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Two estimates on the distribution of zeros of the first derivative of Dirichlet *L*-functions under the generalized Riemann hypothesis

par Ade Irma SURIAJAYA

Résumé. Le nombre de zéros and la distribution des parties réelles des zéros non-réelles de la dérivée de la fonction zêta de Riemann a été étudiée par B. C. Berndt, N. Levinson, H. L. Montgomery, H. Akatsuka et l'auteure. Berndt, Levinson et Montgomery ont étudié le cas inconditionnel, alors qu'Akatsuka et l'auteure ont donné de meilleures estimations sous l'hypothèse de Riemann. Récemment F. Ge a amélioré l'estimation du nombre de zéros par Akatsuka. Dans cet article nous montrons des résultats similaires relatifs à la dérivée des fonctions L de Dirichlet associées aux caractères primitifs de Dirichlet, sous l'hypothèse de Riemann généralisée.

Abstract. The number of zeros and the distribution of the real part of non-real zeros of the derivatives of the Riemann zeta function have been investigated by B. C. Berndt, N. Levinson, H. L. Montgomery, H. Akatsuka, and the author. Berndt, Levinson, and Montgomery investigated the unconditional case, while Akatsuka and the author gave sharper estimates under the truth of the Riemann hypothesis. Recently, F. Ge improved the estimate on the number of zeros shown by Akatsuka. In this paper, we prove similar results related to the first derivative of Dirichlet L-functions associated with primitive Dirichlet characters under the assumption of the generalized Riemann hypothesis.

1. Introduction

Zeros of the Riemann zeta function are also related to those of its derivatives $\zeta^{(k)}(s)$ for positive integer k. For example, A. Speiser [9] proved that the Riemann hypothesis is equivalent to the statement that $\zeta'(s)$ has no

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non-real zeros to the left of the critical line. In 2012, assuming the Riemann hypothesis, H. Akatsuka [1, Theorems 1 and 3] showed that we can approximate the distribution of zeros of $\zeta'(s)$ as follows:

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ \zeta'(\rho')=0,\, 0<\gamma'\le T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - \log\log 2\right)T - \mathrm{Li}\left(\frac{T}{2\pi}\right) + O((\log\log T)^2),$$

where the sum is counted with multiplicity and

$$\mathrm{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t},$$

and

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{(\log\log T)^{1/2}}\right), \tag{1.1}$$

where $N_1(T)$ denotes the number of zeros of $\zeta'(s)$ with $0 < \mathrm{Im}(s) \le T$, counted with multiplicity. These results are also extended to higher order derivatives by the author [10, Theorems 1 and 3].

These results of Akatsuka [1, Theorems 1 and 3] and the author [10, Theorems 1 and 3], under the truth of the Riemann hypothesis, improve the error term $O(\log T)$ in the unconditional results obtained by N. Levinson and H. L. Montgomery [5, Theorem 10] and by B. C. Berndt [3, Theorem]. Recently, F. Ge [4, Theorem 1] showed that we can improve the error term in (1.1) shown by Akatsuka [1, Theorem 3] to

$$O\left(\frac{\log T}{\log\log T}\right).$$

This result is the current best estimate on the number of zeros of $\zeta'(s)$ under the Riemann hypothesis.

We are interested in extending these results of Akatsuka [1, Theorems 1 and 3] and Ge [4, Theorem 1] to Dirichlet L-functions. In this paper we consider only Dirichlet L-functions associated with primitive Dirichlet characters χ modulo $q > 1$, $L(s,\chi)$. Note that there exists only one Dirichlet character modulo 1 and the associated Dirichlet L-function is the Riemann zeta function, whose results are given in [1]. The *generalized Riemann hypothesis* states that both $\zeta(s)$ and $L(s,\chi)$ satisfy the Riemann hypothesis, that is, all nontrivial zeros lie on the critical line $\mathrm{Re}(s) = 1/2$.

Zeros of $L^{(k)}(s,\chi)$ have been studied by C. Y. Yıldırım [13] in 1996 including zero-free regions and the number of zeros. Akatsuka and the author in their recent preprint [2, Theorems 1, 2, 4, and 5] improved the zero-free region on the left half-plane [13, Theorem 3] and the number of zeros [13, Theorem 4] shown by Yıldırım, for the case $k = 1$. We also obtained a result [2, Theorem 6] on the distribution of the real part of zeros and proved results [2, Theorems 8 and 9], analogous to Speiser's theorem [9], for Dirichlet L-functions.

Throughout this paper, for a given integer $q > 1$, we denote by m the smallest prime number that does not divide q. Recall that we can show $m = O(\log q)$. We also determine a number κ as

$$\kappa := \begin{cases} 0, & \chi(-1) = 1; \\ 1, & \chi(-1) = -1. \end{cases} \tag{1.2}$$

Next, we let $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ denote the zeros of $L(s,\chi)$ and $L'(s,\chi)$ in the right half-plane $\mathrm{Re}(s) > 0$. We know that $L(s,\chi)$ has only trivial zeros in $\mathrm{Re}(s) \le 0$. We remark that zeros of $L'(s,\chi)$ satisfying $\mathrm{Re}(s) \le 0$ can also be regarded as "trivial" (see [2, Theorems 1, 2, and 4]). Then we define $N_1(T,\chi)$ for $T \ge 2$ as the number of zeros of $L'(s,\chi)$ satisfying $\mathrm{Re}(s) > 0$ and $|\,\mathrm{Im}(s)| \le T$, counted with multiplicity.

Our main theorems are as follows:

Theorem 1.1. *Assume that the generalized Riemann hypothesis is true, then for $T \ge 2$, we have*

$$\sum_{\substack{\rho' = \beta' + i\gamma', \\ |\gamma'| \le T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{\pi} \log\log \frac{qT}{2\pi} + \frac{T}{\pi}\left(\frac{1}{2}\log m - \log\log m\right) - \frac{2}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O\Big(m^{1/2}(\log\log(qT))^2 + m\log\log(qT) + m^{1/2}\log q\Big),$$

where the sum is counted with multiplicity.

Theorem 1.2. *Assume that the generalized Riemann hypothesis is true, then for $T \ge 2$, we have*

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + O\left(A(q,T)\frac{m^{1/2}\log(qT)}{\log\log(qT)} + \log q\right),$$

where

$$A(q,T) := \min\left\{(\log\log(qT))^{1/2}, 1 + \frac{m^{1/2}}{\log\log(qT)}\right\}.$$

In this paper, we first review some basic estimates related to $\log L(s,\chi)$ near the critical line and zero-free regions of $L'(s,\chi)$ in Section 2. In Section 3, we show important lemmas crucial for the proofs of our main theorems and finally prove them in Section 4. For convenience, we use variables s and z as complex numbers, with $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$. Finally, we abbreviate the generalized Riemann hypothesis as GRH.

2. Preliminaries

2.1. Bounds related to $\log L(s,\chi)$ near the critical line. In this section we give some bounds related to $\log L(s,\chi)$ which can be found in [7, Sections 12.1, 13.2, 14.1]. Only for this subsection, we put $\tau := |t|+4$.

Lemma 2.1. *Assume GRH, then*

$$\log L(\sigma+it,\chi) = O\left(\frac{(\log(q\tau))^{2(1-\sigma)}}{(1-\sigma)\log\log(q\tau)} + \log\log\log(q\tau)\right)$$

holds uniformly for $1/2 + (\log\log(q\tau))^{-1} \le \sigma \le 3/2$.

Proof. This is straightforward from the inequalities in exercise 6 of [7, Section 13.2] (see also page 3 of [6] for the corrected exercise 6 (b) and (c)). □

Lemma 2.2. *Assume GRH, then*

$$\arg L(\sigma+it,\chi) = O\left(\frac{\log(q\tau)}{\log\log(q\tau)}\right)$$

holds uniformly for $\sigma \ge 1/2$.

Proof. See [8, Section 5] or exercise 11 of [7, Section 13.2]. □

With the above lemma and [7, Corollary 14.6], we obtain the following estimate on the number of zeros of $L(s,\chi)$ under GRH:

Proposition 2.3. *Assume GRH and let $N(T,\chi)$ denote the number of zeros of $L(s,\chi)$ satisfying* $\mathrm{Re}(s) > 0$ *and* $|\mathrm{Im}(s)| \le T$*, counted with multiplicity. Then for* $T \ge 2$*,*

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi} - \frac{T}{\pi} + O\left(\frac{\log(qT)}{\log\log(qT)}\right).$$

Proof. This is a straightforward consequence of [7, Corollary 14.6] and [8, Theorem 6] (see exercise 1 of [7, Section 14.1]). □

Lemma 2.4.

$$\frac{L'}{L}(\sigma+it,\chi) = \sum_{\substack{\rho=\beta+i\gamma,\\ |\gamma-t|\le 1}} \frac{1}{\sigma+it-\rho} + O(\log(q\tau))$$

holds uniformly for $-1 \le \sigma \le 2$.

Proof. See [7, Lemma 12.6]. □

2.2. Zero-free regions of $L'(s,\chi)$. We begin with a zero-free region of $L'(s,\chi)$ to the right of the critical line.

Proposition 2.5. *$L'(s,\chi)$ has no zeros when*

$$\sigma > 1 + \frac{m}{2}\left(1+\sqrt{1+\frac{4}{m\log m}}\right).$$

Proof. See [13, Theorem 2] for $k=1$. □

From the above proposition, it is not difficult to check that $L'(s,\chi)\neq 0$ when $\sigma \geq 1+3m/2$. Next we introduce a zero-free region of $L'(s,\chi)$ to the left of the critical line.

Proposition 2.6. *$L'(s,\chi)$ has no zeros when $\sigma \leq 0$ and $|t| \geq 6$. Furthermore, assuming GRH,*

(1) *if $\kappa = 0$ and $q \geq 216$, then $L'(s,\chi)$ has a unique zero in $0 < \mathrm{Re}(s) < 1/2$;*
(2) *if $\kappa = 1$ and $q \geq 23$, then $L'(s,\chi)$ has no zeros in $0 < \mathrm{Re}(s) < 1/2$.*

Thus under GRH, for any fixed $\epsilon > 0$, there are only possibly finitely many zeros in the region defined by $0 < \sigma < 1/2$ and $|t| \leq \epsilon$ for any $L'(s,\chi)$.

Proof. See [2, Theorems 1, 8, and 9] and note that $q \geq 3$ in our case. □

3. Key lemmas

For convenience, we define the function $F(s,\chi)$ as follows:

$$F(s,\chi) := \varepsilon(\chi)2^s\pi^{s-1}q^{\frac{1}{2}-s}\sin\left(\frac{\pi(s+\kappa)}{2}\right)\Gamma(1-s), \tag{3.1}$$

where $\varepsilon(\chi)$ is a factor that depends only on χ, satisfying $|\varepsilon(\chi)| = 1$, and recall that κ is determined as in (1.2). Thus from the functional equation for $L(s,\chi)$, we have $L(s,\chi) = F(s,\chi)L(1-s,\overline{\chi})$. We also define the function $G_1(s,\chi)$ associated with $L'(s,\chi)$ as follows:

$$G_1(s,\chi) := -\frac{m^s}{\chi(m)\log m}L'(s,\chi). \tag{3.2}$$

Lemma 3.1. *For $\sigma \geq 2$, we have*

$$|G_1(\sigma+it,\chi)-1| \leq 2\left(1+\frac{8m}{\sigma}\right)\left(1+\frac{1}{m}\right)^{-\sigma}$$

and

$$\left|\frac{G_1}{L}(\sigma+it,\chi)-1\right| \leq 2\left(1+\frac{8m}{\sigma}\right)\left(1+\frac{1}{m}\right)^{-\sigma}$$

Proof. Let $\sigma \geq 2$. Then from (3.2) and by using the Dirichlet series expression of $L'(s,\chi)$, we can calculate

$$\begin{aligned}
|G_1(s,\chi)-1| &= \left| -\frac{m^s}{\chi(m)\log m}\left(-\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n^s}\right) - 1\right| \\
&\leq \frac{m^\sigma}{\log m}\sum_{n=m+1}^{\infty}\frac{\log n}{n^\sigma} \\
&\leq \frac{m^\sigma}{\log m}\frac{\log (m+1)}{(m+1)^\sigma} + \frac{m^\sigma}{\log m}\int_{m+1}^{\infty}\frac{\log x}{x^\sigma}\,\mathrm{d}x \\
&= \frac{m^\sigma}{\log m}\frac{\log (m+1)}{(m+1)^\sigma}\left(1+\frac{m+1}{\sigma-1}+\frac{m+1}{(\sigma-1)^2\log (m+1)}\right) \\
&\leq \frac{m^\sigma}{\log m}\frac{2\log m}{(m+1)^\sigma}\left(1+\frac{4m}{\sigma-1}\right) \leq 2\left(\frac{m}{m+1}\right)^\sigma\left(1+\frac{8m}{\sigma}\right),
\end{aligned}$$

where we have used $m+1 \leq 2m \leq m^2$ and $\sigma - 1 \geq \sigma/2$ in the last two inequalities.

By using the Dirichlet series expansion of $(L'/L)(s,\chi)$, with calculation similar to the above, we can show the second inequality in the lemma. □

Applying Stirling's formula of the following form

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \int_0^\infty \frac{[u]-u+\frac{1}{2}}{u+z}\,\mathrm{d}u \tag{3.3}$$
$$(-\pi+\delta \leq \arg z \leq \pi-\delta, \quad \text{for any } \delta > 0),$$

we can define the holomorphic function

$$\log F(s,\chi) := \log\varepsilon(\chi) + \left(\frac{1}{2}-s\right)\log\frac{q}{2\pi} + \frac{1}{2}\log\frac{2}{\pi} + \log\sin\frac{\pi}{2}(s+\kappa) + \log\Gamma(1-s) \tag{3.4}$$

for $\sigma < 1$ and $|t| > 1$, where $0 \leq \arg\varepsilon(\chi) < 2\pi$ and $\log\sin\frac{\pi}{2}(s+\kappa)$ is the holomorphic function on $\sigma < 1, |t| > 1$ satisfying

$$\log\sin\frac{\pi}{2}(s+\kappa) := \begin{cases} \dfrac{(1-s-\kappa)\pi}{2}i - \log 2 - \displaystyle\sum_{n=1}^{\infty}\frac{e^{\pi i(s+\kappa)n}}{n}, & t > 1; \\ \dfrac{(s+\kappa-1)\pi}{2}i - \log 2 - \displaystyle\sum_{n=1}^{\infty}\frac{e^{-\pi i(s+\kappa)n}}{n}, & t < -1. \end{cases}$$

Under the above definitions, we can show the following lemma.

Lemma 3.2. *For $\sigma < 1$ and $\pm t > 1$, we have*

$$\frac{F'}{F}(s,\chi) = -\log\left(q(1-s)\right) + \log 2\pi \mp \frac{\pi i}{2} + \frac{1}{2(1-s)} + O\left(\frac{1}{|1-s|^2}\right) + O\left(e^{-\pi|t|}\right),$$

where $-\pi/2 < \arg(1-s) < \pi/2$.

Proof. Applying Stirling's formula (3.3) to $\log\Gamma(z)$ for $\arg z \in (-\pi/2, \pi/2)$, we have

$$\log\Gamma(1-s) = \left(\frac{1}{2} - s\right)\log(1-s) - (1-s) + \frac{1}{2}\log 2\pi + \int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+1-s}\,\mathrm{d}u$$

in the region $\sigma < 1$, $|t| > 1$. From (3.4), we can show that

$$\begin{aligned}\log F(s,\chi) = \log\varepsilon(\chi) + \frac{\pi}{2}\left(\frac{1}{2} - \kappa\right)i - 1 \\ + \left(\frac{1}{2} - s\right)\left(\log\left(q(1-s)\right) - \log 2\pi + \frac{\pi i}{2}\right) \\ + s + \int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+1-s}\,\mathrm{d}u - \sum_{n=1}^\infty \frac{e^{\pi i(s+\kappa)n}}{n}\end{aligned}$$

holds when $\sigma < 1$ and $t > 1$. Differentiating both sides of the above equation with respect to s, we obtain

$$\frac{F'}{F}(s,\chi) = -\log\left(q(1-s)\right) + \log 2\pi - \frac{\pi i}{2} + \frac{1}{2(1-s)} + O\left(\frac{1}{|1-s|^2}\right) + O\left(e^{-\pi|t|}\right)$$

for $\sigma < 1$ and $t > 1$. We can show similarly for $\sigma < 1$ and $t < -1$. □

Lemma 3.3. *There exists a $\sigma_1 \leq -1$ such that*

$$\left|\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right| < 2^\sigma$$

holds for any s with $\sigma \leq \sigma_1$ and $|t| \geq 2$.

Proof. Using Lemma 3.2 and the fact that

$$\frac{L'}{L}(1-s,\overline{\chi}) = O(2^\sigma) \qquad (\sigma \leq -1,\ |t| \geq 2),$$

Lemma 3.3 follows. □

Lemma 3.4. *Fix a* σ_1 *that satisfies Lemma 3.3. Then there exists a* $t' > -\sigma_1$ *such that*

(1) *for any* s *satisfying* $\sigma \leq 1/2$ *and* $|t| \geq t' - 1$,

$$\left|\frac{F'}{F}(s,\chi)\right| \geq 1$$

holds and we can take the logarithmic branch of $\log(F'/F)(s,\chi)$ *in that region such that it is holomorphic there and*

$$5\pi/6 < \arg(F'/F)(s,\chi) < 7\pi/6$$

holds;

(2) *assuming GRH, for any* s *satisfying* $\sigma_1 \leq \sigma < 1/2$ *and* $|t| \geq t' - 1$,

$$\frac{L'}{L}(s,\chi) \neq 0$$

holds and we can take the logarithmic branch of $\log(L'/L)(s,\chi)$ *in that region such that it is holomorphic there and*

$$\pi/2 < \arg(L'/L)(s,\chi) < 3\pi/2$$

holds.

Proof.

(1) It immediately follows from Lemma 3.2.

(2) Corollary 10.18 of [7] allows us to show that

$$\operatorname{Re}\left(\frac{L'}{L}(s,\chi)\right) < -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right)$$

holds for $\sigma < 1/2$, under GRH.

For any small $\delta > 0$, let $|t| > \sigma_1 \tan\delta$. Stirling's formula (3.3) implies

$$\frac{1}{2}\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right) = \frac{1}{2}\log\left|\frac{s+\kappa}{2}\right| + O\left(\frac{1}{|s|}\right).$$

Hence we can find some t' large enough so that

$$\operatorname{Re}\left(\frac{L'}{L}(s,\chi)\right) < 0$$

holds for $\sigma_1 \leq \sigma < 1/2$ and $|t| \geq t' - 1$ and hence $(L'/L)(s,\chi) \neq 0$.

Moreover, we can define a branch of $\log(L'/L)(s,\chi)$ so that it is holomorphic in $\sigma_1 \leq \sigma < 1/2$, $|t| \geq t' - 1$ and

$$\frac{\pi}{2} < \arg\frac{L'}{L}(s,\chi) < \frac{3\pi}{2}$$

holds there. □

Now we fix a sufficiently large t' which satisfies Lemma 3.4.

3.1. The function $\mathcal{F}(t,\chi)$. Now we define a few new functions and show a lemma needed to prove the second estimate in Theorem 1.2. We begin by setting

$$\mathcal{F}_1(t,\chi) := -\sum_{\beta'>1/2} \mathrm{Re}\left(\frac{1}{1/2+it-\rho'}\right),$$

$$\mathcal{F}_2(t,\chi) := \sum_{0<\beta'<1/2} \mathrm{Re}\left(\frac{1}{1/2+it-\rho'}\right),$$

$$h(s,\chi) := \left(\frac{q}{\pi}\right)^{(s+\kappa)/2} \Gamma\left(\frac{s+\kappa}{2}\right),$$

$$\xi(s,\chi) := e^{-i\theta_\chi/2} h(s,\chi) L(s,\chi), \quad \eta(s,\chi) := e^{-i\theta_\chi/2} h(s,\chi) L'(s,\chi)$$

where θ_χ is determined as $e^{i\theta_\chi} := \varepsilon(\chi)$ with $\varepsilon(\chi)$ in (3.1). Putting

$$\mathcal{F}(t,\chi) := -\mathrm{Re}\left(\frac{\eta'}{\eta}(1/2+it,\chi)\right)$$

when $\eta(1/2+it,\chi) \neq 0$ and

$$\mathcal{F}(t,\chi) := \lim_{v\to t} \mathcal{F}(v,\chi)$$

when $\eta(1/2+it,\chi) = 0$, we have

$$\mathcal{F}(t,\chi) = \mathcal{F}_1(t,\chi) - \mathcal{F}_2(t,\chi) + O(\log q).$$

Assuming GRH we have $\mathcal{F}_2(t,\chi) = O(1)$, thus

$$\mathcal{F}(t,\chi) = \mathcal{F}_1(t,\chi) + O(\log q). \tag{3.5}$$

This can be shown by using Hadamard's product expansion similar to [14, Lemma 2]. Then we have the following result.

Lemma 3.5. *Assume GRH. Order the nontrivial zeros ρ of $L(s,\chi)$ as $\rho_n = 1/2 + i\gamma_n$ (if it is a multiple zero, we only count it once: namely we only order the zeros according to their locations) with*

$$\cdots < \gamma_{-l} < \cdots < \gamma_{-2} < \gamma_{-1} < \gamma_0 = 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_l < \cdots$$

(if γ_0 exists.) Let $\tilde{n} := \min\{n \in \mathbb{Z} \mid |\gamma_{\pm n}| \geq t''\}$ for a constant $t'' \geq t'$. Then for $|n| \geq \tilde{n}$ we have

$$\int_{\gamma_n}^{\gamma_{n+1}} \mathcal{F}(t,\chi)\,\mathrm{d}t \leq \pi$$

which by (3.5) immediately implies

$$\int_{\gamma_n}^{\gamma_{n+1}} \mathcal{F}_1(t,\chi)\,\mathrm{d}t = O(\log q).$$

Proof. The proof follows those of [14, Lemmas 1 and 4]. We first show that we can find a $t'' \geq t'$ such that when $|t| \geq t''$, $L(1/2+it,\chi) = 0$ holds if and only if $\operatorname{Re}(\eta(1/2+it,\chi)) = 0$. Since

$$\xi'(s,\chi) = \frac{h'}{h}(s,\chi)\xi(s,\chi) + \eta(s,\chi)$$

and from the functional equation $\xi(s,\chi) = \xi(1-s,\overline{\chi})$ which gives $\xi(1/2 + it,\chi), i\xi'(1/2+it,\chi) \in \mathbb{R}$, we can show that

$$\operatorname{Re}\left(\eta\left(\frac{1}{2}+it,\chi\right)\right) = -\operatorname{Re}\left(\frac{h'}{h}\left(\frac{1}{2}+it,\chi\right)\right)\xi\left(\frac{1}{2}+it,\chi\right). \tag{3.6}$$

By taking the logarithmic derivative of the Hadamard product expression for $\Gamma(z)$ (see [7, (C.10) in p. 522]):

$$\frac{\Gamma'}{\Gamma}(z) = -c_E - \sum_{n=0}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n+1}\right)$$

where c_E is the Euler–Mascheroni constant. Thus when $t \neq 0$, we can show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\left(\frac{h'}{h}\left(\frac{1}{2}+it,\chi\right)\right)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\kappa}{2}+i\frac{t}{2}\right)\right)\right)\\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\kappa}{2}\right)\right.\\
&\qquad\qquad \left. + \frac{t^2}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1/4+\kappa/2)\{(n+1/4+\kappa/2)^2+t^2/4\}}\right)\\
&\quad = \frac{1}{t^3}\sum_{n=0}^{\infty}\frac{(2n+1/2+\kappa)^2}{(n+1/4+\kappa/2)(1+((2n+1/2+\kappa)/t)^2)^2}.
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\left(\frac{h'}{h}\left(\frac{1}{2}+it,\chi\right)\right) > 0 \quad \text{when } t > 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}\left(\frac{h'}{h}\left(\frac{1}{2}+it,\chi\right)\right) < 0 \quad \text{when } t < 0.$$

Therefore we can find a $t'' \geq t'$ such that

$$\operatorname{Re}\left(\frac{h'}{h}\left(\frac{1}{2}+it,\chi\right)\right) \neq 0 \quad \text{for all } |t| \geq t''.$$

From (3.6), we easily see that

$$L(1/2+it,\chi) = 0 \Longleftrightarrow \operatorname{Re}(\eta(1/2+it,\chi)) = 0 \tag{3.7}$$

for all $|t| \geq t''$.

With the above result, we can then define $\arg \eta(1/2+it,\chi)$ such that it is continuous on the interval (γ_n, γ_{n+1}). For any t_1, t_2 satisfying $\gamma_n < t_1 < t_2 < \gamma_{n+1}$, we have

$$\int_{t_1}^{t_2} \mathcal{F}(t,\chi)\,\mathrm{d}t = \arg \eta\left(\frac{1}{2}+it_1,\chi\right) - \arg \eta\left(\frac{1}{2}+it_2,\chi\right) \leq \pi$$

by (3.7). □

From now on we fix a $t_0 \in [t''+1, t''+2]$ such that

$$L(\sigma \pm it_0,\chi) \neq 0,\ L'(\sigma \pm it_0,\chi) \neq 0 \tag{3.8}$$

for all $\sigma \in \mathbb{R}$.

3.2. Bounds related to $\log G_1(s,\chi)$. In this subsection, we give bounds for $\arg(G_1/L)(s,\chi)$ and $\arg G_1(s,\chi)$. We take the logarithmic branches so that $\log L(s,\chi)$ and $\log G_1(s,\chi)$ tend to 0 as $\sigma \to \infty$ and are holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid L(z,\chi)=0, \lambda \leq 0\}$ and $\mathbb{C}\backslash\{z+\lambda \mid L'(z,\chi)=0, \lambda \leq 0\}$, respectively. We write

$$-\arg L(\sigma \pm i\tau,\chi) + \arg G_1(\sigma \pm i\tau,\chi) = \arg \frac{G_1}{L}(\sigma \pm i\tau,\chi)$$

and take the argument on the right-hand side so that $\log(G_1/L)(s,\chi)$ tends to 0 as $\sigma \to \infty$ and is holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid (L'/L)(z,\chi) = 0 \text{ or } \infty, \lambda \leq 0\}$.

Lemma 3.6. *Assume GRH and let $\tau > 1$. Then we have for $1/2 < \sigma \leq 10m$,*

$$\arg \frac{G_1}{L}(\sigma \pm i\tau,\chi) \ll \begin{cases} \dfrac{m}{\sigma} & 3 \leq \sigma \leq 10m, \\ \dfrac{m^{1/2}\log\log(q\tau)+m}{\sigma-1/2} & 1/2 < \sigma \leq 3. \end{cases}$$

Proof. Let $\tau > 1$ and $1/2 < \sigma \leq 10m$. Let

$$u_{G_1/L} = u_{G_1/L}(\sigma,\tau;\chi) := \#\left\{u \in [\sigma, 11m] \,\middle|\, \mathrm{Re}\left(\frac{G_1}{L}(u \pm i\tau,\chi)\right) = 0\right\},$$

then

$$\left|\arg \frac{G_1}{L}(\sigma \pm i\tau,\chi)\right| \leq \left(u_{G_1/L}+1\right)\pi.$$

To estimate $u_{G_1/L}$, we set

$$H_1(z,\chi) := \frac{1}{2}\left(\frac{G_1}{L}(z \pm i\tau,\chi) + \frac{G_1}{L}(z \mp i\tau,\overline{\chi})\right)$$

and

$$n_{H_1}(r,\chi) := \#\{z \in \mathbb{C} \mid H_1(z,\chi) = 0, |z-11m| \leq r\}.$$

Since $H_1(x,\chi) = \mathrm{Re}((G_1/L)(x \pm i\tau,\chi))$ for $x \in \mathbb{R}$, we have $u_{G_1/L} \le n_{H_1}(11m-\sigma,\chi)$ for $1/2 < \sigma \le 10m$.

Now we estimate $n_{H_1}(11m-\sigma,\chi)$. We take $\epsilon = \epsilon_{\sigma,\tau} > 0$. It is easy to show that

$$n_{H_1}(11m-\sigma,\chi) \le \frac{1}{\log\left(1+\epsilon/(11m-\sigma)\right)} \int_0^{11m-\sigma+\epsilon} \frac{n_{H_1}(r,\chi)}{r}\,\mathrm{d}r.$$

Applying using Jensen's theorem (cf. [11, Section 3.61]), we have

$$\int_0^{11m-\sigma+\epsilon} \frac{n_{H_1}(r,\chi)}{r}\,\mathrm{d}r = \frac{1}{2\pi}\int_0^{2\pi} \log\left|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)\right|\mathrm{d}\theta - \log\left|H_1(11m,\chi)\right|.$$

We can easily see that $\log|H_1(11m,\chi)| = O(1)$ by applying the second inequality in Lemma 3.1. Therefore

$$\left|\arg\frac{G_1}{L}(\sigma \pm i\tau,\chi)\right| \le \frac{1}{\log\left(1+\epsilon/(11m-\sigma)\right)} \times \left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)\right|\mathrm{d}\theta + C\right)$$

for some absolute constant $C > 0$.

Now we divide the rest of the proof in two cases:

(a) For $3 \le \sigma \le 10m$, we restrict ϵ to satisfy $0 < \epsilon \le \sigma - 2$. Then $11m + (11m-\sigma+\epsilon)\cos\theta \ge 2$. Applying the second inequality in Lemma 3.1, we can easily obtain

$$\left|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)\right| \le \frac{100m}{11m+(11m-\sigma+\epsilon)\cos\theta}.$$

Applying Cauchy's theorem and the fact that for $a > 1$,

$$\log\left(2a+2\cos\theta\right) = \log\left(1+\left(a-\sqrt{a^2-1}\right)e^{-i\theta}\right) + \log\left(e^{i\theta}+a+\sqrt{a^2-1}\right),$$

we can show that for $c > r > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|c + r\cos\theta|\,\mathrm{d}\theta = \log\frac{c+\sqrt{c^2-r^2}}{2} \tag{3.9}$$

holds. By using (3.9), we can easily show that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log\left|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)\right|\mathrm{d}\theta \\
&\quad\le \log(100m) - \frac{1}{2\pi}\int_0^{2\pi} \log\left(11m+(11m-\sigma+\epsilon)\cos\theta\right)\mathrm{d}\theta \\
&\quad= \log(100m) - \log\frac{11m+\sqrt{11m^2-(11m-\sigma+\epsilon)^2}}{2} \\
&\quad\le \log(100m) - \log\frac{11m}{2} \ll 1.
\end{aligned}$$

Note that $\epsilon/(11m-\sigma) \le 10$, thus $\log(1+\epsilon/(11m-\sigma)) \gg \epsilon/(11m-\sigma)$. Hence

$$\arg \frac{G_1}{L}(\sigma \pm i\tau, \chi) \ll \frac{11m-\sigma}{\epsilon} \ll \frac{m}{\epsilon}.$$

By taking $\epsilon = \sigma - 2$, we obtain

$$\arg \frac{G_1}{L}(\sigma \pm i\tau, \chi) \ll \frac{m}{\sigma}.$$

This is the first inequality in Lemma 3.6.

(b) For $1/2 < \sigma \le 3$, we restrict ϵ to satisfy $0 < \epsilon < \sigma - 1/2$ and we divide the interval of integration into
- $\mathcal{I}_1 := \{\theta \in [0, 2\pi] \mid 11m + (11m - \sigma + \epsilon)\cos\theta \ge 2\}$ and
- $\mathcal{I}_2 := \{\theta \in [0, 2\pi] \mid 11m + (11m - \sigma + \epsilon)\cos\theta < 2\}$.

Since $11m + (11m-\sigma+\epsilon)\cos\theta > 1/2$ and $11m - \sigma + \epsilon < 11m$, on $\mathcal{I}_1$, as in the calculation of case (a), we can show that

$$\begin{aligned}\frac{1}{2\pi}\int_{\theta\in\mathcal{I}_1} &\log|H_1(11m + (11m-\sigma+\epsilon)e^{i\theta}, \chi)| \, \mathrm{d}\theta \\ &\le \frac{1}{2\pi}\int_{\theta\in\mathcal{I}_1} \log \frac{100m}{11m + (11m-\sigma+\epsilon)\cos\theta} \, \mathrm{d}\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \log \frac{100m}{11m + (11m-\sigma+\epsilon)\cos\theta} \, \mathrm{d}\theta \ll 1.\end{aligned}$$

Now we estimate the integral on $\mathcal{I}_2$. Setting

$$\cos\theta_0 := \frac{11m-2}{11m-\sigma+\epsilon}$$

for $\theta_0 \in (0, \pi/2)$, we have $\mathcal{I}_2 = (\pi - \theta_0, \pi + \theta_0)$. Applying Lemma 2.4 and Proposition 2.3, and noting that $(L'/L)(x+iy,\chi) = O(1)$ when $x \ge 2$, we have

$$\frac{L'}{L}(x+iy, \chi) = O\left(\frac{\log(q(|y|+1))}{x - 1/2}\right)$$

for $1/2 < x \le A$, for any fixed $A \ge 2$. Thus,

$$|H_1(11m + (11m-\sigma+\epsilon)e^{i\theta}, \chi)| \le C_1 \frac{m^2}{\log m} \frac{\log(q(\tau + 11m))}{11m + (11m-\sigma+\epsilon)\cos\theta - 1/2}$$

for some absolute constant $C_1 > 0$. Hence

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\theta\in\mathcal{I}_2}\log|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta\\
&\qquad\le\frac{1}{2\pi}\int_{\pi-\theta_0}^{\pi+\theta_0}\log\frac{C_1m^2}{\log m}\frac{\log\left(q(\tau+11m)\right)}{11m-1/2+(11m-\sigma+\epsilon)\cos\theta}\,\mathrm{d}\theta\\
&\qquad=\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0}\log\frac{C_1m^2}{\log m}\frac{\log\left(q(\tau+11m)\right)}{11m-1/2-(11m-\sigma+\epsilon)\cos\theta}\,\mathrm{d}\theta\\
&\qquad=\frac{\theta_0}{\pi}\log\frac{C_1m^2\log\left(q(\tau+11m)\right)}{\log m}\\
&\qquad\qquad-\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0}\log\left(11m-\frac{1}{2}-(11m-\sigma+\epsilon)\cos\theta\right)\mathrm{d}\theta.
\end{aligned}$$

We note that $\cos\theta_0 = 1 + O(1/m)$. By using $1-\cos\theta_0 = 2\sin^2(\theta_0/2)$, we can show

$$\theta_0 \ll \left|\sin\frac{\theta_0}{2}\right| \ll \frac{1}{m^{1/2}}.$$

Hence,

$$\begin{aligned}
&\int_{-\theta_0}^{\theta_0}\log\left(11m-\frac{1}{2}-(11m-\sigma+\epsilon)\cos\theta\right)\mathrm{d}\theta\\
&=\int_{-\theta_0}^{\theta_0}\log\frac{11m-1/2-(11m-\sigma+\epsilon)\cos\theta}{11m-1/2}\,\mathrm{d}\theta+\int_{-\theta_0}^{\theta_0}\log\left(11m-1/2\right)\mathrm{d}\theta\\
&=\int_{-\theta_0}^{\theta_0}\log\left(1-\frac{11m-\sigma+\epsilon}{11m-1/2}\cos\theta\right)\mathrm{d}\theta+O\left(\frac{\log m}{m^{1/2}}\right)
\end{aligned}$$

Recalling that $\sigma-\epsilon > 1/2$ and $\theta_0\in(0,\pi/2)$, we have

$$\int_{-\theta_0}^{\theta_0}\log\left(1-\cos\theta\right)\mathrm{d}\theta\le\int_{-\theta_0}^{\theta_0}\log\left(1-\frac{11m-\sigma+\epsilon}{11m-1/2}\cos\theta\right)\mathrm{d}\theta\le 0.$$

Meanwhile,

$$\begin{aligned}
\int_{-\theta_0}^{\theta_0}\log\left(1-\cos\theta\right)\mathrm{d}\theta&=\int_{-\theta_0}^{\theta_0}\log\left(2\sin^2\frac{\theta}{2}\right)\mathrm{d}\theta\\
&=2\theta_0\log 2+4\int_0^{\theta_0}\log\left(\sin\frac{\theta}{2}\right)\mathrm{d}\theta\\
&=2\theta_0\log 2+4\int_0^{\theta_0}\log\frac{\sin\left(\theta/2\right)}{\theta/2}\,\mathrm{d}\theta+4\int_0^{\theta_0}\log\frac{\theta}{2}\,\mathrm{d}\theta\\
&=O\left(\theta_0\right)+O\left(\theta_0^3\right)+O\left(\theta_0\log\theta_0^{-1}\right)=O\left(\frac{\log m}{m^{1/2}}\right).
\end{aligned}$$

Therefore when $1/2<\sigma\le 3$, recalling that $m\ll \log q\le \log(q\tau)$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&\log|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta\\&=\frac{1}{2\pi}\left(\int_{\theta\in\mathcal{I}_1}+\int_{\theta\in\mathcal{I}_2}\right)\log|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta\\&\ll 1+\frac{\log\log(q(\tau+11m))}{m^{1/2}}+\frac{\log m}{m^{1/2}}\ll 1+\frac{\log\log(q\tau)}{m^{1/2}}.\end{aligned}$$

To obtain the last inequality, we note that it is obvious if $\tau\gg m$, otherwise if $\tau\ll m$, then $\log\log(q(\tau+11m))\ll\log\log(qm)\ll \log\log q\ll\log\log(q\tau)$ since $\tau>1$.

Since $0<\epsilon/(11m-\sigma)<1$, we have $\log(1+\epsilon/(11m-\sigma))\gg \epsilon/m$, thus

$$\arg\frac{G_1}{L}(\sigma\pm i\tau,\chi)\ll\frac{m}{\epsilon}\left(1+\frac{\log\log(q\tau)}{m^{1/2}}\right).$$

Taking $\epsilon=(\sigma-1/2)/2$, we obtain the second inequality in Lemma 3.6. ☐

Lemma 3.7. *Assume GRH and let $A\ge 2$ be fixed. Then there exists a constant $C_0>0$ such that*

$$|L'(\sigma+it,\chi)|\le\exp\left(C_0\left(\frac{(\log q\tau)^{2(1-\sigma)}}{\log\log(q\tau)}+(\log(q\tau))^{1/10}\right)\right)$$

holds for $1/2-1/\log\log(q\tau)\le\sigma\le A$ and $\tau=|t|+4$.

Proof. Applying Lemma 2.1 and Cauchy's integral formula, Lemma 3.7 follows. ☐

Lemma 3.8. *Assume GRH. Then for any $1/2\le\sigma\le 3/4$, we have*

$$\begin{aligned}\arg G_1(\sigma\pm i\tau,\chi)=O\Bigg(&m^{1/2}(\log\log(q\tau))\\&\times\left(m^{1/2}+(\log(q\tau))^{1/10}+\frac{(\log(q\tau))^{2(1-\sigma)}}{(\log\log(q\tau))^{3/2}}\right)\Bigg).\end{aligned}$$

Proof. The proof is similar to that of Lemma 3.6 but we provide the details for clarity. Let $1/2\le\sigma\le 3/4$ and $\tau>1$ be large. Put

$$u_{G_1}=u_{G_1}(\sigma,\tau;\chi):=\#\left\{u\in[\sigma,4m]\mid \mathrm{Re}\,(G_1(u\pm i\tau,\chi))=0\right\},$$

then

$$|\arg G_1(\sigma\pm i\tau,\chi)|\le(u_{G_1}+1)\,\pi.$$

To estimate u_{G_1}, we set

$$X_1(z,\chi) := \frac{G_1(z \pm i\tau, \chi) + G_1(z \mp i\tau, \overline{\chi})}{2}$$

and

$$n_{X_1}(r,\chi) := \#\{z \in \mathbb{C} \mid X_1(z,\chi) = 0, |z - 4m| \le r\}.$$

Then we have $u_{G_1} \le n_{X_1}(4m - \sigma, \chi)$.

Now we estimate $n_{X_1}(4m-\sigma,\chi)$. For each $\sigma \in [1/2, 3/4]$, we take $\epsilon = \epsilon_{\sigma,\tau}$ satisfying $0 < \epsilon \le \sigma - 1/2 + (\log\log(q\tau))^{-1}$. It is easy to show that

$$n_{X_1}(4m-\sigma,\chi) \le \frac{1+3m}{\epsilon} \int_0^{4m-\sigma+\epsilon} \frac{n_{X_1}(r,\chi)}{r}\,\mathrm{d}r.$$

Applying Jensen's theorem (cf. [11, Section 3.61]), we have

$$\begin{aligned}&\int_0^{4m-\sigma+\epsilon} \frac{n_{X_1}(r,\chi)}{r}\,\mathrm{d}r \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} \log|X_1(4m + (4m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta - \log|X_1(4m,\chi)|.\end{aligned}$$

By using the first inequality in Lemma 3.1, we can easily show

$$\log|X_1(4m,\chi)| = O(1).$$

As in the proof of Lemma 3.6, we divide the interval of integration into

- $\mathcal{J}_1 := \{\theta \in [0,2\pi] \mid 4m + (4m-\sigma+\epsilon)\cos\theta \ge 2\}$ and
- $\mathcal{J}_2 := \{\theta \in [0,2\pi] \mid 4m + (4m-\sigma+\epsilon)\cos\theta < 2\}$.

Then similarly, applying the first inequality in Lemma 3.1 and (3.9), we can show that

$$\frac{1}{2\pi}\int_{\theta\in\mathcal{J}_1} \log|X_1(4m + (4m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta = O(1).$$

Next we estimate the integral on $\mathcal{J}_2$. Setting

$$\cos\theta_0 := \frac{4m-2}{4m-\sigma+\epsilon}$$

for $\theta_0 \in (0,\pi/2)$, we have $\mathcal{J}_2 = (\pi-\theta_0, \pi+\theta_0)$ and $\theta_0 = O(m^{-1/2})$. Applying Lemma 3.7, we have

$$\begin{aligned}&|X_1(4m + (4m-\sigma+\epsilon)e^{i\theta},\chi)| \\ &\quad \le \frac{m^2}{\log m}\exp\left(C_0'\left(\frac{(\log(q\tau))^{-3m-2(4m-\sigma+\epsilon)\cos\theta}}{\log\log(q\tau)} + (\log(q\tau))^{1/10}\right)\right)\end{aligned}$$

for some absolute constant $C_0' > 0$. Thus,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\theta\in\mathcal{J}_2}\log|X_1(4m+(4m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta\\
&\qquad\le \theta_0\left(\log\frac{m^2}{\log m}+C_0'(\log(q\tau))^{1/10}\right)\\
&\qquad\qquad+\frac{C_0'(\log(q\tau))^{-3m}}{2\pi\log\log(q\tau)}\int_{\pi-\theta_0}^{\pi+\theta_0}(\log(q\tau))^{-2(4m-\sigma+\epsilon)\cos\theta}\,\mathrm{d}\theta\\
&\qquad\le \theta_0\left(\log\frac{m^2}{\log m}+C_0'(\log(q\tau))^{1/10}\right)\\
&\qquad\qquad+\frac{C_0'(\log(q\tau))^{-3m}}{2\pi\log\log(q\tau)}\int_{0}^{2\pi}(\log(q\tau))^{-2(4m-\sigma+\epsilon)\cos\theta}\,\mathrm{d}\theta.
\end{aligned}$$

As in [1, pp. 2252–2253], we use

$$\int_0^{2\pi} e^{-x\cos\theta}\,\mathrm{d}\theta = 2\pi I_0(x),$$

where I_ν is the Bessel function and

$$I_0(x)=\frac{e^x}{\sqrt{2\pi x}}(1+o(1)).$$

Then there exists a constant $C_1' > 0$ such that

$$I_0(2(4m-\sigma+\epsilon)\log\log(q\tau)) \le C_1'\frac{(\log(q\tau))^{2(4m-\sigma+\epsilon)}}{(m\log\log(q\tau))^{1/2}}.$$

Hence,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\theta\in\mathcal{J}_2}\log|X_1(4m+(4m-\sigma+\epsilon)e^{i\theta},\chi)|\,\mathrm{d}\theta\\
&\qquad\qquad\ll\frac{1}{m^{1/2}}\left((\log(q\tau))^{1/10}+\frac{(\log(q\tau))^{2(1-\sigma+\epsilon)}}{(\log\log(q\tau))^{3/2}}\right).
\end{aligned}$$

Concluding the above, we have

$$\begin{aligned}
\arg G_1(\sigma\pm i\tau,\chi) &\ll n_{X_1}(4m-\sigma,\chi)\\
&\ll\frac{m}{\epsilon}\left(1+\frac{1}{m^{1/2}}\left((\log(q\tau))^{1/10}+\frac{(\log(q\tau))^{2(1-\sigma+\epsilon)}}{(\log\log(q\tau))^{3/2}}\right)\right).
\end{aligned}$$

Taking $\epsilon=(\log\log(q\tau))^{-1}$ completes the proof. □

4. Proof of theorems

In this section, we prove Theorems 1.1 and 1.2. We put the proof of each theorem in a separate subsection. In each subsection, we first prove a proposition which states out the main term of the equation in our main theorem. We use the functions $F(s,\chi)$ and $G_1(s,\chi)$ defined in the previous section (see (3.1) and (3.2)).

4.1. Proof of Theorem 1.1. The following proposition states out the main term of the equation in Theorem 1.1.

Proposition 4.1. *Assume GRH. Take t_0 as in* (3.8). *From Proposition 2.5, we note that $L'(s,\chi) \neq 0$ when $\sigma \geq 4m$. Then for $T \geq t_0$ which satisfies $L(\sigma \pm iT,\chi) \neq 0$ and $L'(\sigma \pm iT,\chi) \neq 0$ for any $\sigma \in \mathbb{R}$, we have*

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<|\gamma'|\leq T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{\pi}\log\log\frac{qT}{2\pi} + \frac{T}{\pi}\left(\frac{1}{2}\log m - \log\log m\right) - \frac{2}{q}\operatorname{Li}\left(\frac{qT}{2\pi}\right)$$
$$- \mathcal{I}(t_0,\chi) + \mathcal{I}(-t_0,\chi) + \mathcal{I}(T,\chi) - \mathcal{I}(-T,\chi)$$
$$+ O(\log\log q) + O(m),$$

where

$$\mathcal{I}(\tau,\chi) := \frac{1}{2\pi}\int_{1/2}^{4m} \left(-\arg L(\sigma+i\tau,\chi) + \arg G_1(\sigma+i\tau,\chi)\right)\mathrm{d}\sigma$$

and the logarithmic branches are taken as in Section 3.2.

Proof. We take σ_1 which satisfies Lemma 3.3 and fix it. Take $T \geq t_0$ such that $L(\sigma \pm iT,\chi) \neq 0$ and $L'(\sigma \pm iT,\chi) \neq 0$ for all $\sigma \in \mathbb{R}$. Let $\delta \in (0,1/2]$ and put $b := 1/2 - \delta$.

Applying Littlewood's lemma (cf. [11, Section 3.8]) to $G_1(s,\chi)$ on the rectangles with vertices $b \pm it_0$, $4m \pm it_0$, $4m \pm iT$, and $b \pm iT$, we obtain

$$(4.1)\quad 2\pi \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<\pm\gamma'\leq T}} (\beta' - b)$$
$$= \int_{t_0}^{T} \log|G_1(b \pm it,\chi)|\,\mathrm{d}t - \int_{t_0}^{T} \log|G_1(4m \pm it,\chi)|\,\mathrm{d}t$$
$$\mp \int_{b}^{4m} \arg G_1(\sigma \pm it_0,\chi)\,\mathrm{d}\sigma \pm \int_{b}^{4m} \arg G_1(\sigma \pm iT,\chi)\,\mathrm{d}\sigma$$
$$=: I_1^{\pm} + I_2^{\pm} + I_3^{\pm} + I_4^{\pm}.$$

Applying the first inequality in Lemma 3.1, we can show that $I_2^+ = I_2^- = O(m)$. Below we estimate I_1^+.

$$
\begin{aligned}
I_1^+ &= \int_{t_0}^{T} \log |G_1(b+it,\chi)|\,\mathrm{d}t = \int_{t_0}^{T} \log\left(\frac{m^b}{\log m}|L'(b+it,\chi)|\right)\mathrm{d}t \\
&= \int_{t_0}^{T} \log\frac{m^b}{\log m}\,\mathrm{d}t + \int_{t_0}^{T} \log|L'(b+it,\chi)|\,\mathrm{d}t \\
&= (b\log m - \log\log m)T + \int_{t_0}^{T}\log|F(b+it,\chi)|\,\mathrm{d}t \\
&\quad + \int_{t_0}^{T}\log\left|\frac{F'}{F}(b+it,\chi)\right|\mathrm{d}t + \int_{t_0}^{T}\log|L(1-b-it,\overline{\chi})|\,\mathrm{d}t \\
&\quad + \int_{t_0}^{T}\log\left|1-\frac{1}{\frac{F'}{F}(b+it,\chi)}\frac{L'}{L}(1-b-it,\overline{\chi})\right|\mathrm{d}t + O(t_0\log m) \\
&=: I_{11}+I_{12}+I_{13}+I_{14}+I_{15}+O(\log m).
\end{aligned}
\tag{4.2}
$$

From (3.4) and Stirling's formula (3.3), we have

$$
\begin{aligned}
I_{12} &= \int_{t_0}^{T}\log|F(b+it,\chi)|\,\mathrm{d}t = \int_{t_0}^{T}\left(\left(\frac{1}{2}-b\right)\log\frac{qt}{2\pi} + O\left(\frac{1}{t^2}\right)\right)\mathrm{d}t \\
&= \left(\frac{1}{2}-b\right)\left(T\log\frac{qT}{2\pi} - T - t_0\log\frac{qt_0}{2\pi} + t_0\right) + O(1).
\end{aligned}
$$

Lemma 3.2 gives us

$$
\log\left|\frac{F'}{F}(b+it,\chi)\right| = \mathrm{Re}\left(\log\frac{F'}{F}(b+it,\chi)\right) = \log\log\frac{q|t|}{2\pi} + O\Big(\frac{1}{t^2\log(q|t|)}\Big),
$$

thus we have

$$
\begin{aligned}
I_{13} &= \int_{t_0}^{T}\log\left|\frac{F'}{F}(b+it,\chi)\right|\mathrm{d}t = \int_{t_0}^{T}\left(\log\log\frac{qt}{2\pi} + O\left(\frac{1}{t^2\log(q|t|)}\right)\right)\mathrm{d}t \\
&= T\log\log\frac{qT}{2\pi} - t_0\log\log\frac{qt_0}{2\pi} - \int_{t_0}^{T}\frac{1}{\log\frac{qt}{2\pi}}\,\mathrm{d}t + O(1) \\
&= T\log\log\frac{qT}{2\pi} - t_0\log\log\frac{qt_0}{2\pi} - \frac{2\pi}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(t_0) \\
&= T\log\log\frac{qT}{2\pi} - \frac{2\pi}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(\log\log q).
\end{aligned}
$$

Next, we estimate I_{14}. We note that $\overline{L(\overline{s},\overline{\chi})} = L(s,\chi)$, hence $|L(1-b-it,\overline{\chi})| = |L(1-b+it,\chi)|$. Take the logarithmic branch of $\log L(s,\chi)$ so that $\log L(s,\chi) = \sum_{n=2}^{\infty}\chi(n)\Lambda(n)(\log n)^{-1}n^{-s}$ holds for $\mathrm{Re}(s)>1$ and that it is holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid L(z,\chi)=0, \lambda\le 0\}$. Then applying Cauchy's integral theorem to $\log L(s,\chi)$ on the rectangle with vertices

$1-b+it_0$, $4m+it_0$, $4m+iT$, $1-b+iT$ and taking the imaginary part, we can show that

$$\begin{aligned} I_{14} &= \int_{t_0}^{T} \log |L(1-b-it,\overline{\chi})|\,\mathrm{d}t = \int_{t_0}^{T} \log |L(1-b+it,\chi)|\,\mathrm{d}t \\ &= \int_{1-b}^{4m} \arg L(\sigma+it_0,\chi)\,\mathrm{d}\sigma - \int_{1-b}^{4m} \arg L(\sigma+iT,\chi)\,\mathrm{d}\sigma + O(1). \end{aligned}$$

Finally we estimate I_{15}. Since $L(s,\chi) = F(s,\chi)L(1-s,\overline{\chi})$, we have

$$\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(s,\chi) = 1 - \frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi}). \tag{4.3}$$

From Lemma 3.4, the function on the left-hand side of (4.3) is holomorphic and has no zeros in $\sigma_1 \le \sigma < 1/2, |t| \ge t' - 1$. From Lemma 3.3, the function on the right-hand side of (4.3) is holomorphic and has no zeros in $\sigma \le \sigma_1, |t| \ge 2$. Thus we can determine

$$\log\left(1 - \frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)$$

so that it tends to 0 as $\sigma \to -\infty$ which follows from Lemma 3.3, and that it is holomorphic in $\sigma < 1/2, |t| > t_0 - 1 (> t' - 1)$. Now we apply Cauchy's integral theorem to it on the trapezoid with vertices $-t_0+it_0$, $b+it_0$, $b+iT$, and $-T+iT$. Lemma 3.3 allows us to show

$$\left(\int_{\sigma_1+iT}^{-T+iT} + \int_{-T+iT}^{-t_0+it_0} + \int_{-t_0+it_0}^{\sigma_1+it_0}\right) \log\left(1 - \frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)\mathrm{d}s = O(1).$$

Thus taking the imaginary part, we obtain

$$\begin{aligned} &\int_{t_0}^{T} \log\left|1 - \frac{1}{\frac{F'}{F}(b+it,\chi)}\frac{L'}{L}(1-b-it,\overline{\chi})\right|\mathrm{d}t \\ &\quad = \int_{\sigma_1}^{b} \arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma+iT,\chi)}\frac{L'}{L}(1-\sigma-iT,\overline{\chi})\right)\mathrm{d}\sigma \\ &\qquad - \int_{\sigma_1}^{b} \arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma+it_0,\chi)}\frac{L'}{L}(1-\sigma-it_0,\overline{\chi})\right)\mathrm{d}\sigma + O(1) \end{aligned}$$

Now we let

$$\log\left(\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(s,\chi)\right) = \log\left(1 - \frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)$$

and determine the logarithmic branch of $\log(F'/F)(s,\chi)$ and $\log(L'/L)(s,\chi)$ in the region $\sigma_1 \le \sigma < 1/2, |t| \ge t_0 - 1$ as in Lemma 3.4. Note that both of them and the functions on both sides of (4.3) are all continuous with respect to s in $\sigma_1 \le \sigma < 1/2, |t| \ge t_0 - 1$. Furthermore, the two regions $\sigma_1 \le \sigma < 1/2, t \ge t_0 - 1$ and $\sigma_1 \le \sigma < 1/2, -t \ge t_0 - 1$ are connected. Thus we have

$$\arg\left(1 - \frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right) = -\arg\frac{F'}{F}(s,\chi) + \arg\frac{L'}{L}(s,\chi) + 2\pi n_q$$

for some $n_q \in \mathbb{Z}$ that depends only at most on q. From our choice of logarithmic branch, we have $n_q = 0$. Thus,

$$-\frac{2\pi}{3} < \arg\left(1 - \frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right) < \frac{2\pi}{3} \tag{4.4}$$

for $\sigma_1 \le \sigma < 1/2, |t| \ge t_0 - 1$. Therefore we obtain

$$I_{15} = \int_{t_0}^{T} \log\left|1 - \frac{1}{\frac{F'}{F}(b+it,\chi)}\frac{L'}{L}(1-b-it,\overline{\chi})\right| \mathrm{d}t = O(1).$$

Collecting the above calculations, we have

$$\begin{aligned} I_1^+ &= T\log\log\frac{qT}{2\pi} + (b\log m - \log\log m)\,T - \frac{2\pi}{q}\operatorname{Li}\left(\frac{qT}{2\pi}\right) \\ &\quad + \left(\frac{1}{2} - b\right)\left(T\log\frac{qT}{2\pi} - T - t_0\log\frac{qt_0}{2\pi} + t_0\right) \\ &\quad + \int_{1-b}^{4m} \arg L(\sigma + it_0,\chi)\,\mathrm{d}\sigma - \int_{1-b}^{4m} \arg L(\sigma + iT,\chi)\,\mathrm{d}\sigma + O(\log\log q). \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} I_1^- &= T\log\log\frac{qT}{2\pi} + (b\log m - \log\log m)\,T - \frac{2\pi}{q}\operatorname{Li}\left(\frac{qT}{2\pi}\right) \\ &\quad + \left(\frac{1}{2} - b\right)\left(T\log\frac{qT}{2\pi} - T - t_0\log\frac{qt_0}{2\pi} + t_0\right) \\ &\quad - \int_{1-b}^{4m} \arg L(\sigma - it_0,\chi)\,\mathrm{d}\sigma + \int_{1-b}^{4m} \arg L(\sigma - iT,\chi)\,\mathrm{d}\sigma + O(\log\log q). \end{aligned}$$

Thus we have

$$\begin{aligned} 2\pi \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<\gamma'\le T}} (\beta' - b) \\ &= T\log\log\frac{qT}{2\pi} + (b\log m - \log\log m)\,T - \frac{2\pi}{q}\operatorname{Li}\left(\frac{qT}{2\pi}\right) \\ &\quad + \left(\frac{1}{2} - b\right)\left(T\log\frac{qT}{2\pi} - T - t_0\log\frac{qt_0}{2\pi} + t_0\right) \\ &\quad + \int_{1-b}^{4m} \arg L(\sigma + it_0,\chi)\,\mathrm{d}\sigma - \int_{1-b}^{4m} \arg L(\sigma + iT,\chi)\,\mathrm{d}\sigma \\ &\quad - \int_{b}^{4m} \arg G_1(\sigma + it_0,\chi)\,\mathrm{d}\sigma + \int_{b}^{4m} \arg G_1(\sigma + iT,\chi)\,\mathrm{d}\sigma \\ &\quad + O(\log\log q) + O(m) \end{aligned}$$

and

$$\begin{aligned}2\pi \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<-\gamma'\le T}} (\beta'-b) &= T\log\log\frac{qT}{2\pi} + (b\log m - \log\log m)\,T - \frac{2\pi}{q}\operatorname{Li}\left(\frac{qT}{2\pi}\right)\\ &\quad + \left(\frac{1}{2}-b\right)\left(T\log\frac{qT}{2\pi} - T - t_0\log\frac{qt_0}{2\pi} + t_0\right)\\ &\quad - \int_{1-b}^{4m} \arg L(\sigma - it_0,\chi)\,\mathrm{d}\sigma + \int_{1-b}^{4m} \arg L(\sigma - iT,\chi)\,\mathrm{d}\sigma\\ &\quad + \int_{b}^{4m} \arg G_1(\sigma - it_0,\chi)\,\mathrm{d}\sigma - \int_{b}^{4m} \arg G_1(\sigma - iT,\chi)\,\mathrm{d}\sigma\\ &\quad + O(\log\log q) + O(m).\end{aligned}$$

Taking $\delta \to 0$, we obtain Proposition 4.1. □

Now we are ready to complete the proof of Theorem 1.1.

Referring to [2, Theorem 6], we have

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ |\gamma'|\le t_0}} (\beta'-1/2) \ll m^{1/2}\log q. \tag{4.5}$$

This also implies that when $2 \le T < t_0$,

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ |\gamma'|\le T}} (\beta'-1/2) \ll m^{1/2}\log q.$$

Next, we estimate

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<|\gamma'|\le T}} (\beta'-1/2).$$

We divide the proof in two cases.

Case 1. For $T \ge t_0$ which satisfies $L(\sigma \pm iT,\chi) \ne 0, L'(\sigma \pm iT,\chi) \ne 0$ for all $\sigma \in \mathbb{R}$.

In this case, we apply Proposition 4.1 and provoke Lemmas 2.2, 3.6, and 3.8 to obtain the error term.

We apply Lemmas 2.2, 3.8, and 3.6 to obtain

$$\int_{1/2}^{1/2+(\log(q\tau))^{-1}} \arg L(\sigma \pm i\tau,\chi)\,\mathrm{d}\sigma \ll 1,$$

$$\int_{3}^{4m} \arg\frac{G_1}{L}(\sigma \pm i\tau,\chi)\,\mathrm{d}\sigma \ll m\log m$$

for $\tau \geq t_0$, and

$$\int_{1/2}^{1/2+(\log(qt_0))^{-1}} \arg G_1(\sigma \pm it_0,\chi)\,\mathrm{d}\sigma \ll \frac{m^{1/2}}{(\log\log q)^{1/2}},$$

$$\int_{1/2}^{1/2+(\log(qT))^{-1}} \arg G_1(\sigma \pm iT,\chi)\,\mathrm{d}\sigma \ll \frac{m^{1/2}}{(\log\log(qT))^{1/2}},$$

$$\int_{1/2+(\log(qt_0))^{-1}}^{3} \arg \frac{G_1}{L}(\sigma \pm it_0,\chi)\,\mathrm{d}\sigma \ll m^{1/2}(\log\log q)^2 + m\log\log q,$$

$$\int_{1/2+(\log(qT))^{-1}}^{3} \arg \frac{G_1}{L}(\sigma \pm iT,\chi)\,\mathrm{d}\sigma \ll m^{1/2}(\log\log(qT))^2 + m\log\log(qT).$$

Inserting the above estimates into the formula given in Proposition 4.1 and adding this to (4.5), we obtain Theorem 1.1 for Case 1.

Case 2. For $T \geq t_0$ such that any of $L(\sigma+iT,\chi)\neq 0$, $L(\sigma-iT,\chi)\neq 0$, $L'(\sigma+iT,\chi)\neq 0$, or $L'(\sigma-iT,\chi)\neq 0$ is *not* satisfied for some $\sigma\in\mathbb{R}$.

In this case, first we look for some small $0<\epsilon<(\log\log(qT))^{-1}$ such that $L(\sigma\pm i(T\pm\epsilon),\chi)\neq 0, L'(\sigma\pm i(T\pm\epsilon),\chi)\neq 0$ holds for all $\sigma\in\mathbb{R}$ and apply the method of Case 1, so we obtain

$$\begin{aligned}\sum_{\substack{\rho'=\beta'+i\gamma',\\ |\gamma'|\leq T\pm\epsilon}}\left(\beta'-\frac{1}{2}\right) &= \frac{(T\pm\epsilon)}{\pi}\log\log\frac{q(T\pm\epsilon)}{2\pi}\\ &\quad+\frac{T\pm\epsilon}{\pi}\left(\frac{1}{2}\log m-\log\log m\right)-\frac{2}{q}\operatorname{Li}\left(\frac{q(T\pm\epsilon)}{2\pi}\right)\\ &\quad+O\Big(m^{1/2}(\log\log(qT))^2+m\log\log(qT)+m^{1/2}\log q\Big).\end{aligned}$$

Noting that

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0-1<|\gamma'|\leq T-\epsilon}}\left(\beta'-\frac{1}{2}\right)\leq\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0-1<|\gamma'|\leq T}}\left(\beta'-\frac{1}{2}\right)\leq\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0-1<|\gamma'|\leq T+\epsilon}}\left(\beta'-\frac{1}{2}\right)$$

together with (4.5), we easily show that the equation in Theorem 1.1 also holds for this case. □

4.2. Proof of Theorem 1.2. The following proposition states out the main term of the equation in Theorem 1.2.

Proposition 4.2. *Assume GRH. Then for $T\geq 2$ which satisfies $L(\sigma\pm iT,\chi)\neq 0$ and $L'(\sigma\pm iT,\chi)\neq 0$ for all $\sigma\in\mathbb{R}$, we have*

$$\begin{aligned}N_1(T,\chi)=\frac{T}{\pi}\log\frac{qT}{2m\pi}-\frac{T}{\pi}+A(T,\chi)+B(T,\chi)-A(-T,\chi)-B(-T,\chi)\\ +O\left(\frac{m^{1/2}\log q}{(\log\log q)^{1/2}}+\log q\right),\end{aligned}$$

where

$$A(\tau,\chi) := \frac{1}{2\pi}\arg G_1\left(\frac{1}{2}+i\tau,\chi\right), \quad B(\tau,\chi) := \frac{1}{2\pi}\arg L\left(\frac{1}{2}+i\tau,\chi\right),$$

and the logarithmic branches are taken as in Section 3.2.

Proof. Take $\sigma_1, t_0, T, \delta, b$ as in the beginning of the proof of Proposition 4.1. Let $b' := 1/2 - \delta/2$. Replacing b by b' in (4.1), we have

$$\begin{aligned}2\pi \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<\pm\gamma'\le T}} (\beta'-b') &= \int_{t_0}^{T} \log|G_1(b'\pm it,\chi)|\,\mathrm{d}t - \int_{t_0}^{T} \log|G_1(4m\pm it,\chi)|\,\mathrm{d}t \\ &\quad \mp \int_{b'}^{4m} \arg G_1(\sigma\pm it_0,\chi)\,\mathrm{d}\sigma \pm \int_{b'}^{4m} \arg G_1(\sigma\pm iT,\chi)\,\mathrm{d}\sigma.\end{aligned}$$

Subtracting these from (4.1), we obtain

$$\begin{aligned}\delta\pi \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<\pm\gamma'\le T}} 1 &= \int_{t_0}^{T} \left(\log|G_1(b\pm it,\chi)| - \log|G_1(b'\pm it,\chi)|\right)\,\mathrm{d}t \\ &\quad \mp \int_{b}^{b'} \arg G_1(\sigma\pm it_0,\chi)\,\mathrm{d}\sigma \pm \int_{b}^{b'} \arg G_1(\sigma\pm iT,\chi)\,\mathrm{d}\sigma \\ &=: J_1^{\pm} + J_2^{\pm} + J_3^{\pm}.\end{aligned}$$

We estimate $J_1^{\pm}$. From (4.2), we have

$$\begin{aligned}J_1^{+} &= \int_{t_0}^{T} \left(\log|G_1(b+it,\chi)| - \log|G_1(b'+it,\chi)|\right)\,\mathrm{d}t \\ &= (b-b')(T-t_0)\log m + \int_{t_0}^{T} \left(\log|F(b+it,\chi)| - \log|F(b'+it,\chi)|\right)\mathrm{d}t \\ &\quad + \int_{t_0}^{T} \left(\log\left|\frac{F'}{F}(b+it,\chi)\right| - \log\left|\frac{F'}{F}(b'+it,\chi)\right|\right)\,\mathrm{d}t \\ &\quad + \int_{t_0}^{T} \left(\log|L(1-b-it,\overline{\chi})| - \log|L(1-b'-it,\overline{\chi})|\right)\,\mathrm{d}t \\ &\quad + \int_{t_0}^{T} \Bigg(\log\left|1 - \frac{1}{\frac{F'}{F}(b+it,\chi)}\frac{L'}{L}(1-b-it,\overline{\chi})\right| \\ &\qquad\qquad - \log\left|1 - \frac{1}{\frac{F'}{F}(b'+it,\chi)}\frac{L'}{L}(1-b'-it,\overline{\chi})\right|\Bigg)\,\mathrm{d}t \\ &=: J_{11} + J_{12} + J_{13} + J_{14} + J_{15}.\end{aligned}$$

Applying Cauchy's theorem to $\log F(s,\chi)$ on the rectangle C with vertices $b+it_0$, $b'+it_0$, $b'+iT$, $b+iT$, and taking the imaginary part, we have

$$J_{12} = \int_{b}^{b'} \arg F(\sigma+it_0,\chi)\,\mathrm{d}\sigma - \int_{b}^{b'} \arg F(\sigma+iT,\chi)\,\mathrm{d}\sigma.$$

From (3.4), we can show that

$$J_{12} = \left(T \log \frac{qT}{2\pi} - T\right)\frac{\delta}{2} - \left(t_0 \log \frac{qt_0}{2\pi} - t_0\right)\frac{\delta}{2} + O(\delta)$$

Next, we take the logarithmic branch of $\log{(F'/F)(s,\chi)}$ as in condition (1) of Lemma 3.4. Applying Cauchy's integral theorem to $\log{(F'/F)(s,\chi)}$ on C taking the imaginary part, we have

$$J_{13} = \int_b^{b'} \arg \frac{F'}{F}(\sigma + it_0, \chi)\, \mathrm{d}\sigma - \int_b^{b'} \arg \frac{F'}{F}(\sigma + iT, \chi)\, \mathrm{d}\sigma = O(\delta)$$

To estimate J_{14}, we define a branch of $\log L(s,\chi)$ as in the estimation of I_{14} in the proof of Proposition 4.1 and apply Cauchy's integral theorem on the rectangle with vertices $1-b'+it_0$, $1-b+it_0$, $1-b+iT$, $1-b'+iT$. Taking the imaginary part we obtain

$$J_{14} = -\int_{1-b'}^{1-b} \arg L(\sigma + it_0, \chi)\, \mathrm{d}\sigma + \int_{1-b'}^{1-b} \arg L(\sigma + iT, \chi)\, \mathrm{d}\sigma.$$

Finally, we define a branch of

$$\log\left(1 - \frac{1}{\frac{F'}{F}(s,\chi)} \frac{L'}{L}(1-s,\overline{\chi})\right)$$

as in the estimation of I_{15} in the proof of Proposition 4.1 and apply Cauchy's integral theorem to it on C. Taking the imaginary part, we have

$$\begin{aligned} J_{15} = \int_b^{b'} &\arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma + it_0,\chi)} \frac{L'}{L}(1-\sigma-it_0,\overline{\chi})\right) \mathrm{d}\sigma \\ &- \int_b^{b'} \arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma + iT,\chi)} \frac{L'}{L}(1-\sigma-iT,\overline{\chi})\right) \mathrm{d}\sigma \\ = O(\delta)& \end{aligned}$$

by (4.4). Then we estimate J_1^- similarly.

We then obtain

$$\begin{aligned} \delta\pi \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_0<\pm\gamma'\leq T}} 1 = &-(T-t_0)\frac{\delta}{2}\log m + \left(T \log \frac{qT}{2\pi} - T\right)\frac{\delta}{2} - \left(t_0 \log \frac{qt_0}{2\pi} - t_0\right)\frac{\delta}{2} \\ &\mp \int_{1-b'}^{1-b} \arg L(\sigma \pm it_0, \chi)\, \mathrm{d}\sigma \pm \int_{1-b'}^{1-b} \arg L(\sigma \pm iT, \chi)\, \mathrm{d}\sigma \\ &\mp \int_b^{b'} \arg G_1(\sigma \pm it_0, \chi)\, \mathrm{d}\sigma \pm \int_b^{b'} \arg G_1(\sigma \pm iT, \chi)\, \mathrm{d}\sigma + O(\delta). \end{aligned}$$

Taking the limit $\delta \to 0$ and applying the mean value theorem, for $\tau = \pm t_0$ and $\tau = \pm T$ we have

$$\lim_{\delta \to 0} \frac{1}{\pi\delta} \int_{1-b'}^{1-b} \arg L(\sigma + i\tau, \chi)\, \mathrm{d}\sigma = B(\tau, \chi)$$

and

$$\lim_{\delta \to 0} \frac{1}{\pi\delta} \int_{b}^{b'} \arg G_1(\sigma + i\tau, \chi)\, \mathrm{d}\sigma = A(\tau, \chi)$$

by noting that $b = 1/2 - \delta$ and $b' = 1/2 - \delta/2$. Hence,

$$\begin{aligned} &N_1(T, \chi) - N_1(t_0, \chi) \\ &\quad = \frac{T}{\pi} \log \frac{qT}{2m\pi} - \frac{T}{\pi} - \left(\frac{t_0}{\pi} \log \frac{qt_0}{2m\pi} - \frac{t_0}{\pi} \right) \\ &\qquad - A(t_0, \chi) - B(t_0, \chi) + A(T, \chi) + B(T, \chi) \\ &\qquad + A(-t_0, \chi) + B(-t_0, \chi) - A(-T, \chi) - B(-T, \chi) + O(1). \end{aligned}$$

Applying Lemma 3.8,

$$A(\pm t_0, \chi) = \frac{1}{2\pi} \arg G_1 \left(\frac{1}{2} \pm it_0, \chi \right) = O\left(\frac{m^{1/2} \log q}{(\log\log q)^{1/2}} \right)$$

and from Lemma 2.2, we have

$$B(\pm t_0, \chi) = \frac{1}{2\pi} \arg L \left(\frac{1}{2} \pm it_0, \chi \right) = O\left(\frac{\log q}{\log\log q} \right).$$

By using the argument principle and [2, Proposition 2.3], we can show that

$$\begin{aligned} N_1(t_0, \chi) &= N(t_0, \chi) + \frac{1}{2\pi} \arg \frac{L'}{L}(-2j - \kappa + 1 + it_0, \chi) \\ &\qquad - \frac{1}{2\pi} \arg \frac{L'}{L}(-2j - \kappa + 1 - it_0, \chi) + O(1) \\ &= \frac{1}{2\pi} \left(\arg \frac{L'}{L}(-2j - \kappa + 1 + it_0, \chi) - \arg \frac{L'}{L}(-2j - \kappa + 1 - it_0, \chi) \right) \\ &\qquad + O(\log q), \end{aligned}$$

where $j \in \mathbb{N}$ is chosen such that $-2j - \kappa + 1 \leq \sigma_1$. From the functional equation for $L(s, \chi)$, we have

$$\frac{L'}{L}(s, \chi) = \frac{F'}{F}(s, \chi) \left(1 - \frac{1}{\frac{F'}{F}(s, \chi)} \frac{L'}{L}(1 - s, \overline{\chi}) \right).$$

Applying Lemmas 3.3 and 3.4 (1), we can show that

$$\arg \frac{L'}{L}(-2j - \kappa + 1 \pm it_0, \chi) = O(1).$$

Hence,

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + A(T,\chi) + B(T,\chi) - A(-T,\chi) - B(-T,\chi) + O\left(\frac{m^{1/2}\log q}{(\log\log q)^{1/2}} + \log q\right).$$

If $2 \le T < t_0$, then $N_1(T,\chi) \le N_1(t_0,\chi) = O(\log q)$, which can be included in the error term. Thus the proof is complete. □

Now we complete the proof of Theorem 1.2. We only need to consider the case when $T \ge t_0$ since we already know that $2 \le T < t_0$, then $N_1(T,\chi) \le N_1(t_0,\chi) = O(\log q)$.

We first prove that $A(q,T)$ in our theorem can be $(\log\log(qT))^{1/2}$.

As in the proof of Theorem 1.1, we also consider two cases. In the first case, for $T \ge t_0$ which satisfies $L(\sigma \pm iT,\chi) \ne 0, L'(\sigma \pm iT,\chi) \ne 0$ for all $\sigma \in \mathbb{R}$, the error terms are estimated as follows: From Lemma 3.8, we have

$$\arg G_1\left(\frac{1}{2} \pm iT,\chi\right) = O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}}\right).$$

From Lemma 2.2, we have

$$\arg L\left(\frac{1}{2} \pm iT,\chi\right) = O\left(\frac{\log(qT)}{\log\log(qT)}\right).$$

Therefore,

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}} + \log q\right) \tag{4.6}$$

for this case.

In the second case, we consider for $T \ge t_0$ such that any of $L(\sigma+iT,\chi) \ne 0$, $L(\sigma - iT,\chi) \ne 0$, $L'(\sigma + iT,\chi) \ne 0$, or $L'(\sigma - iT,\chi) \ne 0$ is *not* satisfied for some $\sigma \in \mathbb{R}$. Similar to the proof of Theorem 1.1, we look for some small $0 < \epsilon < (\log(qT))^{-1}$ such that $L(\sigma \pm i(T \pm \epsilon),\chi) \ne 0, L'(\sigma \pm i(T \pm \epsilon),\chi) \ne 0$ holds for all $\sigma \in \mathbb{R}$. Applying the method of the first case we obtain

$$N_1(T \pm \epsilon,\chi) = \frac{T \pm \epsilon}{\pi}\log\frac{q(T \pm \epsilon)}{2m\pi} - \frac{T \pm \epsilon}{\pi} + O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}} + \log q\right) \tag{4.7}$$

Noting the inequalities

$$N_1(T-\epsilon,\chi) \le N_1(T,\chi) \le N_1(T-\epsilon,\chi) + (N_1(T+\epsilon,\chi) - N_1(T-\epsilon,\chi)),$$

from (4.7) we easily find that (4.6) also holds for this case.

Finally we show that $A(q,T) = 1 + m^{1/2}(\log\log(qT))^{-1}$ in Theorem 1.2. Our aim is to show

$$\arg G_1\left(\frac{1}{2} \pm iT, \chi\right) = O\left(\frac{m^{1/2}\log(qT)}{\log\log(qT)}\left(1 + \frac{m^{1/2}}{\log\log(qT)}\right)\right) + O(\log q)$$

The proof will follow that of [4, Theorem 1]. It is sufficient to prove for $T \geq t_0$ which satisfies $L(\sigma \pm iT, \chi) \neq 0, L'(\sigma \pm iT, \chi) \neq 0$ for all $\sigma \in \mathbb{R}$ and prove for the other case as in the second case above.

We put

$$H_T := \frac{(\log\log(qT))^3}{\log(qT)}.$$

Let $\Delta_1^{\pm}$ denote the change in argument of $G_1(s,\chi)$ along the horizontal line from $\infty \pm iT$ to $1/2 + H_T/\log\log(qT) \pm iT$ and $\Delta_2^{\pm}$ denote that along the horizontal line from $1/2 + H_T/\log\log(qT) \pm iT$ to $1/2 \pm iT$. Then

$$\arg G_1\left(\frac{1}{2} \pm iT, \chi\right) = \Delta_1^{\pm} + \Delta_2^{\pm}.$$

By using Lemma 3.6, we can easily show that when $\sigma \geq 1/2 + H_T/\log\log(qT)$,

$$\arg\frac{G_1}{L}(\sigma \pm iT, \chi) \ll \frac{m^{1/2}\log(qT)}{\log\log(qT)}\left(1 + \frac{m^{1/2}}{\log\log(qT)}\right)$$

which by Lemma 2.2 immediately implies

$$\Delta_1^{\pm} = \arg G_1(\sigma \pm iT, \chi) \ll \frac{m^{1/2}\log(qT)}{\log\log(qT)}\left(1 + \frac{m^{1/2}}{\log\log(qT)}\right).$$

In the rest of this proof, we show that

$$\Delta_2^{\pm} = O\left(\frac{m^{1/2}\log(qT)}{\log\log(qT)}\right) + O(\log q).$$

We first note that

$$\Delta_2^{\pm} = \operatorname{Im}\left(\int_{1/2}^{1/2+H_T/\log\log(qT)} \frac{G_1'}{G_1}(\sigma \pm iT, \chi)\,\mathrm{d}\sigma\right).$$

By using standard methods (see [4, Lemma 6 and p. 5] and [12, Theorem 9.6 (A)]) we can show that

$$\frac{G_1'}{G_1}(\sigma \pm iT, \chi) = \sum_{|\rho' - (1/2 + H_T/(2\log\log(qT)) \pm iT)| \leq 2} \frac{1}{\sigma \pm iT - \rho'} + O(m\log(qT)).$$

For convenience, we write

$$\mathcal{D}(y;\chi) := \left\{ s \in \mathbb{C} \,\middle|\, \left| s - \left(\frac{1}{2} + \frac{H_T}{2\log\log(qT)} + iy\right)\right| \leq 2 \right\}.$$

Thus

$$\begin{aligned}\Delta_2^{\pm} &= \operatorname{Im}\left(\int_{1/2\pm iT}^{1/2+H_T/\log\log(qT)\pm iT}\left(\sum_{\rho'\in\mathcal{D}(\pm T;\chi)}\frac{1}{s-\rho'}+O(m\log(qT))\right)\mathrm{d}s\right)\\ &= \sum_{\rho'\in\mathcal{D}(\pm T;\chi)}\operatorname{Im}\left(\int_{1/2\pm iT}^{1/2+H_T/\log\log(qT)\pm iT}\frac{1}{s-\rho'}\,\mathrm{d}s\right)\\ &\qquad + O\left(m(\log\log(qT))^2\right)\\ &= \sum_{\rho'\in\mathcal{D}(\pm T;\chi)}\left(\arg\left(\frac{1}{2}+\frac{H_T}{\log\log(qT)}\pm iT-\rho'\right)-\arg\left(\frac{1}{2}\pm iT-\rho'\right)\right)\\ &\qquad + O\left(m(\log\log(qT))^2\right).\end{aligned}$$

We set

$$f(\rho') := \left(\arg\left(\frac{1}{2}+\frac{H_T}{\log\log(qT)}\pm iT-\rho'\right)-\arg\left(\frac{1}{2}\pm iT-\rho'\right)\right),$$

and note that $f(\rho')\ll 1$. Now we are left to show

$$\sum_{\rho'\in\mathcal{D}(\pm T;\chi)} f(\rho') = O\left(\frac{m^{1/2}\log(qT)}{\log\log(qT)}\right)+O(\log q).$$

We divide the sum into three regions:

- $\mathcal{D}_1(y;\chi) := \{s\in\mathbb{C} \mid 1/2<\sigma\le 1/2+H_T,\, y-H_T\le t\le y+H_T\}$,
- $\mathcal{D}_2(y;\chi) := \{s\in\mathbb{C} \mid \sigma=1/2,\, y-H_T\le t\le y+H_T\}$,
- $\mathcal{D}_3(y;\chi) := \mathcal{D}(y;\chi)\backslash(\mathcal{D}_1(y;\chi)\cup\mathcal{D}_2(y;\chi))$.

We easily find that

$$\sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)} f(\rho') \ll \sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)} 1.$$

As in the proof of [4, Lemma 8], we first observe that when $T\ge t_0$,

$$\int_{\pm T-H_T}^{\pm T+H_T}\mathcal{F}_1(t,\chi)\,\mathrm{d}t \ll \sum_{\rho\in\mathcal{D}_1(\pm T;\chi)} 1+|O(\log q)|$$

by using Lemma 3.5 (see also [4, proof of Lemma 7]). Meanwhile

$$\mathcal{F}_1(\pm t,\chi) \ge \sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)}\frac{\beta'-1/2}{(\beta'-1/2)^2+(\gamma'\mp t)^2}$$

and so

$$\int_{\pm T-H_T}^{\pm T+H_T} \mathcal{F}_1(t,\chi)\,\mathrm{d}t \geq \sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)} \int_{\pm T-H_T}^{\pm T+H_T} \frac{\beta'-1/2}{(\beta'-1/2)^2+(\gamma'-t)^2}\,\mathrm{d}t$$
$$\geq c \sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)} 1$$

for some absolute constant $c>0$ (cf. [4, proof of Lemma 8]). This implies

$$\sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)} 1 \ll \sum_{\rho\in\mathcal{D}_1(\pm T;\chi)} 1 + |O(\log q)|.$$

By Proposition 2.3, this gives

$$\begin{aligned}\sum_{\rho'\in\mathcal{D}_1(\pm T;\chi)} f(\rho') &\ll \sum_{\rho\in\mathcal{D}_1(\pm T;\chi)} 1 + |O(\log q)| \\ &\leq N(T+H_T,\chi)-N(T-H_T,\chi)+|O(\log q)| \\ &= O\left(\frac{\log(qT)}{\log\log(qT)}\right)+O(\log q).\end{aligned}$$

Similarly we have

$$\sum_{\rho'\in\mathcal{D}_2(\pm T;\chi)} f(\rho') \ll \sum_{\rho'\in\mathcal{D}_2(\pm T;\chi)} 1 \ll \sum_{\rho\in\mathcal{D}_2(\pm T;\chi)} 1$$

by Lemma 7.1 of [2]. Therefore again by Proposition 2.3,

$$\begin{aligned}\sum_{\rho'\in\mathcal{D}_2(\pm T;\chi)} f(\rho') &\ll \sum_{\rho\in\mathcal{D}_2(\pm T;\chi)} 1 \leq N(T+H_T,\chi)-N(T-H_T,\chi) \\ &\ll \frac{\log(qT)}{\log\log(qT)}.\end{aligned}$$

Finally, for $\rho'\in\mathcal{D}_3(\pm T;\chi)$, recalling the definition of $f(\rho')$, we can easily show

$$\begin{aligned}|f(\rho')| &= \left|\mathrm{Im}\left(\log\left(\frac{1}{2}+\frac{H_T}{\log\log(qT)}\pm iT-\rho'\right)-\log\left(\frac{1}{2}\pm iT-\rho'\right)\right)\right| \\ &\leq \left|\log\left(\frac{1}{2}+\frac{H_T}{\log\log(qT)}\pm iT-\rho'\right)-\log\left(\frac{1}{2}\pm iT-\rho'\right)\right| \\ &= \left|\log\left(1+\frac{H_T}{(1/2\pm iT-\rho')\log\log(qT)}\right)\right|.\end{aligned}$$

Since either $\beta' \geq 1/2, |\gamma' \mp T| \geq H_T$ or $\beta' \geq 1/2 + H_T$ holds, we have $|1/2 \pm iT - \rho'| \geq H_T$. Thus

$$\begin{aligned}\sum_{\rho' \in \mathcal{D}_3(\pm T;\chi)} f(\rho') &\ll \frac{1}{\log\log(qT)} \sum_{\rho' \in \mathcal{D}_3(\pm T;\chi)} 1 \\ &\leq \frac{1}{\log\log(qT)} \left(N_1(T+2,\chi) - N_1(T-2,\chi)\right) \\ &\ll \frac{m^{1/2}\log(qT)}{\log\log(qT)}.\end{aligned}$$

Concluding the above, we have

$$\begin{aligned}\sum_{\rho' \in \mathcal{D}(\pm T;\chi)} f(\rho') &= \left(\sum_{\rho' \in \mathcal{D}_1(\pm T;\chi)} + \sum_{\rho' \in \mathcal{D}_2(\pm T;\chi)} + \sum_{\rho' \in \mathcal{D}_3(\pm T;\chi)} \right) f(\rho') \\ &= O\left(\frac{m^{1/2}\log(qT)}{\log\log(qT)} \right) + O(\log q)\end{aligned}$$

and we complete the proof. □

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