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On the Distribution of complex–valued multiplicative functions

par ANTANAS LAURINČIKAS*

In honour of Professor H. Delange on his 80th birthday

RÉSUMÉ. Soient $g_1(m), g_2(m)$ deux fonctions multiplicatives a valeurs complexes. Dans un article on donne les conditions nécessaires et suffisantes de la convergence en un certain sens de

$$\frac{1}{n} N_n((g_1(m), g_2(m)) \in A),\ A \in \mathcal{B}(\mathbb{C}^2),$$

quand $n \to \infty$.

ABSTRACT. Let $g_j(m)$, $j = 1, 2$, be complex–valued multiplicative functions. In the paper the necessary and sufficient conditions are indicated for the convergence in some sense of probability measure

$$\frac{1}{n}\text{card}\{0 \leq m \leq n : (g_1(m), g_2(m)) \in A\},\ A \in \mathcal{B}(\mathbb{C}^2),$$

as $n \to \infty$.

Let $\mathbb{R}, \mathbb{C}, \mathbb{N}$ and $\mathbb{Z}$ denote the sets of all real numbers, of all complex numbers, of all natural numbers and all real integer numbers, respectively. A function $g : \mathbb{N} \to \mathbb{C}$, not identically zero, is called multiplicative if

$$g(mn) = g(m)g(n)$$

for all natural numbers m, n relatively prime to each other. The multiplicative functions play an important role in number theory, therefore its values distribution was studied by many authors. In probabilistic number theory the behaviour of multiplicative functions usually is described by limit theorems in sense of weak convergence of probability measures. Most of the works of this kind are devoted to real–valued multiplicative functions.

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H. Delange was the first to begin to use probabilistic methods to study the distribution of complex–valued multiplicative functions. In [1] he obtained the following result (Theorem A). Let $N_n(\dots)$ denote a number of $m \le n,\ m \in \mathbb{N}$, satisfying the conditions instead of dots, and let

$$\nu_n(\dots) = \frac{1}{n} N_n(\dots).$$

Denote by p a prime number and, for a given complex–valued multiplicative function $g(m)$, define the functions

$$u_g(m) = \begin{cases} \dfrac{g(m)}{\mid g(m) \mid} = e^{i \arg g(m)}, \text{ if } g(m) \neq 0, \\ 0, \text{ if } g(m) = 0, \end{cases}$$

$$v_g(p) = \begin{cases} \ln \mid g(p) \mid, \text{ if } e^{-1} \le \mid g(p) \mid \le e, \\ 1, \text{ if } \mid g(p) \mid < e^{-1} \quad \text{or} \quad \mid g(p) \mid > e. \end{cases}$$

Here $\arg g(m)$ is defined to within the addition of an integer multiple of 2π.

Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S.

THEOREM A. *In order that the sequence of probability measures*

$$\nu_n(g(m) \in A),\ A \in \mathcal{B}(\mathbb{C}),$$

converges "étroitement" to the probability measure not concentrated at $z = 0$, it is necessary and sufficient that the following conditions be satisfied:

1^0 *the series*

$$\sum_p \frac{v_g(p)}{p} \quad \text{and} \quad \sum_p \frac{v_g^2(p)}{p}$$

converge;

2^0 *there exists at least one $m \in \mathbb{N}$ such that the series*

$$\sum_p \frac{1 - u_g^m(p)}{p}$$

converges, or

$$\sum_p \frac{1 - \ Re u_g^m(p) p^{-iu}}{p} = \infty$$

for all $m \in \mathbb{N}$ *and all* $u \in \mathbb{R}$.

In probabilistic number theory the multidimensional theorems are known for additive functions as well as for real–valued multiplicative functions, see, for example [2]–[6]. The aim of this paper is to present a limit theorem which describe the joint distribution of few complex–valued multiplicative functions $g_j(m)$, $j = 1, ..., r$. To do this we can define the probability measure

$$\nu_n((g_1(m), ..., g_r(m)) \in A),\ A \in \mathcal{B}(\mathbb{C}^r),$$

where $\mathbb{C}^r = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r}$, and consider its convergence in some sense to probability measure on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ as $n \to \infty$. This problem is rather complicated, and, for simplicity, we shall limit ourselves to the case $r = 2$ only.

To state our theorem, we will need some notions from [7]. Let P and P_n be probability measures on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$. We say that P_n converges weakly in sence $\mathbb{C}^2$ to P as $n \to \infty$ if P_n converges weakly to P and in addition

$$P_{j;n}(0) \underset{n\to\infty}{\to} P_j(0),\ j = 1, 2,$$

$$P_n(\mathbb{C}^2 \setminus (\mathbb{C} \setminus \{0\})^2) \underset{n\to\infty}{\to} P(\mathbb{C}^2 \setminus (\mathbb{C} \setminus \{0\})^2).$$

Here P_j denote the marginal distributions of P, that is

$$P_1(A) = P(A \times \mathbb{C}),\ P_2(A) = P(\mathbb{C} \times A),\ A \in \mathcal{B}(\mathbb{C}).$$

Let, for simplicity, $0^z = 0$ for all $z \in \mathbb{C}$. Suppose that $| g_j(m) | \not\equiv 1$, $j = 1, 2$.

THEOREM. *In order that the probability measure*

$$P_n(A) \overset{\text{def}}{=} \nu_n((g_1(m), g_2(m)) \in A),\ A \in \mathcal{B}(\mathbb{C}^2),$$

converge weakly in sence $\mathbb{C}^2$ *to a probability measure which marginal distributions are not concentrated on the circle* $| z |= a$, $a \geq 0$, *it is necessary and sufficient that the following conditions be satisfied:*

1^0 *the series* $\sum_p \frac{v_{g_j}(p)}{p}$ *and* $\sum_p \frac{v_{g_j}^2(p)}{p}$, $j = 1, 2$ *converge;*

2^0 *there exists at least one* $k_j \in \mathbb{N}$ *such that the series*

$$\sum_p \frac{1 - u_{g_j}^{k_j}(p)}{p},\ j = 1, 2,$$

converge, or

$$\sum_p \frac{1 - Reu_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-iu}}{p} = \infty$$

for all $k_1, k_2 \in \mathbb{Z}$, $|k_1| + |k_2| \neq 0$, *and all* $u \in \mathbb{R}$.

For the proof of the theorem we will use the characteristic transforms of probability measures on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$ introduced in [7]. In our case $r = 2$ the functions

$$w_j(t_j, k_j) = \int_{\mathbb{C}} r_j^{it_j} e^{ik_j\varphi_j}\, dP_j, \quad j = 1,2,$$

$$w(t_1, t_2, k_1, k_2) = \int_{\mathbb{C}^2} r_1^{it_1} r_2^{it_2} e^{ik_1\varphi_1 + ik_2\varphi_2}\, dP$$

are called the characteristic transforms of probability measure P on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ provided that the integrands are equal to zero when $r_j = 0$ for some $j = 1,2$. Here $r_j = |z_j|$, $\varphi_j = \arg z_j$, $t_j \in \mathbb{R}$, $k_j \in \mathbb{Z}$, $j = 1,2$. We shall require the following continuity theorems for probability measures on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$.

LEMMA 1. *Let* $\{P_n\}$ *be a sequence of probability measures on* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ *and let* $\{w_{j;n}(t_j, k_j),\ j = 1,2,\ w_n(t_1, t_2, k_1, k_2)\}$ *be a sequence of corresponding characteristic transforms.*

Suppose that $w_{j;n}(t_j, k_j) \underset{n\to\infty}{\longrightarrow} w_j(t_j, k_j)$, $j = 1,2$, $w_n(t_1, t_2, k_1, k_2) \underset{n\to\infty}{\longrightarrow} w(t_1, t_2, k_1, k_2)$ *for all* $t_1, t_2 \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}$, *and that the functions* $w_j(t_j, 0)$, $j = 1,2$, $w(t_1, t_2, 0, 0)$ *are continuous at the points* $t_j = 0; t_1 = 0, t_2 = 0$, *respectively.*

Then there exists a probability measure P *on* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ *such that* P_n *converges weakly in sense* $\mathbb{C}^2$ *to* P *as* $n \to \infty$. *In this case* $w_j(t_j, k_j)$, $j = 1,2$, $w(t_1, t_2, k_1, k_2)$ *are the characteristic transforms of the measure* P.

LEMMA 2. *Let* P, P_1, P_2, ... *be probability measures on* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ *and let* $w_j(t_j, k_j)$, $j = 1,2, w(t_1, t_2, k_1, k_2); w_{1;j}(t_j, k_j)$, $j = 1,2, w_1(t_1, t_2, k_1, k_2)$; $w_{2;j}(t_j, k_j)$, $j = 1,2$, $w_2(t_1, t_2, k_1, k_2), \ldots$ *denote their characteristic transforms, respectively.*

If P_n *converges weakly in sense* $\mathbb{C}^2$ *to* P *as* $n \to \infty$ *then* $w_{j;n}(t_j, k_j) \underset{n\to\infty}{\longrightarrow} w_j(t_j, k_j)$, $j = 1,2$, *and* $w_n(t_1, t_2, k_1, k_2) \underset{n\to\infty}{\longrightarrow} w(t_1, t_2, k_1, k_2)$ *for all* $t_1, t_2 \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}$.

Proof. Lemmas 1 and 2 are the case $r = 2$ of Theorems 2 and 3, respectively from [7].

Also we shall use some results on the mean-value

$$\lim_{x\to\infty} \frac{1}{x} \sum_{m\leq x} g(m) \stackrel{\text{def}}{=} M(g)$$

of the multiplicative functions $g(m)$, if this limit exists.

LEMMA 3. *Let $g(m)$ be a multiplicative function satisfying $| g(m) |\leq 1$. If $M(g)$ exists and is not zero, then the series*

$$\sum_p \frac{1-g(p)}{p}$$

converges.

Lemma is the Delange theorem, for the proof see [8], [15].

LEMMA 4. *In order that the mean-value for the multiplicative function $g(m)$, $| g(m) |\leq 1$, exists and be zero, it is necessary and sufficient that one of the following conditions be satisfied:*

1^0 $\sum_p \frac{1- \operatorname{Re} g(p)p^{-iu}}{p} = \infty$ *for every real u;*

2^0 *there exists a real number u_0 such that the series* $\sum_p \frac{1- \operatorname{Re} g(p)p^{-iu_0}}{p}$ *converges and $2^{-riu_0} g(2^r) = -1$ for all $r \in \mathbb{N}$.*

Proof. Lemma is a corollary of the fundamental work of G. Halász [9]. Its statement is given in [10]–[12].

LEMMA 5. *Let $g(m) = g(m; t_1, \dots, t_r)$, $| g(m) |\leq 1$, be a multiplicative function, and the series*

$$\sum_p \frac{1- \operatorname{Re} g(m; t_1, \dots, t_r)p^{-ia(t_1,\dots,t_r)}}{p}$$

converges uniformly in t_j, $| t_j |\leq T$, $j = 1, \dots, r$.

Then, as $x \to \infty$,

$$\frac{1}{x}\sum_{m\le x} g(m;t_1,...,t_r)$$
$$= \frac{x^{ia(t_1,...,t_r)}}{1+a(t_1,...,t_r)}\prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{g(p^\alpha;t_1,...,t_r)}{p^{\alpha(1+ia(t_1,...,t_r))}}\right)+o(1)$$

uniformly in t_j for $| t_j |\le T$, $j=1,...,r$.

Proof. Lemma is a special case of result from [13].

Let

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} u_{g_1}^{k_1}(m)u_{g_2}^{k_2}(m)=0$$

for all $k_1,k_2\in\mathbb{Z}$, $| k_1 | + | k_2 |\ne 0$. By Lemma 4 this is equivalent to the condition

$$\sum_p \frac{1-\ \mathrm{Re}u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-iu}}{p}=\infty$$

for all $u\in\mathbb{R}$, since if

$$\sum_p \frac{1-\ \mathrm{Re}u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-iu_0}}{p}<\infty$$

and

$$2^{-riu_0}u_{g_1}^{k_1}(2^r)u_{g_2}^{k_2}(2^r)=-1,$$

then we have that

$$\sum_p \frac{1-\ \mathrm{Re}u_{g_1}^{2k_1}(p)u_{g_2}^{2k_2}(p)p^{-2iu_0}}{p}<\infty,$$

too, but

$$2^{-2riu_0}u_{g_1}^{2k_1}(2^r)u_{g_2}^{2k_2}(2^r)=1.$$

Let us investigate the set of those triples (k_1,k_2,t), $k_1,k_2\in\mathbb{Z}$, $| k_1 | + | k_2 |\ne 0$, $t\in\mathbb{R}$, for which

$$\sum_p \frac{1-\ \mathrm{Re}u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-it}}{p}<\infty. \tag{1}$$

By a remark above this set is not empty if and only if

$$\frac{1}{n}\sum_{m=1}^{n} u_{g_1}^{k_1}(m)u_{g_2}^{k_2}(m)\underset{n\to\infty}{\not\to} 0 \tag{2}$$

for all $k_1,k_2\in\mathbb{Z}$, $| k_1 | + | k_2 |\ne 0$.

LEMMA 6. *Let the relation (2) be satisfied and* $\sum_{g_1(p)g_2(p)=0} \frac{1}{p} < \infty$. *Then there exist* $q_1, q_2 \in \mathbb{Z}$ *and* $t \in \mathbb{R}$, *or there exist* $a_j, b_j \in \mathbb{Z}$, $j = 1, 2$,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \tag{3}$$

and $t_1, t_2 \in \mathbb{R}$ *such that*

$$\sum_p \frac{1 - \operatorname{Re} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} < \infty \tag{4}$$

if and only if

$$(k_1, k_2) = m(q_1, q_2), \ m \in \mathbb{Z}, \quad \text{and } u = tm,$$

or if and only if

$$(k_1, k_2) = m_1(a_1, a_2) + m_2(b_1, b_2), \ m_1, m_2 \in \mathbb{Z}, \ \text{and} u = m_1 t_1 + m_2 t_2.$$

Proof. Let A be a set of those pairs (k_1, k_2) for which (1) is valid. If $(k_1, k_2) \in A$ then, clearly, $(-k_1, -k_2) \in A$, since $\operatorname{Re} z = \operatorname{Re} \bar{z}$ and (1) remains true with $-t$ instead of t. Obviously $(0,0) \in A$, because in this case one can take $t = 0$. Now let $(k_1', k_2'), (k_1'', k_2'') \in A$, that is

$$\sum_p \frac{1 - \operatorname{Re} u_{g_1}^{k_1'}(p) u_{g_2}^{k_2'}(p) p^{-it'}}{p} < \infty \ \text{ and } \sum_p \frac{1 - \operatorname{Re} u_{g_1}^{k_1''}(p) u_{g_2}^{k_2''}(p) p^{-it''}}{p} < \infty$$

for some real numbers t' and t''. Then in virtue of the inequality

$$1 - \operatorname{Re} z_1 z_2 \le 2(1 - \operatorname{Re} z_1) + 2(1 - \operatorname{Re} z_2) \tag{5}$$

valid for $|z_1| \le 1$, $|z_2| \le 1$, see [14], we find

$$\sum_p \frac{1 - \operatorname{Re} u_{g_1}^{k_1'+k_1''}(p) u_{g_2}^{k_2'+k_2''}(p) p^{-it'-it''}}{p} \le$$

$$\le 2 \sum_p \frac{1 - \operatorname{Re} u_{g_1}^{k_1'}(p) u_{g_2}^{k_2'}(p) p^{-it'}}{p} +$$

$$+2\sum_p \frac{1-\operatorname{Re} u_{g_1}^{k_1''}(p)u_{g_2}^{k_2''}(p)p^{-it''}}{p} < \infty.$$

Consequently, A is a subgroup of the group $\mathbb{Z}^2$. From the general theory of groups it follows that there exists (q_1, q_2) such that each element $(k_1, k_2) \in A$ is presented as

$$(k_1, k_2) = m(q_1, q_2), \quad m \in \mathbb{Z}, \tag{6}$$

or there exist $(a_1, a_2), (b_1, b_2)$ satisfying (3) such that each element $(k_1, k_2) \in A$ is of the form

$$(k_1, k_2) = m_1(a_1, a_2) + m_2(b_1, b_2), \; m_1, m_2 \in \mathbb{Z}. \tag{7}$$

It is well known that for each $(k_1, k_2) \in A$ there exists exactly one real number u satisfying (4). We will find an expression for this u. First let us consider the case (6). Suppose that $u = t$ corresponds to (q_1, q_2). Applying the inequality (5) $\mid m \mid$ times, we easily find that

$$\sum_p \frac{1-\operatorname{Re} u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-imt}}{p} < \infty,$$

so in this case $u = mt$. Now let $(k_1, k_2)$is decribed by (7). Suppose that $u = t_1$, $u = t_2$ correspond to (a_1, a_2) and (b_1, b_2), respectively. Then, reasoning similarly as above, we obtain that

$$\sum_p \frac{1-\operatorname{Re} u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-im_1t_1-im_2t_2}}{p} < \infty,$$

so in this case $u = m_1t_1 + m_2t_2$.

Proof of Theorem. Sufficiency. Denote by $w_{j;n}(t_j, k_j)$, $j = 1,2$, $w_n(t_1, t_2, k_1, k_2)$ the characteristic transforms of the measure P_n. Then we have that

$$w_{j;n}(t_j, k_j) = \frac{1}{n}\sum_{m=1}^{n} \mid g_j(m) \mid^{it_j} u_{g_j}^{k_j}(m), \; j = 1,2,$$

$$w_n(t_1, t_2, k_1, k_2) = \frac{1}{n}\sum_{m=1}^{n} \mid g_1(m) \mid^{it_1} \mid g_2(m) \mid^{it_2} u_{g_1}^{k_1}(m)u_{g_2}^{k_2}(m),$$

$t_1, t_2 \in \mathbb{R},\ k_1, k_2 \in \mathbb{Z}$.

We will begin our investigation from $w_{j;n}(t_j, k_j)$. Obviously, in case of the latter function we can limit ourselves by $k_j \geq 0$ only.

Let the series

$$\sum_p \frac{1 - u_{g_j}^{k_{0j}}(p)}{p},\ j = 1, 2,$$

converge. Using the condition 1^0 and reasoning similarly as in [10], p.224–227, we can prove the existence of such $q_j \in \mathbb{N}$ that the series

$$\sum_p \frac{1 - u_{g_j}^{k_j}(p)}{p},\ j = 1, 2, \tag{8}$$

converge if and only if $q_j \mid k_j$. For these k_j we will examine the convergence of the series

$$\sum_p \frac{1 - \ \mathrm{Re} \mid g_j(p) \mid^{it_j} u_{g_j}^{k_j}(p)}{p}, \quad j = 1, 2. \tag{9}$$

The condition 1^0 shows that

$$\sum_p \frac{1 - \ \mathrm{Re} \mid g_j(p) \mid^{it_j}}{p} = \sum_{g_j(p)=0} \frac{1}{p} + \sum_{g_j(p)\neq 0} \frac{1 - \ \mathrm{Re} e^{it_j \ln |g_j(p)|}}{p} =$$

$$= B + B \sum_{g_j(p)\neq 0} \frac{\sin^2 \frac{t_j}{2} \ln \mid g_j(p) \mid}{p} = B + B \sum_{\substack{g_j(p)\neq 0 \\ |\ln|g_j(p)||>1}} \frac{1}{p} +$$

$$+ B \mid t_j \mid^2 \sum_{\substack{g_j(p)\neq 0 \\ |\ln|g_j(p)||\leq 1}} \frac{\ln^2 \mid g_j(p) \mid}{p} = B \tag{10}$$

uniformly in t_j, $\mid t_j \mid \leq T$. Here and further B is a number (not always the same) bounded by a constant. Thus, by (5), using the convergence of the series (8), we obtain that the series (9) also converges uniformly in t_j, $\mid t_j \mid \leq T$. Therefore, from Lemma 5

$$w_{j;n}(t_j, k_j) = \prod_{p \leq n} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{\mid g_j(p^\alpha) \mid^{it_j} u_{g_j}^{k_j}(p^\alpha)}{p^\alpha}\right) + o(1), \tag{11}$$

as $n \to \infty$, uniformly in t_j, $| t_j | \leq T$. It is easy to see that, for $p \geq p_0$,

$$\ln\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{| g_j(p^\alpha) |^{it_j} u_{g_j}^{k_j}(p^\alpha)}{p^\alpha}\right) = \frac{B}{p^2}+$$

$$+\begin{cases} \dfrac{B}{p}, \text{ if } g_j(p) = 0, \\ \dfrac{B}{p}, \text{ if } | \ln | g_j(p) || > 1 \\ \dfrac{u_{g_j}^{k_j}(p)-1}{p} + it_j\dfrac{\ln | g_j(p) |}{p} + B | t_j |^2 \dfrac{\ln^2 | g_j(p) |}{p} + \\ +B | t_j | \dfrac{1 - \operatorname{Re} u_{g_j}^{k_j}(p)}{p}, \text{ if } | \ln | g_j(m) || \leq 1. \end{cases}$$

From this, (11), using the hypotheses of the theorem, we obtain that

$$w_{j;n}(t_j,k_j) = \prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{| g_j(p^\alpha) |^{it_j} u_{g_j}^{k_j}(p^\alpha)}{p^\alpha}\right) + o(1) \tag{12}$$
$$\stackrel{\text{def}}{=} w_j(t_j,k_j) + o(1)$$

as $n \to \infty$, uniformly in t_j, $| t_j | \leq T$, for every $T > 0$, and for all integers $k_j \geq 0$. Therefore the functions $w_j(t_j, 0)$, $j = 1, 2$, are continuous at $t_j = 0$. Consequently, from Lemma 1 we have that the marginal distributions $P_{j;n}$, $j = 1, 2$, of the measure P_n converge weakly in sense $\mathbb{C}$ to some measure P_j on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $n \to \infty$. It remains to prove that P_j are not concentrated on the circle $| z |= a_j$ for some $a_j \geq 0$. This can be obtained in the following manner. Since $| g_j(m) | \not\equiv 1$, $j = 1, 2$, it is well known [16] that under the hypothesis 1^0 of the theorem a real-valued multiplicative function $| g_j(m) |$ possesses non-degenerate limit distribution, that is the probability measure

$$\nu_n(| g_j(m) | \in A), \ A \in \mathcal{B}(\mathbb{R}),$$

as $n \to \infty$, converges weakly (and at $x = 0$) to some probability measure Q_j not concentrated at $x = a_j$, $a_j \geq 0$. If we suppose that P_j are concentrated on the circle $| z |= a_j$ then it follows that Q_j are degenerate at the point $x = a_j$. Thus, we have obtained a contradiction which shows that P_j, $j = 1, 2$, are not concentrated on the circle.

In the case when $g_j \not| k_j$ it can be proved using a similar method as in [10] that

$$\sum_{p} \frac{1 - \operatorname{Re} u_{g_j}^{k_j}(p) p^{-iu}}{p} = \infty, \quad j = 1, 2, \tag{13}$$

for all $u \in \mathbb{R}$. Let $c_1, c_2, \dots$ be some positive constants. Then, using the identity

$$1 - z_1 \dots z_r = z_2 \dots z_r (1 - z_1) + \dots + z_r (1 - z_{r-1}) + (1 - z_r)$$

and taking into account (10) and (13), we have

$$\sum_{p \le x} \frac{1 - \operatorname{Re} \mid g_j(p) \mid^{it_j} u_{g_j}^{k_j}(p) p^{-iu}}{p} \ge \sum_{p \le x} \frac{1 - \operatorname{Re} u_{g_j}^{k_j}(p) p^{-iu}}{p} -$$

$$-c_1 \sum_{p \le x} \frac{1 - \operatorname{Re} \mid g_j(p) \mid^{it_j}}{p} - c_2 \left(\sum_{p \le x} \frac{1 - \operatorname{Re} \mid g_j(p) \mid^{it_j}}{p} \right)^{\frac{1}{2}} \times$$

$$\times \left(\sum_{p \le x} \frac{1 - \operatorname{Re} u_{g_j}^{k_j}(p) p^{-iu}}{p} \right)^{\frac{1}{2}} \underset{n \to \infty}{\longrightarrow} \infty$$

for all $t_j \in \mathbb{R}$ and all $u \in \mathbb{R}$. Consequently, by Lemma 4 it follows in this case that

$$w_{j;n}(t_j, k_j) = o(1) \tag{14}$$

as $n \to \infty$, for all $t_j \in \mathbb{R}$.

If

$$\sum_{p} \frac{1 - \operatorname{Re} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} = \infty$$

for all $k_1, k_2 \in \mathbb{Z}$, $\mid k_1 \mid + \mid k_2 \mid \ne 0$, and all $u \in \mathbb{R}$, then, taking $k_2 = 0$ and after this $k_1 = 0$, we find that

$$\sum_{p} \frac{1 - \operatorname{Re} u_{g_j}^{k_j}(p) p^{-iu}}{p} = \infty$$

for all $k_j \in \mathbb{N}$ and all $u \in \mathbb{R}$. Therefore, reasoning exactly as in the case $q_j \not| k_j$ considered above, we obtain that

$$w_{j;n}(t_j, k_j) = o(1), \tag{15}$$

as $n \to \infty$, for all $t_j \in \mathbb{R}$ and $k_j \in \mathbb{N}$.

The function $w_{j;n}(t_j, 0)$ already has been considered above. Consequently, we see from (12), (14) and (15) that in all cases

$$w_{j;n}(t_j, k_j) = w_j(t_j, k_j) + o(1), \quad j = 1, 2, \tag{16}$$

as $n \to \infty$, for all $t_j \in \mathbb{R}$ and $k_j \in \mathbb{Z}$, and the function $w_j(t_j, 0)$ is continuous at $t_j = 0$, $j = 1, 2$.

Now we will examine the function $w_n(t_1, t_2, k_1, k_2)$. When $k_1 = k_2 = 0$ we have that

$$w_n(t_1, t_2, 0, 0) = \frac{1}{n} \sum_{m=1}^{n} | g_1(m) |^{it_1} | g_2(m) |^{it_2},$$

and to prove the existence of limit function it suffices to verify the hypothesis of Lemma 5. In view of (5) and (10)

$$\sum_p \frac{1 - \operatorname{Re} | g_1(p) |^{it_1} | g_2(p) |^{it_2}}{p} = B \tag{17}$$

uniformly in t_j, $| t_j | \leq T$, $j = 1, 2$. Thus Lemma 5 yields that

(18)

$$w_n(t_1, t_2, 0, 0) = \prod_{p \leq n} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{| g_1(p^\alpha) |^{it_1} | g_2(p^\alpha) |^{it_2}}{p^\alpha}\right) + o(1),$$

as $n \to \infty$, uniformly in t_j, $| t_j | \leq T$, $j = 1, 2$. It is clear that, for $p \geq p_0$,

$$\ln \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{| g_1(p^\alpha) |^{it_1} | g_2(p^\alpha) |^{it_2}}{p^\alpha}\right) = \frac{B}{p^2} +$$

$$+ \begin{cases} \dfrac{B}{p}, \text{ if } g_1(p) g_2(p) = 0, \\ \dfrac{B}{p}, \text{ if } | \ln | g_1(p) || > 1 \text{ or } | \ln | g_2(p) || > 1 \\ i \displaystyle\sum_{j=1}^{2} t_j \frac{\ln | g_j(p) |}{p} + B \sum_{j=1}^{2} (t_j^4 + 1) \frac{\ln^2 | g_j(p) |}{p} \\ \qquad\qquad\qquad \text{if } | \ln | g_j(m) || \leq 1, \ j = 1, 2. \end{cases}$$

Therefore (18) and the hypotheses of the theorem yield

(19)

$$w_n(t_1,t_2,0,0) = \prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{|\, g_1(p^\alpha)\,|^{it_1}\,|\, g_2(p^\alpha)\,|^{it_2}}{p}\right)+o(1)$$

$$\stackrel{\text{def}}{=} w(t_1,t_2,0,0)+o(1)$$

uniformly in t_j, $|\ t_j\ |\leq T, j=1,2$, and evidently the limit function $w(t_1,t_2,0,0)$ is continuous at $t_1=0, t_2=0$.

Now we will investigate the general case of the function $w_n(t_1,t_2,k_1,k_2)$. First assume there exist $k_{0j}\in\mathbb{N}$ such that the series

$$\sum_p \frac{1-u_{g_j}^{k_{0j}}(p)}{p}, \quad j=1,2,$$

converge. Then, clearly,

$$\sum_p \frac{1-\ \operatorname{Re} u_{g_j}^{k_{0j}}(p)}{p}<\infty, \quad j=1,2,$$

and whence in view of (5)

$$\sum_p \frac{1-\ \operatorname{Re} u_{g_1}^{mk_{01}}(p)u_{g_2}^{mk_{02}}(p)}{p}<\infty$$

for each $m\in\mathbb{N}$. From this and Lemma 4 we have that the relation (2) is satisfied, and therefore Lemma 6 gives

(20) $$(k_{01},k_{02})=m_0(q_1,q_2),\ m_0\in\mathbb{Z},\quad \text{and } u=0,$$

or

(21) $$\begin{cases}(k_{01},k_{02})=m_{01}(a_1,a_2)+m_{02}(b_1,b_2)\\ m_{01},m_{02}\in\mathbb{Z}, \qquad m_{01}t_1+m_{02}t_2=0.\end{cases}$$

By Lemma 6 if

$$(k_1,k_2)\neq m(q_1,q_2),\ m\in\mathbb{Z},$$

and

$$(k_1,k_2)\neq m_1(a_1,a_2)+m_2(b_1,b_2),\ m_1,m_2\in\mathbb{Z},$$

then

$$\sum_p \frac{1 - \operatorname{Re} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} = \infty$$

for all $u \in \mathbb{R}$. For these k_1, k_2 we have by (17) that

$$\sum_{p \le x} \frac{1 - \operatorname{Re} | g_1(p) |^{it_1} | g_2(p) |^{it_2} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} \ge$$

$$\ge \sum_{p \le x} \frac{1 - \operatorname{Re} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} - c_3 \sum_{p \le x} \frac{1 - \operatorname{Re} | g_1(p) |^{it_1} | g_2(p) |^{it_2}}{p} -$$

$$-c_4 \left(\sum_{p \le x} \frac{1 - \operatorname{Re} | g_1(p) |^{it_1} | g_2(p) |^{it_2}}{p} \right)^{\frac{1}{2}} \left(\sum_{p \le x} \frac{1 - \operatorname{Re} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} \right)^{\frac{1}{2}}$$

tends to infinity when x tends to infinity for all $t_j \in \mathbb{R}$, $j = 1, 2$, and all $u \in \mathbb{R}$. In consequence, by Lemma 4 for these k_1, k_2

$$w_n(t_1, t_2, k_1, k_2) = o(1), \tag{22}$$

as $n \to \infty$, for all $t_j \in \mathbb{R}$, $j = 1, 2$.

It remains to study more complicated cases when (k_1, k_2) is presented as

$$(k_1, k_2) = m(q_1, q_2), \ m \in \mathbb{Z}, \tag{23}$$

or

$$(k_1, k_2) = m_1(a_1, a_2) + m_2(b_1, b_2), \ m_1, m_2 \in \mathbb{Z}. \tag{24}$$

First suppose that (20) is true and consider (k_1, k_2) having the form (23). Note that in this case $t = 0$. Then from (4) it follows that

$$\sum_p{}' \frac{1 - \cos m\alpha_p}{p} < \infty \tag{25}$$

for all $m \in \mathbb{N}$. Here $\alpha_p = q_1 \arg g_1(p) + q_2 \arg g_2(p)$, and the dash indicates that the sum is extended over those p for which $g_1(p) g_2(p) \ne 0$. We must prove that the series

$$\sum_p{}' \frac{\sin m\alpha_p}{p} \tag{26}$$

converges, too. To prove this we apply the Delange method used in [10], [14]. Let, for $x \geq 2$, $m, m_1, m_2 \in \mathbb{N}$,

$$A_m(x) = \sum_{p \leq x}{}' \frac{\sin m\alpha_p}{p}$$

and

$$\Phi_{m_1,m_2}(x) = A_{m_1+m_2}(x) - A_{m_1}(x) - A_{m_2}(x). \tag{27}$$

Then, for $x_2 > x_1 \geq 2$,

$$\Phi_{m_1,m_2}(x_2) - \Phi_{m_1,m_2}(x_1) = \sum_{x_1 < p \leq x_2}{}' \frac{\sin(m_1+m_2)\alpha_p - \sin m_1\alpha_p - \sin m_2\alpha_p}{p}$$

$$= B \sum_{j=1}^{2} \sum_{x_1 < p \leq x_2}{}' \frac{1 - \cos m_j\alpha_p}{p} \underset{x_1 \to \infty}{\longrightarrow} 0$$

by (25). Thus, $\Phi_{m_1,m_2}(x)$ has a finite limit as $x \to \infty$. From the convergence of the series

$$\sum_p \frac{1 - u_{g_j}^{k_{0j}}(p)}{p}, \quad j = 1, 2, \tag{28}$$

and (20) we deduce that

$$\lim_{x \to \infty} A_{m_0}(x) = A_{m_0} \neq \infty. \tag{29}$$

From the definition of $\Phi_{m_1,m_2}(x)$ it follows that

$$A_1(x) = \frac{1}{m_0} \left(A_{m_0}(x) - \sum_{j=1}^{m_0-1} \Phi_{j,1}(x) \right).$$

Since $\Phi_{m_1,m_2}(x)$ has a finite limit as $x \to \infty$, from this and (29) we find that

$$\lim_{x \to \infty} A_1(x) = A_1 \neq \infty. \tag{30}$$

Now suppose that

$$\lim_{x \to \infty} A_m(x) = A_m \neq \infty. \tag{31}$$

The equality (27) gives

$$A_{m+1}(x)=\Phi_{m,1}(x)+A_m(x)+A_1(x).$$

Consequently, the properties of $\Phi_{m,1}(x)$ and (30), (31) yield

$$\lim_{x\to\infty} A_{m+1}(x)=A_{m+1}\neq\infty,$$

so by induction the series (26) converges for all $m\in\mathbb{N}$. From this and (25) we have that the series

$$\sum_p \frac{1-u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)}{p} \tag{32}$$

converges for all k_1,k_2 satisfying (23).

Now we assume that (21) is valid and consider (k_1,k_2) having the form (24). Let us put

$$\alpha_{1p}=a_1\ \arg g_1(p)+a_2\ \arg g_2(p)-t_1\ \ln p,$$

$$\alpha_{2p}=b_1\ \arg g_1(p)+b_2\ \arg g_2(p)-t_2\ \ln p.$$

Then from (4)

$$\sum_p{}' \frac{1-\cos(m_1\alpha_{1p}+m_2\alpha_{2p})}{p}<\infty,\quad m_1,m_2\in\mathbb{Z}, \tag{33}$$

and we must prove that the series

$$\sum_p{}' \frac{\sin(m_1\alpha_{1p}+m_2\alpha_{2p})}{p} \tag{34}$$

converges, too. The convergence of the series (28) and (21) imply the convergence of the series

$$\sum_p{}' \frac{\sin(m_{01}a_j+m_{02}b_j)\ \arg g_j(p)}{p},\quad j=1,2.$$

From this, using again the convergence of the series (28) and the equality $m_{01}t_1+m_{02}t_2=0$, we deduce that the series

$$\sum_p{}' \frac{\sin(m_{01}\alpha_{1p}+m_{02}\alpha_{2p})}{p} \tag{35}$$

converges. Let us put, for $x \geq 2$, $m_j, m_j^{'}, m_j^{''} \in \mathbb{Z}$,

$$B_{m_1,m_2}(x) = \sum_{p \leq x}{}' \frac{\sin(m_1\alpha_{1p} + m_2\alpha_{2p})}{p}$$

and

$$\Psi_{m_1^{'},m_1^{''};m_2^{'},m_2^{''}}(x) = B_{m_1^{'}+m_1^{''},m_2^{'}+m_2^{''}}(x) - B_{m_1^{'},m_2^{'}}(x) - B_{m_1^{''},m_2^{''}}(x). \tag{36}$$

Then, for $x_2 > x_1 \geq 2$, in view of (33)

$$\Psi_{m_1^{'},m_1^{''};m_2^{'},m_2^{''}}(x_2) - \Psi_{m_1^{'},m_1^{''};m_2^{'},m_2^{''}}(x_1) =$$

$$= B \sum_{x_1 < p \leq x_2}{}' \frac{1 - \cos(m_1^{'}\alpha_{1p} + m_2^{'}\alpha_{2p})}{p} +$$

$$+ B \sum_{x_1 < p \leq x_2}{}' \frac{1 - \cos(m_1^{''}\alpha_{1p} + m_2^{''}\alpha_{2p})}{p} \underset{x_1 \to \infty}{\to} 0.$$

Thus, the limit

$$\lim_{x \to \infty} \Psi_{m_1^{'},m_1^{''};m_2^{'},m_2^{''}}(x) = \Psi_{m_1^{'},m_1^{''};m_2^{'},m_2^{''}} \neq \infty \tag{37}$$

exists. Now we take $m_2^{'} = m_{02}$, $m_2^{''} = 0$. Clearly, we can suppose that $m_{01} > 0$. Then (36) gives the formula

$$B_{1,0}(x) = \frac{1}{m_{01}} \left(B_{m_{01},m_{02}}(x) - \sum_{j=1}^{m_{01}-1} \Psi_{j,1;m_{02},0}(x) \right). \tag{38}$$

Since the series (35) converges, the limit

$$\lim_{x \to \infty} B_{m_{01},m_{02}}(x) = B_{m_{01},m_{02}} \neq \infty \tag{39}$$

exists. Therefore (37)–(39) yield

$$\lim_{x \to \infty} B_{1,0}(x) = B_{1,0} \neq \infty. \tag{40}$$

Let us suppose that, for $m \in \mathbb{N}$,

$$\lim_{x \to \infty} B_{m,0}(x) = B_{m,0} \neq \infty. \tag{41}$$

Then in virtue of (36)

$$B_{m+1,0}(x) = \Psi_{m,1;0,0}(x) + B_{m,0} + B_{1,0}(x),$$

and therefore by (37), (40) and (41)

$$\lim_{x\to\infty} B_{m,0}(x) = B_{m,0} \neq \infty \tag{42}$$

for all $m \in \mathbb{N}$, and, consequently, for all $m \in \mathbb{Z}$.

Reasoning similarly, we can show that

$$\lim_{x\to\infty} B_{0,m}(x) = B_{0,m} \neq \infty \tag{43}$$

for all $m \in \mathbb{Z}$. Now it is easy to see that (37), (42) and (43) yield the relation

$$\lim_{x\to\infty} B_{m_1,m_2}(x) = B_{m_1,m_2} \neq 0$$

for all $m_1, m_2 \in \mathbb{Z}$, and this shows that the series (34) converges for all $m_1, m_2 \in \mathbb{Z}$.

Thus, we have proved that in considered case the series

$$\sum_p \frac{1 - u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-iu}}{p}, \tag{44}$$

where

$$(k_1, k_2) = m_1(a_1, a_2) + m_2(b_1, b_2),$$
$$u = m_1t_1 + m_2t_2, \; m_1, m_2 \in \mathbb{Z},$$

is convergent.

Now we return to $w_n(t_1, t_2, k_1, k_2)$. The convergence of the series (32) and (17) imply

$$\sum_p \frac{1 - \operatorname{Re} \mid g_1(p) \mid^{it_1} \mid g_2(p) \mid^{it_2} u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)}{p} < \infty$$

for all $t_1, t_2 \in \mathbb{R}$ and k_1, k_2 satisfying (23) (if it take place). Therefore, by Lemma 5

(45)

$$w_n(t_1, t_2, k_1, k_2) =$$
$$\prod_{p\leq n}\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{\mid g_1(p^\alpha) \mid^{it_1}\mid g_2(p^\alpha) \mid^{it_2} u_{g_1}^{k_1}(p^\alpha)u_{g_2}^{k_2}(p^\alpha)}{p^\alpha}\right) + o(1).$$

Since, for $p \geq p_0$,

$$\ln\left(1-\frac{1}{p}\right)\left(1+\sum_{\alpha=1}^{\infty}\frac{|\, g_1(p^\alpha)\,|^{it_1}|\, g_2(p^\alpha)\,|^{it_2}\, u_{g_1}^{k_1}(p^\alpha)u_{g_2}^{k_2}(p^\alpha)}{p^\alpha}\right)=\frac{B}{p^2}+$$

$$+\begin{cases}\frac{B}{p}, \text{ if } g_1(p)g_2(p)=0,\\ \frac{B}{p}, \text{ if } \;|\ln|\, g_1(p)\,||>1 \;\text{ or }\; |\ln|\, g_2(p)\,||>1,\\ \frac{\exp\{ik_1\ \arg g_1(p)+ik_2\ \arg g_2(p)\}-1}{p}+i\sum_{j=1}^{2}t_j\frac{\ln|\, g_j(p)\,|}{p}+\\ +B\sum_{j=1}^{2}(t_j^4+1)\frac{\ln^2|\, g_j(p)\,|}{p}+\\ +B(t_1^2+t_2^2)\frac{1-\cos(k_1\ \arg g_1(p)+k_2\ \arg g_2(p))}{p}\\ \qquad\qquad \text{if } \;|\ln|\, g_j(m)\,||\leq 1,\ j=1,2.\end{cases}$$

Hence and from (45) we have that the limit

$$\lim_{n\to\infty} w_n(t_1,t_2,k_1,k_2)=w(t_1,t_2,k_1,k_2) \tag{46}$$

exists.

The case of convergence of the series (44) is considered similarly, and we again obtain the relation (46).

If the series

$$\sum_p \frac{1-\ \operatorname{Re} u_{g_1}^{k_1}(p)u_{g_2}^{k_2}(p)p^{-iu}}{p}$$

diverges for all $k_1,k_2 \in \mathbb{Z}$, $|\, k_1\,|+|\, k_2\,|\neq 0$, and all $u\in\mathbb{R}$, then clearly, the relation (22) is valid. In consequence, the sufficiency of the theorem follows from (16), (19), (22), (46) and Lemma 1.

Necessity. Suppose that there exists a probability measure P on $(\mathbb{C}^2,\mathcal{B}(\mathbb{C}^2))$ with marginal distributions not concentrated on a circle such that P_n converges weakly in sense $\mathbb{C}^2$ to P as $n\to\infty$. By Lemma 2 the characteristic transforms of P_n converge to those of P as $n\to\infty$. In particular,

$$w_{j;n}(t_j,0)=\frac{1}{n}\sum_{m=1}^{n}|\, g_j(m)\,|^{it_j}\underset{n\to\infty}{\rightarrow} w_j(t_j,0),\ j=1,2,$$

where $w_j(t_j, k_j)$ are the characteristic transforms of P_j. From this we obtain that the multiplicative function $\mid g_j(m) \mid$ possesses a non-degenerate limit distribution. Then it follows from [16] that the hypothesis 1^0 of the theorem is valid.

Moreover

$$w_{j;n}(0, k_j) = \frac{1}{n} \sum_{m=1}^{n} u_{g_j}^{k_j}(m) \underset{n\to\infty}{\longrightarrow} w_j(0, k_j), \; j = 1, 2.$$

Suppose that $w_j(0, k_j) \not\equiv 0$. Let $w_j(0, k_{0j}) \neq 0$. Then by Lemma 3 the series

$$\sum_p \frac{1 - u_{g_j}^{k_{0j}}(p)}{p}, \quad j = 1, 2,$$

converges. From this we deduce that

$$\sum_p \frac{1 - \ \mathrm{Re} u_{g_1}^{mk_{01}}(p) u_{g_2}^{mk_{02}}(p)}{p} < \infty$$

for all $m \in \mathbb{N}$. Consequently, Lemma 4 shows that

$$w_n(0, 0, k_1, k_2) = \frac{1}{n} \sum_{m=1}^{n} u_{g_1}^{k_1}(m) u_{g_2}^{k_2}(m) \underset{n\to\infty}{\not\to} 0$$

for all $k_1, k_2 \in \mathbb{Z}$, $\mid k_1 \mid + \mid k_2 \mid \neq 0$. If $w_n(0, 0, k_1, k_2) \underset{n\to\infty}{\longrightarrow} 0$ for all $k_1, k_2 \in \mathbb{Z}$, $\mid k_1 \mid + \mid k_2 \mid \neq 0$, then Lemma 4 for these k_1, k_2 implies

$$\sum_p \frac{1 - \ \mathrm{Re} u_{g_1}^{k_1}(p) u_{g_2}^{k_2}(p) p^{-iu}}{p} = \infty$$

for all $u \in \mathbb{R}$.

This completes the proof of the theorem.

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