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Confluent Parry numbers, their spectra, and integers in positive- and negative-base number systems

par Daniel DOMBEK, Zuzana MASÁKOVÁ et Tomáš VÁVRA

Résumé. Nous étudions le développement des nombres réels en bases positive et negative, suivant les travaux de Rényi, et Ito & Sadahiro. Nous comparons les ensembles $\mathbb{Z}_\beta^+$ et $\mathbb{Z}_{-\beta}$ des nombres β-entiers non-negatifs et $(-\beta)$-entiers. Nous décrivons les bases $(\pm\beta)$ pour lesquelles $\mathbb{Z}_\beta^+$ et $\mathbb{Z}_{-\beta}$ sont codés par des mots infinis qui sont des points fixes de morphismes conjugés. De plus, nous démontrons que cela se produit précisement pour les nombres β ayant la propriété suivante: toute combinaison linéaire de puissances non-négatives de la base $-\beta$, à coefficients dans $\{0, 1, \dots, \lfloor\beta\rfloor\}$, correspond à un $(-\beta)$-entier, même si la suite donnée de chiffres est interdite comme $(-\beta)$-développement.

Abstract. In this paper we study the expansions of real numbers in positive and negative real base as introduced by Rényi, and Ito & Sadahiro, respectively. In particular, we compare the sets $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ of nonnegative β-integers and $(-\beta)$-integers. We describe all bases $(\pm\beta)$ for which $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ can be coded by infinite words which are fixed points of conjugated morphisms, and consequently have the same language. Moreover, we prove that this happens precisely for β with another interesting property, namely that any linear combination of non-negative powers of the base $-\beta$ with coefficients in $\{0, 1, \dots, \lfloor\beta\rfloor\}$ is a $(-\beta)$-integer, although the corresponding sequence of digits is forbidden as a $(-\beta)$-expansion.

1. Introduction

The expansions of numbers in general real base $\beta > 1$ called β-expansions were introduced by Rényi in [24]. These are a particular case of representations usually written in the form $x_k \cdots x_1 x_0 \bullet x_{-1} \cdots$, assigned to each

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$x \in \mathbb{R}^+$ and satisfying

$$x = x_k\beta^k + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + \cdots\,, \quad \text{where } x_i \in \{0,1,\dots,\lceil\beta\rceil - 1\}\,.$$

As a natural generalization of the set $\mathbb{Z}$ of integers, the set $\mathbb{Z}_\beta$ of β-integers was defined in [7] as the set of real numbers with β-expansions of the form $\pm x_k \cdots x_0 \bullet 0^\omega$, where 0^ω stands for infinite repetition of zeros. The set of nonnegative β-integers is denoted by $\mathbb{Z}_\beta^+$ and the ordering of distances between its consecutive points can be encoded by an infinite word invariant under a morphism.

In the Rényi β-expansions, many properties are specific for the class of confluent Parry numbers, sometimes called generalized multinacci numbers, i.e. zeros > 1 of polynomials

$$x^d - mx^{d-1} - mx^{d-2} - \cdots - mx - n\,, \quad \text{where } \ d \geq 1,\ m \geq n \geq 1\,. \tag{1.1}$$

Note that all such numbers are Pisot numbers, i.e. algebraic integers > 1 with conjugates inside the unit circle. Among exceptional properties of confluent Parry bases is for example the existence of the so-called optimal representations [8], or the fact that the infinite word u_β coding the set $\mathbb{Z}_\beta^+$ is reversal closed and the corresponding Rauzy fractal is centrally symmetric [5]. Linear numeration systems for confluent Parry numbers are also mentioned in [10], in connection to calculating the Garsia entropy.

From our point of view, the most important aspect of confluent Parry bases β is that any integer linear combination of non-negative powers of the base with coefficients in $\{0,1,\dots,\lceil\beta\rceil - 1\}$ is a β-integer, although the sequence of coefficients may be forbidden in the corresponding Rényi number system, as shown in 1992 by Frougny [14]. In other words, rewriting of any representation $y_k \cdots y_0 \bullet 0^\omega$ of $y \in \mathbb{R}^+$ with coefficients $y_i \in \{0,1,\dots,\lceil\beta\rceil - 1\}$ into the β-expansion $\langle y\rangle_\beta = x_\ell \cdots x_0 \bullet x_{-1}x_{-2}\cdots$ does not produce a fractional part, which means that the coefficients $x_{-1}, x_{-2}, \dots$ are all zero. Formally, defining

$$X(\beta) = \Big\{ \sum_{j=0}^{N} a_j\beta^j : N \in \mathbb{N},\, a_j \in \mathbb{Z},\, 0 \leq a_j < \beta \Big\}\,, \tag{1.2}$$

one can write for confluent Parry numbers that $X(\beta) = \mathbb{Z}_\beta^+$.

Recall that the study of sets $X(\beta)$ for $\beta \in (1,2)$ was initiated by Erdős et al. [11]. Later, the problem was generalized to considering coefficients a_j in a more general alphabet $\{0,1,\dots,r\}$. The corresponding set is called the spectrum of β.

Our aim is to obtain a result analogous to that of [14] in case of negative base systems considered by Ito and Sadahiro [17]. Following their work, given $\beta > 1$, one can define the $(-\beta)$-expansion $\langle x\rangle_{-\beta}$ of any real number x. In analogy with β-expansions, real numbers x with $\langle x\rangle_{-\beta} = x_k \cdots x_0 \bullet 0^\omega$

are called $(-\beta)$-integers; they form the set $\mathbb{Z}_{-\beta}$ that can be encoded by a bidirectional infinite word invariant under an antimorphism.

Our question can thus be stated as follows: For which $\beta > 1$ is any linear combination of non-negative powers of $-\beta$ with coefficients in $\{0,1,\ldots,\lfloor\beta\rfloor\}$ a $(-\beta)$-integer? Such a question was already put forth in [22], where it is shown that among quadratic numbers, only zeros of $x^2 - mx - m$ with $m \geq 1$ have the required property. Moreover, these are exactly the bases for which the distances between consecutive points in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ take the same values ≤ 1. Moreover, the infinite words u_β, and $u_{-\beta}$ coding the ordering of distances in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$, respectively, have the same language, which follows from the fact that they are fixed points of conjugated morphisms.

In this paper we solve the question in general. Defining

$$X(-\beta) = \Big\{ \sum_{j=0}^{N} a_j(-\beta)^j : N \in \mathbb{N},\, a_j \in \mathbb{Z},\, 0 \leq a_j \leq \beta \Big\}, \tag{1.3}$$

we show that $X(-\beta) = \mathbb{Z}_{-\beta}$ if and only if β is a zero of (1.1) with the additional requirement that $m = n$ if d is even.

In view of the result in the quadratic case, there is another natural question to ask: For which $\beta > 1$, the sets $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ are in some sense similar? This similarity can be expressed by comparing the values of distances between consecutive points in $\mathbb{Z}_\beta^+$ and in $\mathbb{Z}_{-\beta}$. Or, one can study their ordering in the real line coded by infinite words u_β and $u_{-\beta}$. It turns out that the phenomena satisfied equivalently in the quadratic case are not equivalent for bases of higher degree. However, the property $X(-\beta) = \mathbb{Z}_{-\beta}$ characterizes the bases for which the infinite words u_β and $u_{-\beta}$ are fixed by conjugated morphisms. In particular, we prove the following theorem.

Theorem 1.1. *Let $\beta > 1$. Denote by φ the canonical morphism of β (over a finite or an infinite alphabet) and ψ the antimorphism fixing the infinite word coding $\mathbb{Z}_{-\beta}$. Then the following conditions are equivalent:*

1. *β is a zero of $x^d - mx^{d-1} - \cdots - mx - n$ with $m \geq n \geq 1$, such that $n = m$ for d even.*
2. *φ^2 is conjugated with ψ^2.*
3. *$\mathbb{Z}_{-\beta} = X(-\beta)$.*

The paper is organized as follows. In Section 2 we recall the definition and properties of numeration systems with positive and negative base, as defined by Rényi, and Ito and Sadahiro, respectively. In particular, we focus on the set of β- and $(-\beta)$-integers and their coding by infinite words invariant under morphisms. In Section 3 we consider $\beta > 1$, the root of (1.1) and reproduce the proof that $X(\beta) = \mathbb{Z}_\beta^+$. Then we demonstrate implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) of Theorem 1.1 formulated as Proposition 3.3 and Corollary 3.6, respectively. Implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are given separately in Section 4 (Proposition(4.5) and Section 5 (Proposition(5.5). Let us mention that the equivalence (1) $\Leftrightarrow$ (2) was announced in a conference contribution [9]. Finally, Section 6 contains some remarks and examples.

2. Preliminaries

Given $\alpha \in \mathbb{R}$, $|\alpha| > 1$, every $x \in \mathbb{R}$ can be expressed as series $x = \sum_{i=-\infty}^{N} a_i \alpha^i$, $a_i \in \mathbb{Z}$. Such a series is called an α-representation of x and is usually denoted by $x = a_N a_{N-1} \cdots a_1 a_0 \bullet a_{-1} a_{-2} \cdots$, where $\bullet$ lies between the coefficients at negative and non-negative powers of the base. An α-representation is not unique. A specific type of α-representations are the β-expansions and $(-\beta)$-expansions whose properties we summarize below.

We will also need the notion of finite and infinite words over a given alphabet $\mathcal{A}$. We denote by $\mathcal{A}^*$ the monoid of finite words equipped with the operation of concatenation and the empty word ϵ. A morphism over $\mathcal{A}$ is a mapping $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ that satisfies $\varphi(vw) = \varphi(v)\varphi(w)$ for every pair of finite words $v, w \in \mathcal{A}^*$. An antimorphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ satisfies $\psi(vw) = \psi(w)\psi(v)$ for $v, w \in \mathcal{A}^*$. The action of morphisms and antimorphisms can be extended to infinite words, both onedirectional $u_0 u_1 u_2 \cdots \in \mathcal{A}^{\mathbb{N}}$ and bidirectional $\cdots u_{-2} u_{-1} u_0 u_1 u_2 \cdots \in \mathcal{A}^{\mathbb{Z}}$. In particular, if $u = \cdots u_{-2} u_{-1} | u_0 u_1 u_2 \cdots \in \mathcal{A}^{\mathbb{Z}}$ is a pointed bidirectional infinite word, then

$$\varphi(u) = \cdots \varphi(u_{-2})\varphi(u_{-1}) | \varphi(u_0)\varphi(u_1)\varphi(u_2) \cdots ,$$
$$\psi(u) = \cdots \psi(u_2)\psi(u_1)\psi(u_0) | \psi(u_{-1})\psi(u_{-2}) \cdots .$$

2.1. Rényi $\boldsymbol{\beta}$-expansions. In 1957, Rényi [24] defined the positional numeration system with a positive real base. For $\beta > 1$, any $x \in [0, 1)$ has an expansion of the form $d_\beta(x) = x_1 x_2 x_3 \cdots$ defined by

$$x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \text{ where } T_\beta(x) = \beta x - \lfloor \beta x \rfloor .$$

Any $x \in [0, 1)$ is then represented by an infinite string of non-negative integer digits $< \beta$, formally, by an element of $\mathcal{A}^{\mathbb{N}} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}^{\mathbb{N}}$. Not every infinite word over $\mathcal{A}^{\mathbb{N}}$ plays the role of $d_\beta(x)$ of some $x \in [0, 1)$. Those that do are called admissible (or β-admissible) and their characterization is due to Parry [23]. He proved that a digit string $x_1 x_2 \cdots \in \mathcal{A}^{\mathbb{N}}$ is admissible if and only if it fulfills the lexicographic condition

$$0^\omega \preceq_{\text{lex}} x_i x_{i+1} x_{i+2} \cdots \prec_{\text{lex}} d_\beta^*(1) := \lim_{y \to 1_-} d_\beta(y) \text{ for all } i \geq 1 . \tag{2.1}$$

Here, we write $u^\omega = uuu \cdots$ for infinite repetition of the word u and $\prec_{\text{lex}}$ stands for the standard lexicographic ordering. The limit is taken over the product topology on $\mathcal{A}^{\mathbb{N}}$. Let us point out that the lexicographic ordering on admissible strings corresponds to the ordering on the unit interval $[0, 1)$, i.e. $x < y$ if and only if $d_\beta(x) \prec_{\text{lex}} d_\beta(y)$.

Recall that the so-called Rényi expansion of unity is defined as

$$d_\beta(1) = d_1 d_2 d_3 \cdots , \text{ where } d_1 = \lfloor \beta \rfloor \text{ and } d_2 d_3 \cdots = d_\beta(\beta - \lfloor \beta \rfloor) . \tag{2.2}$$

If $d_\beta(1)$ is eventually periodic, then β is called a Parry number. If, moreover, $d_\beta(1) = d_1 \cdots d_k 0^\omega$, then it is called a simple Parry number.

For the infinite Rényi expansion of unity $d^*_\beta(1)$, one has

$$d^*_\beta(1) = \begin{cases} (d_1 \cdots d_{k-1}(d_k - 1))^\omega & \text{if } d_\beta(1) = d_1 \cdots d_k 0^\omega \text{ with } d_k \neq 0, \\ d_\beta(1) & \text{otherwise.} \end{cases}$$

The notion of β-expansions can be naturally extended from $[0,1)$ to all reals.

Definition 2.1. Let $\beta > 1$, $x \in \mathbb{R}^+$. Let $k \in \mathbb{N}$ be minimal such that $\frac{x}{\beta^k} \in [0,1)$ and $d_\beta\left(\frac{x}{\beta^k}\right) = x_1 x_2 x_3 \cdots$. Then the β-expansion of x is defined as

$$\langle x \rangle_\beta = \begin{cases} x_1 \cdots x_{k-1} x_k \bullet x_{k+1} x_{k+2} \cdots & \text{if } k \geq 1, \\ 0 \bullet x_1 x_2 x_3 \cdots & \text{if } k = 0. \end{cases}$$

For negative real numbers x we use the notation $-\langle |x| \rangle_\beta$.

When β is an integer, the set of numbers whose expansion uses only non-negative powers of β is precisely equal to $\mathbb{Z}$. As a natural generalization of $\mathbb{Z}$, the set $\mathbb{Z}_\beta$ of β-integers can be defined for every $\beta > 1$ using the notion of $\langle x \rangle_\beta$.

Definition 2.2. Let $\beta > 1$. We define $\mathbb{Z}_\beta = \mathbb{Z}^+_\beta \cup (-\mathbb{Z}^+_\beta)$, where

$$\mathbb{Z}^+_\beta = \{x \geq 0 \ : \ \langle x \rangle_\beta = x_k \cdots x_1 x_0 \bullet 0^\omega\} \ = \ \bigcup_{i \geq 0} \beta^i T_\beta^{-i}(0) .$$

The distances between consecutive elements of $\mathbb{Z}_\beta$ take values

$$\Delta_k = \sum_{i \geq 1} \frac{d'_{i+k}}{\beta^i}, \quad k = 0, 1, 2, \dots , \tag{2.3}$$

where $d^*_\beta(1) = d'_1 d'_2 d'_3 \cdots$, see [26]. Since $\Delta_0 = \sum_{i \geq 1} \frac{d'_i}{\beta^i} = 1$ and any suffix of $d^*_\beta(1)$ either fulfills (2.1) or is equal to $d^*_\beta(1)$ itself, we get $\Delta_k \leq 1$ for all k.

We can encode the ordering of distances in

$$\mathbb{Z}^+_\beta = \{z_0 = 0 < z_1 < z_2 < \cdots\}$$

by an infinite word $v_\beta = v_0 v_1 v_2 \cdots$ over the infinite alphabet $\mathbb{N}$. We set $v_j = k$ if k is the greatest index, at which the β-expansions $\langle z_j \rangle_\beta$ and $\langle z_{j+1} \rangle_\beta$ differ. Note that one has $z_{j+1} - z_j = \Delta_k$.

It can be seen that v_β is a fixed point of a morphism $\varphi : \mathbb{N}^* \to \mathbb{N}^*$, defined by

$$\varphi(i) = 0^{d'_{i+1}}(i+1) \quad \text{for all} \quad i \in \mathbb{N}\,. \tag{2.4}$$

If β is a Parry number, it is obvious from (2.3) that the distances between consecutive elements of $\mathbb{Z}_\beta^+$ take only finitely many values. Then both v_β and φ can be projected onto a finite alphabet. In particular, if $d_\beta(1) = d_1 \cdots d_k 0^\omega$, $d_k \neq 0$, then

$$\Delta_{j+k} = \Delta_j \text{ for all } j \geq 0$$

and the infinite word u_β on the restricted alphabet $\{0, \ldots, k-1\}$ is a fixed point of the morphism

$$\begin{aligned} \varphi(i) &= 0^{d_{i+1}}(i+1) \quad \text{for } i \leq k-2\,, \\ \varphi(k-1) &= 0^{d_k}\,. \end{aligned} \tag{2.5}$$

If $d_\beta(1) = d_\beta^*(1) = d_1 \cdots d_k(d_{k+1} \cdots d_{k+p})^\omega$ with k, p minimal, then

$$\Delta_{j+p} = \Delta_j \text{ for all } j \geq k\,,$$

and the infinite word u_β on the restricted alphabet $\{0, \ldots, k+p-1\}$ is a fixed point of the morphism

$$\begin{aligned} \varphi(i) &= 0^{d_{i+1}}(i+1) \quad \text{for } i \leq k+p-2\,, \\ \varphi(k+p-1) &= 0^{d_{k+p}}k\,. \end{aligned} \tag{2.6}$$

These, the so-called canonical morphisms, were given in [12].

2.2. Ito-Sadahiro $(-\beta)$-expansions. In 2009, Ito and Sadahiro [17] considered a numeration system with a base $-\beta < -1$. Any $x \in [\ell, \ell+1)$, where $\ell = \frac{-\beta}{\beta+1}$, has an expansion of the form $d_{-\beta}(x) = x_1x_2x_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ defined by

$$x_i = \lfloor -\beta T_{-\beta}^{i-1}(x) - \ell \rfloor \in \mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}\,,$$

where

$$T_{-\beta}(x) = -\beta x - \lfloor -\beta x - \ell \rfloor\,.$$

Ito and Sadahiro proved that a digit string $x_1x_2x_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ is $(-\beta)$-admissible, i.e. equal to $d_{-\beta}(x)$ for some $x \in [\ell, \ell+1)$, if and only if it fulfills

$$d_{-\beta}(\ell) \preceq_{\text{alt}} x_ix_{i+1}x_{i+2} \cdots \prec_{\text{alt}} d^*_{-\beta}(\ell+1) \text{ for all } i \geq 1\,. \tag{2.7}$$

Here, $d^*_{-\beta}(\ell+1) := \lim_{y \to \ell+1_-} d_{-\beta}(y)$ and $\prec_{\text{alt}}$ stands for alternate lexicographic ordering defined as follows:

$$x_1x_2 \cdots \prec_{\text{alt}} y_1y_2 \cdots \;\Leftrightarrow\; (-1)^k(x_k - y_k) < 0 \text{ for } k \text{ smallest with } x_k \neq y_k\,.$$

The alternate ordering corresponds to the ordering on reals, i.e. $x < y$ if and only if $d_{-\beta}(x) \prec_{\text{alt}} d_{-\beta}(y)$. Note that $\prec_{\text{alt}}$ can also be used to compare finite digit strings when suffix 0^ω is added to them.

In [17] it is also shown that

$$d^*_{-\beta}(\ell+1) = \begin{cases} (0l_1\cdots l_{q-1}(l_q-1))^\omega & \text{if } d_{-\beta}(\ell) = (l_1l_2\cdots l_q)^\omega \text{ for } q \text{ odd},\\ 0d_{-\beta}(\ell) & \text{otherwise.}\end{cases}$$

Now the expansion can be defined for all reals without the need of a minus sign.

Definition 2.3. Let $-\beta < -1$, $x \in \mathbb{R}$. Let $k \in \mathbb{N}$ be minimal such that $\frac{x}{(-\beta)^k} \in (\ell, \ell+1)$ and $d_{-\beta}\Big(\frac{x}{(-\beta)^k}\Big) = x_1x_2x_3\cdots$. Then the $(-\beta)$-expansion of x is defined as

$$\langle x\rangle_{-\beta} = \begin{cases} x_1\cdots x_{k-1}x_k \bullet x_{k+1}x_{k+2}\cdots & \text{if } k \geq 1,\\ 0 \bullet x_1x_2x_3\cdots & \text{if } k = 0.\end{cases}$$

Similarly as in a positive base numeration, the set of $(-\beta)$-integers $\mathbb{Z}_{-\beta}$ can now be defined using the notion of $\langle x\rangle_{-\beta}$. Note that $\mathbb{Z}_{-\beta}$ coincides with $\mathbb{Z}$ if and only if β is an integer.

Definition 2.4. Let $-\beta < -1$. Then the set of $(-\beta)$-integers is defined as

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{R} \;:\; \langle x\rangle_{-\beta} = x_k\cdots x_1x_0 \bullet 0^\omega\} \;=\; \bigcup_{i\geq 0} (-\beta)^i T_{-\beta}^{-i}(0)\,.$$

It can be shown that if $1 < \beta < \tau = \frac{1}{2}(1+\sqrt{5})$, then the set of $(-\beta)$-integers is trivial, $\mathbb{Z}_{-\beta} = \{0\}$. It is therefore reasonable to consider $\beta \geq \tau$. In order to describe the distances between consecutive $(-\beta)$-integers, we will recall some notation from [3]. Let

$$\min(k) = \min\{a_{k-1}\cdots a_1a_0 \;:\; a_{k-1}\cdots a_1a_00^\omega \text{ is admissible}\}\,,$$

where min is taken with respect to the alternate order on finite strings. Similarly we define $\max(k)$. Furthermore, let γ be the "value function" mapping finite digit strings to real numbers,

$$\gamma: \quad x_{k-1}\cdots x_1x_0 \quad\mapsto\quad \gamma(x_{k-1}\cdots x_1x_0) = \sum_{i=0}^{k-1} x_i(-\beta)^i\,.$$

It was shown in [3] that the distance between two consecutive elements $x < y$ of $\mathbb{Z}_{-\beta}$ takes the value

$$y - x = \Delta'_k = \Big|(-\beta)^k + \gamma\big(\min(k)\big) - \gamma\big(\max(k)\big)\Big|\,, \tag{2.8}$$

where k is the greatest index at which $\langle x\rangle_{-\beta}$ and $\langle y\rangle_{-\beta}$ differ.

We can encode the ordering of distances in $\mathbb{Z}_{-\beta} = \{\cdots < z_{-1} < z_0 = 0 < z_1 < \cdots\}$ by a biinfinite word

$$v_{-\beta} = \cdots v_{-3}v_{-2}v_{-1}|v_0v_1v_2\cdots\,, \quad v_i \in \{0,1,2,\ldots\}\,,$$

where $v_j = k$ if k is the greatest index, at which the $(-\beta)$-expansions of z_j and z_{j+1} differ. Note that $z_{j+1} - z_j = \Delta'_k$.

In [3] it is shown that there exists an antimorphism $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that ψ^2 is a non-erasing non-identical morphism and $\psi(v_{-\beta}) = v_{-\beta}$. Moreover, ψ is of the form

$$\psi(k) = \begin{cases} S_k(k+1)\widetilde{R_k} & \text{for } k \text{ even}, \\ R_k(k+1)\widetilde{S_k} & \text{for } k \text{ odd}, \end{cases} \tag{2.9}$$

where $\widetilde{u}$ denotes the mirror image of the word u. The word S_k codes the distances between consecutive $(-\beta)$-integers

$$\{\gamma(\min(k)0), \dots, \gamma(\min(k+1))\}$$

(in given order) and similarly R_k in

$$\{\gamma(\max(k)0), \dots, \gamma(\max(k+1))\}.$$

When $d_{-\beta}(\ell)$ is eventually periodic, then β is called an Yrrap number (or Ito-Sadahiro number). In this case, $v_{-\beta}$ and ψ can be projected to a finite alphabet as the distances of the same length can be coded by the same letter, see [3, 25]. The infinite word over the restricted alphabet is denoted $u_{-\beta}$.

3. Confluent Parry numbers

Let us study the properties of β- and $(-\beta)$-integers in case that $\beta > 1$ is a confluent Parry number, i.e. a zero of

$$x^d - mx^{d-1} - mx^{d-2} - \dots - mx - n,$$

where $d \geq 1,\ m \geq n \geq 1$. Their significance was first observed by Frougny in [14] who shows that the corresponding linear numeration system is confluent. We present the formulation using the spectrum $X(\beta)$, for which we aim to find an analogue in case of negative base. For illustration of the differences between negative and positive base, we include its proof.

Theorem 3.1. *Let $\beta > 1$. Then $X(\beta) = \mathbb{Z}_\beta^+$ if and only if β is a zero of* (1.1).

Proof. Obviously, $X(\beta) \supset \mathbb{Z}_\beta^+$ for every $\beta > 1$. Realize that $\beta > 1$ satisfying (1.1) is equivalent to

$$d_\beta(1) = m^{d-1}n0^\omega, \ \text{i.e.} \quad d_\beta^*(1) = [m^{d-1}(n-1)]^\omega. \tag{3.1}$$

First, let us show that (3.1) implies $X(\beta) \subset \mathbb{Z}_\beta^+$. Let $x = \sum_{i=0}^N a_i\beta^i \in X(\beta)$. If $\langle x\rangle_\beta = a_N \cdots a_1a_0 \bullet 0^\omega$, then obviously $x \in \mathbb{Z}_\beta^+$. In the opposite case, by the Parry condition (2.1), we derive that $a_N \cdots a_1a_0$ contains a substring which is lexicographically greater than or equal to $m^{d-1}n$, consequently, the string $0a_N \cdots a_1a_0$ contains a substring $ym^{d-1}z$, where $0 \leq y < m$ and $n \leq z \leq m$. Since β is a zero of (1.1), replacing the substring $ym^{d-1}z$ in

$0a_N \cdots a_1 a_0$ by the substring $(y+1)0^{d-1}(z-n)$, we find a representation of the same number x with strictly smaller digit sum. Consequently, after a finite number of such steps, we obtain a representation $b_K b_{K-1} \cdots b_1 b_0 \bullet 0^\omega$ of x which does not contain any substring lexicographically greater than or equal to $m^{d-1}n$, which by (2.1) shows that $\langle x\rangle_\beta = b_K b_{K-1} \cdots b_1 b_0 \bullet 0^\omega$, i.e. $x \in \mathbb{Z}_\beta^+$.

It remains to show that (3.1) is necessary for $X(\beta) \subset \mathbb{Z}_\beta^+$. Let $d_\beta(1) = d_1 d_2 d_3 \cdots$ and let $i \geq 2$ be minimal, such that $d_i < d_1$, i.e. $d_\beta(1) = d_1 d_1 \cdots d_1 d_i d_{i+1} \cdots$. Suppose that $d_{i+1} d_{i+2} \cdots \neq 0^\omega$. We find an element of the spectrum $X(\beta)$ which is not a β-integer. Consider

$$z = d_i + 1 + \sum_{j=1}^{i-1} d_1 \beta^j = \underbrace{d_1 d_1 \cdots d_1}_{i-1 \text{ times}}(d_i+1)\bullet$$

By (2.1), we have $d_{i+1} d_{i+2} \cdots \prec_{\text{lex}} d_1 d_2 \cdots$ and the lexicographic ordering corresponds to the natural order in $[0,1]$. Consequently,

$$z - \beta^i = 1 - \sum_{k=1}^{+\infty} \frac{d_{i+k}}{\beta^k} \in (0,1)\,.$$

If $\beta^i < z < \beta^{i+1}$, then the β-expansion of z is of the form

$$\langle z\rangle_\beta = 10^i \bullet z_1 z_2 z_3 \cdots\,,$$

where $\langle z - \beta^i\rangle_\beta = 0 \bullet z_1 z_2 z_3 \cdots \neq 0^\omega$. Therefore $z \in X(\beta)$ but $z \notin \mathbb{Z}_\beta^+$.

It remains to solve the case that $z \geq \beta^{i+1}$. From $\beta^{i+1} \leq z < \beta^i + 1$, we derive that $d_1 = \lfloor\beta\rfloor = 1$ and $\beta^k(\beta-1) = T^k(\beta-1) < 1$ for $0 \leq k \leq i$. This, by definition (2.2) of $d_\beta(1)$ implies $d_{k+1} = \lfloor \beta T^{k-1}(\beta-1)\rfloor = 0$ for $1 \leq k \leq i$. Hence we have $d_\beta(1) = 10^j 1 d_{j+3} d_{j+4} \cdots$, where $j \geq i \geq 2$. Consider $w = \beta^j + 1 = 10^{j-1}1\bullet$, i.e.

$$w - \beta^{j+1} = 1 - \frac{1}{\beta} - \sum_{k=2}^{+\infty} \frac{d_{j+1+k}}{\beta^k} \in (0,1)\,.$$

Since $d_{j+2} = \lfloor \beta^{j+1}(\beta-1)\rfloor = 1$, it follows that

$$1 \leq \beta^{j+2} - \beta^{j+1} < \beta^{j+2} - \beta^j = \beta^{j+2} - w + 1\,,$$

and therefore $w < \beta^{j+2}$. Necessarily, the β-expansion of w is of the form $\langle w\rangle_\beta = 10^{j+1} \bullet w_1 w_2 w_3 \cdots$, where $\langle w - \beta^{j+1}\rangle_\beta = 0 \bullet w_1 w_2 w_3 \cdots \neq 0^\omega$. Therefore $w \in X(\beta)$ but $w \notin \mathbb{Z}_\beta^+$. □

From the above theorem, one can see that for the description of the gap sequence in the spectrum $X(\beta)$, it is sufficient to use the knowledge about

β-integers. Since $d_\beta(1) = m^{d-1}n$, from (2.3) and (2.5), we derive that the gaps in $X(\beta) = \mathbb{Z}_\beta$ take values

$$\Delta_i = m\Big(\frac{1}{\beta} + \cdots + \frac{1}{\beta^{d-1-i}}\Big) + \frac{n}{\beta^{d-i}} \quad \text{for } 0 \le i \le d-1\,.$$

and the gap sequence in $X(\beta) = \mathbb{Z}_\beta^+$ is coded by the infinite word u_β over the alphabet $\{0, \dots, d-1\}$, which is a fixed point of the morphism

$$\varphi(i) = 0^m(i+1) \ \text{ for } i \le d-2\,, \quad \varphi(d-1) = 0^n\,.$$

Let us now study what role confluent Parry numbers β play in systems with negative base $-\beta$. We have

$$d_{-\beta}(\ell) = \begin{cases} (m0)^{k-1}m(m-n)^\omega & \text{if } d = 2k, \\ (m0)^k n^\omega & \text{if } d = 2k+1. \end{cases}$$

Let first d be odd or $m = n$. Then it can be verified using (2.8) that we get the same set of distances in $\mathbb{Z}_{-\beta}$ as in $\mathbb{Z}_\beta^+$, and, moreover, they are "ordered" the same way, i.e. $\Delta_i = \Delta_i'$, where i always corresponds to the greatest index at which the expansions of two neighbors in $\mathbb{Z}_\beta^+$ or $\mathbb{Z}_{-\beta}$ differ. The infinite word $u_{-\beta}$ coding $\mathbb{Z}_{-\beta}$ is a fixed point of the antimorphism

$$\psi(i) = 0^m(i+1) \ \text{ for } 0 \le i \le d-2\,, \quad \psi(d-1) = 0^n\,.$$

Although the prescriptions for the morphism φ and antimorphism ψ coincide, for comparing them, we have to use the second iteration, and compare the morphisms φ^2, ψ^2. We have

i	$\varphi(i)$	$\varphi^2(i)$	$\psi(i)$	$\psi^2(i)$
0	0^m1	$(0^m1)^m0^m2$	0^m1	$0^m2(0^m1)^m$
1	0^m2	$(0^m1)^m0^m3$	0^m2	$0^m3(0^m1)^m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(d-2)$	$0^m(d-1)$	$(0^m1)^m0^n$	$0^m(d-1)$	$0^n(0^m1)^m$
$(d-1)$	0^n	$(0^m1)^n$	0^n	$(0^m1)^n$

wherefrom it can be seen that

$$\varphi^2(i)(0^m1)^m = (0^m1)^m\psi^2(i)\,, \qquad \text{for all } i \in \{0, \dots, d-1\}\,. \tag{3.2}$$

This means that the morphisms φ^2, ψ^2 are conjugated.

Definition 3.2. Let $\mathcal{A}$ be an alphabet (finite or infinite) and $\pi, \rho : \mathcal{A}^* \to \mathcal{A}^*$ be morphisms on $\mathcal{A}$. We say that π and ρ are conjugated if there exists a word $w \in \mathcal{A}^*$ such that either

$$w\pi(a) = \rho(a)w, \ \text{for all } a \in \mathcal{A}\,, \ \text{ or } \pi(a)w = w\rho(a), \ \text{for all } a \in \mathcal{A}\,.$$

We denote $\pi \sim \rho$.

Relation (3.2) thus proves the following proposition which is in fact implication (1) $\Rightarrow$ (2) of Theorem 1.1.

Proposition 3.3. *Let $\beta > 1$ be a zero of $x^d - mx^{d-1} - \cdots - mx - n$ with $m \geq n \geq 1$, such that d is odd or $n = m$, and let φ and ψ be as above. Then φ^2 and ψ^2 are conjugated morphisms.*

It is well known that languages of fixed points of conjugated morphisms coincide. As a consequence, the infinite words u_β and $u_{-\beta}$ have the same language.

Example 3.4. Consider now $\beta > 1$ zero of $x^d - mx^{d-1} - \cdots - mx - n$, where d is even and $1 \leq n < m$. Relation (2.8) implies that not all values of distances in $\mathbb{Z}_{-\beta}$ correspond to their $\mathbb{Z}_\beta^+$ counterparts. In particular,

$$\Delta'_i = \Delta_i\,, \qquad \text{for } i = 0, \ldots, d-2\,,$$
$$\Delta'_{d-1} = 1 + \frac{n}{\beta} = \Delta_{d-1} + 1 > 1.$$

This implies that the morphisms φ^2 and ψ^2 cannot be connected by any similar property as in the previous case.

In analogy with Theorem 3.1, we would like to compare the spectrum $X(-\beta)$ (defined by (1.3)) with the set of $(-\beta)$-integers. The question, however, is much more complicated in negative base. We first describe the gaps in $X(-\beta)$ and give an antimorphism under which the gap sequence of $X(-\beta)$ is invariant (see Proposition(3.5). We will see that the gaps and the antimorphism coincide with those for $\mathbb{Z}_{-\beta}$ when β is a zero of (1.1) with d odd or $m = n$. For such β, this proves that $\mathbb{Z}_{-\beta} = X(-\beta)$, which constitutes implication (1) $\Rightarrow$ (3) of Theorem 1.1. The fact that no other β has this property (i.e. implication (3) $\Rightarrow$ (1)) is more complicated and is demonstrated in Section 5.

Proposition 3.5. *Let $\beta > 1$. The gaps $y_{j+1} - y_j$ in the set*

$$X(-\beta) = \{\cdots < y_{-1} < 0 = y_0 < y_1 < \cdots\}$$

are ≤ 1. In particular, if β is a zero of (1.1), then the gaps take values

$$\Delta_i = m\Big(\frac{1}{\beta} + \cdots + \frac{1}{\beta^{d-1-i}}\Big) + \frac{n}{\beta^{d-i}} \quad \text{for } 0 \leq i \leq d-1\,.$$

Moreover, the infinite word $u = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$ coding the gap sequence of $X(-\beta)$ by $u_j = i$ if $y_{j+1} - y_j = \Delta_i$, is a fixed point of the antimorphism $\psi : \{0, 1, \ldots, d-1\}^* \to \{0, 1, \ldots, d-1\}^*$,

$$\psi(i) = 0^m(i+1), \ \textit{for } 0 \leq i \leq d-2, \qquad \psi(d-1) = 0^n\,.$$

Proof. Let us define the sets

$$A_0 = \{0, 1, \ldots, \lfloor \beta \rfloor\},$$
$$A_{k+1} = (-\beta)A_k + \{0, 1, \ldots, \lfloor \beta \rfloor\}.$$

It can be easily verified that $X(-\beta) = \bigcup_{k \in \mathbb{N}} A_k$ and that the gaps between consecutive points of A_k are ≤ 1 for any $\beta > 1$. Since $A_k \subset \mathcal{A}_{k+1}$, it shows that the gaps in $X(-\beta)$ are also ≤ 1.

Let $\beta > 1$ be a zero of (1.1). We will show by induction that gaps between consecutive points of A_k take only values Δ_i, and that the gap Δ_{d-1} always follows the gap Δ_0. The idea of the proof is illustrated in Figure 3.1.

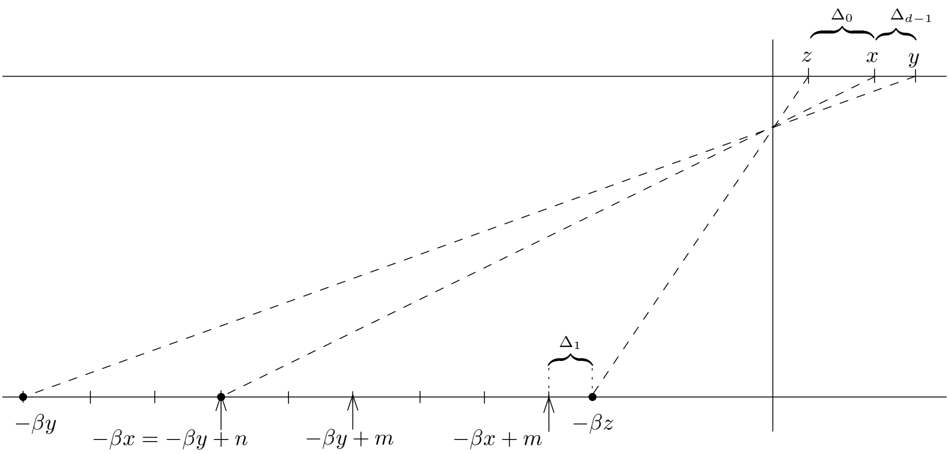

FIGURE 3.1. Construction of the antimorphism ψ for $u_{-\beta}$, where β is the root of (1.1) with $m = 5$, $n = 3$.

Suppose that x, y are consecutive points in A_k such that $y = x + \Delta_i$ for some $0 \leq i < d - 1$. Then

(3.3)
$$\{-\beta y\} + \{0, \ldots, m\} = \{-\beta y, -\beta y + 1, \ldots, -\beta y + m\} = A_{k+1} \cap [-\beta y, -\beta x)\,.$$

Note that all gaps $[-\beta y, -\beta x] \cap A_{k+1}$ are of length $\Delta_0 = 1$, except

$$(-\beta x) - (-\beta y + m) = \beta(y - x - \frac{m}{\beta}) = \beta(\Delta_i - \frac{m}{\beta}) = \Delta_{i+1}.$$

Moreover, Δ_{d-1} appears only if $y - x = \Delta_{d-2}$ and is preceded by $\Delta_0 = 1$.

Suppose now that $z < x < y$ where $y - x = \Delta_{d-1} = \frac{n}{\beta}$ and $x - z = \Delta_0 = 1$. Note that $-\beta y + n = -\beta x$, therefore we have

$$\begin{aligned} \{-\beta y, -\beta x\} + \{0, \ldots, m\} &= \{-\beta y, -\beta y + 1, \ldots, -\beta y + n = -\beta x, \ldots, -\beta x + m\} \\ &= A_{k+1} \cap [-\beta y, -\beta z)\,. \end{aligned} \tag{3.4}$$

Between $-\beta y$ and $-\beta x$ one obtains only gaps Δ_0. Consequently, the gaps between consecutive points of A_k are of the form Δ_i for every k.

From the above relations (3.3) and (3.4), we can also read that $X(-\beta)$ is invariant under the antimorphism

$$0 \to 0^m 1,\ 1 \to 0^m 2,\ \ldots,\ (d-2) \to 0^m(d-1),\ (d-1) \to 0^n\,.$$

$\square$

Corollary 3.6. *Let $\beta > 1$ be a zero of $x^d - mx^{d-1} - \cdots - mx - n$ with $m \geq n \geq 1$, such that d is odd or $n = m$, then $\mathbb{Z}_{-\beta} = X(-\beta)$.*

4. Proof (2) ⇒ (1) of Theorem 1.1

Our aim is to show that the fact that $\varphi^2 \sim \psi^2$ implies that β belongs to a specific class of numbers. In what follows, we denote by φ the canonical morphism of β. In particular, if β is a Parry number, then φ is given by (2.5) or (2.6), and if β is not a Parry number, then φ is the morphism over $\mathbb{N}$, given by (2.4). The antimorphism ψ figuring in $\varphi^2 \sim \psi^2$ is the infinite antimorphism (2.9) fixing $v_{-\beta}$, if φ is infinite; or its projection fixing the infinite word $u_{-\beta}$, otherwise.

Lemma 4.1. *Let $\beta > 1$. Denote by Δ_i the distances in $\mathbb{Z}_\beta^+$ given by* (2.3) *and Δ'_i the distances in $\mathbb{Z}_{-\beta}$ given by* (2.8). *Let φ be the canonical morphism of β and ψ an antimorphism fixing the infinite word coding $\mathbb{Z}_{-\beta}$. If $\varphi^2 \sim \psi^2$, then either both $\{\Delta_0, \Delta_1, \ldots\}$ and $\{\Delta'_0, \Delta'_1, \ldots\}$ are infinite sets or they have the same cardinality. Moreover, $\Delta_i = \Delta'_i$ for all i.*

Proof. We can assume the existence of at least three distinct distances in $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$, since the case with one distance corresponds to integer bases and two distances correspond to quadratic bases, already solved in [22].

Thus φ is defined over at least three letters. Denoting $m = \lfloor\beta\rfloor$, one has $\varphi(0) = 0^m 1$, $\varphi(1) = 0^{d_2} 2$, which implies $\varphi^2(0) = (0^m 1)^m 0^{d_2} 2$. Since $\varphi^2 \sim \psi^2$, the words $\varphi^2(0)$ and $\psi^2(0)$ both contain the same number of zeros and ones and only one letter 2 and we have

$$\beta^2 \Delta_0 = a\Delta_0 + b\Delta_1 + \Delta_2 = a\Delta'_0 + b\Delta'_1 + \Delta'_2\,, \tag{4.1}$$

where $a = m^2 + d_2$ and $b = m$. It holds that $\Delta_0 = \Delta'_0 = 1$, and since $\min(1) \in \{m-1, m\}$, $m = \lfloor\beta\rfloor$, by (2.8), we obtain $\Delta'_1 \in \{\Delta_1, \Delta_1 + 1\}$. Assuming $\Delta'_1 = \Delta_1 + 1$ together with $\Delta_1 < 1$ leads to $\Delta'_2 \leq 0$, which is a contradiction.

Consequently, $\Delta_0 = \Delta'_0$, $\Delta_1 = \Delta'_1$ and (4.1) imply $\Delta_2 = \Delta'_2$. We can obtain the statement by repeating the same process for $\varphi^2(i) \sim \psi^2(i)$, $i \geq 1$. $\square$

In the sequel, we use the following statement from combinatorics on words; its proof can be found in [19]. It will be useful for determining the word w for conjugation of morphisms as in Definition 3.2.

Proposition 4.2. *Let $x, y, w \in \mathcal{A}^*$ satisfy $wx = yw$. Then w is a prefix of y^ω.*

It follows that when morphisms φ and ψ are conjugated, i.e. there exists w such that $w\varphi(a) = \psi(a)w$ for all $a \in \mathcal{A}$, or $\varphi(a)w = w\psi(a)$ for all $a \in \mathcal{A}$, then w is a common prefix of all $(\psi(a))^\omega$ or $(\varphi(a))^\omega$ respectively.

Proposition 4.3. *Let $\varphi^2 \sim \psi^2$. Then β is a Parry number.*

Proof. If the morphisms φ^2 and ψ^2 are conjugated, then the distances between consecutive elements in $\mathbb{Z}_\beta^+$ and in $\mathbb{Z}_{-\beta}$ coincide. This in turn implies that $\Delta_i' \le 1$ for all i. From $\Delta_1' = \beta - \min(1)$ we obtain that $\min(1) = m = \lfloor \beta \rfloor$, hence the string $m0^\omega$ is $(-\beta)$-admissible. Necessarily, either $d_{-\beta}(\ell) = m0^\omega$ and thus β is a quadratic number, zero of $x^2 - mx - m$, or

$$d_{-\beta}(\ell) = m0^{2k-1}a\cdots\,, \ a, k \ge 1\,.$$

In the latter case we have $\min(0) = \epsilon = \max(0)$, $\min(1) = m$, $\max(1) = 0$, $\min(2) = m0$, $\max(2) = 0m$, which, by (2.9), gives $\psi(0) = 0^m1$, $\psi(1) = 0^m2$. Consequently, $\psi^2(0) = 0^m2(0^m1)^m$. From $\varphi^2 \sim \psi^2$, we derive that that the word $\varphi^2(0) = (0^m1)^m0^{d_2}2$ has the same number of occurrences of 0 as the word $\psi^2(0)$, i.e. $d_2 = m$. Since now β is not quadratic, we derive that

$$d_\beta^*(1) = mm\cdots\,. \tag{4.2}$$

Now assume that $\beta > 1$ is not a Parry number, hence $d_\beta^*(1)$ is aperiodic. We compare the morphisms φ^2 and ψ^2,

i	$\varphi(i)$	$\varphi^2(i)$	$\psi(i)$	$\psi^2(i)$
0	0^m1	$(0^m1)^m0^m2$	0^m1	$0^m2(0^m1)^m$
1	0^m2	$(0^m1)^m0^{d_3}3$	0^m2	$\psi(2)(0^m1)^m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$0^{d_{k+1}}(k+1)$	$(0^m1)^{d_{k+1}}0^{d_{k+2}}(k+2)$	$\psi(k)$	$\psi^2(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

If $\varphi^2 \sim \psi^2$, then there exists a word w such that $\varphi^2(i)w = w\psi^2(i)$ for all i or $w\varphi^2(i) = \psi^2(i)w$ for all i. Assume that $w\varphi^2(i) = \psi^2(i)w$. From Proposition 4.2, w is a prefix of $\big(\psi^2(0)\big)^\omega = \big(0^m2(0^m1)^m\big)^\omega$. This is not possible, since from $w\varphi^2(1) = \psi^2(1)w$, the last letter of w is 3. Therefore necessarily $\varphi^2(i)w = w\psi^2(i)$ for all i, and w is a prefix of $\big(\varphi^2(0)\big)^\omega = \big((0^m1)^m0^m2\big)^\omega$, moreover, having suffix $(0^m1)^m$.

Comparing with $\varphi^2(1)w = w\psi^2(1)$, we obtain $w = (0^m1)^m$, which therefore must be a prefix of all $\varphi^2(i)$ for $i \ge 0$. This implies $d_{i+1} = m$ for $i \ge 0$, i.e. $d_\beta(1) = m^\omega$, which gives $\beta \in \mathbb{N}$. This shows that $d_\beta(1)$ cannot be aperiodic. □

Proposition 4.4. *Let $\varphi^2 \sim \psi^2$. Then β is a simple Parry number.*

Proof. Assume that $\beta > 1$ is a non-simple Parry number, i.e. let

$$d_\beta^*(1) = d_1 \cdots d_k (d_{k+1} \cdots d_{k+p})^\omega$$

for $k, p \geq 1$ chosen minimal. As we have shown (cf. (4.2)), $\varphi^2 \sim \psi^2$ implies that $d_\beta^*(1)$ (not purely periodic) has prefix mm, hence $k + p \geq 3$.

We distinguish several subcases. At first, if $k+p = 3$, then either $d_\beta^*(1) = m(md_3)^\omega$ (case A) or $d_\beta^*(1) = mm(d_3)^\omega$ (case B). Hence

i	$\varphi(i)$ (A)	$\varphi^2(i)$ (A)	$\varphi(i)$ (B)	$\varphi^2(i)$ (B)	$\psi(i)$	$\psi^2(i)$
0	0^m1	$(0^m1)^m0^m2$	0^m1	$(0^m1)^m0^m2$	0^m1	$0^m2(0^m1)^m$
1	0^m2	$(0^m1)^m0^{d_3}1$	0^m2	$(0^m1)^m0^{d_3}2$	0^m2	$\psi(2)(0^m1)^m$
2	$0^{d_3}1$	$(0^m1)^{d_3}0^m2$	$0^{d_3}2$	$(0^m1)^{d_3}0^{d_3}2$	$\psi(2)$	$\psi^2(2)$

In both cases, we can use Proposition 4.2 and similarly as in the proof of Proposition 4.3, we derive that $\varphi^2(i)w = w\psi^2(i)$ where $w = (0^m1)^m$. Necessarily $d_3 = m$, which is a contradiction with $d_\beta^*(1)$ not being purely periodic.

Assume that $k+p \geq 4$. Then there are at least four distinct gaps $\Delta_{0,1,2,3}$ in $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$, and we have

(4.3)

i	$\varphi(i)$	$\varphi^2(i)$	$\psi(i)$	$\psi^2(i)$
0	0^m1	$(0^m1)^m0^m2$	0^m1	$0^m2(0^m1)^m$
1	0^m2	$(0^m1)^m0^{d_3}3$	0^m2	$\psi(2)(0^m1)^m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(k+p-3)$	$0^{d_{k+p-2}}(k+p-2)$	$(0^m1)^{d_{k+p-2}}0^{d_{k+p-1}}(k+p-1)$	$\psi(k+p-3)$	$\psi^2(k+p-3)$
$(k+p-2)$	$0^{d_{k+p-1}}(k+p-1)$	$(0^m1)^{d_{k+p-1}}0^{d_{k+p}}k$	$\psi(k+p-2)$	$\psi^2(k+p-2)$
$(k+p-1)$	$0^{d_{k+p}}k$	$(0^m1)^{d_{k+p}}\varphi(k)$	$\psi(k+p-1)$	$\psi^2(k+p-1)$

where $\varphi(k) = 0^{d_{k+1}}k$ if $p = 1$ and $\varphi(k) = 0^{d_{k+1}}(k+1)$ otherwise. As before, we derive that $\varphi^2(i)(0^m1)^m = (0^m1)^m\psi^2(i)$, and therefore $\varphi^2(i)$ has $(0^m1)^m$ as its prefix for all $i \in \{0, \ldots, k+p-1\}$. All but the last two rows in (4.3) then imply $d_3 = \cdots = d_{k+p-2} = m$.

If $k \geq 2$, then also $d_{k+p-1} = d_{k+p} = m$ and we have a contradiction, $d_\beta^*(1) = m^\omega$. If, on the other hand, $k = 1$, the last line of (4.3) then implies

$$\varphi^2(p) = (0^m1)^{d_{p+1}}0^m2$$

and necessarily $d_{p+1} = m$, hence $d_\beta^*(1) = m(mm\cdots md_pm)^\omega$. Either $d_p = m$, which gives $d_\beta^*(1) = m^\omega$, or $d_p < m$ and we get a contradiction with minimality of $k, p \geq 1$. $\square$

Now we can proceed with the proof of the implication (2) $\Rightarrow$ (1) of the main theorem.

Proposition 4.5. *Let $\varphi^2 \sim \psi^2$. Then β is a zero of $x^d - mx^{d-1} - \cdots - mx - n$, $m \geq n \geq 1$, $d \geq 1$, with $m = n$ if d is even.*

Proof. Thanks to Proposition 4.4, relation (4.2), and results in [22], we can consider only simple Parry numbers $\beta > 1$ with

$$d^*_\beta(1) = [mmd_3 \cdots d_{k-1}(d_k - 1)]^\omega\,,\ k \geq 3\,,$$

where $d_i \in \{0, \ldots, m\}$ and $d_k \neq 0$. Clearly, if $k = 3$, $d^*_\beta(1)$ is implies that β is in the desired class of numbers.

Let $k \geq 4$. Then there are at least four distinct gaps $\Delta_{0,1,2,3}$ and from (2.5) and (2.9) we get

i	$\varphi(i)$	$\varphi^2(i)$	$\psi(i)$	$\psi^2(i)$
0	0^m1	$(0^m1)^m0^m2$	0^m1	$0^m2(0^m1)^m$
1	0^m2	$(0^m1)^m0^{d_3}3$	0^m2	$\psi(2)(0^m1)^m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-3$	$0^{d_{k-2}}(k-2)$	$(0^m1)^{d_{k-2}}0^{d_{k-1}}(k-1)$	$\psi(k-3)$	$\psi^2(k-3)$
$k-2$	$0^{d_{k-1}}(k-1)$	$(0^m1)^{d_{k-1}}0^{d_k}$	$\psi(k-2)$	$\psi^2(k-2)$
$k-1$	0^{d_k}	$(0^m1)^{d_k}$	$\psi(k-1)$	$\psi^2(k-1)$

Using again Proposition 4.2, similarly as before, we obtain $\varphi^2(i)(0^m1)^m = (0^m1)^m\psi^2(i)$, which implies that $\varphi^2(i)$ has $(0^m1)^m$ as its prefix for all $i \in \{0, \ldots, k-2\}$. It directly follows that $d^*_\beta(1) = [mm\cdots m(d_k - 1)]^\omega$ and Example 3.4 excludes the case with k even and $d_k < m$. □

5. Proof (3) ⇒ (1) of Theorem 1.1

In this section we will prove that property $X(-\beta) = \mathbb{Z}_{-\beta}$ for $\beta > 1$ can be satisfied only if β is a zero of the polynomial $x^d - mx^{d-1} - \cdots - mx - n$, where $d \geq 1$, $m \geq n \geq 1$, and d is odd or $m = n$.

Lemma 5.1. *Let $d_{-\beta}(\ell) = l_1l_2 \ldots l_j \ldots$. Then*

$$(-\beta)^{j+1} + l_1(-\beta)^j + (l_2 - l_1)(-\beta)^{j-1} + \cdots + (l_j - l_{j-1})(-\beta) - l_j \in [-\beta, 1)\,.$$

Proof. By the definition of the transformation and the expansion $d_{-\beta}(\ell)$ we have

$$T^j(\ell) = (-\beta)^j\ell - (-\beta)^{j-1}l_1 - \cdots - (-\beta)l_{j-1} - l_j \in [\ell, \ell+1)\,.$$

We obtain the statement by multiplying both sides by $(\beta + 1)$. □

The above lemma gives us useful estimations on β when a prefix of $d_{-\beta}(\ell)$ is known. In particular, we will make use of the following implications, valid for $k \geq 1$, $t \geq 1$ and $a, b, c \in \{0, 1, \ldots, m\}$ with $a \neq m$ and $b, c \neq 0$.

$$\text{(5.1)}\quad d_{-\beta}(\ell) = (m\,0)^k \cdots \\ \Rightarrow -\beta^{2k+1} + m\beta^{2k} + \cdots + m\beta + m < m + 1$$

$$\text{(5.2)}\quad d_{-\beta}(\ell) = (m\,0)^k a \cdots \\ \Rightarrow -\beta^{2k+2} + m\beta^{2k+1} + \cdots + m\beta^2 + a\beta + a > -1$$

$$\text{(5.3)}\quad d_{-\beta}(\ell) = (m\,0)^k mb \cdots \\ \Rightarrow -\beta^{2k+2} + m\beta^{2k+1} + \cdots + m\beta + m - b \geq \frac{b}{\beta} - 1$$

$$\text{(5.4)}\quad d_{-\beta}(\ell) = (m\,0)^k 0^{2t-1} c \cdots \\ \Rightarrow \beta^{2k} - m\beta^{2k-1} - \cdots - m\beta - m > -\frac{c}{\beta^{2t}} - \frac{c+1}{\beta^{2t+1}}$$

Assume that β satisfies $X(-\beta) = \mathbb{Z}_{-\beta}$. By Proposition 3.5, the gaps in $\mathbb{Z}_{-\beta}$ are ≤ 1 and it follows from $\Delta'_1 = \beta - \min(1) < 1$ that $\min(1) = m$, hence the string $m0^\omega$ is $(-\beta)$-admissible. Since the string $(m0)^\omega$ is never admissible (otherwise $d_{-\beta}(\ell) = (m0)^\omega$ which is impossible), it makes sense to speak about the greatest index $k \geq 1$ such that $(m0)^k 0^\omega$ is admissible. The following statement provides a necessary condition on the expansion $d_{-\beta}(\ell)$ when $X(\beta) = \mathbb{Z}_{-\beta}$.

Proposition 5.2. *Assume that β satisfies $X(-\beta) = \mathbb{Z}_{-\beta}$. If $k \geq 1$ is maximal such that $(m0)^k 0^\omega$ is admissible, then*

$$d_{-\beta}(\ell) = (m0)^k ab \cdots\,, \ ab \neq m0\,.$$

Proof. From the Ito-Sadahiro admissibility condition (2.7), it is clear that the string $(m\,0)^k$ is a prefix of $d_{-\beta}(\ell)$. When also $(m\,0)^{k+1}$ is a prefix of $d_{-\beta}(\ell)$ then one can derive from (2.7) that the next nonzero digit must be on an even position. Thus we necessarily have

$$d_{-\beta}(\ell) = (m0)^k ab \cdots, ab \neq m0 \ \text{ or } \ d_{-\beta}(\ell) = (m0)^{k+1} 0^{2t-1} c \cdots, c, t \geq 1\,.$$

Let us exclude the latter case of $d_{-\beta}(\ell)$. Assume that it holds $d_{-\beta}(\ell) = (m0)^{k+1} 0^{2t-1} c \cdots$, for some $k, c, t \geq 1$. We will prove that $X(-\beta) \neq \mathbb{Z}_{-\beta}$ by showing that one of the gaps in $\mathbb{Z}_{-\beta}$ is > 1, which, by Proposition 3.5, is impossible in $X(-\beta)$.

Consider extremal strings

$$\begin{array}{rcccccc} \max(2k+2) & = & 0 & (m & 0)^k & (m-1) \\ \min(2k+2) & = & m & (0 & m)^k & 1 \end{array}$$

Therefore the value of Δ'_{2k+2} is

$$\Delta'_{2k+2} = \Big| (-\beta)^{2k+2} + \gamma(\min(2k+2)) - \gamma(\max(2k+2)) \Big| \\ = \beta^{2k+2} - m\beta^{2k+1} - \cdots - m\beta - m + 2$$

and can be estimated using (5.4) by

$$\Delta'_{2k+2} \geq 2 - \frac{c}{\beta^{2t}} - \frac{c+1}{\beta^{2t+1}} \geq 2 - \frac{c}{\beta^2} - \frac{c+1}{\beta^3}\,.$$

If β is greater than the Tribonacci constant $\beta_0 = 1.83929\cdots$, zero of $x^3 - x^2 - x - 1$, then we use $c < \beta$ to conclude that

$$\Delta'_{2k+2} \geq 2 - \frac{c}{\beta^2} - \frac{c+1}{\beta^3} \geq 2 - \frac{\beta}{\beta^2} - \frac{\beta+1}{\beta^3} = 2 - \frac{1}{\beta} - \frac{1}{\beta^2} - \frac{1}{\beta^3} > 1.$$

If, on the other hand, $\tau = \frac{1}{2}(1+\sqrt{5}) \leq \beta \leq \beta_0 < 2$, then $c = 1$, and we have

$$\Delta'_{2k+2} \geq 2 - \frac{c}{\beta^2} - \frac{c+1}{\beta^3} \geq 2 - \frac{1}{\tau^2} - \frac{2}{\tau^3} = 1 + \frac{1}{\tau^4} > 1.$$

This concludes the proof, since for $\beta < \tau$, we have $\mathbb{Z}_{-\beta} = \{0\} \neq X(-\beta)$. $\square$

Remark 5.3. In the proof of the above proposition, we have shown that if $d_{-\beta}(\ell) = (m0)^j 0^{2t-1} c \cdots$, then $\Delta'_{2j} > 1$, and therefore by Proposition 3.5, $\mathbb{Z}_{-\beta} \neq X(-\beta)$. This is shown for $j = k+1 \geq 2$, but in fact, the argument works for $j = 1$, as well.

Lemma 5.4. *Let $d_{-\beta}(\ell)$ have prefix $(m0)^k ab$, $k \geq 1$, $ab \neq m0$, and let $\mathbb{Z}_{-\beta} \neq \{0\}$. Let z be a real number with a $(-\beta)$-representation $z = 1(m\,0)^k m \bullet 0^\omega$. Then the most significant digit in the $(-\beta)$-expansion $\langle z \rangle_{-\beta}$ of z is at the position of $(-\beta)^n$ where $n \leq 2k-1$.*

Proof. By Lemma 3 from [20], the statement is true, if we show that

$$0 > z = -\beta^{2k+1} + m\beta^{2k} + m\beta^{2k-2} + \cdots + m\beta^2 + m > \frac{-\beta^{2k+1}}{\beta+1}$$

We prove $\frac{\beta+1}{\beta} z < 0$ for the general case $d_{-\beta}(\ell) = (m\,0)^k \cdots$. We have

$$\frac{\beta+1}{\beta} z = -\beta^{2k} \underbrace{-\beta^{2k+1} + m\beta^{2k} + \ldots + m}_{<1+m,\text{ see (5.1)}} + \frac{m}{\beta} < -\beta^2 + 1 + m + \frac{m}{\beta} = H\,.$$

For $\beta > 2$, we use $m < \beta$ to further estimate $H < -\beta^2 + 1 + \beta + 1 = 2 + \beta - \beta^2 < 0$. For $\beta \in (1,2)$ we have that $m = \lfloor \beta \rfloor = 1$, which gives

$$H = -\beta^2 + 2 + \frac{1}{\beta} = -(\beta^2 - \beta - 1)\frac{\beta+1}{\beta}.$$

The last expression is non-positive for $\beta \geq \tau$, i.e. such that $\mathbb{Z}_{-\beta}$ is nontrivial.

We will verify the second inequality, namely $(\beta+1)z > -\beta^{2k+1}$, separately in two cases, dependently on the form of $d_{-\beta}(\ell)$.

(1) For $d_{-\beta}(\ell) = (m\,0)^k a\cdots$, $a < m$ we have

$$(\beta+1)z = -\beta^{2k+1} \underbrace{-\beta^{2k+2} + m\beta^{2k+1} + \cdots + m\beta^2 + a\beta + a}_{>-1,\text{ see (5.2)}} + m\beta + m - a\beta - a$$
$$> -\beta^{2k+1} + (m-a)\beta + m - (a+1) > -\beta^{2k+1}\,.$$

(2) In case that $d_{-\beta}(\ell) = (m\,0)^k mb\cdots$, with $b > 0$ we have

$$(\beta+1)z = -\beta^{2k+1} \underbrace{-\beta^{2k+2} + m\beta^{2k+1} + \cdots + m\beta + m - b}_{>\frac{b}{\beta}-1,\text{ see (5.3)}} + b$$
$$> -\beta^{2k+1} + \frac{b}{\beta} - 1 + b > -\beta^{2k+1}\,.$$

□

Now we are in a state to prove the remaining implication (3) ⇒ (1) of Theorem 1.1.

Proposition 5.5. *Let $\beta > 1$. If $X(-\beta) = \mathbb{Z}_{-\beta}$, then β is a zero of*

$$x^d - mx^{d-1} - \cdots - mx - n$$

with $m \ge n \ge 1$, such that d is odd or $n = m$.

Proof. According to Proposition 5.2, if $X(-\beta) = \mathbb{Z}_{-\beta}$, then $d_{-\beta}(\ell) = (m\,0)^k ab0^\omega$, $ab \neq m0$, where $k \ge 1$ is such that $(m\,0)^{k+1}0^\omega$ is forbidden while $(m\,0)^k 0^\omega$ is admissible. Consider the number z represented by the forbidden string $1(m\,0)^k m \bullet 0^\omega$. According to Lemma 5.4, the most significant digit of the $(-\beta)$-expansion $\langle z\rangle_{-\beta}$ of z is at the position $n \le 2k-1$. Since $X(-\beta) = \mathbb{Z}_{-\beta}$, z is a $(-\beta)$-integer, denote its $(-\beta)$-expansion by $\langle z\rangle_{-\beta} = z_{2k-1}z_{2k-2}\cdots z_0 \bullet 0^\omega$. We have

$$\begin{aligned} 0 &= \gamma\big(1(m\,0)^k m\big) - \gamma\big(z_{2k-1}z_{2k-2}\cdots z_0\big) \\ &= -\beta^{2k+1} + m\beta^{2k} + z_{2k-1}\beta^{2k-1} + (m - z_{2k-2})\beta^{2k-2} + \cdots + z_1\beta + m - z_0\,. \end{aligned}$$

Denote by j the maximal index at which the string $z_{2k-1}z_{2k-2}\cdots z_0$ differs from the string $(m0)^k$, i.e. $z_j \le m-1$ if j is odd and $z_j \ge 1$ if j is even. We estimate

(5.5)
$$\begin{aligned} 0 &= \gamma\big(1(m\,0)^k m\big) - \gamma\big(z_{2k-1}z_{2k-2}\cdots z_0\big) \\ &\le \underbrace{-\beta^{2k+1} + m\beta^{2k} + \cdots + (m-1)\beta^j + \cdots + m\beta + m}_{<m+1-\beta^j} - (m-z_1)\beta - z_0 \\ &\le m + 1 - \beta^j - (m - z_1)\beta - z_0\,, \end{aligned}$$

where we have used (5.1). If $j \ge 2$, we further estimate

$$m + 1 - \beta^j - (m - z_1)\beta - z_0 < \beta + 1 - \beta^2 \le 0\,, \tag{5.6}$$

where the last inequality holds since β is necessarily $\geq \tau = \frac{1}{2}(1+\sqrt{5})$. However, inequalities (5.5) and (5.6) are in contradiction. Therefore $j \in \{0,1\}$, i.e. the expansion of z can differ from the string $(m\,0)^k$ only at the positions of $(-\beta)^1$ and $(-\beta)^0$. We therefore have

$$\begin{aligned}
0 &= \gamma\big(1(m\,0)^k m\big) - \gamma\big(z_{2k-1}z_{2k-2}\cdots z_0\big) = \gamma\big(1(m\,0)^k m\big) - \gamma\big((m0)^{k-1}z_1 z_0\big) \\
&= \underbrace{-\beta^{2k+1} + m\beta^{2k} + \cdots + m\beta^2 + m\beta + m}_{<m+1} - (m-z_1)\beta - z_0 \\
&< m+1-(m-z_1)\beta - z_0\,.
\end{aligned}$$

It is easy to verify that $0 < m+1-(m-z_1)\beta - z_0$ cannot be satisfied if $z_1 \leq m-2$. Further, we derive that $z_1 = m-1$ implies $z_0 = 0$. So the possible pairs (z_1, z_0) are (m,n), $0 \leq n \leq m$, and $(m-1,0)$. Equation $0 = \gamma\big(1(m\,0)^k m\big) - \gamma\big((m0)^{k-1}z_1 z_0\big)$ for these pairs implies that β is a zero of the polynomial

$$\begin{array}{ll}
x^{2k+1}-mx^{2k}-\cdots-mx-n & \text{when } (z_1,z_0)=(m,m-n),\ 0\leq n<m\,, \\
x^{2k}-mx^{2k-1}-\cdots-mx-m & \text{when } (z_1,z_0)=(m,m)\,, \\
x^{2k+1}-mx^{2k}-\cdots-(m-1)x-m & \text{when } (z_1,z_0)=(m-1,0)\,.
\end{array}$$

In order to conclude the proof, we have to show that the latter case does not occur. In fact, the irreducible polynomial $x^{2k+1}-mx^{2k}-\cdots-(m-1)x-m$ has a unique zero $\beta > 1$, for which $d_{-\beta}(\ell) = [(m\,0)^k(0\,m)^k]^\omega$. However, by Remark 5.3, $d_{-\beta}(\ell) = (m0)^k 0^{2t-1}c\cdots$, $c,t \geq 1$, implies $X(-\beta) \neq \mathbb{Z}_{-\beta}$. $\square$

6. Comments and examples

Confluent Parry numbers of an even degree: Let β be a confluent Parry number of an even degree $d \geq 2$ with minimal polynomial $p(x) = x^d - mx^{d-1} - \cdots - mx - n$, $m > n \geq 1$. We already know from Example 3.4 that the sets of distances in $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ do not coincide, hence $\varphi^2 \nsim \psi^2$. However, one can use similar approach as in [22] to show that cutting every distance $\Delta'_{d-1} > 1$ in $\mathbb{Z}_{-\beta}$ into $\Delta'_{d-1} = 1 + \frac{n}{\beta} = \Delta_0 + \Delta_{d-1}$, we obtain a structure that can be coded by an infinite word with the same language as the language of u_β. Formally, we apply on $u_{-\beta}$ a morphism $\pi : \{0,\ldots,d-1\}^* \to \{0,\ldots,d-1\}^*$,

$$\pi(i) = \begin{cases} i & \text{if } i \in \{0,\ldots,d-2\}, \\ 0(d-1) & \text{if } i = d-1. \end{cases}$$

Then it can be verified that the words u_β and $\pi(u_{-\beta})$ have the same language. Indeed, one can show that $\pi(u_{-\beta})$ is a fixed point of a morphism $\widetilde{\psi}$, which is the unique morphism for which $\pi \circ \psi^2 = \widetilde{\psi} \circ \pi$, and $\varphi^2 \sim \widetilde{\psi}$.

Quadratic and cubic numbers: The comparison of $(\pm\beta)$-integers for quadratic numbers β was done in [22]. It was shown that the sets of distances in $\mathbb{Z}_\beta^+$ and in $\mathbb{Z}_{-\beta}$ coincide if and only if $\varphi^2 \sim \psi^2$, which is equivalent to $\mathbb{Z}_{-\beta} = X(-\beta)$. This happens precisely for the class of zeros $\beta > 1$ of $x^2 - mx - m$, $m \geq 1$. For other quadratic Parry numbers, $\mathbb{Z}_{-\beta}$ always contains distances > 1.

Considering cubic Parry numbers, we see a difference. There may be bases β, for which the sets of distances in $\mathbb{Z}_\beta^+$ and $\mathbb{Z}_{-\beta}$ coincide, however, $\varphi^2 \not\sim \psi^2$. We may study the possible situations starting with the classification of cubic Pisot numbers, which can be derived from [1] and [4]. For illustration, we restrict ourselves to cubic Pisot units, i.e. $\beta > 1$ with minimal polynomial $p(x) = x^3 - ax^2 - bx - c$, $c = \pm 1$.

- If $p(x) = x^3 - mx^2 - mx - 1 = 0$, $m \geq 1$, then β is a confluent Parry number for which $\mathbb{Z}_{-\beta} = X(-\beta)$, and from
$$d_\beta(1) = mm10^\omega\,, \quad d_{-\beta}(\ell) = m01^\omega\,,$$
we may derive for the sets of distances that
$$\{\Delta_1, \Delta_2, \Delta_3\} = \{\Delta_1', \Delta_2', \Delta_3'\}$$
and that $\varphi^2 \sim \psi^2$.
- If $p(x) = x^3 - (m+1)x^2 + x - 1 = 0$, $m \geq 1$, or $p(x) = x^3 - (m+1)x^2 + 1 = 0$, $m \geq 2$, then $\{\Delta_1, \Delta_2, \Delta_3\} = \{\Delta_1', \Delta_2', \Delta_3'\}$, however, the morphisms are not conjugated.
- For all the other cases of cubic Pisot units, we derive from $d_\beta(1)$ and $d_{-\beta}(\ell)$ that $\mathbb{Z}_{-\beta}$ contains a distance > 1.

More about spectra of Pisot numbers: The set $X(\beta)$ is a special case of a more general notion of a spectrum of a real number $\beta > 1$, defined as the set of $p(\beta)$ where p ranges over all polynomials with coefficients restricted to a finite set of integers. In particular,
$$X^r(\beta) = \Big\{ \sum_{j=0}^{N} a_j \beta^j : N \in \mathbb{N},\, a_j \in \{0, 1, \ldots, r\} \Big\}\,.$$
For an extensive overview of the problem of spectra, see for example [2]. A general result by Feng and Wen [13] states that for a Pisot number β and $r + 1 > \beta$, the sequence of distances in $X^r(\beta)$ can be generated by a substitution. However, neither an explicit prescription for the substitution, nor the values of distances and their frequencies are known in general. Bugeaud in 2002 [6] gives these in case that β is a multinacci number, i.e. zero of (1.1) for $m = n = 1$, and $r = \lfloor\beta\rfloor = 1$. Garth and Hare in 2006 [16] provide the substitution for any zero of (1.1) and $r = \lfloor\beta\rfloor$. Notably, in both cases, the substitution can be simply obtained by observing that

$X^{\lfloor\beta\rfloor}(\beta) = \mathbb{Z}_\beta^+$ and using the canonical substitution for β-integers given in 1995 by Fabre [12].

Arithmetics: Among the desired properties of a number system is that the set of numbers with finite expansions is closed under arithmetical operations, in particular, under addition and subtraction. For bases $\beta > 1$, this the so-called finiteness property rewrites as $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta, \beta^{-1}]$. It was shown in [15] that the finiteness property holds for all confluent Parry numbers. An analogous finiteness property for negative base system has not yet been sufficiently explored. So far, the only known class of numbers $\beta > 1$ such that $\mathrm{Fin}(-\beta) = \mathbb{Z}[\beta, \beta^{-1}]$ were the zeros of $x^2 - mx + n$, $m - 2 \geq n \geq 1$, as shown in [21].

One could have expected that the bases for which $\mathbb{Z}_{-\beta} = X(-\beta)$ are proper candidates for the negative finiteness property. However, for almost all confluent Parry numbers, this is not valid. To see this, it is sufficient to consider the following examples of sum of numbers in $\mathbb{Z}_{-\beta}$ with infinite expansions:

- For β zero of $x^{2k} - mx^{2k-1} - \cdots - mx - m$, $m \geq 1$, one has
$$\langle m+1\rangle_{-\beta} = 1m0 \bullet 0^{2k-3}11m\left[0^{2k-3}110\right]^\omega;$$
- for β zero of $x^3 - mx^2 - mx - n$, $m > n \geq 1$, one has
$$\langle -\beta + m + 1\rangle = 0 \bullet 0(m-n+1)(m-n+1)1(n+1)^\omega;$$
- for β zero of $x^{2k+1} - mx^{2k} - \cdots - mx - n$, $m > n \geq 1$, $k \geq 2$, one has
$$\langle -\beta + m + 1\rangle_{-\beta} = 0 \bullet 0^{2k-1}(m-n+1)(m-n+1)0\left[0^{2k-3}1(n+1)n\right]^\omega.$$

On the other hand, in case when β is of odd degree and $n = m$, we conjecture that the finiteness property is satisfied. The proof for cubic β, namely the zeros of $x^3 - mx^2 - mx - m$, obtained recently by one of the authors, can be found in [27].

Measurably isomorphic transformations: An interesting comparison of $(\pm\beta)$-numeration is done by Kalle in [18], where the similarity of the transformations T_β and $T_{-\beta}$ is studied. It is shown that although the transformations cannot be isomorphic (since they have different number of fixed points), one can sometimes find a measurable isomorphism between $(\pm\beta)$-transformations. Among all bases $\beta \in (1,2)$, this happens precisely if β is a multinacci number.

Note that for $\beta \in (1,2)$ the notions of multinacci numbers and confluent Parry numbers coincide. Hence for bases $\beta < 2$ we get that the existence of a measurable isomorphism between $(\pm\beta)$-transformations is equivalent to all conditions in Theorem 1.1. One could expect this measurable isomorphism property to be related to properties in Theorem 1.1 also for $\beta > 2$.

Nevertheless, it is conjectured in [18] that among all $\beta > 1$, the measurable isomorphism property will hold exactly for the zeros of polynomials of the form $x^d - mx^{d-1} - \ldots - mx - n$, where $m \geq n \geq 1$ and $d \geq 1$ is arbitrary. As we exclude in our results the subclass of β's with minimal polynomial of even degree d with $m > n$, it seems that properties in Theorem 1.1 are, in general, not equivalent to T_β and $T_{-\beta}$ being measurably isomorphic.

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