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Corrigendum to "On traces of the Brandt-Eichler matrices"

par Juliusz BRZEZIŃSKI

Résumé. Cette note est une correction de mon article [1]. Elle corrige une formule dans la proposition 2.2. D'après le résultat corrigé, le nombre $\iota(n,m)$ d'idéaux principaux à gauche de norme q^m dans l'ordre de Eichler de niveau n sur un anneau de valuation discrète R dont le corps résiduel est de cardinalité q est $\iota(n,m) = (m+1)q^m$ si $m < n$ et

$$\iota(n,m) = (n+1)q^m + 2q^{m-1} + \cdots + 2q^n + q^{n-1}$$

lorsque $m \geq n$. La démonstration de la proposition n'était pas donnée dans mon article (étant "pénible mais sans obstacle"). Malheureusement, certains coefficients dans le second cas étaient erronés. Une démonstration complète suit ci-dessous.

Abstract. This is a correction to my paper [1]. It corrects a formula in Proposition 2.2. The corrected result says that the number $\iota(n,m)$ of principal left ideals with norm q^m in the Eichler order of level n over a discrete valuation ring R with residue field of cardinality q is $\iota(n,m) = (m+1)q^m$ if $m < n$ and

$$\iota(n,m) = (n+1)q^m + 2q^{m-1} + \cdots + 2q^n + q^{n-1}$$

when $m \geq n$. The proof of the Proposition was not given in my paper (as "tedious but straightforward"). Unfortunately, some coefficients in the second case were erroneous. A complete proof follows below.

Let R be a discrete valuation ring with maximal ideal (π). Assume that the residue ring $R/(\pi)$ is finite and let $q = |R/(\pi)|$ be its cardinality. The discrete valuation defined by R will be denoted by v. Thus $v(\pi^m) = m$. Let

$$\Lambda_n = \begin{pmatrix} R & R \\ \pi^n R & R \end{pmatrix}$$

be an Eichler order in the matrix algebra $M_2(K)$ over the quotient field K of R.

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Our purpose is to find the number of principal left ideals in Λ_n with norm q^m $(m > 0)$, which will be denoted by $\iota(n,m)$. Since each such ideal has form $\Lambda_n M$, where $M \in \Lambda_n$ and the determinant of M is a generator of (π^m), we want to find a set of representatives for the orbits $\Lambda_n^* M$ of the unit group Λ_n^* of Λ_n acting on the set of matrices in Λ_n having norm q^m (recall that the norm of $\Lambda_n M$ is the cardinality of $R/(\det(M))$).

We denote by

$$\varepsilon = \begin{pmatrix} e_{11} & e_{12} \\ \pi^n e_{21} & e_{22} \end{pmatrix}$$

the elements of Λ_n^*, and by

$$M = \begin{pmatrix} m_{11} & m_{12} \\ \pi^n m_{21} & m_{22} \end{pmatrix}$$

the matrices in Λ_n.

Our purpose is to choose a "canonical" set of representatives of the orbits $\Lambda_n^* M$.

We split the orbits into 3 types: Type 1 are those having a representant with $m_{11} = 0$, Type 2 those having a representant with $m_{21} = 0$ and Type 3 those which can not be represented by a matrix with $m_{11}m_{21} = 0$.

First we give a canonical choice of matrices representing each type and when this is done, we count the number of orbits by counting the number of representatives of each kind.

Type 1. If a matrix M has $m_{11} = 0$, then $\det M = \pi^n m_{21} m_{12} \neq 0$, so multiplying M from the left by a suitable diagonal unit matrix, we may assume that

$$M = \begin{pmatrix} 0 & \pi^s \\ \pi^{n+r} & m_{22} \end{pmatrix}.$$

Now, we can take a product

$$\varepsilon M = \begin{pmatrix} 1 & 0 \\ \pi^n q & 1 \end{pmatrix} M = \begin{pmatrix} 0 & \pi^s \\ \pi^{n+r} & m_{22} + q\pi^{n+s} \end{pmatrix},$$

so we can assume that m_{22} is reduced modulo π^{n+s}. Thus we arrive to the description of the matrices of Type 1:

$$M^{(1)}_{r,s,c} = \begin{pmatrix} 0 & \pi^s \\ \pi^{n+r} & c \end{pmatrix},$$

where c is reduced modulo π^{n+s}. We check easily that two matrices $M^{(1)}_{r,s,c}$ and $M^{(1)}_{r',s',c'}$ define the same orbit if and only if $r = r'$, $s = s'$ and $c = c'$ $(\mathrm{mod}\ \pi^{n+s})$.

The number $\iota_1(n,m)$ of matrices of Type 1 with norm q^m is equal to all possible choices of r, s, c such that $r + s + n = m$ and c is reduced modulo π^{n+s}. Thus, we have such matrices only if $m \geq n$ and for each $s = 0, \ldots, m-n$ we have q^{s+n} such matrices, that is,

$$\iota_1(n,m) = q^n + \cdots + q^m. \tag{0.1}$$

Type 2. Since this time, we have $\det M = m_{11}m_{22} \neq 0$, we can multiply M by a diagonal unit matrix, so that

$$M = \begin{pmatrix} \pi^r & m_{12} \\ 0 & \pi^s \end{pmatrix}.$$

Now, we can take a product

$$\varepsilon M = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} M = \begin{pmatrix} \pi^r & m_{12} + q\pi^s \\ 0 & \pi^s \end{pmatrix},$$

so we can assume that m_{12} is reduced modulo π^s. Thus, the matrices of Type 2 are

$$M^{(2)}_{r,s,c} = \begin{pmatrix} \pi^r & c \\ 0 & \pi^s \end{pmatrix},$$

where c is reduced modulo π^s. We check easily that two matrices $M^{(2)}_{r,s,c}$ and $M^{(2)}_{r',s',c'}$ define the same orbit if and only if $r = r'$, $s = s'$ and $c = c'$ (mod π^s).

The number $\iota_2(n,m)$ of matrices of Type 2 with norm q^m is equal to all possible choices of r, s, c such that $r + s = m$ and c is reduced modulo π^s. Thus, for each $s = 0, \ldots, m$ we have q^s such matrices, that is,

$$\iota_2(n,m) = 1 + q + \cdots + q^m. \tag{0.2}$$

Type 3. This time, we assume that there is no representant of the orbit $\Lambda_n M$ with $m_{11}m_{21} = 0$, so we can start with a representant

$$M = \begin{pmatrix} \pi^k & m_{12} \\ \pi^N & m_{22} \end{pmatrix},$$

where $N \geq n$, which we obtain multiplying M by a suitable diagonal unit matrix. First of all, we will show that it is possible to choose a representant of the orbit $\Lambda_n^* M$ such that $N - k \in \{1, \ldots, n-1\}$ (that is, $n + k > N$). In fact, if $N - k \geq n$, then we can multiply M by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\pi^{N-k} & 1 \end{pmatrix},$$

which gives a representant with 0 in the left lower position.

Let now $r = v(m_{22} - \pi^{N-k}m_{12})$, so $\det M = \pi^k(m_{22} - \pi^{N-k}m_{12})$ and $v(\det M) = k + r$. It is easy to check that two matrices

$$M = \begin{pmatrix} \pi^k & m_{12} \\ \pi^N & m_{22} \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} \pi^{k'} & m'_{12} \\ \pi^{N'} & m'_{22} \end{pmatrix},$$

with $r = v(m_{22} - \pi^{N-k} m_{12})$, $r' = v(m'_{22} - \pi^{N'-k'} m'_{12})$ represent the same orbit of Λ_n^* if and only if $(N, k, r) = (N', k', r')$ and $m_{12} = m'_{12} \pmod{\pi^r}$ and $m_{22} = m'_{22} \pmod{\pi^{(n+k)-N+r}}$. Moreover, a product $\varepsilon M_{N,k,r}$, where $\varepsilon \in \Lambda_n^*$ never has 0 in the first column, that is, these matrices can not represent an orbit of Type 1 or 2. Thus the matrices of Type 3 are

$$M^{(3)}_{N,k,r,a,b} = \begin{pmatrix} \pi^k & b \\ \pi^N & a \end{pmatrix},$$

where $N \geq n$, $N - n < k < N$, $r = v(a - \pi^{N-k} b)$, a is reduced modulo $\pi^{(n+k)-N+r}$ and b is reduced modulo π^r. Notice that $v(\det M^{(3)}_{N,k,r,a,b}) = k + r$.

In the sequel, we will use the following observations: If $r < N - k$, then $a = \pi^r a_0$, $\pi \nmid a_0$. If $r \geq N - k$, then $a = \pi^{N-k} a_0$, $\pi^{r-(N-k)} \mid a_0 - b$ and $\pi^{r-(N-k)+1} \nmid a_0 - b$.

Now we want to compute the number $\iota_3(n, m)$ of matrices of Type 3 with norm q^m. This is a little more complicated than the corresponding task in cases 1 and 2. First of all, notice that fixing n, m, we have $n \leq N < n + m$ (that is, N assumes m values). In fact, if $N \geq m + n$, then $k > N - n \geq m$, which is impossible, since $k + r = m$. If we fix N, then the number of possible pairs (k, r) such that $k + r = m$ and $N - n < k < N$ will be denoted by c_N. It is easy to check that

$$c_N = n - max\ (1, N - m).$$

In order to compute $\iota_3(n, m)$, we start counting the contribution coming from matrices of Type 3 with fixed N. For each r (and $k = m - r$), we count the number of corresponding matrices of Type 3.

We have two cases. First we consider the case $r < N - k$. As we know, we have q^r possibilities for b. As regards a, we have $a = \pi^r a_0$, $\pi \nmid a_0$, so the number of possibilities for a is given by the number of possible residues a_0 modulo $\pi^{(n+k)-N}$, which are invertible in $R/(\pi^{(n+k)-N})$, that is, we get $q^{(n+k)-N} - q^{(n+k)-N-1}$ possibilities for a_0. The number of possible matrices is

$$\begin{aligned} q^r(q^{(n+k)-N} - q^{(n+k)-N-1}) &= q^{r+(n+k)-N} - q^{r+(n+k)-N-1} \\ &= q^{(n+m)-N} - q^{(n+m)-N-1}. \end{aligned} \tag{0.3}$$

The second case is $r \geq N - k$. This time, we have as before that b is reduced modulo π^r, but $a = \pi^{N-k} a_0$, where $\pi^{r-(N-k)} \mid a_0 - b$ and $\pi^{r-(N-k)+1} \nmid a_0 - b$. Since a is reduced modulo $\pi^{n+k-N+r}$, we have to count the number of pairs (a_0, b) such that a_0 is reduced modulo $\pi^{n+2k-2N+r}$, b is reduced modulo π^r and $\pi^{r-(N-k)} \mid a_0 - b$, $\pi^{r-(N-k)+1} \nmid a_0 - b$. These configuration seems to be rather messy, but the situation is essentially very simple: we have residues modulo some π^x and π^y and we have to count the number of pairs of residues whose difference is divisible by π^z and not divisible by π^{z+1} for some $z \le \min(x,y)$. If we use q as before, then the answer is

$$q^x q^{y-z} - q^x q^{y-z-1} = q^{x+y-z} - q^{x+y-z-1},$$

which can be easily checked (take q^x residues $a_0 = 0, 1, \dots, q^x - 1$ and for each of them, those residues among $b = 0, 1, \dots, q^y - 1$ for which q^z divides $a_0 - b$; then repeat the counting looking at those for which q^{z+1} divides $a_0 - b$ and subtract their number).

In our case, we have $x = n + 2k - 2N + r$, $y = r$, $z = r + k - N$. Thus $x + y - z = n + k - N + r = n + m - N$, which means that the number of matrices is exactly the same as in the case $r < N - k$ and is given by (0.3).

Now, it remains to compute the number $\iota(n,m)$ of all matrices of Types 1, 2, 3. We have

$$\begin{aligned}\iota(n,m) &= \iota_1(n,m) + \iota_2(n,m) + \iota_3(n,m)\\ &= \sum_{i=n}^{m} q^i + \sum_{i=0}^{m} q^i + \sum_{N=n}^{n+m-1} c_N (q^{n+m-N} - q^{n+m-N-1}),\end{aligned}$$

where the first sum is 0 when $m < n$.

If $m < n$, then it is easy to check that $c_N = m + n - N$ for $N = n, \dots, n+m-1$. Thus, we have the sum:

$$\iota(n,m) = \sum_{i=0}^{m} q^i + \sum_{N=n}^{n+m-1} (n+m-N)(q^{n+m-N} - q^{n+m-N-1}) = (m+1)q^m.$$

If $m \ge n$, then $c_N = n - 1$ for $N = n, \dots, m$ and $c_N = n + m - N$ for $N = m+1, \dots, n+m-1$. Hence, we have

$$\begin{aligned}\iota(n,m) &= \sum_{i=n}^{m} q^i + \sum_{i=0}^{m} q^i + \sum_{N=n}^{m} (n-1)(q^{n+m-N} - q^{n+m-N-1})\\ &\quad + \sum_{N=m+1}^{n+m-1} (n+m-N)(q^{n+m-N} - q^{n+m-N-1})\\ &= (n+1)q^m + 2q^{m-1} + \cdots + 2q^n + q^{n-1}.\end{aligned}$$

References

[1] J. Brzeziński, "On traces of the Brandt-Eichler matrices", *Journal de Théorie des Nombres de Bordeaux* **10** (1998), no. 2, p. 273-285.

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