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On a construction of $C^1(\mathbb{Z}_p)$ functionals from $\mathbb{Z}_p$-extensions of algebraic number fields

par Timothy ALL et Bradley WALLER

Résumé. Soit k un corps de nombres et k_∞/k une $\mathbb{Z}_p$-extension. Nous construisons un $\mathbb{Z}_p[\![T-1]\!]$-morphisme naturel de $\varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ dans un sous-ensemble particulier de $C^1(\mathbb{Z}_p)^*$, le dual de l'espace vectoriel sur $\mathbb{C}_p$ des fonctions continûment dérivables de $\mathbb{Z}_p \to \mathbb{C}_p$. Nous appliquons les résultats au problème d'interpolation des sommes de Gauss attachées aux caractères de Dirichlet.

Abstract. Let k be any number field, and let k_∞/k be any $\mathbb{Z}_p$-extension. We construct a natural $\mathbb{Z}_p[\![T-1]\!]$-morphism from $\varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ into a special subset of $C^1(\mathbb{Z}_p)^*$, the dual of the $\mathbb{C}_p$-vector space of continuously differentiable functions from $\mathbb{Z}_p \to \mathbb{C}_p$. We apply the results to the problem of interpolating Gauss sums attached to Dirichlet characters.

1. Introduction

Fix an odd prime p and let d be a positive integer co-prime to p. For an integer n, we let $\zeta_n = e^{2\pi i/n}$ so that $\zeta_n^x = \zeta_{n/x}$ for every $x \mid n$. Let $k_{(n)} = \mathbb{Q}(\zeta_{dp^{n+1}})$, and let $G_n = \mathrm{Gal}(k_{(n)}/k_{(0)})$.

We take a moment to review some classical theory from which this paper draws inspiration. Let $\theta_n \in \mathbb{Q}[\mathrm{Gal}(k_{(n)}/\mathbb{Q})]$ denote the classical Stickelberger element attached to the number field $k_{(n)}$. Recall that θ_n, once properly made integral, annihilates the class group of $k_{(n)}$ (see [9]). Suppose φ is a non-trivial even Dirichlet character of conductor dp^{n+1} taking values in Ω_p, an algebraic closure of $\mathbb{Q}_p$. The character φ decomposes uniquely into a product of a tame character χ and a wild character ψ. Suppose $\chi \neq 1$ and let $\theta_n(\chi^{-1}\omega) \in \Omega_p[G_n]$ denote the $\chi^{-1}\omega$-part of θ_n, where ω denotes the Teichmüler character. In a celebrated work [5], Iwasawa showed that the sequence $(\theta_n(\chi^{-1}\omega)) \in \varprojlim \Omega_p[G_n]$ (the projective limit taken with respect to the natural maps) is associated in a natural way to a function

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$F_\chi(T) \in \Omega_p[\![T-1]\!]$ whose coefficients are integral and lie in a finite extension of $\mathbb{Q}_p$. What's more, this function is essentially the p-adic L-function of Leopoldt and Kubota. In fact, we have

$$L_p(s, \chi\psi) = F_\chi\left(\zeta_\psi(1+p)^s\right)$$

where $\zeta_\psi = \overline{\psi}(1+p)$.

Unfortunately, if one restricts the action of θ_n to $k_{(n)}^+$, the maximal real subfield of $k_{(n)}$, it reduces to a multiple of the norm. With $\log_p$ denoting the Iwasawa logarithm, non-trivial explicit elements such as

$$\vartheta_n = \sum_{\sigma \in G(k_{(n)}/\mathbb{Q})} \log_p(1 - \zeta_{dp^{n+1}}^\sigma)\sigma^{-1}$$

were shown in [1], once properly made integral, to annihilate $\mathrm{Cl}(k_{(n)}^+) \otimes_{\mathbb{Z}} \mathcal{O}$ where $\mathcal{O}$ is the ring of integers of the topological closure of $k_{(n)} \hookrightarrow \Omega_p$. Since $(1 - \zeta_{dp^{n+1}}) \in \varprojlim k_{(n)}^\times$ where the projective limit is taken with respect to the norm maps, it follows that $(\vartheta_n(\chi)) \in \varprojlim \Omega_p[G_n]$. This article was born out of considering what analytic functions were naturally associate to the non-trivial sequences $(\vartheta_n(\chi))$ (or more generally, to elements in $\varprojlim k_{(n)}^\times$) in analogy with Iwasawa's construction of p-adic L-functions from $\left(\theta_n(\chi^{-1}\omega)\right)$.

Towards that end, let k be any number field, and let

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset \bigcup_{n=0}^\infty k_n = k_\infty$$

denote a $\mathbb{Z}_p$-extension of k. So $\Gamma := \mathrm{Gal}(k_\infty/k)$ is topologically isomorphic to $\mathbb{Z}_p$, and $\Gamma_n = \mathrm{Gal}(k_n/k_0) \simeq \Gamma/\Gamma^{p^n}$. Let γ_0 be a fixed topological generator for Γ and associate Γ with $\mathbb{Z}_p$ via the isomorphism $\gamma_0^a \mapsto a$. Let $\mathfrak{p}$ be a prime of k such that the inertia subgroup of $\mathfrak{p}$ is $\mathrm{Gal}(k_\infty/k_j)$ for some $j \geq 0$. This necessitates $\mathfrak{p} \mid p$, and the valuation $v_\mathfrak{p}$ extends to $k_\infty \hookrightarrow \Omega_p$.

Let $\mathbb{C}_p$ denote the topological closure of Ω_p. Let μ be a collection of maps $\{\mu_n : \Gamma_n \to \mathbb{C}_p\}_{n=0}^\infty$ with the following property:

$$\mu_n(x) = \sum_{y \mapsto x} \mu_{n+1}(y)$$

where $\Gamma_{n+1} \to \Gamma_n$ naturally. We call such a collection of maps a *distribution* on Γ. We denote the ring (under convolution) of all $\mathbb{C}_p$-valued distributions on Γ by $\mathcal{D}(\Gamma)$, and we write $\mu(a + p^n\mathbb{Z}_p)$ in place of the more cumbersome $\mu_n(\gamma_0^a \bmod \Gamma^{p^n})$.

We write $\mathbb{C}_p[\![\Gamma]\!]$ for the inverse limit of $\mathbb{C}_p[\Gamma_n]$ (with respect to the natural maps). The rings $\mathbb{C}_p[\![\Gamma]\!]$ and $\mathcal{D}(\Gamma)$ are isomorphic via the map

$$\begin{aligned} \beta : \mathcal{D}(\Gamma) &\to \mathbb{C}_p[\![\Gamma]\!] \\ \mu &\mapsto \beta(\mu) = \left(\sum_{a=0}^{p^n-1} \mu(a + p^n\mathbb{Z}_p)\gamma_0^{-a} \right). \end{aligned}$$

We also have the following Γ-maps relating $\varprojlim k_n^\times$ to $\mathcal{D}(\Gamma)$ and $\mathbb{C}_p[\![\Gamma]\!]$:

$$\rho : \begin{cases} \varprojlim k_n^\times & \to \mathcal{D}(\Gamma) \\ (\ell_n) = \ell & \mapsto \rho(\ell) : a + p^n\mathbb{Z}_p \mapsto -\log_p\left(\ell_n^{\gamma_0^a}\right), \end{cases}$$

$$\alpha : \begin{cases} \varprojlim k_n^\times & \to \mathbb{C}_p[\![\Gamma]\!] \\ (\ell_n) = \ell & \mapsto \alpha(\ell) = \left(-\sum_{a=0}^{p^n-1} \log_p\left(\ell_n^{\gamma_0^a}\right)\gamma_0^{-a} \right). \end{cases}$$

Taken together, we have the commutative diagram of Γ-maps:

$$\begin{tikzcd} \varprojlim k_n^\times \arrow[r, "\rho"] \arrow[dr, "\alpha"'] & \mathcal{D}(\Gamma) \arrow[d, "\beta"] \\ & \mathbb{C}_p[\![\Gamma]\!] \end{tikzcd}$$

We write $\mathcal{M}(\Gamma)$ for the sub-ring of $\mathcal{D}(\Gamma)$ consisting of those distributions that are $\mathbb{Z}_p$-valued, and let $\mathcal{I}(\Gamma) \subset \mathcal{D}(\Gamma)$ denote the $\mathcal{M}(\Gamma)$-module generated by the image of ρ.

What does one do with distributions anyway? For $\mu \in \mathcal{D}(\Gamma)$, we say that a function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is *μ-integrable* to mean that the limit

$$\int_{\mathbb{Z}_p} f(x)\ d\mu(x) := \lim_{n\to\infty} \sum_{a=0}^{p^n-1} f(a)\mu(a + p^n\mathbb{Z}_p)$$

exists. We call this limit the *Volkenborn integral of f with respect to μ*. The distinguishing feature of Volkenborn integration is the uniform choice of representatives from the classes $a + p^n\mathbb{Z}_p$ where $0 \le a < p^n - 1$ (namely, the choosing of a itself).

Thus distributions give rise to linear functionals on appropriate function spaces. For example, it's well known that every continuous function is μ-integrable for every $\mu \in \mathcal{M}(\Gamma)$. So every $\mu \in \mathcal{M}(\Gamma)$ determines a linear functional on $C(\mathbb{Z}_p)$, the collection of continuous functions on $\mathbb{Z}_p$, where

$$\mu(f) := \int_{\mathbb{Z}_p} f(x)\ \mathrm{d}\mu(x).$$

What's more, the Fourier transform $\mathcal{M}(\Gamma) \to \Lambda := \mathbb{Z}_p[\![T-1]\!]$ given by $\mu \mapsto \widehat{\mu}(T)$ where

$$\widehat{\mu}(T) = \mu(T^x) = \int_{\mathbb{Z}_p} T^x \, \mathrm{d}\mu(x) = \sum_{m=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{m} \, \mathrm{d}\mu(x) \right) (T-1)^m$$

is a well-defined isomorphism. Taken together in this setting, we have the commuting diagram of isomorphisms

$$\begin{tikzcd} \mathcal{M}(\Gamma) \arrow[r] \arrow[d, "\beta"'] & \Lambda \arrow[dl] \\ \mathbb{Z}_p[\![\Gamma]\!] & \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} \mu \arrow[r, mapsto] \arrow[d, mapsto] & \widehat{\mu} \arrow[dl, mapsto] \\ \beta(\mu) & \end{tikzcd}$$

If M is a module over $\mathcal{M}(\Gamma)$ or $\mathbb{Z}_p[\![\Gamma]\!]$ naturally, then we consider it a module over Λ (or any of the others for that matter) through the above diagram.

In particular, consider $k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as a $\mathbb{Z}_p[\Gamma_n]$-module in the natural way. Then $\varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a $\mathbb{Z}_p[\![\Gamma]\!]$-module. Now, extend the Iwasawa logarithm $\log_p$ to a function $\operatorname{Log}_p : k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{C}_p$ in the natural way: $\operatorname{Log}_p(\ell \otimes x) = x \log_p(\ell)$. It follows that the map

$$\begin{aligned} \varrho : \varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p &\to \mathcal{I}(\Gamma) \\ (\mathfrak{l}_n) = \mathfrak{l} &\mapsto \varrho(\mathfrak{l}) = \mathfrak{L} : a + p^n\mathbb{Z}_p \mapsto \operatorname{Log}_p\left(\mathfrak{l}_n^{\gamma_0^a}\right) \end{aligned}$$

is an onto Λ-morphism.

Our main result is that if $f : \mathbb{Z}_p \to \mathbb{C}_p$ is continuously differentiable, then f is λ-integrable for every $\lambda \in \mathcal{I}(\Gamma)$, in other words

Theorem 1.1. *Let $\lambda \in \mathcal{I}(\Gamma)$. Then λ defines a linear functional on $C^1(\mathbb{Z}_p)$:*

$$\lambda(f) := \int_{\mathbb{Z}_p} f(x) \, \mathrm{d}\lambda(x).$$

In particular, the Fourier transform $\widehat{\lambda}(T) \in \mathbb{C}_p[\![T-1]\!]$ exists and has radius of convergence ≥ 1. The analytic functions $\widehat{\lambda}(T)$ are like L-functions for the underlying norm coherent sequence. For example, consider the following special case. Suppose $k/\mathbb{Q}$ is an abelian number field whose conductor is not divisible by p^2, and let F be any abelian number field linearly disjoint from k and of conductor co-prime to p. If k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension of k, then the tower of number fields Fk_n forms the cyclotomic $\mathbb{Z}_p$-extension of Fk, and we consider Γ_n (resp. $\Delta := \mathrm{Gal}(k_0/\mathbb{Q})$) as being contained in (resp. a quotient of) the set of automorphisms of $\mathrm{Gal}(Fk_n/\mathbb{Q})$ fixing F.

For a character χ of Δ, let ρ_χ be the Γ-map

$$\rho_\chi : \varprojlim (Fk_n)^\times \to \mathcal{D}(\Gamma)$$
$$(\ell_n) \mapsto \lambda_\chi : a + p^n\mathbb{Z}_p \mapsto -\sum_{\delta\in\Delta} \log_p(\ell_n^{\gamma_0^a\delta})\overline{\chi}(\delta).$$

Let $\mathcal{I}_\chi^F(\Gamma)$ denote the $\mathcal{M}(\Gamma)$-module generated by the image of ρ_χ. The functions $\widehat{\lambda}_\chi(T)$ (or $\widehat{\lambda}(T)$, for that matter) interpolate values reminiscent of those found in the formula for $L_p(1,\varphi)$, the p-adic L-function of Leopoldt, Kubota, Iwasawa, et al. As a straightforward consequence of the above theorem, we have

Theorem 1.2. *Let* $\lambda_\chi \in \mathcal{I}_\chi^F(\Gamma)$*. Then* λ_χ *defines a linear functional on* $C^1(\mathbb{Z}_p)$ *where*

$$\lambda_\chi(f) := \int_{\mathbb{Z}_p} f(x)\ \mathrm{d}\lambda_\chi(x).$$

If ψ *is a character of* Γ_n *with* $\zeta_\psi = \overline{\psi}(\gamma_0)$ *and* $(\ell_n) \xrightarrow{\rho_\chi} \lambda_\chi$*, then*

$$\widehat{\lambda}_\chi(\zeta_\psi) = -\sum_\sigma \log_p(\ell_n^\sigma)\overline{\varphi}(\sigma)$$

where the sum runs over all $\sigma \in \Gamma_n \times \Delta = \mathrm{Gal}(k_n/\mathbb{Q})$ *and* $\varphi = \chi\psi$*.*

We apply the above results to the problem of interpolating Gauss sums attached to a Dirichlet character. Particularly interesting is the case when the tamely ramified character χ is of conductor p. In this case, the Gauss sums

$$\tau(\chi\psi) = \sum_{a=1}^{p^{n+1}} \chi\psi(a)\zeta_{p^{n+1}}^a$$

are essentially interpolated from the Fourier transform of $\lambda_\chi \in \mathcal{I}_\chi^{\mathbb{Q}(\zeta_{p-1})}(\Gamma)$ where the underlying norm coherent sequence generates the projective limit of principal units of $\mathbb{Q}_p(\zeta_{p^{n+1}})$. Since it's peripheral to the interpolation problem, we also show how to use the special values of the functions $\widehat{\lambda}_\chi(T)$ to construct an explicit sequence $(\vartheta_n) \in \mathbb{Z}_p[\Gamma_n]$ such that ϑ_n annihilates the χ-part of the Sylow p-subgroup of $\mathrm{Cl}(k_{(n)}^+)$ for every $n \geq 0$.

2. Volkenborn Distributions

In this section we give an overview of the theory of *Volkenborn distributions*. C. Barbacioru [3] developed the general notion of a Volkenborn distribution in his doctoral dissertation. This section is largely an overview of the tools from [3] that will be needed in the sequel wherein we show that distributions in $\mathcal{I}(\Gamma)$ are, in fact, Volkenborn.

Definition. A distribution μ, on $\mathbb{Z}_p$, is said to be Volkenborn if there exists $B(\mu) \in \mathbb{R}_{\geq 0}$ such that

$$|p\mu(a + p^{n+1}\mathbb{Z}_p) - \mu(a + p^n\mathbb{Z}_p)|_p \leq B(\mu)$$

for all $a \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 0}$

Note that all distributions that are bounded in value are necessarily Volkenborn, but a distribution need not be bounded to be Volkenborn. In fact, the prototype Volkenborn distribution is the Haar distribution: $a + p^n\mathbb{Z}_p \mapsto p^{-n}$.

Lemma 2.1. *Let μ be a Volkenborn distribution and let $f_n : \mathbb{Z}_p \to \mathbb{C}_p$ be defined by $f_n : x \mapsto p^n\mu(x + p^n\mathbb{Z}_p)$. Then there exists a continuous and bounded function $f : \mathbb{Z}_p \to \mathbb{C}_p$ such that $f_n \rightrightarrows f$ uniformly on $\mathbb{Z}_p$.*

Proof. Note that

$$p^n\mu(a + p^n\mathbb{Z}_p) = \left(\sum_{j=1}^{n} p^{j-1}(p\mu(a + p^j\mathbb{Z}_p) - \mu(a + p^{j-1}\mathbb{Z}_p))\right) + \mu(\mathbb{Z}_p).$$

The terms of the sum go to zero as $j \to \infty$ since μ is Volkenborn. It follows that the sum converges. Define $f : \mathbb{Z}_p \to \mathbb{C}_p$ by $x \mapsto \lim p^n\mu(x+p^n\mathbb{Z}_p)$. Note the above shows that f is bounded, in fact, $|f(x)| \leq \max\{B(\mu), \mu(\mathbb{Z}_p)\}$ for all $x \in \mathbb{Z}_p$.

Now, let $x \in \mathbb{Z}_p$ be arbitrary. Let $m > n$ be sufficiently large so that

$$\begin{aligned} |f(x) - f_n(x)|_p &\leq \max\left\{\{|f_{j+1}(x) - f_j(x)|_p\}_{j=n}^{m-1} \cup \{|f(x) - f_m(x)|_p\}\right\} \\ &\leq \max\{|f_{j+1}(x) - f_j(x)|_p\}_{j=n}^{m-1} \\ &\leq \frac{B(\mu)}{p^n}. \end{aligned}$$

The above bound does not depend on x, so $f_n \rightrightarrows f$. The function f is continuous since it is a uniform limit of continuous functions on a compact set. □

For a Volkenborn distribution μ, we want to show that all C^1 functions are μ-integrable. The strategy will be to first show that polynomials are μ-integrable. This, in conjunction with properties of Mahler series of C^1 functions, will give us the μ-integrability of C^1 functions.

Proposition 2.2. *Let μ be a Volkenborn distribution and P be a polynomial. Then P is μ-integrable.*

Proof. Since limits are finitely additive, it suffices to show that $P(x) = x^m$ is μ-integrable for all $m \in \mathbb{Z}_{\geq 0}$. We proceed by induction. For $P(x) = 1$, we have

$$\int_{\mathbb{Z}_p} \mathrm{d}\mu(x) = \lim_{n\to\infty} \sum_{a=0}^{p^n-1} \mu(a + p^n\mathbb{Z}_p) = \mu(\mathbb{Z}_p).$$

Now, let $S_{n,m} := \sum_{j=0}^{p^n-1} j^m \mu(j + p^n\mathbb{Z}_p)$ for $m, n \in \mathbb{Z}_{\geq 0}$. We wish to show that for a fixed $m \geq 1$ that $S_{n,m}$ is a Cauchy sequence. Note that

$$(\star) \qquad \begin{aligned} S_{n+1,m} - S_{n,m} &= \sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1}((j + kp^n)^m - j^m)\mu(j + kp^n + p^{n+1}\mathbb{Z}_p) \\ &= \sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1}\sum_{l=1}^{m}\binom{m}{l}(kp^n)^l j^{m-l}\mu(j + kp^n + p^{n+1}\mathbb{Z}_p). \end{aligned}$$

By Lemma 2.1 we only need to show that the $l = 1$ term from $(\star)$ is small. To do so, we will rewrite that term as follows:

$$\sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1} mkp^n j^{m-1}\mu(j + kp^n + p^{n+1}\mathbb{Z}_p) = a_n + b_n$$

where

$$\begin{aligned} a_n &= \sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1} mkp^n j^{m-1}(\mu(j + kp^n + p^{n+1}\mathbb{Z}_p) - \frac{1}{p}\mu(j + p^n\mathbb{Z}_p)), \\ b_n &= \sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1} mkp^{n-1} j^{m-1}\mu(j + p^n\mathbb{Z}_p). \end{aligned}$$

It remains to show that both a_n and b_n go to zero as $n \to \infty$. For a_n, we have

$$\begin{aligned} |a_n|_p &= \left|\sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1} mkp^n j^{m-1}(\mu(j + kp^n + p^{n+1}\mathbb{Z}_p) - \frac{1}{p}\mu(j + p^n\mathbb{Z}_p))\right|_p \\ &\leq \left|mp^n(\mu(j + kp^n + p^{n+1}\mathbb{Z}_p) - \frac{1}{p}\mu(j + p^n\mathbb{Z}_p))\right|_p \\ &\leq p^{1-n}B(\mu). \end{aligned}$$

It follows that $a_n \to 0$ as $n \to \infty$. For b_n, we have

$$b_n = \sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1} mkp^{n-1} j^{m-1}\mu(j + p^n\mathbb{Z}_p) = \frac{p-1}{2}mp^n S_{n,m-1}.$$

By the inductive hypothesis $\{S_{n,m-1}\}_{n=0}^{\infty}$ is a bounded sequence (since it is a convergent sequence). It follows that $b_n \to 0$ as $n \to \infty$. This shows that $S_{n,m}$ is a Cauchy sequence, so $\lim_{n\to\infty} S_{n,m}$ converges. $\square$

Since C^1 functions are determined by their Mahler series, it is important to know bounds on $\left|\int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x)\right|$. The next proposition gives such a bound.

Proposition 2.3. *Let μ be a Volkenborn distribution. Then there exists $c \in \mathbb{R}_{\geq 0}$ such that for all $m \in \mathbb{Z}_{\geq 0}$ we have*

$$\left|\int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x)\right|_p \leq cm$$

Proof. For $m = 0$, we know that $\int_{\mathbb{Z}_p}\mathrm{d}\mu(x)$ exists and equals $\mu(\mathbb{Z}_p)$. From this point on let $m \in \mathbb{Z}_{\geq 1}$. By Proposition 2.2 we know that $\binom{x}{m}$ is μ integrable. The proof of the inequality proceeds in a similar manner to the proof of Proposition 2.2, and we will use the sequence $\{T_{n,m}\}_{n=0}^{\infty}$ where

$$T_{n,m} := \sum_{j=0}^{p^n-1}\binom{j}{m}\mu(j+p^n\mathbb{Z}_p).$$

Note that

$$\begin{aligned}(2.1)\quad & |T_{n+1,m}-T_{n,m}|_p \\ &= \left|\sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1}\left(\binom{j+kp^n}{m}-\binom{j}{m}\right)\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)\right|_p.\end{aligned}$$

To estimate Equation (2.1), we use the binomial identity

$$\binom{j+kp^n}{m} = \sum_{l=0}^{m}\binom{j}{l}\binom{kp^n}{m-l}.$$

The right hand side of Equation (2.1) becomes

$$\left|\sum_{j=0}^{p^n-1}\sum_{k=0}^{p-1}\sum_{l=0}^{m-1}\binom{j}{l}\binom{kp^n}{m-l}\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)\right|_p. \tag{2.2}$$

We can bound each term of the sum from Equation (2.2) as follows:

$$\left|\binom{j}{l}\binom{kp^n}{m-l}\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)\right|_p \leq \left|\binom{kp^n}{m-l}\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)\right|_p,$$

where the right hand side of the above inequality equals

$$\begin{aligned}\left|\frac{kp^n}{m-l}\binom{kp^n-1}{m-l-1}\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)\right|_p &\leq \left|\frac{kp^n}{m-l}\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)\right|_p \\ &\leq p^{-n}m|\mu(j+kp^n+p^{n+1}\mathbb{Z}_p)|_p \\ &\leq Cpm \qquad\qquad \text{by Lemma 2.1.}\end{aligned}$$

This estimate gives us that Equation (2.2) is bounded above by Cpm. In other words,

$$|T_{n+1,m} - T_{n,m}|_p \leq Cpm. \tag{2.3}$$

Now we are in position to prove the result.

$$\begin{aligned}|T_{n,m}|_p &= \left|\sum_{j=0}^{n}(T_{n,m} - T_{n-1,m}) + T_{0,m}\right|_p \\ &\leq \max\{Cpm, |T_{0,m}|\} \\ &= \max\{Cpm, |\mu(\mathbb{Z}_p)|_p\}.\end{aligned}$$

Letting $c = \max\{Cp, |\mu(\mathbb{Z}_p)|_p\}$, we see that $|T_{n,m}|_P \leq cm$. This gives us that $|\int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x)|_p \leq cm$, as claimed. □

It is important to note that c from Proposition 2.3 is independent of m.

Theorem 2.4 (Barbacioru [3])**.** *Let* $f \in C^1(\mathbb{Z}_p)$ *and* μ *be a Volkenborn distribution. Then* f *is* μ*-integrable.*

Proof. Since $f \in C^1$, we know that the Mahler series of f is of the form

$$\sum_{m=0}^{\infty} a_m \binom{\cdot}{m} \quad \text{where} \quad \lim_{m\to\infty} m|a_m|_p = 0$$

(see [2]). We will show that

$$\int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu(x) = \sum_{m=0}^{\infty} a_m \int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x). \tag{2.4}$$

By Proposition 2.3 we know that $\left|\int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x)\right|_p \leq cm$. This tells us that

$$\lim_{m\to\infty} a_m \int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x) = 0.$$

Thus the right hand side of Equation (2.4) converges.

Now we will show that the left hand side of Equation (2.4) exists and equals the right hand side of the same equation. To do so we will use the sequence $\{T_{n,m}\}_{m=0}^{\infty}$ from Proposition 2.3. The proof of Proposition 2.3 showed that there exists $c \in \mathbb{R}_{\geq 0}$ such that $|T_{n,m}|_p \leq cm$.

Let $\epsilon > 0$. Then there exists $M \in \mathbb{Z}_{>0}$ such that for all $m \geq M$ we have that $|a_m T_{n,m}|_p < \epsilon$. Also, there exists $N \in \mathbb{Z}_{>0}$ such that for all $0 \leq m \leq M$ and $n \geq N$ we have that $|a_m(T_{n,m} - \int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x))|_p < \epsilon$.

Let $n \geq N$. Then

$$\sum_{j=0}^{p^n-1} f(j)\mu(j+p^n\mathbb{Z}_p) - \sum_{m=0}^{\infty} a_m \int_{\mathbb{Z}_p}\binom{x}{m}\mathrm{d}\mu(x) = a_M + b_M$$

where

$$\begin{aligned} a_M &= \sum_{j=0}^{p^n-1} \sum_{m=0}^{M} a_m \binom{j}{m} \mu(j+p^n\mathbb{Z}_p) - \sum_{m=0}^{M} a_m \int_{\mathbb{Z}_p} \binom{x}{m} \mathrm{d}\mu(x) \\ &= \sum_{m=0}^{M} a_m \left(T_{n,m} - \int_{\mathbb{Z}_p} \binom{x}{m} \mathrm{d}\mu(x) \right) \end{aligned}$$

and

$$\begin{aligned} b_M &= \sum_{j=0}^{p^n-1} \sum_{m=M}^{\infty} a_m \binom{j}{m} \mu(j+p^n\mathbb{Z}_p) - \sum_{m=M}^{\infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} \mathrm{d}\mu(x) \\ &= \sum_{m=M}^{\infty} a_m T_{n,m} - \sum_{m=M}^{\infty} a_m \int_{\mathbb{Z}_p} \binom{x}{m} \mathrm{d}\mu(x). \end{aligned}$$

We have $|a_M|_p < \epsilon$ by our choice of n (which depends on M), and $|b_M|_p < \epsilon$ by our choice of M. It follows that $\int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu(x)$ exists, so f is μ-integrable. $\square$

3. The module of Volkenborn Distributions

Let $\mathcal{V}(\Gamma)$ denote the subgroup (under addition) of $\mathcal{D}(\Gamma)$ of Volkenborn distributions. Recall that $\Lambda \simeq \mathcal{M}(\Gamma)$ acts on $\mathcal{D}(\Gamma)$ by convolution. In this section, we show that $\mathcal{V}(\Gamma)$ is closed under that action so we may view $\mathcal{V}(\Gamma)$ as a Λ-module. We then investigate the effect that the action of Λ on $\mathcal{V}(\Gamma)$ has on the Fourier transform of distributions contained in $\mathcal{V}(\Gamma)$, and most importantly, we will show that $\mathcal{I}(\Gamma)$ is a sub-module of $\mathcal{V}(\Gamma)$.

Lemma 3.1. *$\mathcal{V}(\Gamma)$ is a Λ-module.*

Proof. Let $\nu \in \mathcal{V}(\Gamma)$ and μ a bounded distribution, i.e., a distribution such that there exists $B \in \mathbb{R}_{\geq 0}$ satisfying

$$|\mu(a+p^n\mathbb{Z}_p)|_p \leq B$$

for all a and n. We show more generally that $\nu * \mu \in \mathcal{V}(\Gamma)$. By the definition for convolution, we have

$$(\nu*\mu)(a+p^{n+1}\mathbb{Z}_p) = \sum_{j=0}^{p^n-1} \sum_{k=0}^{p-1} \nu(j+kp^n+p^{n+1}\mathbb{Z}_p)\mu(a-j-kp^n+p^{n+1}\mathbb{Z}_p),$$

and similarly

$$(\nu*\mu)(a+p^n\mathbb{Z}_p) = \sum_{j=0}^{p^n-1} \nu(j+p^n\mathbb{Z}_p) \sum_{k=0}^{p-1} \mu(a-j-kp^n+p^{n+1}\mathbb{Z}_p).$$

So we see that $p \cdot (\nu * \mu)(a + p^{n+1}\mathbb{Z}_p) - (\nu * \mu)(a + p^n\mathbb{Z}_p)$ equals

$$\sum_{j=0}^{p^n-1} \sum_{k=0}^{p-1} \mu(a-j-kp^n+p^{n+1}\mathbb{Z}_p) \cdot \Big(p \cdot \nu(j+kp^n+p^{n+1}\mathbb{Z}_p) - \nu(j+p^n\mathbb{Z}_p)\Big).$$

Since μ is bounded and

$$p \cdot \nu(j + kp^n + p^{n+1}\mathbb{Z}_p) - \nu(j + p^n\mathbb{Z}_p)$$

is bounded independently from j and k, we see that $\nu * \mu \in \mathcal{V}(\Gamma)$. □

The Fourier transform of a Volkenborn distribution is guaranteed to exist from Theorem 2.4. We now study how convolution by $\mu \in \mathcal{M}(\Gamma)$ affects the Fourier transform of $\nu \in \mathcal{V}(\Gamma)$. For a Volkenborn distribution ν, let f_ν denote the function defined by $x \mapsto \lim p^n \nu(x + p^n\mathbb{Z}_p)$. Recall that f_ν is a bounded continuous function by Lemma 2.1. Let $\mathbf{S}$ denote the indefinite-sum operator. For $f \in C(\mathbb{Z}_p)$, the action of $\mathbf{S}$ on f simply shifts the Mahler expansion in the following way:

$$\mathbf{S}f = \mathbf{S} \sum_{m=0}^{\infty} \binom{\cdot}{m} (\nabla^m f)(0) = \sum_{m=0}^{\infty} \binom{\cdot}{m+1} (\nabla^m f)(0) \in C(\mathbb{Z}_p)$$

where $(\nabla f)(x) = f(x+1) - f(x)$ is the finite-difference operator. The reader should consult [8, Chapter V] for more details.

Proposition 3.2. *Let* $\mu \in \mathcal{M}(\Gamma)$*. For every* $\nu \in \mathcal{V}(\Gamma)$*, we have*

$$(\widehat{\nu * \mu})(T) = \widehat{\nu}(T) \cdot \widehat{\mu}(T) - \log_p(T) \cdot \sum_{m=0}^{\infty} \mu\left(\mathbf{S}^{m+1}(f_\nu \circ \iota)\right)(T-1)^m$$

where $\iota : x \mapsto -1 - x$ *is the canonical involution of* $\mathbb{Z}_p$*.*

Proof. Note that

$$\sum_{a,b=0}^{p^n-1} T^{a+b} \nu(a + p^n\mathbb{Z}_p)\mu(b + p^n\mathbb{Z}_p) \xrightarrow{n\to\infty} \int_{\mathbb{Z}_p} T^x \, \mathrm{d}\nu(x) \cdot \int_{\mathbb{Z}_p} T^x \, \mathrm{d}\mu(x).$$

Consider the sum on the left. Collecting all terms such that $a + b \equiv c \bmod p^n$, we see that it equals

$$\sum_{c=0}^{p^n-1} T^c(\mu * \nu)(c + p^n\mathbb{Z}_p) + \sum_{c=0}^{p^n-2} \sum_{d=1}^{p^n-c-1} (T^{c+p^n} - T^c)\mu(c+d+p^n\mathbb{Z}_p)\nu(-d+p^n\mathbb{Z}_p).$$

As $n \to \infty$, the term on the left converges to $(\widehat{\mu * \nu})(T)$ since $\mu * \nu$ is Volkenborn. Hence the term on the right converges. We rewrite that term as

$$\frac{T^{p^n} - 1}{p^n} \sum_{m=0}^{\infty} \sum_{c=0}^{p^n-2} \binom{c}{m} \sum_{d=1}^{p^n-c-1} \mu(c + d + p^n\mathbb{Z}_p) \cdot p^n \nu(-d + p^n\mathbb{Z}_p) \cdot (T-1)^m.$$

This expression converges to $\log_p(T) \cdot G(T)$ where the m-th coefficient of $G(T)$ equals

$$g_m := \lim_{n\to\infty} \sum_{c=0}^{p^n-2} \binom{c}{m} \sum_{d=1}^{p^n-c-1} \mu(c+d+p^n\mathbb{Z}_p) \cdot p^n\nu(-d+p^n\mathbb{Z}_p).$$

We collect terms according to $\mu(j+p^n\mathbb{Z}_p)$ obtaining

$$g_m = \lim_{n\to\infty} \sum_{j=0}^{p^n-1} \left(\binom{\cdot}{m} \circledast (f_n \circ \iota) \right)(j) \cdot \mu(j+p^n\mathbb{Z}_p)$$

where $f_n : x \mapsto p^n\nu(x+p^n\mathbb{Z}_p)$ and $\circledast$ is the shifted-convolution product. We now use the fact that $\binom{\cdot}{m} \circledast g = \mathbf{S}^{m+1}g$ and $f_n \rightrightarrows f_\nu$ to obtain

$$\begin{aligned} g_m &= \lim_{n\to\infty} \sum_{j=0}^{p^n-1} \mathbf{S}^{m+1}(f_\nu \circ \iota)(j)\mu(j+p^n\mathbb{Z}_p) \\ &= \int_{\mathbb{Z}_p} \mathbf{S}^{m+1}(f_\nu \circ \iota)(x)\, \mathrm{d}\mu(x). \end{aligned}$$

This completes the proof of the proposition. □

Remark. Note that if $\nu \in \mathcal{M}(\Gamma)$ (or even if ν is merely bounded in value), then $f_\nu \equiv 0$. So we recover the well-known fact that $\widehat{\nu * \mu} = \widehat{\nu} * \widehat{\mu}$, the Fourier transform of a convolution of measures equals the convolution of Fourier transforms.

We now show that $\mathcal{I}(\Gamma)$ is contained in $\mathcal{V}(\Gamma)$. This is the key to our main result.

Theorem 3.3. *$\mathcal{I}(\Gamma)$ is a sub-module of $\mathcal{V}(\Gamma)$.*

Proof. In light of Lemma 3.1, it suffices to prove that the generators of $\mathcal{I}(\Gamma)$ reside in $\mathcal{V}(\Gamma)$. Let $\ell = (\ell_n) \in \varprojlim k_n^\times$ and $\lambda = \rho(\ell)$. We have

$$p\lambda(a+p^n\mathbb{Z}_p) - \lambda(a+p^{n-1}\mathbb{Z}_p) = \log_p\left(\frac{\ell_{n-1}^{\gamma_0^a}}{\ell_n^{\gamma_0^a p}}\right).$$

Observe that

$$\frac{\ell_{n-1}^{\gamma_0^a}}{\ell_n^{\gamma_0^a p}} \xrightarrow{N_{n-1}^n} 1$$

where N_{n-1}^n is the norm from k_n to k_{n-1}. Since k_n/k_{n-1} is a cyclic extension, Hilbert's Theorem 90 gives an element $\alpha_n \in k_n^\times$ such that

$$\frac{\ell_{n-1}^{\gamma_0^a}}{\ell_n^{\gamma_0^a p}} = \alpha_n^{\gamma_0^a(\gamma_n-1)} \quad \text{where} \quad \gamma_n = \gamma_0^{p^{n-1}}.$$

It remains to show that $\log_p\left(\alpha_n^{\gamma_0^a(\gamma_n-1)}\right)$ is bounded independent of a and n. In fact, we need only show that it is bounded independent of a and n for all n sufficiently large.

Assume that the inertia subgroup for $\mathfrak{p}$ of k is $\mathrm{Gal}(k_\infty/k_m)$ and let $n \geq m$. Fix an embedding $k_\infty \hookrightarrow \Omega_p$, and let π_n be a local parameter for K_n, the topological closure of k_n. Since K_n/K_m is totally ramified, it follows that

$$N_m^n(\pi_n) = \pi_m$$

is a local parameter for K_m. Moreover, we get that

$$N_m^n\left(K_n^\times\right) = \langle \pi_m \rangle \times N_m^n(U_n)$$

where U_n denotes the units of K_n. Note that

$$[U_m : N_m^n(U_n)] = p^{n-m}$$

since K_n/K_m is cyclic and totally ramified. Let $U_m^{(j)}$ denote the j-th group of principal units of K_m, so $U_m^{(j)} = 1 + (\pi_m)^j \subset U_m$. Let q denote the order of the residue class field for K_m, and recall the filtration

$$U_m \supset U_m^{(1)} \supset U_m^{(2)} \supset \cdots$$

where

$$[U_m^{(j)} : U_m^{(j+1)}] = \begin{cases} q-1 & j = 0 \\ q & \text{else.} \end{cases}$$

Let r be the smallest positive integer such that $U_m^{(r)} \subset N_m^n(U_n)$, so

$$\langle \pi_m \rangle \times U_m^{(r)} \subseteq N_m^n(K_n^\times). \tag{3.1}$$

From the above filtration, we see that as n increases so must r. Let n be large enough so that $r > 1$.

From Equation (3.1), local class field theory gives us that $K_n \subseteq L_r$ where L_r is the field of π_m^r-division points of some Lubin-Tate module for π_m (see [6, 7]). For a real number $s \geq -1$, we define the s-th ramification group

$$G_s(L_r/K_m) = \{\sigma \in \mathrm{Gal}(L_r/K_m) : w(\sigma(a) - a) \geq s + 1 \quad \forall a \in \mathcal{O}\}$$

where $\mathcal{O}$ is the valuation ring of L_r and w is the valuation associate to its maximal ideal. The Lubin-Tate extensions have the property that

$$G_{q^{r-1}-1}(L_r/K_m) = \mathrm{Gal}(L_r/L_{r-1}).$$

Let $H \subset \mathrm{Gal}(L_r/K_m)$ such that K_n is the fixed field of H. A theorem of Herbrand (see [7, II.10.7]) gives us that

$$G_s(L_r/K_m)H/H = G_t(K_n/K_m), \text{ with } t = \int_0^s \frac{dx}{[G_0(L_r/K_n) : G_x(L_r/K_n)]}.$$

By the minimality of r, we have that

$$G_{q^{r-1}-1}(L_r/K_m) = \mathrm{Gal}(L_r/L_{r-1}) \not\subseteq H,$$

so for $s = q^{r-1} - 1$, we have $G_t(K_n/K_m)$ is non-trivial. We now obtain a crude but functional lower bound for the value t. Since L_r/K_m is totally ramified, we have

$$t = \frac{[K_n : K_m]}{[L_r : K_m]} \sum_{j=1}^{q^{r-1}-1} \#G_j(L_r/K_n) \geq \frac{p^{n-m}}{q-1} \cdot \frac{q^{r-1}-1}{q^{r-1}} \geq \frac{p^{n-m-1}}{q-1} = t(n)$$

where the last inequality follows because $r > 1$. It follows that

$$\gamma_n \in \mathrm{Gal}(K_n/K_{n-1}) \subseteq G_t(K_n/K_m) \subseteq G_{t(n)}(K_n/K_m).$$

Let $e(\pi_n : p)$ denote the ramification index of π_n over p. For all n sufficiently large, we have $\alpha_n^{\gamma_0^a(\gamma_n-1)} - 1 \in (\pi_n)^{t(n)}$ so

$$v_p\left(\alpha_n^{\gamma_0^a(\gamma_n-1)} - 1\right) \geq \frac{t(n)}{e(\pi_n : p)} = \frac{1}{p(q-1)e(\pi : p)}.$$

Whence $\log_p(\alpha_n^{\gamma_0^a(\gamma_n-1)})$ is bounded independent of n and a for all n sufficiently large. This proves the theorem. $\square$

We may now give

Proofs of Theorems 1.1 and 1.2. Theorem 1.1 follows from Theorems 2.4 and 3.3. It's straightforward to verify that if $\lambda_\chi \in \mathcal{I}_\chi^F(\Gamma)$, then λ_χ is Volkenborn. So the first statement of Theorem 1.2 also follows from Theorems 2.4 and 3.3. For the second statement of Theorem 1.2, note that $\psi(\gamma_0)$ is a p^n-th root of unity so

$$\begin{aligned}\widehat{\lambda}_\chi(\zeta_\psi) &= \lim_{j\to\infty} -\sum_{a=0}^{p^j-1} \overline{\psi}(\gamma_0^a) \sum_{\delta\in\Delta} \log_p\left(\ell_n^{\gamma_0^a\delta}\right)\overline{\chi}(\delta) \\ &= -\sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a) \sum_{\delta\in\Delta} \log_p\left(\ell_n^{\gamma_0^a\delta}\right)\overline{\chi}(\delta).\end{aligned}$$

The second statement now follows. $\square$

Remark. Let $D \subset \mathbb{C}_p$ be the open disk of radius 1 centered about 1, and let $H(D)$ denote the ring of power series in $\mathbb{C}_p[\![T-1]\!]$ convergent on D. Let $\mathcal{F}$ be the composition of the map ϱ with the Fourier transform:

$$\begin{tikzcd} \varprojlim k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[r, "\varrho"] \arrow[rr, bend right, "\mathcal{F}"'] & \mathcal{I}(\Gamma) \arrow[r] & H(D)/\big(\log_p T\big) \end{tikzcd}$$

Given Proposition 3.2 and Theorem 3.3, we get that $\mathcal{F}$ is a well-defined Λ-morphism. If $(\mathfrak{l}_n) = \mathfrak{l} \in \ker \mathfrak{F}$ and $\varrho(\mathfrak{l}) = \mathfrak{L}$, then for every $n \geq 0$, for every character ψ of Γ_n, we have

$$0 = \widehat{\mathfrak{L}}(\zeta_\psi) = \sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a) \operatorname{Log}_p(\mathfrak{l}_n^{\gamma_0^a})$$

where

$$e_\psi \cdot \sum_{a=0}^{p^n-1} \operatorname{Log}_p(\mathfrak{l}_n^{\gamma_0^a})\gamma_0^{-a} = \sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a) \operatorname{Log}_p(\mathfrak{l}_n^{\gamma_0^a}) \cdot e_\psi \in \mathbb{C}_p[\Gamma_n]$$

and $e_\psi \in \mathbb{C}_p[\Gamma_n]$ is the idempotent associate to ψ. Since $\mathbb{C}_p[\Gamma_n] = \bigoplus_\psi \mathbb{C}_p e_\psi$, it follows that

$$\begin{aligned}\widehat{\mathfrak{L}} \equiv 0 \bmod (\log_p T) &\Leftrightarrow 0 = \sum_{a=0}^{p^n-1} \operatorname{Log}_p(\mathfrak{l}_n^{\gamma_0^a})\gamma_0^{-a}, \quad \forall n \geq 0 \\ &\Leftrightarrow \mathfrak{L} = 0.\end{aligned}$$

Whether $\mathfrak{L}$ is the 0-distribution is a more delicate question. For suppose $\mathfrak{l}_n = \sum(\ell_j \otimes x_j)$, then

$$\mathfrak{L}(p^n\mathbb{Z}_p) = \operatorname{Log}_p(\mathfrak{l}_n) = \sum x_j \log_p(\ell_j).$$

We now need to know whether the terms $\log_p(\ell_j)$ are p-adically independent, a question related to Leopoldt's conjecture.

4. Applications to Cyclotomic Fields

In this section we specialize to the case when k is the cyclotomic field $\mathbb{Q}(\zeta_{pd})$ and k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension of k. Recall that d is a positive integer co-prime to p. We apply the previous results to the problem of interpolating Gauss sums attached to Dirichlet characters. Despite the specialization to cyclotomic fields, we continue to use the more generic notation, i.e., we write k_n in place of $k_{(n)}$ and Γ_n in place of G_n.

For a Dirichlet character φ, let f_φ denote the conductor of φ and let $\tau(\varphi)$ denote the Gauss sum

$$\tau(\varphi) = \sum_{a=1}^{f_\varphi} \varphi(a)\zeta_{f_\varphi}^a.$$

We associate Dirichlet characters of conductor dividing dp^{n+1} to characters of $\operatorname{Gal}(\mathbb{Q}(\zeta_{dp^{n+1}})/\mathbb{Q})$ in the obvious way: $\varphi(\sigma_a) = \varphi(a)$ where $\sigma_a : \zeta_{dp^{n+1}} \mapsto \zeta_{dp^{n+1}}^a$.

Fix a character χ of $\Delta = \operatorname{Gal}(k/\mathbb{Q})$ with $d \mid f_\chi$, and let $F = \mathbb{Q}(\zeta_{p-1})$. For any integer t, let ℓ_t denote the norm coherent sequence of elements

$$\ell_t = (\zeta_{p-1}^t - \zeta_{dp^{n+1}}) \in \varprojlim (Fk_n)^\times.$$

Note that the norm coherency of ℓ_t follows from the fact that $\zeta_n^x = \zeta_{n/x}$ for all $x \mid n$. As in Section 1, we consider F inert under the action of Δ and Γ_n.

4.1. For arbitrary $d > 0$. In this section, we assume that d is an arbitrary positive integer co-prime to p, and we evaluate the Fourier transform of the distributions associate to the norm coherent sequences ℓ_t at p-power roots of unity.

Theorem 4.1. *If $\lambda_\chi = \rho_\chi(\ell_t) \in \mathcal{I}_\chi^F(\Gamma)$ and ψ is a character of Γ_n with $\zeta_\psi = \overline{\psi}(\gamma_0)$, then*

$$\widehat{\lambda}_\chi(\zeta_\psi) = -(1 - \overline{\varphi}(p)) \sum_{a=1}^{f_\varphi} \log_p(\zeta_{p-1}^t - \zeta_{f_\varphi}^a)\overline{\varphi}(a)$$

where $\varphi = \chi\psi$. [1]

Proof. Note that λ_χ satisfies

$$\widehat{\lambda}_\chi(\zeta_\psi) = \int_{\mathbb{Z}_p} \overline{\psi}(\gamma_0)^x \, \mathrm{d}\lambda_\chi(x) = \sum_{a=0}^{p^n-1} \overline{\psi}(\gamma_0^a)\lambda_\chi(a + p^n\mathbb{Z}_p)$$

where the last equality follows since ψ is a character of Γ_n. Hence

$$\begin{aligned}\widehat{\lambda}_\chi(\zeta_\psi) &= -\sum_{\substack{b=1\\(b,mp)=1}}^{dp^{n+1}} \log_p(\zeta_{p-1}^t - \zeta_{dp^{n+1}}^b)\overline{\varphi}(b)\\ &= -(1 - \overline{\varphi}(p)) \sum_{b=1}^{dp^{n+1}} \log_p(\zeta_{p-1}^t - \zeta_{dp^{n+1}}^b)\overline{\varphi}(b)\\ &= -(1 - \overline{\varphi}(p)) \sum_{b=1}^{f_\varphi} \log_p(\zeta_{p-1}^t - \zeta_{f_\varphi}^b)\overline{\varphi}(b),\end{aligned}$$

since $f_\varphi \mid dp^{n+1}$. This completes the proof of the theorem. □

Corollary 4.2. *If $t \equiv 0 \bmod p-1$, then*

$$\widehat{\lambda}_\chi(\zeta_\psi) = \frac{1 - \overline{\varphi}(p)}{1 - \varphi(p)/p} \tau(\overline{\varphi}) L_p(1, \varphi).$$

where $L_p(s, \varphi)$ is the Leopoldt–Kubota p-adic L-function. In particular, if $p \mid f_\varphi$, then

$$\widehat{\lambda}_\chi(\zeta_\psi) = \tau(\overline{\varphi}) L_p(1, \varphi).$$

[1] As usual, we interpret $\log_p(0) \cdot 0$ to be 0.

Proof. The first fact follows immediately from the formula (see [10])

$$L_p(1,\varphi) = -\left(1 - \frac{\varphi(p)}{p}\right) \frac{1}{\tau(\overline{\varphi})} \sum_{a=1}^{f_\varphi} \log_p(1-\zeta_{f_\varphi}^a)\overline{\varphi}(a).$$

The second fact follows from the first since if $p \mid f_\varphi$, then $\varphi(p) = 0$. □

4.2. For $\boldsymbol{d = 1}$. Combining Corollary 4.2 with results from Iwasawa [4] allow us to view the Gauss sums $\tau(\overline{\chi\psi})$ in an interesting light when the conductor of $\chi\psi$ is a p-power and χ is even. Essentially, they arise as special values of the Fourier transform of a distribution associate to a generating sequence for the projective limit of principal units of $\mathbb{Q}_p(\zeta_{p^{n+1}})$.

Accordingly, we restrict ourselves to the scenario when $d = 1$ so $k_n = \mathbb{Q}(\zeta_{p^{n+1}})$. Consequently, we have $K_n = \mathbb{Q}_p(\zeta_{p^{n+1}})$. We make heavy use of the maps ρ, α and β introduced in Section 1. The maps ρ and α obviously extend to $\varprojlim K_n^\times$. For any $x = (x_n) \in \varprojlim K_n^\times$, we write the components of $\alpha(x)$ as $\alpha_n(x_n) \in \mathbb{C}_p[\Gamma_n]$. Let U_n denote the principal units of K_n, and set $U = \varprojlim U_n$. Let χ be an even character, and consider the following diagram of maps:

$$\begin{tikzcd} e_\chi U \arrow[r, "\rho"] \arrow[dr, "\alpha"'] & \rho(e_\chi U) \arrow[r] \arrow[d, "\beta"] & \mathbb{C}_p[\![T-1]\!] \\ & \alpha(e_\chi U) & \end{tikzcd}$$

where the far right map is the Fourier transform. Since $\log_p$ has no roots on $e_\chi U_n$ (since χ is not the Teichmüller character), the maps ρ, β, and α form a commutative diagram of Λ-isomorphisms (see Section 3).

Theorem 4.3. *If χ is a non-trivial even character of Δ, then there exists an integer t, dependent on χ, such that the distribution*

$$\lambda_\chi = \frac{1}{p-1} \cdot \rho_\chi(\ell_t) \in \mathcal{I}_\chi^F(\Gamma)$$

satisfies

$$\widehat{\lambda}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}) \cdot \textit{(unit)}$$

for every wildly ramified character ψ.

Proof. Let $C_n \subseteq U_n$ denote the topological closure of the cyclotomic units of k_n congruent to 1 mod π_n, and let $C = \varprojlim C_n$ with respect to the norm maps. Recall that $e_\chi C_n$ is generated by

$$c_n = \left(\zeta_{p^{n+1}}^{(1-\delta_0\gamma_0)/2} \frac{\zeta_{p^{n+1}}^{\delta_0\gamma_0} - 1}{\zeta_{p^{n+1}} - 1}\right)^{(p-1)e_\chi}$$

where δ_0 generates Δ. Let $c = (c_n) \in e_\chi C$, and let $\xi_\chi = \rho(c)$ denote the associated distribution. What's more, there exists a $p-1$-st root of unity $\zeta_\chi \in \mathbb{Z}_p$ not equal to 1 such that $e_\chi U_n$ is generated by

$$u_n = \left(\frac{\zeta_\chi - \zeta_{p^{n+1}}}{\omega(\zeta_\chi - 1)}\right)^{e_\chi}$$

where ω is the Teichmüller character (see [10, Theorem 13.54]). Let $u = (u_n) \in e_\chi U$, and let $\lambda_\chi = \rho(u)$ denote the associated distribution. Then

$$\begin{aligned}\lambda_\chi(a + p^n\mathbb{Z}_p) &= -\log_p\left(u_n^{\gamma_0^a}\right)\\ &= \frac{-1}{p-1}\sum_{\delta\in\Delta}\log_p\left(\zeta_\chi - \zeta_{p^{n+1}}^{\gamma_0^a\delta}\right)\overline{\chi}(\delta).\end{aligned}$$

So $\lambda_\chi = \rho_\chi(\ell_t)/(p-1)$ for some integer t. From Theorem 4.1, it follows that

$$\widehat{\lambda}_\chi(\zeta_\psi) = \frac{-1}{p-1}\sum_{b=1}^{f_\varphi}\log_p\left(\zeta_\chi - \zeta_{f_\varphi}^b\right)\overline{\varphi}(b)$$

where $\varphi = \chi\psi$. Since the terms $\log_p(\zeta_\chi - \zeta_{f_\varphi}^b)$ for $(b, f_\varphi) = 1$ are linearly independent over $\mathbb{Q}$, they must also be linearly independent over $\mathbb{Q}^{\mathrm{alg}}$ by a theorem of Brumer. Hence $\widehat{\lambda}_\chi(\zeta_\psi) \neq 0$.

Now, there exists a distribution $\mu_\chi \in \mathcal{M}(\Gamma)$ such that $\xi_\chi = \mu_\chi * \lambda_\chi$. Specifically, μ_χ satisfies $u^{\beta(\mu_\chi)} = c$. Also, there exists $H_\chi(T) \in \Lambda^\times$ such that

$$\widehat{\mu}_\chi(T) \cdot H_\chi(T) = G_\chi(T) \in \Lambda$$

where

$$G_\chi\big((1+p)^s\big) = L_p(1-s, \chi),$$

(see [4] or [10, Theorem 13.56]). Let $F_\chi(T) \in \Lambda$ such that $L_p(s, \chi\psi) = F_\chi\big(\zeta_\psi(1+p)^s\big)$. Then F_χ and G_χ are related via the formula

$$F_\chi(T) = G_\chi\left(\frac{1+p}{T}\right).$$

Since $\widehat{\lambda}_\chi(\zeta_\psi) \neq 0$, by Proposition 3.2 we have that

$$\left[\widehat{\mu}_\chi(T) = \frac{\widehat{\xi}_\chi(T)}{\widehat{\lambda}_\chi(T)}\right]_{T=\zeta} \tag{4.1}$$

for any p^n-th root of unity ζ. It follows that

$$L_p(1, \chi\psi) = F_\chi\big(\zeta_\psi(1+p)\big) = G_\chi(\zeta_{\overline{\psi}}) = \frac{\widehat{\xi}_\chi(\zeta_{\overline{\psi}})}{\widehat{\lambda}_\chi(\zeta_{\overline{\psi}})} \cdot H_\chi(\zeta_{\overline{\psi}}).$$

On the other hand, by Corollary 4.2 we have

$$L_p(1,\chi\overline{\psi}) = \frac{\widehat{\xi}_\chi(\zeta_{\overline{\psi}})}{\tau(\overline{\chi}\psi)(\chi(\delta_0)\zeta_\psi - 1)}.$$

Dividing the formula for $L_p(1,\chi\overline{\psi})$ by the formula for $L_p(1,\chi\psi)$ yields

$$\frac{L_p(1,\chi\overline{\psi})}{L_p(1,\chi\psi)} = \frac{\widehat{\lambda}_\chi(\zeta_{\overline{\psi}})}{\tau(\overline{\chi}\psi)(\chi(\delta_0)\zeta_\psi - 1)H_\chi(\zeta_{\overline{\psi}})}$$

whence

$$\widehat{\lambda}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi})\left[(\chi(\delta_0)\zeta_{\overline{\psi}} - 1)H_\chi(\zeta_\psi)F_\chi\big(\zeta_\psi(1+p)\big)^{1-\sigma}\right] \tag{4.2}$$

where $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta_\psi)/\mathbb{Q}_p)$ is defined by $\sigma : \zeta_\psi \mapsto \zeta_{\overline{\psi}}$. Since χ is non-trivial, the term above in brackets is a unit of $\mathbb{Z}_p[\zeta_\psi]$. □

Keeping notation from the proof of Theorem 4.3, let $\mathrm{R}_p(e_\chi U_n)$ denote the p-adic regulator of $e_\chi U_n$. Specifically, for any set of elements $\epsilon_1, \epsilon_2, \ldots, \epsilon_{p^n} \in e_\chi U_n$ that generate $e_\chi U_n$ as a $\mathbb{Z}_p$-module, set

$$\mathrm{R}_p(e_\chi U_n) = \det\big(\log_p(\epsilon_j^\gamma)\big)_{j,\gamma}$$

where γ ranges over Γ_n. Note that $\mathrm{R}_p(e_\chi U_n)$ is determined only up to a unit of $\mathbb{Z}_p$.

Corollary 4.4. *There exists $y \in e_\chi U$ such that y generates $e_\chi U$ (as a $\mathbb{Z}_p[\![\Gamma]\!]$-module) and the associated distribution $\upsilon_\chi = \rho(y)$ satisfies*

$$\prod_{\psi\in\widehat{\Gamma_n}} \widehat{\upsilon}_\chi(\zeta_\psi) = \prod_{\psi\in\widehat{\Gamma_n}} \tau(\overline{\chi\psi}) = \mathrm{R}_p(e_\chi U_n).$$

Proof. Keeping notation from the previous theorem, let $\varepsilon \in \mathcal{M}(\Gamma)$ such that

$$\widehat{\varepsilon}(T) = \left[\left(\frac{\chi(\delta_0)}{T} - 1\right)H_\chi(T)\right]^{-1} \in \Lambda^\times.$$

Let $y = u^{\beta(\varepsilon)}$ where u is defined as in the proof of Theorem 4.3. Since β and the Fourier transform are isomorphisms (when restricted to $\mathcal{M}(\Gamma)$), it follows that $y = (y_n)$ also generates $e_\chi U$. Moreover, letting $\rho(y) = \upsilon_\chi \in \mathcal{I}_\chi^F(\Gamma)$, from Equations (4.1) and (4.2) we get that

$$\widehat{\upsilon}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}) \cdot F_\chi\big(\zeta_\psi(1+p)\big)^{1-\sigma}.$$

Note that $\widehat{\upsilon}_\chi(\zeta_\psi)$ is the $\chi\psi$-part of $\mathrm{R}_p(e_\chi U_n)$. To be precise, let $\Upsilon_\chi^{(n)} = \alpha_n(y_n)$, so that

$$\Upsilon_\chi^{(n)} = -\sum_{a=0}^{p^n-1}\sum_{\delta\in\Delta} \log_p(y_n^{\gamma_0^a})\overline{\chi}(\delta)\gamma_0^{-a} \in K_n[\Gamma_n].$$

Now let $\Upsilon^{(n)}_{\chi\psi} \in K_n$ be defined by

$$e_{\chi\psi}\Upsilon^{(n)} = \Upsilon^{(n)}_{\chi\psi} e_{\chi\psi},$$

and note that $\Upsilon^{(n)}_{\chi\psi} = \widehat{\upsilon}_\chi(\zeta_\psi)$. Then $R_p(e_\chi U_n)$ equals

$$\prod_{\psi\in\widehat{\Gamma_n}} \Upsilon^{(n)}_{\chi\psi} = \prod_{\psi\in\widehat{\Gamma_n}} \widehat{\upsilon}_\chi(\zeta_\psi) = \prod_{\psi\in\widehat{\Gamma_n}} \tau(\overline{\chi\psi}) \cdot \frac{F_\chi\big(\zeta_\psi(1+p)\big)}{F_\chi\big(\zeta_{\overline{\psi}}(1+p)\big)} = \prod_{\psi\in\widehat{\Gamma_n}} \tau(\overline{\chi\psi}). \quad \square$$

Under the additional assumption that p is regular, the above equality of products can be refined into an equality of components. In particular, keeping notation from Theorem 4.3 and Corollary 4.4, we have

Corollary 4.5. *If p is a regular prime, then there exists $v \in e_\chi U$ such that $v = (v_n)$ generates $e_\chi U$ and the associated distribution $\nu_\chi = \rho(v)$ satisfies*

$$\widehat{\nu}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}) = \mathrm{N}^{(n)}_{\chi\psi},$$

where $\mathrm{N}^{(n)}_\chi = \alpha_n(v_n)$ and $N^{(n)}_{\chi\psi}$ is its ψ-part.

Proof. If p is a regular prime, then $F_\chi(T) \in \Lambda^\times$, hence

$$G_\chi(T),\ G_\chi\left(\frac{1+p}{T}\right) \in \Lambda^\times$$

as well. So the values $F_\chi\big(\zeta_\psi(1+p)\big)^{1-\sigma}$ are interpolated by a power series in $\Lambda^\times$, namely,

$$F_\chi\big(\zeta_\psi(1+p)\big)^{1-\sigma} = \left.\frac{G_\chi(T)}{G_\chi((1+p)/T)}\right|_{T=\zeta_\psi}.$$

Let $\varepsilon \in \mathcal{M}(\Gamma)$ such that

$$\widehat{\varepsilon}(T) = \left[\frac{G_\chi(T)}{G_\chi((1+p)/T)}\right]^{-1}.$$

Let $v = y^{\beta(\varepsilon)}$ where y is as in Corollary 4.4. In a similar fashion, we get that $v = (v_n)$ also generates $e_\chi U$, and if $\nu_\chi = \rho(v)$, then

$$\mathrm{N}^{(n)}_{\chi\psi} = -\sum_{a=0}^{p^n-1} \overline{\psi}(a) \sum_{\delta\in\Delta} \log_p\big(v_n^{\gamma_0^a\delta}\big)\overline{\chi}(\delta) = \widehat{\nu}_\chi(\zeta_\psi) = \tau(\overline{\chi\psi}). \qquad \square$$

Remark. Note that $\mu_\chi \in \mathcal{M}(\Gamma)$ from the proof of Theorem 4.3 can be given explicitly in terms of $\widehat{\lambda}_\chi(T)$ and $\widehat{\xi}_\chi(T)$ from Theorem 4.3. In particular,

$$\mu_\chi(a + p^n\mathbb{Z}_p) = \frac{1}{p^n}\sum_{j=0}^{p^n-1} \zeta_{p^n}^{-ja} \left.\frac{\widehat{\xi}_\chi(T)}{\widehat{\lambda}_\chi(T)}\right|_{T=\zeta_{p^n}^j}.$$

Let $E_n \subseteq U_n$ denote the topological closure of the units of k_n congruent to 1 mod π_n. The map

$$\begin{aligned} \alpha_n : e_\chi U_n &\to K_n[\Gamma_n] \\ \epsilon &\mapsto \sum_{a=0}^{p^n-1} \log_p\left(\epsilon^{\gamma_0^a}\right)\gamma_0^{-a} \end{aligned}$$

is a $\mathbb{Z}_p[\Gamma_n]$-module map. Since $\widehat{\lambda}_\chi(\zeta_\psi)$ doesn't vanish, the element $\alpha_n(u_n)$ is invertible in $K_n[\Gamma_n]$. Since U_n is generated by u_n, the element $\alpha_n(u_n)^{-1}$ acts as an *integralizer* for the map $\alpha_n|_{E_n}$ in the sense of [1, Definition 2.5]. Let $A_n : E_n \to \mathbb{Z}_p[\Gamma_n]$ be the map defined by

$$A_n(\epsilon) = \alpha_n(u_n)^{-1} \cdot \alpha_n(\epsilon).$$

It follows from [1, Theorem 3.1] that the image of the cyclotomic units of k_n under A_n annihilates the χ-part of the Sylow p-subgroup of $\mathrm{Cl}(k_n^+)$. Let $\beta(\mu_\chi) = \left(M_\chi^{(n)}\right) \in \mathbb{Z}_p[\![\Gamma]\!]$, so

$$M_\chi^{(n)} = \sum_{a=0}^{p^n-1} \mu_\chi(a + p^n\mathbb{Z}_p)\gamma_0^{-a} \in \mathbb{Z}_p[\Gamma_n].$$

Since $\beta(\mu_\chi) = \alpha(u)^{-1} \cdot \alpha(c)$, it follows that

$$M_\chi^{(n)} = \alpha_n(u_n)^{-1} \cdot \alpha_n(c_n).$$

So $M_\chi^{(n)}$ is indeed in the image of the cyclotomic units of the map A_n. Therefore $\left(M_\chi^{(n)}\right) \in \mathbb{Z}_p[\![\Gamma]\!]$ is a coherent sequence of explicit annihilators of the χ-part of the Sylow p-subgroup of $\mathrm{Cl}(k_n^+)$.

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