} I_{+}$. It is the universal $C^{*}$-algebra generated by $\{e(x): x \in A / R\}$ and $\left\{\mu_{a}: a \in I_{+}\right\}$ subject to the relations

$$
\begin{aligned}
\mu_{a}^{*} \mu_{a} & =1, \quad a \in I_{+} \\
\mu_{a} \mu_{b} & =\mu_{a b}, \quad a, b \in I_{+} \\
e(0) & =1, e(x)^{*}=e(-x), e(x) e(y)=e(x+y), \quad x, y \in A / R \\
\frac{1}{N(\mathfrak{A})} & \sum_{[x: a \cdot x=y]} e(x)=\mu_{a} e(y) \mu_{a}^{*}, \quad a \in I_{+}, y \in A / R
\end{aligned}
$$

where $\mathfrak{A}$ is the ideal associated to $a$.

Let $\left\{\sigma_{t} ; t \in \mathbb{R}\right\}$ be the 1-parameter automorphism group of $C_{K}$ given by the following action on the symbols $e(x), x \in A / R$ and $\mu_{a}, a \in I_{+}$,

$$
\sigma_{t}(e(x))=e(x), \quad \sigma_{t}\left(\mu_{a}\right)=N(\mathfrak{A})^{i t} \mu_{a}, \quad x \in A / R, a \in I_{+}, t \in \mathbb{R}
$$

where $\mathfrak{A}$ is the ideal associated to $a$. Consider the Hilbert space $l^{2}\left(I_{+}\right)$ and let $\left(\varepsilon_{a}\right)_{a \in I_{+}}$be the standard orthonormal basis. Define an unbounded positive operator $H$ on $l^{2}\left(I_{+}\right)$by

$$
H \varepsilon_{a}=\log (N(\mathfrak{A})) \varepsilon_{a}, \quad a \in I_{+}
$$

Notice that $H \varepsilon_{1}=0$ where $1=(1)_{v \in M_{K}^{o}}$. Consider, for a fixed admissible character $\chi$ (see §5) on $A / R$ and any $u \in W$, the involutive representation $\rho_{u}$ of $C_{K}$ on $l^{2}\left(I_{+}\right)$which is the unique extension of the representation defined by

$$
\begin{gathered}
\rho_{u}\left(\mu_{a}\right) \varepsilon_{b}=\varepsilon_{a b}, \quad a, b \in I_{+} \\
\rho_{u}(e(y)) \varepsilon_{b}=\chi((u b) \cdot y) \varepsilon_{b}, \quad b \in I_{+}, \quad y \in A / R
\end{gathered}
$$

We have

$$
\rho_{u}\left(\sigma_{t}(x)\right)=e^{i t H} \rho_{u}(x) e^{-i t H}, \quad x \in C_{K}
$$

The main result of this paper is the following generalization of Theorem 5 of [BC] and Théorème 0.1 of [HaLe], see also Proposition 46 of [L]. Its proof follows closely the treatments of [BC] and [HaLe], although differences do arise and we content ourselves in $\S 5$ with an explanation of how to handle them, leaving the remaining details to the reader.
Theorem. Let $K$ be a number field. The $C^{*}$-dynamical system $\left(C_{K}, \sigma_{t}\right)$ has symmetry group $W$, with the action $[u] \in \operatorname{Aut}\left(C_{K}\right)$ of $u \in W$ given on $e(x), x \in A / R$ and $\mu_{a}, a \in I_{+} b y$

$$
[u]: e(x) \mapsto e(u \cdot x), x \in A / R \quad, \quad[u]: \mu_{a} \mapsto \mu_{a}, a \in I_{+}
$$

This action commutes with $\sigma$, so that $[u] \circ \sigma_{t}=\sigma_{t} \circ[u]$ for $u \in W, t \in \mathbb{R}$. Moreover,
(1) for $0<\beta \leq 1$, there is a unique $K M S_{\beta}$ state $\Phi_{\beta}$. It is a factorial state of Type $I I I_{1}$ and the associated factor is the Araki-Woods factor $R_{\infty}$.
(2) for $\beta>1$ and $u \in W$, the state

$$
\Phi_{\beta, u}(x)=\zeta_{K}(\beta)^{-1} \operatorname{Trace}\left(\rho_{u}(x) e^{-\beta H}\right), \quad x \in C_{K}
$$

is a $K M S_{\beta}$ state on $\left(C_{K}, \sigma_{t}\right)$ which is factorial of Type $I_{\infty}$ where

$$
\zeta_{K}(\beta)=\sum_{\mathfrak{A} \in \mathcal{I}} \frac{1}{N(\mathfrak{A})^{\beta}}=\operatorname{Trace}\left(e^{-\beta H}\right)
$$

is the Dedekind zeta function (at $\beta$ ) of the number field $K$. The action of $W$ on $C_{K}$ induces an action on these $K M S_{\beta}$ states which permutes them transitively and the map $u \mapsto \Phi_{\beta, u}$ is a homeomorphism of the compact group $W$ into the space $\mathcal{E}\left(K_{\beta}\right)$ of extreme points of the convex compact Choquet simplex $K_{\beta}$ of $K M S_{\beta}$ states on $\left(C_{K}, \sigma_{t}\right)$.
(3) the Dedekind zeta function $\zeta_{K}$ of the number field $K$ is the partition function of $\left(C_{K}, \sigma_{t}\right)$.

## 3. A SYSTEM WITHOUT INTERACTION

Generalising $\S 2$ of [BC], we can construct a non-interactive system, which will be useful in the sequel, as follows. Let $\mathcal{P}$ be the set of prime ideals of $\mathcal{O}$ and S be the second quantisation functor (as in [BC], p416).

Proposition 1. (a) For every prime ideal $\mathfrak{P}$, let $\mu_{\mathfrak{P}}$ be the isometry in $\mathrm{Sl}^{2}(\mathcal{P})=l^{2}(\mathcal{I})$ given by the polar decomposition of the creation operator associated to the unit vector $\varepsilon_{\mathfrak{P}} \in \operatorname{Sl}^{2}(\mathcal{P})$, where $\left\{\varepsilon_{\mathfrak{B}}, \mathfrak{B} \in \mathcal{I}\right\}$ denotes the standard orthonormal basis of $l^{2}(\mathcal{I})$. The $C^{*}$-algebra $C^{*}(\mathcal{I})$ generated by the $\mu_{\mathfrak{P}}$ with $\mathfrak{P}$ prime is the same as that generated by the isometries $\mu_{\mathfrak{A}}, \mathfrak{A} \in \mathcal{I}$ defined by,

$$
\mu_{\mathfrak{A}} \varepsilon_{\mathfrak{B}}=\varepsilon_{\mathfrak{A} \mathfrak{B}}, \quad \mathfrak{B} \in \mathcal{I}
$$

The $C^{*}$-algebra $C^{*}\left(I_{+}\right)$generated by the $\mu_{a}, a \in I_{+}$is isomorphic to $C^{*}(\mathcal{I})$.
(b) Let $\tau_{\mathfrak{P}}$ be the Toeplitz $C^{*}$-algebra generated by $\mu_{\mathfrak{P}}$. Then $C^{*}(\mathcal{I})$ is the infinite tensor product

$$
C^{*}(\mathcal{I})=\otimes_{\mathfrak{P} \in \mathcal{P}} \tau_{\mathfrak{P}}
$$

(c) Let $H$ be the operator in $l^{2}(\mathcal{I})$ given by

$$
H \varepsilon_{\mathfrak{B}}=\log (N(\mathfrak{B})) \varepsilon_{\mathfrak{B}}, \quad \mathfrak{B} \in \mathcal{I}
$$

then the equality

$$
\sigma_{t}(x)=e^{i t H} x e^{-i t H}, \quad x \in C^{*}(\mathcal{I})
$$

defines a 1-parameter group of automorphisms of $C^{*}(\mathcal{I})$ which may be factorised as

$$
\sigma_{t}=\otimes_{\mathfrak{P} \in \mathcal{P}} \sigma_{t, \mathfrak{P}}
$$

where

$$
\sigma_{t, \mathfrak{P}}\left(\mu_{\mathfrak{P}}\right)=N(\mathfrak{P})^{i t} \mu_{\mathfrak{P}} .
$$

Proof. By analogy with the proof of Proposition 7 of [BC].
Notice that the dynamical system $\sigma_{t}$, once transported to $C^{*}\left(I_{+}\right)$, is the restriction to that algebra of the dynamical system $\sigma_{t}$ on $C_{K}$ defined in $\S 2$.

Similar arguments to that given in [BC], Proposition 8 for the case $K=$ $\mathbb{Q}$ give the following result which underlines the fact that, as the above system is an infinite tensor product of non-interacting systems, it displays no phenomenon of spontaneous symmetry breaking at a critical temperature.

Proposition 2. (a) For every $\beta>0$, there is a unique $K M S_{\beta}$ state on $\left(C^{*}(\mathcal{I}), \sigma_{t}\right)$. It is the infinite tensor product

$$
\Phi_{\beta}=\otimes_{\mathfrak{P} \in \mathcal{P}} \Phi_{\beta, \mathfrak{P}}
$$

where $\Phi_{\beta, \mathfrak{P}}$ is the unique $K M S_{\beta}$ state on $\left(\tau_{\mathfrak{P}}, \sigma_{t, \mathfrak{P}}\right)$. The eigenvalue list of $\Phi_{\beta, \mathfrak{P}}$ is

$$
\left\{\left(1-N(\mathfrak{P})^{-\beta}\right) N(\mathfrak{P})^{-n \beta}: n \in \mathbb{N}\right\}
$$

(b) For $\beta>1$, the state $\Phi_{\beta}$ is of Type $I_{\infty}$ and is given by

$$
\Phi_{\beta}(x)=\zeta_{K}(\beta)^{-1} \operatorname{Trace}\left(x e^{-\beta H}\right), \quad x \in C^{*}(\mathcal{I})
$$

(c) For $0<\beta \leq 1$ the state $\Phi_{\beta}$ is of Type $I I_{1}$ and the associated factor is the Araki-Woods factor $R_{\infty}$.

## 4. SEmigroup CROSSED PRODUCTS AND A SYSTEM WITH Interaction

In this section, we recall some basic facts on semigroup crossed product $C^{*}$-algebras needed for the construction of $C_{K}$. We use as references the articles [LR1], [LR2] and [ALR] (see [LR1] and [L] for some historical background including the relation to work of Nica [Ni]). Our semigroups will all be abelian without zero divisors. A semigroup system is a triple $(\mathcal{A}, S, \alpha)$ consisting of a separable unital $C^{*}$-algebra $\mathcal{A}$, a semigroup $S$ and an action $\alpha$ of $S$ by endomorphisms on $\mathcal{A}$. These endomorphisms need not be unital. Define a covariant representation of $(\mathcal{A}, S, \alpha)$ to be a pair $(\pi, V)$ consisting of a unital representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an isometric representation $V$ of $S$ on $\mathcal{H}$ such that

$$
\pi\left(\alpha_{a}(x)\right)=V_{a} \pi(x) V_{a}^{*}, \quad a \in S, x \in \mathcal{A}
$$

Lemma 1. Suppose $(\mathcal{A}, S, \alpha)$ is a semigroup system which has a non-trivial covariant representation. Then there is a triple $\left(\mathcal{B}, \iota_{\mathcal{A}}, \iota_{S}\right)$ consisting of a $C^{*}$-algebra $\mathcal{B}$, a unital homomorphism $\iota_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{B}$ and a semigroup homomorphism $\iota_{S}$ of $S$ into the isometries of $\mathcal{B}$ such that
(1) $\iota_{\mathcal{A}}\left(\alpha_{a}(x)\right)=\iota_{S}(a) \iota_{\mathcal{A}}(x) \iota_{S}(a)^{*}$ for $a \in S$ and $x \in \mathcal{A}$,
(2) for every covariant representation $(\pi, V)$ of $(\mathcal{A}, S, \alpha)$ there is a unital representation $\pi \times V$ of $\mathcal{B}$ with $(\pi \times V) \circ \iota_{\mathcal{A}}=\pi$ and $(\pi \times V) \circ \iota_{S}=V$,
(3) the $C^{*}$-algebra $\mathcal{B}$ is generated by $\left\{\iota_{\mathcal{A}}(x): x \in \mathcal{A}\right\} \cup\left\{\iota_{S}(a): a \in S\right\}$. The triple $\left(\mathcal{B}, \iota_{\mathcal{A}}, \iota_{S}\right)$ is unique up to isomorphism.

Proof. This is a direct application of Proposition 2.1 of [LR1].
We define the crossed product of $\mathcal{A}$ by $S$ to be the unital $C^{*}$-algebra $\mathcal{B}$ together with the pair $\left(\iota_{\mathcal{A}}, \iota_{S}\right)$. We denote this crossed product by $\mathcal{A} \rtimes_{\alpha} S$, or by $\mathcal{A} \rtimes S$ when $\alpha$ is understood.

Returning to the situation of $\S 2$, the semigroup $I=J \cap R$ acts on $C^{*}(A / R)$ via $C^{*}$-endomorphisms defined by

$$
\gamma_{a}: \delta_{x} \mapsto \delta_{a x}, \quad a \in I, x \in A / R
$$

Right inverses for these endomorphisms are given by the action $\alpha$ of the semigroup $I$ on $C^{*}(A / R)$ defined by

$$
\alpha_{a}\left(\delta_{y}\right)=\frac{1}{N(\mathfrak{A})} \sum_{[x: a \cdot x=y]} \delta_{x}, \quad a \in I, y \in A / R
$$

where $\mathfrak{A}$ is the ideal associated to $a$. The $\alpha_{a}, a \in I$ are indeed $C^{*}$ endomorphisms of $C^{*}(A / R)$, as one checks by a straightforward computation.

A non-trivial covariant representation of $\left(C^{*}(A / R), I, \alpha\right)$ on $l^{2}(A / R)$ is given by $(\lambda, L)$, with $\lambda$ the left regular representation of $C^{*}(A / R)$ on $l^{2}(A / R)$ and

$$
L_{a} \varepsilon_{y}=\frac{1}{\sqrt{N(\mathfrak{A})}} \sum_{[x: a \cdot x=y]} \varepsilon_{x}, \quad a \in I, y \in A / R
$$

where $\mathfrak{A}$ is the ideal associated to $a$. Here $\left\{\varepsilon_{y} ; y \in A / R\right\}$ is the usual orthonormal basis of $l^{2}(A / R)$. Hence the system $\left(C^{*}(A / R), I, \alpha\right)$ has a crossed product $C^{*}(A / R) \rtimes_{\alpha} I$.

The $C^{*}$-algebra $C^{*}(K / \mathcal{O}) \rtimes \mathcal{O}^{\times}$of [ALR], $\S 1$, where $\mathcal{O}^{\times}$is the semigroup of non-zero integers of $K$, is obtained by embedding $\mathcal{O}^{\times}$in $I$ diagonally by $a \mapsto(a)_{v \in M_{K}^{o}}, a \in \mathcal{O}^{\times}$and considering the restriction of $\alpha$ to an action of $\mathcal{O}^{\times}$.

Let $\mathcal{T}$ be any sub-semigroup of $I$ and let $C^{*}(A / R) \rtimes_{\alpha} \mathcal{T}$ be the semigroup crossed product obtained by restricting $\alpha$ to an action of $\mathcal{T}$. We have the following generalization of Proposition 2.1 of [ALR] which is a reformulation of the universal property of $C^{*}(A / R) \rtimes_{\alpha} \mathcal{T}$ in terms of generators and relations.

Lemma 2. The semigroup crossed product $C^{*}(A / R) \rtimes_{\alpha} \mathcal{T}$ is the universal $C^{*}$-algebra generated by elements $\left\{\mu_{a}: a \in \mathcal{T}\right\}$ and $\{e(x) ; x \in A / R\}$ subject to the relations

$$
\begin{aligned}
\mu_{a}^{*} \mu_{a} & =1, \quad a \in \mathcal{T} \\
\mu_{a} \mu_{b} & =\mu_{a b}, \quad a, b \in \mathcal{T} \\
e(0) & =1, e(x)^{*}=e(-x), e(x) e(y)=e(x+y), \quad x, y \in A / R \\
\frac{1}{N(\mathfrak{A})} & \sum_{[x: a \cdot x=y]} e(x)=\mu_{a} e(y) \mu_{a}^{*}, \quad a \in \mathcal{T}, y \in A / R
\end{aligned}
$$

where $\mathfrak{A}$ is the ideal associated to $a$.

Now consider, as in §2, the semigroup crossed product obtained by restricting $\alpha$ to an action of the image $I_{+} \subset I$ of a fixed interactive splitting of the short exact sequence

$$
1 \rightarrow W \rightarrow I \rightarrow \mathcal{I} \rightarrow 1
$$

We form the corresponding semigroup crossed product $C_{K}=C^{*}(A / R) \rtimes_{\alpha}$ $I_{+}$. The presentation of this algebra as the universal $C^{*}$-algebra generated by elements $\left\{\mu_{a}: a \in I_{+}\right\}$and $\{e(x) ; x \in A / R\}$ subject to the relations of Lemma 2, with $\mathcal{T}=I_{+}$, was given in $\S 2$. One can deduce from these relations two further ones, given by

$$
\begin{aligned}
\mu_{a} \mu_{b}^{*} & =\mu_{b}^{*} \mu_{a}, \quad a, b \in I_{+}, \quad \mathfrak{A}+\mathfrak{B}=\mathcal{O} \\
e(x) \mu_{a} & =\mu_{a} e(a \cdot x), \quad a \in I_{+}, \quad x \in A / R
\end{aligned}
$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are the ideals associated to $a$ and $b$. The analogous observation for $K=\mathbb{Q}$ was made in [ALR]. Compare with Proposition 18 of [BC]. Moreover, arguments similar to those of [BC], p. 433, show that the universal involutive algebra $\mathcal{C}$ generated by the $\left\{\mu_{a}: a \in I_{+}\right\}$and $\{e(x) ; x \in A / R\}$ subject to the relations of Lemma 2 , with $\mathcal{T}=I_{+}$, is a dense sub-algebra of $C_{K}$ spanned linearly by the (independent) monomials of the form $\mu_{a} e(x) \mu_{b}^{*}$ where $a, b \in I_{+}$have coprime associated ideals and $x \in A / R$.

It is now easy to see how the action of the symmetry group $W$ on $C_{K}$ arises. The group $W$ acts by outer automorphisms on $C_{K}$. To compute this action, consider $C_{K}$ as a subalgebra of $C^{*}(A / R) \rtimes_{\alpha} I$ and let $u \in W$ act by $[u] \in \operatorname{Aut}\left(C_{K}\right)$ where

$$
[u](x)=\mu_{u}^{*} x \mu_{u}, \quad x \in C_{K} .
$$

The right hand side is computed within the algebra $C^{*}(A / R) \rtimes_{\alpha} I$. That this is the action of $W$ on $C_{K}$ described in the Theorem is immediate.

Lemma 3. The fixed point algebra $C_{K}^{W}$ of the action of $W$ on $C_{K}$ is the $C^{*}$-algebra $C^{*}\left(I_{+}\right)$generated by the $\mu_{a}, a \in I_{+}$.

Proof. One adapts easily the proof of [BC], Proposition 21 (b).
Let $J_{+}=I_{+}^{-1} I_{+}$. The $C^{*}$-algebra $C_{K}$ is isomorphic to $C_{r}^{*}\left(P_{A}^{+}, P_{R}^{+}\right)$in the sense of $[\mathrm{BC}]$, the $C^{*}$-Hecke algebra associated to the almost normal inclusion $P_{R}^{+} \subset P_{A}^{+}$where

$$
P_{R}^{+}=\left(\begin{array}{cc}
1 & R \\
0 & 1
\end{array}\right), \quad P_{A}^{+}=\left(\begin{array}{cc}
1 & A \\
0 & J_{+}
\end{array}\right)
$$

The dense involutive Hecke algebra associated to this inclusion as defined in $[\mathrm{BC}], \S 1$, is isomorphic to the algebra $\mathcal{C}$ introduced above.

We can use the identification of $C_{K}$ as a $C^{*}$-Hecke algebra and define the dynamical system in the same way as in [BC], Proposition 4. This gives the same dynamical system $\left(C_{K}, \sigma_{t}\right)$ as that of $\S 2$. It is the 1-parameter group $\left\{\sigma_{t}: t \in \mathbb{R}\right\}$ of automorphisms of $C_{K}$ determined by their values on the spanning monomials,

$$
\sigma_{t}\left(\mu_{a} e(x) \mu_{b}^{*}\right)=N(\mathfrak{A})^{i t} N(\mathfrak{B})^{-i t} \mu_{a} e(x) \mu_{b}^{*}
$$

for $a, b \in I_{+}$with associated coprime ideals $\mathfrak{A}, \mathfrak{B}$ and for $x \in A / R$. Compare with [L], Proposition 7.

Notice that the algebra $C^{*}(A / R) \rtimes_{\alpha} I$ is not the one giving rise to interaction. It is a restricted product over the primes. It is the splitting $+: \mathcal{I} \rightarrow I$ with the appropriate properties which gives rise to interaction, as we shall see in $\S 5$.

## 5. Outline of the proof of the Theorem

It remains to comment on parts (1) and (2) of the Theorem. Let us suppose first that $\beta>1$. Recall from [HaLe], $\S 5$ the following facts about characters of $A / R$. A character $\chi_{v}$ of $K_{v} / \mathcal{O}_{v}, v \in M_{K}^{o}$ is said to be admissible if it is non-trivial on $\pi_{v}^{-1} \mathcal{O}_{v} / \mathcal{O}_{v}$, where $\pi_{v}$ is a local uniformiser at $v$. An admissible character always exists for all $v \in M_{K}^{o}$. A character $\chi$ of $A / R$ is said to be admissible if there exists for all $v \in M_{K}^{o}$ an admissible character $\chi_{v}$ such that, for all $y=\left(y_{v}\right) \in A / R$,

$$
\chi(y)=\prod_{v \in M_{K}^{o}} \chi_{v}\left(y_{v}\right)
$$

The group of characters of $A / R$ is isomorphic to $R$. Indeed, on fixing an admissible character $\chi$ this isomorphism is given by

$$
y \mapsto \chi(y \cdot), \quad y \in R
$$

and the application

$$
u \mapsto \chi(u \cdot), \quad u \in W
$$

of $W$ into the characters of $A / R$ is injective.
One verifies easily and in a similar way to Proposition 23 of [BC] that the maps $\rho_{u}$ of $C_{K}$ on $l^{2}\left(I_{+}\right)$given in $\S 2$ are indeed representations and that the operator $H$ implements the dynamical system $\sigma_{t}$,

$$
\rho_{u}\left(\sigma_{t}(x)\right)=e^{i t H} \rho_{u}(x) e^{-i t H}, \quad x \in C_{K}
$$

It is clear that the states $\Phi_{\beta, u}$ of part (2) of the Theorem are $\mathrm{KMS}_{\beta}$-states for $\beta>1$.

We now want to study the map $u \mapsto \Phi_{\beta, u}, u \in W$. For this, we adapt the arguments of [HaLe] §5.3. Consider the representation

$$
\rho_{\beta, u}: C_{K} \rightarrow \mathcal{L}\left(l^{2}\left(I_{+}\right) \otimes l^{2}\left(I_{+}\right)\right)
$$

$$
\rho_{\beta, u}(x)(\xi \otimes \eta)=\rho_{u}(x)(\xi) \otimes \eta
$$

Let $\Omega_{\beta, u}$ be the unit vector of $l^{2}\left(I_{+}\right) \otimes l^{2}\left(I_{+}\right)$defined by

$$
\Omega_{\beta, u}=\zeta_{K}(\beta)^{-1 / 2} \sum_{\mathfrak{A} \in \mathcal{I}} N(\mathfrak{A})^{-\beta / 2} \varepsilon_{a} \otimes \varepsilon_{a}
$$

where $\mathfrak{A}$ is the ideal associated to $a$. For each $x \in C_{K}$ we have

$$
\Phi_{\beta, u}(x)=\left\langle\rho_{\beta, u}(x) \Omega_{\beta, u}, \Omega_{\beta, u}\right\rangle
$$

Let $C^{*}\left(I_{+}\right)$denote the $C^{*}$-algebra generated by the $\mu_{a}, a \in I_{+}$. Each vector $\varepsilon_{a} \otimes \varepsilon_{b}$ of the basis of $l^{2}\left(I_{+}\right) \otimes l^{2}\left(I_{+}\right)$belongs to the closure of $\rho_{\beta, u}\left(C^{*}\left(I_{+}\right)\right)\left(\Omega_{\beta, u}\right)$. Therefore $\rho_{\beta, u}\left(C_{K}\right)\left(\Omega_{\beta, u}\right)$ is a dense sub-vector space of $l^{2}\left(I_{+}\right) \otimes l^{2}\left(I_{+}\right)$and $\left(\rho_{\beta, u}, \Omega_{\beta, u}\right)$ defines the GNS representation of $\Phi_{\beta, u}$. The representation $\rho_{u}$ is irreducible, so that the commutant of $\rho_{\beta, u}\left(C_{K}\right)$ is $\operatorname{Id} \otimes \mathcal{L}\left(l^{2}\left(I_{+}\right)\right)$. The von Neumann algebra $M$ generated by $\rho_{\beta, u}\left(C_{K}\right)$ is thus $\mathcal{L}\left(l^{2}\left(I_{+}\right)\right) \otimes$ Id and $M \cap M^{\prime}$ has trivial centre. Therefore $\Phi_{\beta, u}$ is a factor state of Type $\mathrm{I}_{\infty}$, with list of eigenvalues $\left\{\zeta_{K}(\beta)^{-1} N(\mathfrak{A})^{-\beta}, \mathfrak{A} \in \mathcal{I}\right\}$. As $\Phi_{\beta, u}$ is factorial, it is an extremal $\mathrm{KMS}_{\beta}$ state.

Notice that $\Phi_{\beta, u}$ determines a unique state $\widetilde{\Phi}_{\beta, u}$ on the von-Neumann algebra $M=\mathcal{L}\left(l^{2}\left(I_{+}\right)\right) \otimes$ Id generated by $\rho_{\beta, u}\left(C_{K}\right)$,

$$
\widetilde{\Phi}_{\beta, u}(X \otimes \operatorname{Id})=\left\langle X\left(\Omega_{\beta, u}\right), \Omega_{\beta, u}\right\rangle, \quad X \otimes \operatorname{Id} \in M
$$

For each $s>0$, the operator $e^{-s H} \in M$. Hence given $\Phi_{\beta, u}$ we can determine, for each $x \in A / R$, the value of

$$
\begin{aligned}
\lim _{s \rightarrow \infty} \widetilde{\Phi}_{\beta, u}\left(\rho_{u}(e(x)) e^{-s H} \otimes \mathrm{Id}\right) & =\zeta_{K}(\beta)^{-1}\left\langle\rho_{u}(e(x)) \varepsilon_{1}, \varepsilon_{1}\right\rangle \\
& =\zeta_{K}(\beta)^{-1} \chi(u \cdot x) .
\end{aligned}
$$

Therefore $\Phi_{\beta, u}$ determines uniquely the character $x \mapsto \chi(u \cdot x)$ of $A / R$ and as we remarked already, this map from $W$ to the characters of $A / R$ is injective. It follows that the map $u \mapsto \Phi_{\beta, u}$ is an injective continuous map of the compact group $W$ into the space $\mathcal{E}\left(K_{\beta}\right)$ of extreme points of the convex compact Choquet simplex $K_{\beta}$ of $\mathrm{KMS}_{\beta}$ states on $\left(C_{K}, \sigma_{t}\right)$.

For any $u \in W$, we have

$$
\Phi_{\beta, 1} \circ[u]=\Phi_{\beta, u} .
$$

Let $\Psi$ be an extremal $\mathrm{KMS}_{\beta}$ state on $\left(C_{K}, \sigma_{t}\right)$. The following two $\mathrm{KMS}_{\beta}$ states

$$
\int_{W} \Psi \circ[u] d u, \quad \int_{W} \Phi_{\beta, u} d u
$$

are invariant under the action of $W$. They are therefore completely determined by their restriction to $C_{K}^{W}=C^{*}\left(I_{+}\right)$. By Proposition 2 of $\S 3$, the
system $\left(C^{*}\left(I_{+}\right), \sigma_{t}\right)$ is without interaction and has a unique $\mathrm{KMS}_{\beta}$ state denoted by $\Phi_{\beta}$. Therefore,

$$
\int_{W} \Psi \circ[u] d u=\int_{W} \Phi_{\beta, u} d u=\int_{W} \Phi_{\beta, 1} \circ[u] d u
$$

As $\Psi$ is extremal, this gives two decompositions of the same state as a barycenter of measures over $\mathcal{E}\left(K_{\beta}\right)$, and as $K_{\beta}$ is a Choquet simplex,

$$
\Psi \circ[u] \in\left\{\Phi_{\beta, v}: v \in W\right\}
$$

for almost all $u \in W$. Hence, for some $u, v \in W$ we have $\Psi=\Phi_{\beta, v} \circ\left[u^{-1}\right]$ and $\Psi$ is in the image of $u \mapsto \Phi_{\beta, u}$. Since $u \mapsto \Phi_{\beta, u}$ is continuous and bijective and $W$ is compact, it is a homeomorphism with range $\mathcal{E}\left(K_{\beta}\right)$. This concludes our treatment of part (2) of the Theorem.

Now suppose $0<\beta \leq 1$. The key steps in the proof of part (1) of the Theorem are the generalizations of Lemma 27 and Corollary 29 of [BC] to the dynamical system $\left(C_{K}, \sigma_{t}\right)$. It is here that the assumptions on the interactive splitting $+: \mathcal{I} \rightarrow I_{+}$play a crucial role. We shall again make use of the $C^{*}$ - dynamical system $\left(C^{*}\left(I_{+}\right), \sigma_{t}\right)$ with its unique $\mathrm{KMS}_{\beta}$ state $\Phi_{\beta}$. We consider as in $[\mathrm{BC}]$ for the case $K=\mathbb{Q}$, the spectral subspaces $C_{K, \chi}$ for each character $\chi$ of the abelian compact group $W$,

$$
C_{K, \chi}=\left\{x \in C_{K}:[u](x)=\chi(u)(x), \text { all } u \in W\right\}
$$

Therefore $C_{K, 1}=C^{*}\left(I_{+}\right)$by Lemma 3 .
Lemma 4. Let $0<\beta \leq 1$ and let $\Psi$ be a $K M S_{\beta}$ state on $\left(C_{K}, \sigma_{t}\right)$. Then:
(1) The restriction of $\Psi$ to $C^{*}\left(I_{+}\right)$is $\Phi_{\beta}$.
(2) The restriction of $\Psi$ to the spectral subspace $C_{K, \chi}$ is zero when $\chi$ is non-trivial.

Proof. Part (1) is clear. For part (2), we need to generalise the proof of Lemma 27, (b) and (c) of [BC]. We say $V \in C^{*}(A / R)=C(R)$ is localised in a finite subset $F$ of finite places if

$$
V \in\left(\otimes_{v \in F} C\left(\mathcal{O}_{v}\right)\right) \otimes 1 \subset C(R)
$$

Similarly, given a character $\chi$ of $W$, we say that it is localised in $F$ if it factors through the projection $W \rightarrow \prod_{v \in F} K_{v}^{*}$. Let $w \in M_{K}^{o} \backslash F$ and let $\mathfrak{P}_{w}$ be the corresponding prime ideal of $\mathcal{O}$. Let $p_{w}=\left(p_{w, v}\right)_{v}=\mathfrak{P}_{w}^{+}$be the image of $\mathfrak{P}_{w}$ under the given interactive splitting + . To $p_{w}$ we associate the following element $g_{w} \in W$. Writing $g_{w}=\left(g_{w, v}\right)_{v}$ we let

$$
g_{w, v}=p_{w} \in \mathcal{O}_{w}^{*} \quad \text { if } v \neq w, \quad g_{w, w}=1
$$

Our assumptions on the interactive splitting ensure that, at least on the images under + of the principal prime ideals coprime to $F$, the map $p_{w} \mapsto$ $g_{w}$ is not trivial. By Dirichlet's Density Theorem (see [N], Theorem 6.2,
p131), we know that there are infinitely many principal prime ideals coprime to $F$. For any $f \in C(R)$ and $a \in I_{+}$, we have (see $\left.\S 4\right) f \mu_{a}=\mu_{a} f_{a}$ where $f_{a}(x)=f(a x)$ for $x \in R$. If $f$ is localised in $F$ and $w \notin F$ we have $\left[g_{w}\right](f)=f_{p_{w}}$ so that

$$
f \mu_{p_{w}}=\mu_{p_{w}}\left[g_{w}\right](f)
$$

Let $V$ be a partial isometry in $C^{*}(A / R)$ and $\chi$ be a non-trivial character of $W$ both localised in $F$ and such that

$$
[g](V)=\chi(g) V, \quad \text { for all } g \in W
$$

Then, letting $f=V$ we have

$$
V \mu_{p_{w}}=\chi\left(g_{w}\right) \mu_{p_{w}} V
$$

which gives the analogue of equation (5) of the Proof of Lemma 27 in [BC], p. 445 . By continuity, we may view $\chi$ as a character of $G=\prod_{v \in F} \mathcal{O}_{v}^{*} /(1+$ $\pi_{v}^{n_{v}} \mathcal{O}_{v}$ ) for certain minimal integers $n_{v}>0, v \in F$. Let $\mathfrak{C}$ be the (finite) cycle $\prod_{v \in F} \mathfrak{P}_{v}^{n_{v}}$. Then $G \simeq(\mathcal{O} / \mathfrak{C O})^{*}$. An interactive splitting + can be extended multiplicatively to a group homomorphism from the fractional ideals $\mathcal{F}=\mathcal{I}^{-1} \mathcal{I}$ of $K$ to $J$. Let $\mathcal{F}(\mathfrak{C})$ be the group of fractional ideals prime to $\mathfrak{C}$. Let $P(\mathfrak{C})$ be the group of principal ideals prime to $\mathfrak{C}$ and $P_{\mathfrak{C}}$ be the subgroup of principal ideals generated by elements $\alpha \in K^{*}$ with $\alpha \equiv 1$ modulo $\mathfrak{P}_{v}^{n_{v}}$ for all $v \in F$. There are natural projections $p_{1}: J \rightarrow \prod_{v \in F} K_{v}^{*}$ and $p_{2}: \prod_{v \in F} \mathcal{O}_{v}^{*} \rightarrow G$ with $\left(p_{1} \circ+\right)(\mathcal{F}(\mathfrak{C}))$ contained in $\prod_{v \in F} \mathcal{O}_{v}^{*}$. By our assumptions on the interactive splitting, any prime ideal in $P(\mathfrak{C})$ which is non-trivial $\bmod P_{\mathfrak{C}}$ has non-zero image under the $\operatorname{map} p_{2} \circ p_{1} \circ+$ Let $h=\operatorname{Card}(\mathcal{F}(\mathfrak{C}) / \mathfrak{P}(\mathfrak{C}))$ and $h_{\mathfrak{C}}=\operatorname{Card}\left(\mathcal{F}(\mathfrak{C}) / \mathfrak{P}_{\mathfrak{C}}\right)$. As $\mathfrak{C}$ is non-trivial the quotient group $P(\mathfrak{C}) / \mathfrak{P}_{\mathfrak{C}}$ is non-trivial and we have $h_{\mathfrak{C}}>h$. The Dirichlet density ([Lg], p167) of each class in $\mathcal{F}(\mathfrak{C}) / \mathfrak{P}_{\mathfrak{C}}$ is $\frac{1}{h_{\mathfrak{C}}}$, so that there are infinitely many prime ideals in each class of $P(\mathfrak{C}) / \mathfrak{P}_{\mathfrak{C}}$. Having checked these points, the rest of the proof of Lemma 27 of [BC] adapts easily to our situation.

Part (1) of the Theorem is now an immediate consequence of Lemma 4. One can develop an analogous discussion to that of $[\mathrm{BC}], \S 3$ relating the $C^{*}$ Hecke algebra $C_{K}$ to products of trees. This enables one to compute explicitly the unique $\mathrm{KMS}_{\beta}$ state for $0<\beta \leq 1$. To define the generalization of the function $\Psi_{\beta}$ of $[\mathrm{BC}]$, Theorem 5 , write $y \in A / R$ as $y=a / b$ with $a=\left(a_{v}\right)_{v}, b=\left(b_{v}\right)_{v} \in R$. For all $v$ with $y_{v} \neq 0$ take $a_{v}, b_{v} \in \mathcal{O}_{v}$ non-zero with either $a_{v}$ or $b_{v}$ in $\mathcal{O}_{v}^{*}$. If $y_{v}=0$, let $a_{v}=0$ and $b_{v}=1$. Then, if $\mathfrak{B}$ is the ideal associated to $b$, write its prime factorisation as $\mathfrak{B}=\prod_{\mathfrak{P}} \mathfrak{P}^{n_{\mathfrak{P}}}$. Set

$$
\Psi_{b}(y)=\prod_{\mathfrak{P}, n_{\mathfrak{P}} \neq 0} N(\mathfrak{P})^{-n_{\mathfrak{P}} \beta}\left(1-N(\mathfrak{P})^{\beta-1}\right)\left(1-N(\mathfrak{P})^{-1}\right)^{-1}
$$

The discussion of $[\mathrm{BC}], \S 3$ goes through with $P_{\mathbb{Q}}^{+}$replaced by $P_{A}^{+}$and $P_{\mathbb{Z}}^{+}$ replaced by $P_{R}^{+}$(defined in $\S 4$ of this paper), the Hilbert space $\mathcal{H}_{\beta}$ as in [BC], Proposition 32 having now a natural basis indexed by $P_{A}^{+} / P_{R}^{+}$with an inner product invariant under left translation by $P_{A}^{+}$and given by

$$
\left\langle\left(\begin{array}{ll}
1 & y \\
0 & j
\end{array}\right) \varepsilon_{0}, \varepsilon_{0}\right\rangle=0, \quad j \neq 1 ; \quad\left\langle\left(\begin{array}{ll}
1 & y \\
0 & 1
\end{array}\right) \varepsilon_{0}, \varepsilon_{0}\right\rangle=\Psi_{\beta}(y)
$$

Here, the vector $\varepsilon_{0}$ is the class of $P_{R}^{+}$. Let $\delta$ be the module on $P_{A}^{+}$defined by

$$
\delta\left(\begin{array}{ll}
1 & y \\
0 & j
\end{array}\right)=|j| .
$$

One shows that the opposite $C^{*}$-algebra $C_{K}^{o}$ admits a representation $\rho$ in $\mathcal{H}_{\beta}$ given by the right convolution with $\delta^{\beta / 2} f$ for any $P_{R}^{+}$-bi-invariant function $f$ on $P_{A}^{+}$, using the description of $C_{K}$ as a Hecke algebra. Finally, one checks that the vector $\varepsilon_{0}$ defines a $\mathrm{KMS}_{\beta}$ state on $\left(C_{K}, \sigma_{t}\right)$, which gives therefore the unique one,

$$
\Phi_{\beta}(x)=\left\langle\rho(x) \varepsilon_{0}, \varepsilon_{0}\right\rangle
$$

The proofs of all these statements are straightforward generalizations of those for $K=\mathbb{Q}$ given in [BC].

## 6. Concluding remarks

As we said in the Introduction, our treatment differs from those of [HaLe] and [ALR] in that we view the ideals rather than just the principal ideals as playing the role of the positive integers in [BC]. By choosing our interactive splitting + in such a way that for $K=\mathbb{Q}$ we have $I_{+}=\mathbb{N}_{>0}$, the positive integers, we recover the $C^{*}$-Hecke algebra dynamical system $\left(C^{*}\left(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}^{+}\right), \sigma_{t}\right)$ of $[\mathrm{BC}]$. Let $\mathcal{S}_{+}$denote the semigroup given by the image of the semigroup $\mathcal{S}$ of principal integral ideals of $\mathcal{O}$ under an interactive splitting $+: \mathcal{I} \rightarrow I$. Notice that when $\mathcal{O}$ is not principal $\mathcal{S}_{+}$need not be a sub-semigroup of the principal idèles. This is because the map + is built up multiplicatively from its value on the prime ideals, and not all of the prime ideals are principal. What happens when we consider the algebra $C_{K}^{\prime}=C^{*}(K / \mathcal{O}) \rtimes_{\alpha} \mathcal{S}_{+}$and restrict $\sigma$ and the action of $W$ to form the $C^{*}$-dynamical system $\left(C_{K}^{\prime}, \sigma_{t}\right)$ ? The group $W$ acts on $C_{K}^{\prime}$ by symmetries commuting with the action of the $\sigma_{t}$. Recall that if $\mathcal{F}$ denotes the group of fractional ideals of $K$ and $P$ denotes the subgroup of principal ideals, then the cardinality $h$ of the ideal class group $\mathcal{F} / P$ is called the class number of $K$. We denote by $\mathcal{R}_{1}, \ldots, \mathcal{R}_{h}$ the ideal classes of $\mathcal{F}$ modulo $P$, with $\mathcal{R}_{1}=P=\mathcal{S}^{-1} \mathcal{S}$. The Hilbert space $l^{2}\left(I_{+}\right)$is the direct sum of the Hilbert spaces $\mathcal{H}_{i}=l^{2}\left(\left(\mathcal{I} \cap \mathcal{R}_{i}\right)_{+}\right), i=1, \ldots h$, and each $\mathcal{H}_{i}$ is a cyclic representation space for $C_{K}^{\prime}$. It is easy to see, for example when $\beta>1$, that one has
on $\left(C_{K}^{\prime}, \sigma_{t}\right)$ a larger family of extremal Type $\mathrm{I}_{\infty} \mathrm{KMS}_{\beta}$-states, given by

$$
\Phi_{\beta, u, i}(x)=\zeta_{i}(\beta)^{-1} \operatorname{Trace}_{\mathcal{H}_{i}}\left(\rho_{u}(x) e^{-\beta H_{i}}\right), \quad x \in C_{K}, \quad u \in W, \quad i=1, \ldots, h
$$

where $H_{i}$ is the restriction of $H$ to $\mathcal{H}_{i}$ and

$$
\zeta_{i}(\beta)=\sum_{\mathfrak{A} \in \mathcal{I} \cap \mathcal{R}_{i}} \frac{1}{N(\mathfrak{A})^{\beta}}=\operatorname{Trace}_{\mathcal{H}_{i}}\left(e^{-\beta H_{i}}\right)
$$

is the partial Dedekind zeta function (at $\beta$ ) of the number field $K$, associated to $\mathcal{R}_{i}$. For $\beta>1$, a similar family of $\mathrm{KMS}_{\beta}$ states indexed by the ideal class group occurs even for the analogue for number fields of the noninteractive system of Proposition 8 of [BC] as defined in $\S 3$ of the present paper (see also [L], Remark 47).

The question as to what extent $W$ can be interpreted as a Galois group is treated in [HaLe], once appropriate modifications are made to account for the fact that we work here with the full set of finite places of $K$. The conclusion of that discussion is that only in the case $K=\mathbb{Q}$ can one identify as in [BC] the symmetry group $W$ with a true Galois group, in that case the Galois group of the maximal abelian extension of $\mathbb{Q}$. As pointed out by $R$. Langlands, one can view the Artin correspondence in class field theory as an equality between appropriate Artin and Dirichlet L-functions (see [N], Chapter V, §5). One can set up these L-functions (for the case of an abelian extension, where they coincide) using the language of second quantisation inherent in the setting up of the non-interactive system $\left(C^{*}(\mathcal{I}), \sigma_{t}\right)$ (compare with $\left.[\mathrm{BC}], \S 2\right)$. It would be interesting to construct more general types of Euler products using second quantisation.

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Paula B. Cohen
UMR Arithmétique - Géométrie - Analyse - Topologie
Université des Sciences et Technologies de Lille F- 59655 Villeneuve d'Ascq cedex
E-mail : Paula.Cohen@univ-lille1.fr

