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Résumé. Nous montrons que, si $J_n(\mathcal{G})$ est l'ensemble des réels dans $[0,1]$ dont la fraction continue infinie est constituée de nombres entiers compris entre 1 et n, alors $\lim_{n\to\infty} H_{h_n}(J_n(\mathcal{G})) = 1 = \mathrm{H}_1(J(\mathcal{G}))$, où h_n est la dimension de Hausdorff de $J_n(\mathcal{G})$, H_{h_n} est la mesure de Hausdorff correspondant et où $J(\mathcal{G})$ est l'ensemble de tous les nombres irrationnels de $[0,1]$, i.e. ceux dont la fraction continue est infinie. Nous montrons aussi que cette propriété n'est pas générale en construisant une classe de systèmes de fonctions itérées $\mathcal{S}$ sur $[0,1]$, formés de similarités, pour lesquels $\underline{\lim}_{F\to E} \mathrm{H}_{h_F}(J_F) < \mathrm{H}_{h_\mathcal{S}}(J_\mathcal{S})$; cette limite inférieure s'étend sur les sous-ensembles finis de l'alphabet infini E.

Abstract. We prove that if by $J_n(\mathcal{G})$ we denote the set of all numbers in $[0,1]$ whose infinite continued fraction expansions have all entries in the finite set $\{1,2,\dots,n\}$, then $\lim_{n\to\infty} \mathrm{H}_{h_n}(J_n(\mathcal{G})) = 1 = \mathrm{H}_1(J(\mathcal{G}))$, where h_n is the Hausdorff dimension of $J_n(\mathcal{G})$, H_{h_n} is the corresponding Hausdorff measure, and $J(\mathcal{G})$ denotes the set of all irrational numbers in $[0,1]$, i .e. those whose continued fraction expansion is infinite. We also show that this property is not too common by constructing a class of infinite iterated function systems $\mathcal{S}$ on $[0,1]$, consisting of similarities, for which $\underline{\lim}_{F\to E} \mathrm{H}_{h_F}(J_F) < \mathrm{H}_{h_\mathcal{S}}(J_\mathcal{S})$; the lower limit is taken over finite subsets of the countable infinite alphabet E.

1. Introduction

Let (X,ρ) be metric space and let $A\subset X$. Given $t\geq 0$ we define

$$\mathrm{H}_t(A) := \lim_{\delta\to 0} \inf\Big\{ \sum_{n=1}^\infty \mathrm{diam}^t(U_n) : \bigcup_{n=1}^\infty U_n \supset A, \mathrm{diam}(U_n)\leq \delta \ \forall\ n\geq 1\Big\}$$

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and $\mathrm{H}_t(A)$ is called the t-dimensional (outer) Hausdorff measure of A. The function $A \mapsto \mathrm{H}_t(A)$ restricted to the σ-algebra of Borel sets of X is (an ordinary non-negative σ-additive) measure. The number

$$\mathrm{HD}(A) = \inf\{t > 0 : \mathrm{H}_t(A) = 0\}$$

is called the Hausdorff dimension of A. Frequently, especially in dynamics, if $0 < \mathrm{H}_t(X) < +\infty$, one considers also normalized Hausdorff measure, i.e. the function

$$A \mapsto \mathrm{H}_t^1(A) := \mathrm{H}_t(A)/\mathrm{H}_t(X).$$

In order to avoid any confusion as to which Hausdorff measure we mean, we frequently refer to $\mathrm{H}_t(A)$ as the numerical value of the Hausdorff measure of A. In this paper we always consider the Hausdorff measure (and dimension) with respect to the standard (Euclidean) metric on the ambient space which is with no exception $\mathbb{R}^q$ with some integer $q \geq 1$.

Passing to a number theoretical context, in agreement with notation of Section 3 by $J_n(\mathcal{G})$ we denote the set of all numbers in $[0,1]$ whose infinite continued fraction expansions have all entries in the finite set $\{1, 2, \dots, n\}$. It is well-known (see [1], comp. [3] and [4], where an analogous statement is proved for all conformal iterated function systems), that

$$\lim_{n\to\infty} \mathrm{HD}\big(J_n(\mathcal{G})\big) = 1. \tag{1.1}$$

D. Hensley was even able to show in [1] that

$$\lim_{n\to\infty} n\big(1 - \mathrm{HD}\big(J_n(\mathcal{G})\big)\big) = \frac{6}{\pi^2}.$$

Motivated by such results and some continuity properties of the numerical value of the Hausdorff measure of the limit sets in conformal dynamics (see [5] and [6]), we asked ourselves whether a continuity like in (1.1) holds on a deeper level of Hausdorff measures. Armed with the theory of iterated function systems it can be relatively easy to show that the continuity holds for normalized Hausdorff measures in the weak* topology on Borel probability measures on the unit interval $[0,1]$. For the numerical values of Hausdorff measures the positiver answer is given in Section 3 below; see Theorem 3.1. Its proof is in its majority number theoretical slightly touching on iterated function systems. However, this result fits well into the context of both: number theory and iterated function systems. Section 4 briefly describes the latter and recalls Bowen's formula expressing the Hausdorff dimension of the limit set in dynamical terms. If $\mathcal{S} = \{\phi_e\}e \in E$ is a conformal iterated function systems satisfying the Open Set Condition, then (see [4])

$$\sup\{\mathrm{HD}(J_F) : F \subset E\} = \mathrm{HD}(J_{\mathcal{S}})$$

where the supremum is taken over all finite subsets F of E and J_F is the limit set of the iterated function system $\{\phi_e\}_{e\in F}$. Motivated by this fact and Theorem 3.1 we asked ourselves whether

$$\lim_{F\to E} \mathrm{H}_{h_F}(J_F) = \mathrm{H}_{h_{\mathcal{S}}}(J_{\mathcal{S}})$$

for all conformal iterated function systems satisfying the Open Set Condition; in here $h_F = \mathrm{HD}(J_F)$, $h_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$ and H_t denotes always t-dimensional Hausdorff measure. We show in Section 6 that the answer is in general negative. It is negative already in the simplest possible situation to think about: linear (similarity), so no distortion of derivative, IFS on $[0,1]$ whose limit set is all of $[0,1]$ but a countable set (as is also the case for continued fractions). This shows that being of number theoretical origin, continued fractions are rather special amongst IFSs on $[0,1]$. It also shows that bounded distortion of derivative, one of the main technical issues in the proof of Theorem 3.1, is by no means all what counts for the proof of this theorem. As a convenient tool to prove discontinuity in the counterexample constructed in Section 6, we derived in Section 5 a simple formula to express the Hausdorff measure of iterated function systems consisting of similarities; this formula is of interest on its own.

2. Selected Preliminaries from Geometric Measure Theory

In this section we collect some well-known general density theorems which ultimately express the numerical value of Hausdorff measures in the form suitable for our continuity considerations in the following sections. We start with the following density theorem for Hausdorff measures (see [2] for example).

Fact 2.1. *Let X be a metric space, with $\mathrm{HD}(X) = h$, such that $\mathrm{H}_h(X) < +\infty$. Then (see p. 91 in [2]),*

$$\lim_{r\to 0} \sup\left\{\frac{\mathrm{H}_h(F)}{\mathrm{diam}^h(F)} : x \in F,\ \overline{F} = F,\, \mathrm{diam}(F) \le r\right\} = 1$$

for H_h–a.e. $x \in X$.

As an immediate consequence of this, we get the following, fundamental for us, fact, which was extensively explored in [5] and [6].

Theorem 2.2. *If X is a metric space and $0 < \mathrm{H}_h(X) < +\infty$, then*

$$\mathrm{H}_h(X) = \lim_{r\to 0} \inf\left\{\frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} : x \in F,\ \overline{F} = F,\, \mathrm{diam}(F) \le r\right\}$$

for H_h^1–a.e. $x \in X$.

Since in all Euclidean metric spaces the diameter of the closed convex hull of every set A is the same as the diameter of A, as an immediate consequence of this theorem, we get the following.

Corollary 2.3. *If X is a subset of a Euclidean metric space $\mathbb{R}^d$ and $0 < \mathrm{H}_h(X) < +\infty$, then for H_h^1–a.e. $x \in X$ we have that*

$$\mathrm{H}_h(X) = \lim_{r\to 0} \inf \left\{ \frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} \right\}$$

where, given $r > 0$, the supremum is taken over all closed and convex sets $F \subset \mathbb{R}^d$ such that $x \in F$ and $\mathrm{diam}(F) \le r$.

Being even more specific, we get the following consequence.

Corollary 2.4. *If X is a subset of an interval $\Delta \subset \mathbb{R}$ and $0 < \mathrm{H}_h(X) < +\infty$, then for H_h^1–a.e. $x \in X$ we have that*

$$\mathrm{H}_h(X) = \lim_{r\to 0} \inf \left\{ \frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} \right\},$$

where, given $r > 0$, the supremum is taken over all closed intervals $F \subset \mathbb{R}^d$ such that $x \in F$ and $\mathrm{diam}(F) \le r$.

3. Continued Fractions

For every integer $n \ge 1$ let $g_n : [0,1] \to [0,1]$ be given by the formula

$$g_n(x) = \frac{1}{n+x}. \tag{3.1}$$

Note that there exists $\xi > 0$ such that for all $n \ge 1$,

$$g_n\left(B_{\mathbb{C}}\left(\frac{1}{2}, \frac{1}{2}+\xi\right)\right) \subset B_{\mathbb{C}}\left(\frac{1}{2}, \frac{1}{2}+\xi\right) \tag{3.2}$$

and $g_n : B_{\mathbb{C}}\left(\frac{1}{2}, \frac{1}{2}+\xi\right) \to B_{\mathbb{C}}\left(\frac{1}{2}, \frac{1}{2}+\xi\right)$ is a univalent map. The collection of maps $\mathcal{G} := \{g_n\}_{n=1}^{\infty}$, acting on both $[0,1]$ and $B_{\mathbb{C}}\left(\frac{1}{2}, \frac{1}{2}+\xi\right)$, forms a conformal iterated function system in the sense of [4] and [3]. It is called the Gauss system. In view of (3.2), for every $\omega \in \mathbb{N}_1^* := \bigcup_{n\ge 1} \mathbb{N}^n$, say $\omega \in \mathbb{N}^n$, the composition

$$g_\omega := g_{\omega_1} \circ g_{\omega_2} \circ \ldots \circ g_{\omega_n}$$

is a well-define self-map of both $\overline{B}_{\mathbb{C}}\left(\frac{1}{2}, \frac{1}{2}+\xi\right)$ and $[0,1]$. The map $G : (0,1] \to (0,1]$, defined by the formula,

$$G(x) = \frac{1}{x} - n \quad \text{if} \quad x \in \Big(\frac{1}{n+1}, \frac{1}{n}\Big],$$

is called the Gauss map. Of course

$$G \circ g_n|_{[0,1)} = \mathrm{Id}|_{[0,1)},$$

and iterating this formula,

$$G^{|\omega|} \circ g_\omega|_{[0,1)} = \mathrm{Id}|_{[0,1)}$$

for every finite word $\omega \in \mathbb{N}^*$, where the latter throughout this section includes the symbol "0" and $g_0 := \mathrm{Id}$. So, for every $k \geq 0$, and every irrational number $x \in [0,1]$,

$$G^k(x) \in \left[\frac{1}{\omega(x)_{k+1}+1}, \frac{1}{\omega(x)_{k+1}}\right],$$

where $\omega(x)_{k+1}$ is the $(k+1)$th digit of the continued fraction expansion of x. Given an arbitrary non-empty subset E of $\mathbb{N}$ we denote by $J_E(\mathcal{G})$ the set of all numbers in $[0,1]$ whose infinite continued fraction expansions have all entries in E. Of course

$$g_\omega(J_E(\mathcal{G})) \subset J_E(\mathcal{G})$$

for all $\omega \in E^*$, and moreover

$$J_E(\mathcal{G}) = \bigcup_{\omega \in E^n} g_\omega(J_E(\mathcal{G})).$$

Anticipating the terminology of the next section we call $J_E(\mathcal{G})$ the limit set of the system $\mathcal{G}_E := \{g_n : n \in E\}$, which is also a conformal iterated function system in the sense of [4] and [3]. In this section we exclusively consider only those systems where the set E is of the form $\mathbb{N}_n := \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. We then abbreviate $\mathcal{G}_{\mathbb{N}_n}$ and $J_{\mathbb{N}_n}(\mathcal{G})$ to $\mathcal{G}_n$ and $J_n(\mathcal{G})$ respectively. We also write $J(\mathcal{G})$ for $J_{\mathbb{N}}(\mathcal{G})$, i. e. the set of all irrational numbers in the interval $[0,1]$. Let

$$h_n := \mathrm{HD}(J_n(\mathcal{G}))$$

be the Hausdorff dimension of the limit set $J_n(\mathcal{G})$. It follows from Theorem 1 (formula 7.11) in [1], comp. [4], that

$$\lim_{n\to\infty} h_n = \mathrm{HD}(J(\mathcal{G})) = 1, \tag{3.3}$$

In fact Theorem 1 in [1] provides the rate of convergence of the sequence $(h_n)_{n=1}^\infty$ to 1:

$$\lim_{n\to\infty} n(1-h_n) = \frac{6}{\pi^2}. \tag{3.4}$$

Since each $\mathcal{G}_n$, $n \geq 2$, is a finite conformal iterated function system consisting of at least two elements, we have (see [1] or [4] for instance) the following well-known result.

$$0 < \mathrm{H}_{h_n}(J_n(\mathcal{G})) < +\infty.$$

The main, number theoretical, result of this section is this.

Theorem 3.1.

$$\lim_{n\to\infty} \mathrm{H}_{h_n}(J_n(\mathcal{G})) = 1 = \mathrm{H}_1(J(\mathcal{G})).$$

Of course $\mathrm{H}_1(J(\mathcal{G})) = \mathrm{H}_1([0,1]) = 1$. So, only the first equality is to be proved. We start it with a long series of lemmas.

If $g : \Delta_1 \to \Delta_2$ is a differentiable diffeomorphism, we define

$$\kappa(g) := \sup\left\{\frac{|g'(y)|}{|g'(x)|} : x, y \in \Delta_1\right\}$$

and call this number the distortion of the map $g : \Delta_1 \to \Delta_2$. We say that g has bounded distortion if $\kappa(g) < +\infty$. The following lemma collects the basic, straightforward to prove, properties of the concept of distortion.

Lemma 3.2. *Let Δ_i, $i = 1,2,3$, be some three intervals in $\mathbb{R}$. Let also* $\mathrm{Diff}(\Delta_i, \Delta_j)$, $1 \le i,j \le 3$ *be the set of all diffeomorphisms from Δ_i onto Δ_j. Then*

(a) *If $g \in \mathrm{Diff}(\Delta_i, \Delta_j)$, then $\kappa(g) = \kappa(g^{-1})$*
(b) *If $g \in \mathrm{Diff}(\Delta_i, \Delta_j)$, then $\kappa(g) \ge 1$*
(c) *If $g_1 \in \mathrm{Diff}(\Delta_1, \Delta_2)$ and $g_2 \in \mathrm{Diff}(\Delta_2, \Delta_3)$, then $\kappa(g_2 \circ g_1) \le \kappa(g_1)\kappa(g_2)$*
(d) *If $g \in \mathrm{Diff}(\Delta_i, \Delta_j)$ and Δ is an interval contained in Δ_1, then*

$$\kappa^{-1}(g)|g'(x)| \cdot |\Delta| \le |g(\Delta)| \le \kappa(g)|g'(x)| \cdot |\Delta|$$

for every $x \in \Delta_i$. In particular

$$\kappa^{-1}(g)\sup\{|g'|\} \cdot |\Delta| \le |g(\Delta)| \le \kappa(g)\inf\{|g'|\} \cdot |\Delta|.$$

It follows from (3.2) and (3.1) that

$$g_\omega\left(B_{\overline{\hat{\mathbb{C}}}}\left(\frac{1}{2}, \frac{1}{2} + \xi\right)\right) \subset B_{\overline{\hat{\mathbb{C}}}}\left(\frac{1}{2}, \frac{1}{2} + \xi\right)$$

for all $\omega \in \mathbb{N}^*$, and that all maps $g_\omega\left(B_{\overline{\hat{\mathbb{C}}}}\left(\frac{1}{2}, \frac{1}{2} + \xi\right)\right) \to \mathbb{C}$ are 1-to-1 and holomorphic. As an immediate consequence of Koebe's Distortion Theorem we get therefore the following.

Lemma 3.3.

$$\lim_{t \to 0^+} \sup\{\kappa(g_\omega|_\Delta) : \omega \in \mathbb{N}^*, \ \textit{intervals } \Delta \subset [0,1] \textit{ with } |\Delta| \le t\} = 1.$$

Since

$$\lim_{n \to \infty} \sup\{|g_\omega([0,1])| : \omega \in \mathbb{N}^n\} = 0 \tag{3.5}$$

(the convergence is even exponentially fast), as an immediate consequence of this lemma we get the following.

Lemma 3.4.

$$\lim_{q \to \infty} \sup\{\kappa(g_\omega|_{g_\tau([0,1])}) : \omega \in \mathbb{N}^*, |\tau| = q\} = 1.$$

We shall prove the following.

Lemma 3.5.

$$\lim_{n\to\infty} \kappa(g_n) = 1.$$

Proof. We have

$$|g_n'(x)| = \frac{1}{(x+n)^2},$$

and therefore,

$$\kappa(g_n) = \frac{(n+1)^2}{n^2} \to 1 \quad \text{as} \quad n \to \infty.$$

□

Since $|g_n([0,1])| = \frac{1}{n(n+1)} \to 0$ as $n \to \infty$, as an immediate consequence of this lemma, Lemma 3.3, and Lemma 3.2, we get the following.

Lemma 3.6.

$$\lim_{n\to\infty} \sup\{\kappa(g_\omega \circ g_n) : \omega \in \mathbb{N}^*\} = 1.$$

We now pass to examine normalized Hausdorff measures. For every $n \geq 2$ let

$$m_n := \mathrm{H}_{h_n}^{-1}(J_n(\mathcal{G})) \cdot \mathrm{H}_{h_n}|_{J_n(\mathcal{G})}.$$

We also frequently consider m_n as a Borel probability measure on $[0,1]$, i.e.

$$m_n(A) = \mathrm{H}_{h_n}^{-1}(J_n(\mathcal{G})) \cdot \mathrm{H}_{h_n}(J_n(\mathcal{G}) \cap A)$$

for Borel subsets A of $[0,1]$. It follows from [4] that m_n is the unique (probability) h_n-conformal measure on $J_n(\mathcal{G})$, meaning that

$$m_n(g_\omega(A)) = \int_A |g_\omega'|^{h_n} dm_n$$

for every Borel set $A \subset [0,1]$ and all $\omega \in \mathbb{N}_n^*$.

We start with the following definition.

Definition 3.7. A family $\mathcal{R}$ of closed subintervals of $[0,1]$ is called extremal if

$$\liminf_{n\to\infty} \inf\left\{\frac{|\Delta|^{h_n}}{m_n(\Delta)} : \Delta \in \mathcal{R}\right\} \geq 1.$$

Lemma 3.8. *For every $\delta > 0$ the family $\mathcal{R}_\delta$ of all closed intervals $\Delta \subset [0,1]$ with $|\Delta| \geq \delta$ is extremal.*

Proof. Suppose on the contrary that for some $\delta > 0$ the family $\mathcal{R}_\delta$ is not extremal. This means that there exist $\eta \in [0,1)$, an increasing sequence $(n_j)_1^\infty$ of positive integers, and a sequence $(\Delta_j)_1^\infty$ of closed intervals in $\mathcal{R}_\delta$ such that

$$\lim_{j\to\infty} \frac{|\Delta_j|^{h_{n_j}}}{m_{n_j}(\Delta_j)} = \eta. \tag{3.6}$$

Passing to a subsequence we may assume that the left-hand endpoints and the right-hand endpoints of Δ_j converge respectively to a and b in $[0,1]$ with $b-a \geq \delta$. Let $\Delta := [a,b] \in \mathcal{R}_\delta$. Since the sequence $(m_{n_j})_1^\infty$ converges weakly to m, the Lebesgue measure on $[0,1]$, we get from (3.6) that

$$1 = \frac{|\Delta|}{m(\Delta)} \leq \frac{\lim_{j\to\infty} |\Delta_j|^{h_{n_j}}}{\limsup_{j\to\infty} m_{n_j}(\Delta_j)} = \liminf_{j\to\infty} \frac{|\Delta_j|^{h_{n_j}}}{m_{n_j}(\Delta_j)} = \eta < 1.$$

This contradiction finishes the proof. □

Lemma 3.9.

$$\liminf_{\substack{n\to\infty \\ r\to 0}} \left\{ \frac{r^{h_n}}{m_n([0,r])} \right\} \geq 1.$$

Proof. Fix $N \geq 2$ so large that $h_N \geq 3/4$ and keep always $n \geq N$. For every $r \in (0,1/2)$ let $s_r \geq 1$ be the unique integer such that

$$\frac{1}{s_r+1} < r \leq \frac{1}{s_r}.$$

We then have

$$\begin{aligned} m_n([0,r]) &\leq \sum_{j=s_r}^\infty m_n(g_j([0,1])) \\ &\leq \sum_{j=s_r}^\infty ||g_j'||_\infty^{h_n} m_n([0,1]) \\ &\leq \sum_{j=s_r}^\infty j^{-2h_n} \\ &\leq \int_{s_r-1}^\infty x^{-2h_n}\,dx \\ &= (2h_n-1)^{-1}(s_r-1)^{1-2h_n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{m_n([0,r])}{r^{h_n}} &\leq (2h_n-1)^{-1}(s_r+1)^{h_n}(s_r-1)^{1-2h_n} \\ &= (2h_n-1)^{-1}\left(\frac{s_r+1}{s_r-1}\right)^{h_n}(s_r-1)^{1-h_n} \\ &= (2h_n-1)^{-1}\left(1+\frac{2}{s_r-1}\right)^{h_n}(s_r-1)^{1-h_n} \\ &\leq (2h_n-1)^{-1}(1+4r)^{h_n}(s_r-1)^{1-h_n} \end{aligned} \tag{3.7}$$

Of course if $0 < r \leq \frac{1}{n+1}$, then

$$\frac{m_n([0,r])}{r^{h_n}} = 0.$$

Hence, we can continue (3.7) assuming that $r > \frac{1}{n+1}$. Then $s_r < n+1$, and (3.7) along with (3.4) yield for all $n \geq 2$ large enough the following.

$$\begin{aligned}\frac{m_n([0,r])}{r^{h_n}} &\leq (2h_n-1)^{-1}(1+4r)n^{1-h_n}\\ &\leq (2h_n-1)^{-1}(1+4r)n^{\frac{7}{\pi^2 n}}\\ &= (2h_n-1)^{-1}(n^{\frac{1}{n}})^{\frac{7}{\pi^2}}(1+4r).\end{aligned}$$

Since $\lim_{n\to\infty} h_n = 1$ and $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, this formula gives

$$\limsup_{\substack{n\to\infty\\ r\to 0}} \left\{\frac{m_n([0,r])}{r^{h_n}}\right\} \leq 1.$$

The proof is complete. □

For every $\omega \in \mathbb{N}^*$ let

$$B_*(g_\omega(0), r) = [g_\omega(0), g_\omega(0)+r] \quad\text{and}\quad B_*(g_\omega(0), r) = [g_\omega(0)-r, g_\omega(0)]$$

respectively if $|\omega|$ is even or odd. Let

$$B_*(\omega) := \{B_*(g_\omega(0), r) : r \in (0,1]\}$$

We shall prove the following.

Lemma 3.10. *For every $\omega \in \mathbb{N}^*$ the family $B_*(\omega)$ is extremal.*

Proof. For all $r \in (0, |g_\omega([0,1])|]$ there exists a unique $\hat{r} \in (0,1]$ such that

$$B_*(g_\omega(0), r) = g_\omega([0,\hat{r}]).$$

By virtue of Lemma 3.2(d) this gives,

$$\kappa^{-1}(g_\omega|_{[0,\hat{r}]})|g_\omega'(0)|\hat{r} \leq r \leq \kappa(g_\omega|_{[0,\hat{r}]})|g_\omega'(0)|\hat{r}.$$

Hence,

$$\begin{aligned}\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))} &\geq \frac{\kappa^{-h_n}(g_\omega|_{[0,\hat{r}]})|g_\omega'(0)|^{h_n}\hat{r}^{h_n}}{\kappa^{h_n}(g_\omega|_{[0,\hat{r}]})|g_\omega'(0)|^{h_n}m_n([0,r])}\\ &= \kappa^{-2h_n}(g_\omega|_{[0,\hat{r}]})\frac{\hat{r}^{h_n}}{m_n([0,r])}.\end{aligned} \tag{3.8}$$

Since $\lim_{r\to 0}\hat{r} = 0$, as an immediate consequence of both, Lemma 3.9 and Lemma 3.3, we get the following.

$$\liminf_{\substack{n\to\infty\\ r\to 0}} \frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))} \geq 1.$$

This means that for every $\varepsilon > 0$ there exist an integer $N_\varepsilon \geq 2$ and a radius $R_\varepsilon \in (0,1]$ such that

$$\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))} \geq 1-\varepsilon$$

for all $n \ge N_\varepsilon$ and all $0 < r \le R_\varepsilon$. Invoking now Lemma 3.8, we therefore get

$$\begin{aligned}
\varliminf_{n\to\infty} \inf_{r\in(0,1]} &\left\{\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))}\right\} \\
&= \varliminf_{n\to\infty} \min\left\{\inf_{r\in(0,R_\varepsilon]}\left\{\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))}\right\},\right. \\
&\qquad\qquad\left.\inf_{r\in(R_\varepsilon,1]}\left\{\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))}\right\}\right\} \\
&\ge \min\left\{\varliminf_{n\to\infty}\inf_{r\in(0,R_\varepsilon]}\left\{\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))}\right\},\right. \\
&\qquad\qquad\left.\varliminf_{n\to\infty}\inf_{r\in(R_\varepsilon,1]}\left\{\frac{r^{h_n}}{m_n(B_*(g_\omega(0),r))}\right\}\right\} \\
&\ge \min\{1-\varepsilon,1\} = 1-\varepsilon.
\end{aligned}$$

Letting now $\varepsilon \to 0^+$ our lemma follows. $\square$

Now consider an arbitrary finite word $\omega \in \mathbb{N}^*$. Put $k = |\omega|$. Since

$$g_\omega(0) = g_{\omega|_{k-1}(\omega_k-1)1}(0)$$

if $\omega_k \ge 2$, and

$$g_\omega(0) = g_{\omega|_{k-2}(\omega_{k-1}+1)}(0)$$

if $\omega_k = 1$, and since $|\omega|_{k-1}(\omega_k-1)1| = |\omega|+1$ and $|\omega|_{k-2}(\omega_{k-1}+1)| = |\omega|-1$, as an immediate consequence of Lemma 3.10, we get the following.

Corollary 3.11. *For every $\omega \in \mathbb{N}^*$ let $\mathcal{R}_e(\omega)$ be the collection of all closed intervals Δ in $[0,1]$ having $g_\omega(0)$ as one of its endpoints. Then each family $\mathcal{R}_e(\omega)$, $\omega \in \mathbb{N}^*$, is extremal.*

If $\mathcal{R}$ and $\mathcal{S}$ are two families of closed subintervals of $[0,1]$, then

$$\mathcal{R} * \mathcal{S} := \{\Delta \cup \Gamma : \Delta \in \mathcal{R}, \Gamma \in \mathcal{S}, \text{ and } \#(\Delta \cap \Gamma) = 1\}.$$

Of course the operation "$*$" is is associative and commutative. Generalizing Definition 3.7 we introduce the following.

Definition 3.12. A sequence $(\mathcal{R}_k)_1^\infty$ of families of closed subintervals of $[0,1]$ is called extremal if

$$\liminf_{\substack{n\to\infty\\ k\to\infty}} \inf\left\{\frac{|\Delta|^{h_n}}{m_n(\Delta)} : \Delta \in \mathcal{R}_k\right\} \ge 1.$$

The first obvious observations are these.

Lemma 3.13. *If for every $k \geq 1$, $\mathcal{R}_k \subset \mathcal{S}_k$ and the sequence $(\mathcal{S}_k)_1^\infty$ is extremal, then the sequence $(\mathcal{R}_k)_1^\infty$ is also extremal.*

Lemma 3.14. *A sequence $(\mathcal{R}_k)_1^\infty$ is extremal if and only if the sequence $(\bigcup_{l=k} \mathcal{R}_l)_{k=1}^\infty$ is extremal.*

Lemma 3.15. *If $(\mathcal{R}_k)_1^\infty$ and $(\mathcal{S}_k)_1^\infty$ are two extremal sequences, then the sequence $(\mathcal{R}_k \cup \mathcal{S}_k)_1^\infty$ is also extremal.*

Lemma 3.16. *If $(\mathcal{R}_k)_{k=1}^\infty$ is an extremal sequence of extremal families of sets, then the family $\bigcup_{n=1}^\infty \mathcal{R}_k$ is extremal.*

Now we shall prove the following slightly more involved lemma.

Lemma 3.17. *If $(\mathcal{R}_k)_1^\infty$ and $(\mathcal{S}_k)_1^\infty$ are two extremal sequences, then the sequence $(\mathcal{R}_k * \mathcal{S}_k)_1^\infty$ is also extremal.*

Proof. Fix $\varepsilon > 0$. By our hypothesis there exists $N_\varepsilon \geq 2$ such that

$$\frac{|\Gamma|^{h_n}}{m_n(\Gamma)} \geq 1 - \varepsilon \tag{3.9}$$

for all $n, k \geq N_\varepsilon$ and all $\Gamma \in \mathcal{R}_k \cup \mathcal{S}_k$. Fix $n, k \geq N_\varepsilon$ and $\Delta \in \mathcal{R}_k * \mathcal{S}_k$. This means that

$$\Delta = \Delta_- \cup \Delta_+$$

with some $\Delta_- \in \mathcal{R}_k$ and $\Delta_+ \in \mathcal{S}_k$ such that $\mathbb{D}_- \cap \Delta_-$ is a singleton. Now the standard calculus argument shows that

$$x^t + (1-x)^t \leq 2^{1-t}$$

for all $t, x \in [0,1]$. Therefore we get

$$\frac{|\Delta_-|^{h_n} + |\Delta_+|^{h_n}}{(|\Delta_-| + |\Delta_+|)^{h_n}} = \left(\frac{|\Delta_-|}{|\Delta_-| + |\Delta_+|}\right)^{h_n} + \left(\frac{|\Delta_-|}{|\Delta_+| + |\Delta_+|}\right)^{h_n} \leq 2^{1-h_n}.$$

Hence, using also (3.9), we get

$$\begin{aligned}
\frac{|\Delta|^{h_n}}{m_n(\Delta)} &= \frac{(|\Delta_-| + |\Delta_+|)^{h_n}}{m_n(\Delta_-) + m_n(\Delta_+)} \\
&\geq 2^{h_n - 1} \frac{|\Delta_-|^{h_n} + |\Delta_+|^{h_n}}{m_n(\Delta_-) + m_n(\Delta_+)} \\
&\geq 2^{h_n - 1} \min\left\{ \frac{|\Delta_-|^{h_n}}{m_n(\Delta_-)}, \frac{|\Delta_+|^{h_n}}{m_n(\Delta_+)} \right\} \\
&\geq (1-\varepsilon) 2^{h_n - 1}
\end{aligned}$$

Invoking (3.3) this completes the proof. ☐

If $\mathcal{R}$ and $\mathcal{S}$ are two families of closed subintervals of $[0,1]$, we define

$$\mathcal{R} \otimes \mathcal{S} := \mathcal{R} \cup (\mathcal{R} * \mathcal{S}) \cup \mathcal{S}.$$

As an immediate consequence of Lemma 3.15 and Lemma 3.17 we get the following.

Corollary 3.18. *If $(\mathcal{R}_k)_1^\infty$ and $(\mathcal{S}_k)_1^\infty$ are two extremal sequences, then the sequence $(\mathcal{R}_k \otimes \mathcal{S}_k)_1^\infty$ is also extremal.*

An immediate induction then yields the following.

Lemma 3.19. *If T is a finite set and for every $t \in T$ a sequence $(\mathcal{R}_k(t))_{k=1}^\infty$ is extremal, then the sequence $\left(\otimes_{t\in T}\mathcal{R}_k(t)\right)_{k=1}^\infty$ is also extremal.*

Applying this lemma to a constant sequence we get the following.

Corollary 3.20. *If T is a finite set and for every $t \in T$ a family $\mathcal{R}(t)$ is extremal, then the family $\otimes_{t\in T}\mathcal{R}(t)$ is also extremal.*

For every $\omega \in \mathbb{N}^*$ let $\mathcal{R}(\omega)$ be the collection of all closed intervals Δ in $[0,1]$ containing $g_\omega(0)$. We can now easily upgrade Corollary 3.11 to the following.

Lemma 3.21. *For every $\omega \in \mathbb{N}^*$ the family $\mathcal{R}(\omega)$, $\omega \in \mathbb{N}^*$, is extremal.*

Proof. It suffices to notice that $\mathcal{R}(\omega) = \mathcal{R}_e(\omega) \otimes \mathcal{R}_e(\omega)$ and to apply Corollary 3.11 along with Lemma 3.19. $\square$

Now for every integer $k \geq 1$ let $\mathcal{S}_k^-$ be the family of all intervals of the form

$$\left[\frac{1}{k} - r, \frac{1}{k}\right], \ r \in \left[0, \frac{1}{k(k+1)}\right].$$

We shall prove the following.

Lemma 3.22. *The sequence $(\mathcal{S}_k^-)_1^\infty$ is extremal.*

Proof. We start the proof in the same way as the proof of Lemma 3.10 with $\omega = k$. Formula (3.8) then says that

$$\frac{r^{h_n}}{m_n\left(\left[\frac{1}{k} - r, \frac{1}{k}\right]\right)} \geq \kappa^{-2}(g_k|_{0,\hat{r}]})\frac{\hat{r}^{h_n}}{m_n([0,\hat{r}])} \geq \kappa^{-2}(g_k)\frac{\hat{r}^{h_n}}{m_n([0,\hat{r}])}$$

for all $r \in \left[0, \frac{1}{k(k+1)}\right]$. Invoking now Lemma 3.10 and Lemma 3.5 completes the proof. $\square$

Now for every integer $k \geq 1$ let $\mathcal{S}_k^+$ be the family of all intervals of the form

$$\left[\frac{1}{k+1}, \frac{1}{k+1} + r\right], \ r \in \left[0, \frac{1}{k(k+1)}\right].$$

We shall prove the following.

Lemma 3.23. *The sequence $(\mathcal{S}_k^+)_1^\infty$ is extremal.*

Proof. Observe that for every $r \in \left[0, \frac{1}{k(k+1)}\right]$ there exists a unique $\tilde{r} \in [0,1]$ such that

$$\left[\frac{1}{k+1}, \frac{1}{k+1} + r\right] = g_k([1-\tilde{r}, 1]).$$

Proceeding now in the same way as that leading to (3.8), we get the following.

$$\begin{aligned}\frac{r^{h_n}}{m_n\left(\left[\frac{1}{k+1}, \frac{1}{k+1} + r\right]\right)} &\geq \kappa^{-2}\left(g_k|_{[1-\tilde{r},1]}\right)\frac{\tilde{r}^{h_n}}{m_n([1-\tilde{r},1])} \\ &\geq \kappa^{-2}(g_k)\frac{\tilde{r}^{h_n}}{m_n([1-\tilde{r},1])}.\end{aligned}$$

Invoking now Lemma 3.5 and Corollary 3.11 (with $\omega = 1$), completes the proof. □

Now we shall prove a purely computational lemma.

Lemma 3.24.

$$\overline{\lim_{k\to\infty}} \sup\left\{\frac{(k-1)^{-a} - (k-1+q)^{-a}}{k^{-a} - (k+q)^{-a}} : \alpha \in [1/2, 1],\ q \geq 1\right\} \leq 1.$$

Proof. We have for all $\alpha \in [1/2, 1]$ all $q \geq 1$, and all $k \geq 2$ that

$$\begin{aligned}\frac{(k-1)^{-a} - (k-1+q)^{-a}}{k^{-a} - (k+q)^{-a}} &= \left(\frac{k-1}{k}\right)^{-\alpha}\frac{\left(1 - \left(\frac{k-1+q}{k-1}\right)^{-\alpha}\right)}{\left(1 - \left(\frac{k+q}{k}\right)^{-\alpha}\right)} \\ &= \left(\frac{k}{k-1}\right)^{\alpha}\frac{\left(1 - \left(1 + \frac{q}{k-1}\right)^{-\alpha}\right)}{\left(1 - (1 + \frac{q}{k})^{-\alpha}\right)} \\ &\leq \left(\frac{k}{k-1}\right)\frac{1 - \left(1 + \frac{q}{k-1}\right)^{-\alpha}}{1 - (1 + \frac{q}{k})^{-\alpha}}.\end{aligned}$$

Since $\lim_{k\to\infty}\frac{k}{k-1} = 1$, it is therefore enough to show that

$$\overline{\lim_{k\to\infty}} \sup\left\{\frac{1 - \left(1 + \frac{q}{k-1}\right)^{-\alpha}}{1 - (1 + \frac{q}{k})^{-\alpha}} : \alpha \in [1/2, 1],\ q \geq 1\right\} \leq 1. \tag{3.10}$$

With $\alpha \in [1/2, 1]$ let

$$\psi_\alpha(t) = 1 - (1+t)^{-\alpha}, \quad t \geq 0.$$

The Mean Value Theorem then gives

$$\begin{aligned}\psi_\alpha\left(\frac{q}{k-1}\right)-\psi_\alpha\left(\frac{q}{k}\right)&=\alpha\left(\frac{q}{k-1}-\frac{q}{k}\right)(1+\xi)^{-(1+\alpha)}\\&\le\frac{q}{k(k-1)}(1+\xi)^{-(1+\alpha)}\\&\le\frac{2q}{k^2}(1+\xi)^{-(1+\alpha)}\\&\le\frac{2q}{k^2}\left(1+\frac{q}{k}\right)^{-(1+\alpha)}\end{aligned}\tag{3.11}$$

for some $\xi\in\left[\frac{q}{k},\frac{q}{k-1}\right]$ and all $k\ge 2$. Now, if $q\ge k$, then $\psi_\alpha(q/k)\ge 1-2^{-\alpha}\ge 1-2^{-\frac{1}{2}}=1-\frac{\sqrt{2}}{2}>0$. Hence,

$$\begin{aligned}\frac{\psi_\alpha\left(\frac{q}{k-1}\right)-\psi_\alpha\left(\frac{q}{k}\right)}{\psi_\alpha\left(\frac{q}{k}\right)}&\le 2\left(1-\frac{\sqrt{2}}{2}\right)^{-1}\frac{q}{k^2}\left(\frac{q}{k}\right)^{-(1+\alpha)}\\&=2\left(1-\frac{\sqrt{2}}{2}\right)^{-1}q^{-\alpha}k^{\alpha-1}\\&\le 2\left(1-\frac{\sqrt{2}}{2}\right)^{-1}q^{-\alpha}\\&\le 2\left(1-\frac{\sqrt{2}}{2}\right)^{-1}k^{-\alpha}\\&\le 2\left(1-\frac{\sqrt{2}}{2}\right)^{-1}k^{-\frac{1}{2}}.\end{aligned}$$

Equivalently,

$$\frac{\psi_\alpha\left(\frac{q}{k-1}\right)}{\psi_\alpha\left(\frac{q}{k}\right)}\le 1+2\left(1-\frac{\sqrt{2}}{2}\right)^{-1}k^{-\frac{1}{2}}.\tag{3.12}$$

So, assume that $q\le k$. Applying the Mean Value Theorem once more, we get

$$\psi_\alpha\left(\frac{q}{k}\right)=\alpha\frac{q}{k}(1+\gamma)^{-(1+\alpha)}\ge\alpha\frac{q}{k}2^{-(1+\alpha)}\ge\frac{1}{2}2^{-2}a\frac{q}{k}=\frac{1}{8}\frac{q}{k}$$

for some $\gamma\in[0,q/k]\subset[0,1]$. Therefore, using also (3.11), we get

$$\frac{\psi_\alpha\left(\frac{q}{k-1}\right)-\psi_\alpha\left(\frac{q}{k}\right)}{\psi_\alpha\left(\frac{q}{k}\right)}\le 16\frac{q}{k^2}\left(1+\frac{q}{k}\right)^{-(1+\alpha)}\frac{k}{q}\le\frac{16}{k}.$$

Equivalently,

$$\frac{\psi_\alpha\left(\frac{q}{k-1}\right)}{\psi_\alpha\left(\frac{q}{k}\right)} \le 1 + \frac{16}{k}.$$

Along with (3.12) this shows that (3.10) holds, and the proof is complete. $\square$

Now for every $k \ge 2$ let

$$\mathcal{M}_k^+ = \left\{\left[\frac{1}{l+q}, \frac{1}{l}\right] : k \le l,\, q \ge 1\right\}.$$

We shall prove the following.

Lemma 3.25. *The sequence $(\mathcal{M}_k^+)_{k=2}^\infty$ is extremal.*

Proof. Since $m_n\left(\left[0, \frac{1}{n+1}\right]\right) = 0$ we are to show that

$$\varliminf_{\substack{k\to\infty\\ n\to\infty}} \inf\left\{\frac{\left(\frac{1}{l} - \frac{1}{l+q}\right)^{h_n}}{m_n\left(\left[\frac{1}{l+q}, \frac{1}{l}\right]\right)} : q \ge 1,\ k \le l \le l+q \le n+1\right\} \ge 1.$$

Equivalently,

$$\varlimsup_{\substack{k\to\infty\\ n\to\infty}} \sup\left\{\frac{m_n\left(\left[\frac{1}{l+q}, \frac{1}{l}\right]\right)}{\left(\frac{1}{l} - \frac{1}{l+q}\right)^{h_n}} : q \ge 1,\ k \le l \le l+q \le n+1\right\} \le 1.$$

But

$$\begin{aligned}
m_n\left(\left[\frac{1}{l+q}, \frac{1}{l}\right]\right) &= \sum_{j=l}^{l+q-1} m_n\left(\left[\frac{1}{j+1}, \frac{1}{j}\right]\right) \\
&\le \sum_{j=l}^{l+q-1} \frac{1}{j^{2h_n}} \\
&\le \int_{l-1}^{l+q-1} x^{-2h_n}\,dx \\
&= \frac{1}{2h_n - 1}\left((l-1)^{1-2h_n} - (l-1+q)^{1-2h_n}\right).
\end{aligned}$$

So, it is enough to show that

(3.13)

$$\varlimsup_{\substack{k\to\infty\\ n\to\infty}} \sup\left\{\frac{(l-1)^{1-2h_n} - (l-1+q)^{1-2h_n}}{\left(\frac{1}{l} - \frac{1}{l+q}\right)^{h_n}} : q \ge 1,\ k \le l \le l+q \le n+1\right\} \le 1.$$

Since $1/2 \le 2h_n - 1 \le 1$ for all $n \ge 2$ large enough, by virtue of Lemma 3.24, it thus suffices to show that

$$\varlimsup_{\substack{k\to\infty\\ n\to\infty}} \sup\left\{\frac{l^{1-2h_n}-(l+q)^{1-2h_n}}{\left(\frac{1}{l}-\frac{1}{l+q}\right)^{h_n}} : q \ge 1,\ k \le l \le l+q \le n+1\right\} \le 1.$$

We have

$$\begin{aligned}
\frac{l^{1-2h_n}-(l+q)^{1-2h_n}}{\left(\frac{1}{l}-\frac{1}{l+q}\right)^{h_n}} &= \frac{(l+q)^{1-2h_n}\left(\left(\frac{l}{l+q}\right)^{1-2h_n}-1\right)}{(l+q)^{-h_n}\left(\frac{l}{l+q}-1\right)^{h_n}} \\
&= (l+q)^{1-h_n}\frac{\left(\frac{l+q}{l}\right)^{2h_n-1}-1}{\left(\frac{l+q}{l}-1\right)^{h_n}} \\
&\le (l+q)^{1-h_n}\frac{\frac{l+q}{l}-1}{\left(\frac{l+q}{l}-1\right)^{h_n}} \\
&= (l+q)^{1-h_n}\left(\frac{l+q}{l}-1\right)^{1-h_n} \\
&= (l+q)^{1-h_n}\left(\frac{q}{l}\right)^{1-h_n} \\
&\le (q(l+q))^{1-h_n} \\
&\le (l+q)^{2(1-h_n)} \\
&\le (n+1)^{2(1-h_n)} \\
&\le \left((n+1)^{\frac{1}{n}}\right)^{\frac{14}{\pi^2}},
\end{aligned}$$

where the last inequality holds for all $n \ge 2$ large enough due to (3.4). Since $\lim_{n\to\infty}(n+1)^{\frac{1}{n}} = 1$, formula (3.13) is established and the proof is complete. □

For every $k \ge 1$ let $\mathcal{N}_k^+$ be the family of all closed intervals contained in $[0, 1/k]$ that intersect the set $\{1/l : l \in \mathbb{N}\}$ or equivalently, the set $\{1/l : l \ge k\}$. We shall prove the following.

Lemma 3.26. *The sequence $(\mathcal{N}_k^+)_1^\infty$ is extremal.*

Proof. Since $\mathcal{N}_k^+ \subset \bigcup_{k=l}^\infty \mathcal{S}_l^- \otimes \mathcal{M}_l^+ \otimes \mathcal{S}_l^+$, the proof is concluded by invoking Lemma 3.25, Lemma 3.22, Lemma 3.23, and Lemma 3.19 and Lemma 3.14. □

Along with Lemma 3.21, restricted to the subfamily generated by words of length one, and Lemma 3.16, Lemma 3.26 yields the following.

Lemma 3.27. *The family* $\mathcal{N} := \bigcup_{k=1}^{\infty} \mathcal{N}_k^+$ *precisely consisting of all those closed intervals that intersect the set* $\{1/n : n \in \mathbb{N}\}$, *is extremal.*

Let $\mathcal{F}$ be the family of all closed intervals in $[0,1]$. We shall prove the following.

Proposition 3.28. *The family* $\mathcal{F}$ *is extremal.*

Proof. Proceeding by contradiction suppose that the family $\mathcal{F}$ is not extremal. This means that there are $\eta \in (0,1)$ and two sequences, $(n_j)_1^\infty$ of strictly increasing positive integers, and $(F_j)_1^\infty$ of closed intervals in $[0,1]$ such that

$$\frac{|F_j|^{h_{n_j}}}{m_{n_j}(F_j)} < \eta \tag{3.14}$$

for all $j \geq 1$. For every $j \geq 1$ let $\omega^{(j)} \in \mathbb{N}^*$ be the longest word such that

$$F_j \subset g_{\omega^{(j)}}([0,1]). \tag{3.15}$$

Denote

$$l_j := |\omega^{(j)}|.$$

Fix

$$\eta < \xi < 1.$$

Formula (3.15) means that

$$G^{l_j}(F_j) \cap \{1/n : n \geq 2\} \neq \emptyset. \tag{3.16}$$

By Lemma 3.27, formula (3.16) implies that

$$\varliminf_{j\to\infty} \frac{|G^{l_j}(F_j)|^{h_{n_j}}}{m_{n_j}(G^{l_j}(F_j))} \geq 1. \tag{3.17}$$

By Lemma 3.3 there exists $s \in (0,1]$ so small that

$$\kappa(g_\omega|_\Delta) < \xi/\eta \tag{3.18}$$

for all $\omega \in \mathbb{N}^*$ and all intervals $\Delta \subset [0,1]$ with $|\Delta| \leq s$. Now we shall show that

$$\varliminf_{j\to\infty} |G^{l_j}(F_j)| = 0. \tag{3.19}$$

Indeed, assume on the contrary that this lower limit is positive. This means that there exist $\theta > 0$ and an integer $P_1 \geq 1$ such that

$$|G^{l_j}(F_j)| > \theta$$

for all $j \geq P_1$. This in turn implies that for every $j \geq P_1$ there exists a least $q_j \in \{0, \ldots, l_j\}$ such that

$$|G^{q_j}(F_j)| > \min\{\theta, s\}. \tag{3.20}$$

By Lemma 3.8 and by (3.14),

$$\lim_{j\to\infty} |F_j| = 0.$$

Therefore there exists $P_2 \geq P_1$ such that $q_j \geq 1$ for all $j \geq P_2$. It then follows from the definition of q_j (the least one) that either

$$\gamma \leq |G^{q_j-1}(F_j)| \leq \min\{\theta, s\} \leq s \tag{3.21}$$

for all $j \geq P_2$ and some $\gamma > 0$ or $\overline{\lim}_{j\to\infty} \omega^{(j)}_{q_j+1} = \infty$. In the former case denote $q_j - 1$ by p_j, while in the latter case denote q_j by p_j. In either case, respectively by (3.18) or Lemma 3.6, for infinitely many js large enough

$$\kappa(G^{p_j}|_{F_j}) < \xi/\eta. \tag{3.22}$$

On the other hand, in view of Lemma 3.8 again and of (3.20) along with (3.21), we have for all $j \geq P_2$ large enough that

$$\frac{|G^{p_j}(F_j)|^{h_{n_j}}}{m_{n_j}(G^{p_j}(F_j))} > \xi.$$

Making use of this formula, along with (3.14) and (3.22), we get for infinitely many js that

$$\begin{aligned} \eta = \frac{\eta}{\xi}\cdot\xi &< \kappa^{-1}(G^{p_j}|_{F_j})\frac{|G^{p_j}(F_j)|^{h_{n_j}}}{m_{n_j}(G^k(F_j))} \\ &\leq \frac{|F_j|^{h_{n_j}}\inf\{|(G^{p_j})'||_{F_j}\}}{m_{n_j}(F_j)\inf\{|(G^{p_j})'||_{F_j}\}} = \frac{|F_j|^{h_{n_j}}}{m_{n_j}(F_j)} < \eta. \end{aligned} \tag{3.23}$$

This contradiction finishes the proof of (3.19).

Now, because of (3.16), Lemma 3.27 implies that

$$\varliminf_{j\to\infty} \frac{|G^{l_j}(F_j)|^{h_{n_j}}}{m_{n_j}(G^{l_j}(F_j))} \geq 1.$$

So there exists $P_3 \geq P_2$ so large that

$$\frac{|G^{l_j}(F_j)|^{h_{n_j}}}{m_{n_j}(G^{l_j}(F_j))} > \xi \tag{3.24}$$

for all $j \geq P_3$. On the other hand, formula (3.19) entails

$$\kappa(G^{l_j}|_{F_j}) < \xi/\eta$$

for infinitely many js . Having this, (3.24) and (3.14)), we get a contradiction in the same way as in the one involving (3.23). We are done. $\square$

As an immediate consequence of this proposition and Corollary 2.4, we get the following.

$$\varliminf_{n\to\infty} \mathrm{H}_{h_n}(J_n(\mathcal{G})) \geq 1. \tag{3.25}$$

In order to complete the proof Theorem 3.1, we also need the following, much easier to prove, formula.

$$\varlimsup_{n\to\infty} \mathrm{H}_{h_n}(J_n(\mathcal{G})) \le 1. \tag{3.26}$$

Indeed, let $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the shift map, i.e. $\sigma(\omega)$ is uniquely defined be declaring that for every $n \in \mathbb{N}$ the nth coordinate of ω is equal to ω_{n+1}. We denote by $\pi(\omega)$ the unique element of $[0,1]$ whose continued fraction representation is equal to ω. So, we have defined an injective Borel map $\pi : \mathbb{N}^{\mathbb{N}} \to [0,1]$. Its restriction to $\mathbb{N}_n^{\mathbb{N}}$ is then a Borel bijection onto $J_n(\mathcal{G})$. Denote by $\tilde{m}_n$ the image of m_n under the inverse of $\pi|_{\mathbb{N}_n^{\mathbb{N}}}$. It is known from [4] that there exists $\tilde{\mu}_n$, a unique Borel probability measure σ-invariant measure on $\mathbb{N}_n^{\mathbb{N}}$, absolutely continuous with respect to $\tilde{m}_n$. In addition, $\tilde{\mu}_n$ is ergodic with respect to $\sigma : \mathbb{N}_n^{\mathbb{N}} \to \mathbb{N}_n^{\mathbb{N}}$ and equivalent to $\tilde{m}_n$. Now for every $\omega \in \{1,2,\dots,n\}^{\mathbb{N}}$ let

$$Z(\omega) := \{j \ge 1 : \omega_j = (\sigma^{j-1}(\omega))_1 = n\}.$$

Because of Birkhoff's Ergodic Theorem, ergodicity of the measure $\tilde{\mu}_n$, and positivity of $\tilde{\mu}_n\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right)$, there exists a Borel set $\Gamma_n \subset \{1,2,\dots,n\}^{\mathbb{N}}$ with $\tilde{\mu}_n(\Gamma_n) = 1$ (equivalently $\tilde{m}_n(\Gamma_n) = 1$) such that for every $\omega \in \Gamma_n$ the set

$$Z_n(\omega) := \{j \ge 1 : \omega_j = (\sigma^{j-1}(\omega))_1 = n\}$$

is infinite. Now fix $\varepsilon > 0$. By virtue of Lemma 3.6 there exists $N_\varepsilon \ge 1$ such

$$\kappa(g_{\omega|_j}) \le 1 + \varepsilon$$

for $n \ge N_\varepsilon$, all $\omega \in \Gamma_n$, and all $j \in Z_n(\omega)$. But then, using Lemma 3.2(d), we get

$$\begin{aligned}
\frac{\mathrm{diam}^{h_n}(g_{\omega|_j}([0,1]))}{m_n(g_{\omega|_j}([0,1]))} &\le \frac{\kappa^{h_n}(g_{\omega|_j}) \inf^{h_n}\{|g'|_{\omega|_j}|\}}{\inf^{h_n}\{|g'|_{\omega|_j}|\}} = \kappa^{h_n}(g_{\omega|_j}) \\
&\le \kappa(g_{\omega|_j}) \\
&\le 1 + \varepsilon.
\end{aligned}$$

Along with (3.5) this implies that

$$\lim_{r\to 0} \inf \left\{ \frac{\mathrm{diam}^{h_n}(F)}{m_n(F)} : \omega \in \Gamma_n,\ \pi(\omega) \in F,\ \mathrm{diam}(F) \le r \right\} \le 1 + \varepsilon.$$

As $m_n(\pi(\Gamma_n)) \ge \tilde{m}_n(\Gamma_n) = 1$ by Corollary 2.4, this gives that $\mathrm{H}_{h_n}(J_n(\mathcal{G})) \le 1 + \varepsilon$ for all $n \ge N_\varepsilon$. The formula (3.26) is proved.

Now, formulas (3.25) and (3.26) taken together, prove Theorem 3.1.

4. Short Preliminaries on Conformal Iterated Function Systems

Let (X, ρ) be a compact metric space. Let E be a countable set, either finite or infinite, called in the sequel an alphabet. Fix a number $s \in (0,1)$. Suppose that for every $e \in E$ there is given an injective contraction $\phi_i : X \to X$. with a Lipschitz constant $\leq s$. The collection

$$S = \{\phi_e : X \to X\}_{e \in E}$$

is called an iterated function system; briefly an IFS. Our main object of interest is the limit set of the system S. We will now define it. For each $\omega \in E^*$, say $\omega \in E^n$, we consider the map coded by ω:

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X \to X.$$

For every $\omega \in E^{\mathbb{N}}$, the sets $\{\phi_{\omega|_n}(X)\}_{n \geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n \geq 1} \phi_{\omega|_n}(X) \neq \emptyset$. Since for every $n \geq 1$,

$$\mathrm{diam}(\phi_{\omega|_n}(X)) \leq s^n \mathrm{diam}(X),$$

we conclude that the intersection

$$\bigcap_{n \geq 1} \phi_{\omega|_n}(X)$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined the coding map π the coding map from the coding space to the limit set π:

$$\pi : E^{\mathbb{N}} \to X$$

from $E^{\mathbb{N}}$ to X. The set

$$J := J_S = \pi(E^{\mathbb{N}})$$

will be called the limit set of the IFS S. An IFS S is called conformal if the following conditions are satisfied.

(a) X is a compact connected subset of a Euclidean space $\mathbb{R}^d$ and $X = \overline{\mathrm{Int}(X)}$.

(b) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, u, \alpha) \subset \mathrm{Int}(X)$ with vertex x, the symmetry axis determined by the vector $u \in \mathbb{R}^d$ of length l and a central angle of Lebesgue measure α. Here $\mathrm{Con}(x, u, \alpha, l) = \{y : 0 < (y - x, u) \leq \cos\alpha ||y - x|| \leq l\}$.

(c) (Open set Condition; OSC). For all $a, b \in E$, $a \neq b$, it holds

$$\phi_a(\mathrm{Int}(X) \cap \phi_b(\mathrm{Int}(X) = \emptyset.$$

(d) There exists an open connected set $\mathbb{R}^d \supset W \supset X$ such that for every $e \in E$, the map ϕ_e extends to a C^1 conformal diffeomorphism of W into W.

(e) (Bounded Distortion Property) There exist a constant $K \geq 1$ and $\alpha \in (0,1]$such that

$$\left| \frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} - 1 \right| \leq K||y-x||^\alpha$$

and

$$K^{-1} \leq \frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} \leq K$$

for every $\omega \in E^*$ and every pair of points $x, y \in X$.

Remark 4.1. Observe that the Cone condition is automatically satisfied if $d = 1$. Also, (see [4]) the Bounded Distortion Property is satisfied if either $d \geq 2$, or else if $d = 1$ and the alphabet E is finite. It is also trivially satisfied whenever the system S consists of similarities only. Finally, decreasing a constant K if necessary, the latter property in (e) follows from the former.

For every $t \geq 0$ define

$$\mathrm{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E^n} \|\phi'_\omega\|_\infty^t.$$

The limit exists indeed since the corresponding sequence is subadditive. It is called the topological pressure of t. If the system S consists of similarities only, then the pressure is easy to calculate. We have,

$$\mathrm{P}(t) = \log \sum_{e \in E} |\phi'_e|^t.$$

The following formula, called Bowen's formula, was proved in [4].

$$\mathrm{HD}(J_S) = \inf\{t \geq 0 : \mathrm{P}(t) \leq 0\} = \sup\{\mathrm{HD}(J_F) : F \subset E \text{ is finite}\}. \tag{4.1}$$

J_F in here is the limit set of the iterated function system $\{\phi_e : X \to X\}_{e \in F}$. If all elements of the system S are similarities, then this formula simplifies to read the following.

(4.2)

$$\mathrm{HD}(J_S) = \inf\left\{ t \geq 0 : \sum_{e \in E} |\phi'_e|^t \leq 1 \right\} = \sup\{\mathrm{HD}(J_F) : F \subset E \text{ is finite}\}.$$

Remark 4.2. If there exists a parameter $t \geq 0$ such that $\mathrm{P}(t) = 0$, meaning that

$$\sum_{e \in E} |\phi'_e|^t = 1$$

in case of similarities, then this t is unique and is equal to $\mathrm{HD}(J_S)$. The system S is then called regular. All finite alphabet systems are obviously regular.

5. Hausdorff Measures for Similarity IFSs

In this section we prove a considerably simplified formulas for the numerical value of the Hausdorff measure of the limit set of a conformal (either finite or infinite) IFS consisting of similarities only. It will be extensively used in the next section, where a counterexample for continuity of Hausdorff measure is constructed.

Theorem 5.1. *If $\mathcal{S} = \{\phi_e : X \to X\}_{e\in E}$ is a conformal (either finite or infinite) IFS consisting of similarities only, and $\mathrm{H}_h(J_{\mathcal{S}}) > 0$, then*

$$\mathrm{H}_h(J_{\mathcal{S}}) = \inf\left\{\frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} : F \subset X,\, \overline{F} = F\right\}.$$

Proof. Since

$$\inf\left\{\frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} : F \subset X,\, \overline{F}\right\} \le \lim_{r\to 0}\inf\left\{\frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} : x \in F,\; \overline{F} = F,\, \mathrm{diam}(F) \le r\right\}$$

for every $x \in X$, as an immediate consequence of Theorem 2.2, we get that

$$\inf\left\{\frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} : F \subset X,\, \overline{F} = F\right\} \le \mathrm{H}_h(J_{\mathcal{S}}). \tag{5.1}$$

In order to prove the opposite inequality fix $\varepsilon > 0$. Denote the left-hand side of (5.1) by L. Fix a closed subset F of X such that

$$\frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} \le L + \varepsilon \tag{5.2}$$

and

$$\mathrm{H}_h^1(F) > 0.$$

Given $\omega \in E^{\mathbb{N}}$ let

$$Z(\omega) := \{j \ge 0 : \sigma^j(\omega) \in \pi^{-1}(F)\}.$$

Since $\tilde{\mu}_h(\pi^{-1}(F)) = \mu_h(F) > 0$, it follows from Birkhoff's Ergodic Theorem (and ergodicity of $\tilde{\mu}_h$ with respect to the the shift map $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$) that $\tilde{\mu}_h(\Gamma) = 1$, where

$$\Gamma := \{\omega \in^{\mathbb{N}} : Z(\omega) \text{ is infinite}\}.$$

Let $\omega \in \Gamma$ and $j \in Z(\omega)$. Then

$$\pi(\omega) = \phi_{\omega|_j}(\pi(\sigma^j(\omega)) \in \phi_{\omega|_j}(F)$$

and, using (5.2),

$$\frac{\operatorname{diam}^h(\phi_{\omega|_j}(F))}{\mathrm{H}^1_h(\phi_{\omega|_j}(F))} = \frac{|\phi'_{\omega|_j}|^h \operatorname{diam}^h(F)}{|\phi'_{\omega|_j}|^h \mathrm{H}^1_h(F)} = \frac{\operatorname{diam}^h(F)}{\mathrm{H}^1_h(F)} \le L + \varepsilon.$$

Since $Z(\omega)$ is unbounded and since $\mathrm{H}^1_h(\pi(\Gamma)) \ge \widetilde{\mathrm{H}}^1_h(\Gamma) \ge 1$, the last two formulas, in conjunction with Theorem 2.2 ($\pi(\omega)$ plays the role of x appearing there), imply that $\mathrm{H}_h(J_\mathcal{S}) \le L + \varepsilon$. Letting $\varepsilon \to 0^+$, this yields $\mathrm{H}_h(J_\mathcal{S}) \le L$. Along with (5.1) this completes the proof. □

Since in all Euclidean metric spaces the diameter of the closed convex hull of every set A is the same as the diameter of A, as an immediate consequence of this theorem, we get the following.

Corollary 5.2. *If $\mathcal{S} = \{\phi_e : X \to X\}_{e\in E}$ is a conformal (either finite or infinite), IFS consisting of similarities only, $\mathrm{H}_h(J_\mathcal{S}) > 0$, and X is a convex set, then*

$$\mathrm{H}_h(J_\mathcal{S}) = \inf\left\{\frac{\operatorname{diam}^h(F)}{\mathrm{H}^1_h(F)} : F \subset X \textit{ is closed and convex}\right\}.$$

Being even more specific, we get the following consequence.

Corollary 5.3. *If $\mathcal{S} = \{\phi_e : X \to X\}_{e\in E}$ is a conformal (either finite or infinite) IFS consisting of similarities only, $\mathrm{H}_h(J_\mathcal{S}) > 0$, and X is a closed bounded subinterval of $\mathbb{R}$, then*

$$\mathrm{H}_h(X) = \inf\left\{\frac{\operatorname{diam}^h(F)}{\mathrm{H}^1_h(F)} : F \subset X \textit{ is a closed interval}\right\}.$$

6. One Dimensional Linear Counterexample

One of the major technical issues in the proof of Theorem 3.1 was to have the derivative distortion so close to one as desired. As the counterexample, for continuity of the Hausdorff measure, described below shows, this was not the only problem.

Example 6.1. We will construct by induction an infinite iterated function system $\mathcal{S} = \{\phi_n : X \to X\}_{n\in\mathbb{N}}$ with the following properties.

(a) $X = [0, 1]$.

(b) $\mathcal{S}$ consists of decreasing similarities only.

(c) $$\bigcup_{n=0}^{\infty} \phi_n([0, 1]) = (0, 1]$$

and, consequently,

$$J_\mathcal{S} = [0, 1] \setminus \bigcup_{\omega\in\mathbb{N}^*} \phi_\omega(0).$$

(d) $$\varliminf_{n\to\infty} \mathrm{H}_{h_n}(J_n) = 0 < 1 = \mathrm{H}_1(J_{\mathcal{S}}),$$

where $J_n = J_{\mathcal{S}_n}$ is the limit set of the iterated function system $\mathcal{S}_n := \{\phi_0, \phi_1, \ldots, \phi_n\}$, and $h_n := \mathrm{HD}(J_n)$.

We define $I_1 := \{1\}$ and $\phi_1 : [0,1] \to [0,1]$ to be the unique linear (decreasing) map such that

$$\phi_1(0) = 1 \text{ and } \phi_1(1) = 1/2.$$

Proceeding inductively suppose that $n \geq 2$ and I_{n-1}, an initial finite block of $\mathbb{N}$ has been defined along with the linear decreasing maps $\phi_i : [0,1] \to [0,1]$, $i \in I_{n-1}$, satisfying the following properties

(e) $$\left[\frac{1}{n-1}, 1\right] \subset \bigcup_{i \in I_{n-1}} \phi_i([0,1]) \subset \left(\frac{1}{n}, 1\right]$$

(f) $\phi_i(0) = \phi_{i-1}(1)$ for all $i \in I_{n-1} \setminus \{1\}$.

Let N_{n-1} be the largest number in I_{n-1}. Fix a point $\xi_n \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, for example $\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right)$. Let $\phi_{N_{n-1}+1} : [0,1] \to [0,1]$ be the unique linear (decreasing) map such that

(g) $\phi_{N_{n-1}+1}(1) = \xi_n$ and $\phi_{N_{n-1}+1}(0) := \phi_{N_{n-1}}(1)$ is the left-hand endpoint of $\bigcup_{i \in I_{n-1}} \phi_i([0,1])$.

Let

$$\mathcal{R}_n^* := \{\phi_j : 1 \leq j \leq N_{n-1} + 1\}$$

and let

$$s_n^* := \mathrm{HD}(J_{\mathcal{R}_n^*}).$$

Fix

$$\gamma_n \in (s_n^*, 1).$$

Take an integer $k_n \geq 1$ so large that

$$(1 - \gamma_n) \log k_n \geq \log n. \tag{6.1}$$

Since, by Remark 4.2 ,

$$\sum_{i=1}^{N_{n-1}+1} |\phi_i'|^{s_n^*} = 1,$$

there exists $a_n \in (0,1)$ so small that

$$\sum_{i=1}^{N_{n-1}+1} |\phi_i'|^{\gamma_n} + k_n a_n^{\gamma_n} = \sum_{i \in I_{n-1} \cup \{N_{n-1}+1\}} |\phi_i'|^{\gamma_n} + k_n a_n^{\gamma_n} < 1. \tag{6.2}$$

Let

$$I_n^* := \{N_{n-1} + 2, N_{n-1} + 3, \ldots, N_{n-1} + k_n + 2\}$$

and let

$$I_n := I_{n-1} \cup \{N_{n-1} + 1\} \cup I_n^* = \{1, 2 \ldots, N_{n-1} + k_n + 2\}.$$

Now, for every $N_{n-1} + 2 \le i \le N_{n-1} + k_n + 2$, let $\phi_i : [0,1] \to [0,1]$ be a linear (decreasing) map with the following properties:

(h) The scaling factor of ϕ_i is equal to a_n for all $i = N_{n-1}+2, \ldots, N_{n-1}+k_n+2$,

(i) $\phi_{N_{n-1}+2}(0) = \phi_{N_{n-1}+1}(1)$,

(j) $\phi_{i+1}(0) = \phi_i(1)$ for all $i = N_{n-1} + 2, \ldots, N_{n-1} + k_n + 1$.

We set

$$\mathcal{R}_n := \{\phi_i\}_{i \in I_n}.$$

Formula (6.2) implies that

$$s_n := \mathrm{HD}\big(J_{\mathcal{R}_n}\big) < \gamma_n. \tag{6.3}$$

Now let

$$\Delta_n := \bigcup_{i \in I_n} \phi_i([0,1]).$$

By our construction $\Delta_n \subset (0,1]$ is a closed interval and we have

$$\frac{\mathrm{diam}^{s_n}(\Delta_n)}{\hat{m}_n(\Delta_n)} = \frac{(k_n a_n)^{s_n}}{k_n a_n^{s_n}} = k_n^{s_n - 1},$$

where $\hat{m}_n$ is the only s_n-conformal measure for the system $\mathcal{R}_n$. By (6.3) and (6.1), we get

$$\log\big(k_n^{s_n-1}\big) = (s_n - 1)\log k_n < (\gamma_n - 1)\log k_n < -\log n = \log(1/n),$$

and therefore,

$$\frac{\mathrm{diam}^{s_n}(\Delta_n)}{\hat{m}_n(\Delta_n)} < \frac{1}{n}. \tag{6.4}$$

By construction, $(I_n)_1^\infty$ is an ascending sequence of initial blocks of $\mathbb{N}$, $\bigcup_{n=1}^\infty I_n = \mathbb{N}$, $\mathcal{R}_{n+1}|_{I_n} = \mathcal{R}_n$, and we define

$$\mathcal{S} = \bigcup_{n=1}^\infty \mathcal{R}_n.$$

The required properties (a) and (b) then trivially hold for the system $\mathcal{S}$. The property (c) holds by virtue of (e), and (d) holds because of (6.2), which because of Theorem 5.1, implies that $\mathrm{H}\big(J_{\mathcal{R}_n}\big) < 1/n$.

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