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On the Hausdorff dimension of countable intersections of certain sets of normal numbers

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Résumé. On démontre que l'ensemble des nombres qui sont Q-normaux en distribution mais pas simplement Q-normaux en ratio est de dimension de Hausdorff maximale. Sous certaines conditions, on peut aussi démontrer que les intersections dénombrables de ces ensembles sont encore de dimension maximale, en dépit du fait qu'elles ne sont pas gagnantes (au sens de W. Schmidt). En conséquence, nous pouvons construire plusieurs exemples explicites de nombres qui sont simultanément normaux en distribution mais pas simplement normaux en ratio par rapport à certaines familles dénombrables de suites de base. De plus, on démontre que certains ensembles connexes sont soit gagnants, soit de première catégorie.

Abstract. We show that the set of numbers that are Q-distribution normal but not simply Q-ratio normal has full Hausdorff dimension. It is further shown under some conditions that countable intersections of sets of this form still have full Hausdorff dimension even though they are not winning sets (in the sense of W. Schmidt). As a consequence of this, we construct many explicit examples of numbers that are simultaneously distribution normal but not simply ratio normal with respect to certain countable families of basic sequences. Additionally, we prove that some related sets are either winning sets or sets of the first category.

1. Introduction

The Q-Cantor series expansion, first studied by G. Cantor in [2][1], is a natural generalization of the b-ary expansion. $Q = (q_n)_{n=1}^{\infty}$ is a *basic sequence* if each q_n is an integer greater than or equal to 2. Given a basic

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[1] G. Cantor's motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [5]. See also [21] and [6].

sequence Q, the *Q-Cantor series expansion* of a real x in $\mathbb{R}$ is the (unique)[2] expansion of the form

$$x = \lfloor x \rfloor + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n}, \tag{1.1}$$

where $E_0 = \lfloor x \rfloor$ and E_n is in $\{0, 1, \dots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (1.1) with the notation $x = E_0.E_1E_2E_3\dots$ w.r.t. Q.

Clearly, the b-ary expansion is a special case of (1.1) where $q_n = b$ for all n. If one thinks of a b-ary expansion as representing an outcome of repeatedly rolling a fair b-sided die, then a Q-Cantor series expansion may be thought of as representing an outcome of rolling a fair q_1 sided die, followed by a fair q_2 sided die and so on.

For a given basic sequence Q, let $N_n^Q(B, x)$ denote the number of times a block B occurs starting at a position no greater than n in the Q-Cantor series expansion of x. Additionally, define[3]

$$Q_n^{(k)} = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \dots q_{j+k-1}} \text{ and } T_{Q,n}(x) = \left(\prod_{j=1}^{n} q_j \right) x \pmod 1.$$

A. Rényi [17] defined a real number x to be *normal* with respect to Q if for all blocks B of length 1,

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(1)}} = 1. \tag{1.2}$$

If $q_n = b$ for all n and we restrict B to consist of only digits less than b, then (1.2) is equivalent to *simple normality in base b*, but not equivalent to *normality in base b*. A basic sequence Q is *k-divergent* if $\lim_{n \to \infty} Q_n^{(k)} = \infty$. Q is *fully divergent* if Q is k-divergent for all k and *k-convergent* if it is not k-divergent. A basic sequence Q is *infinite in limit* if $q_n \to \infty$.

Definition. A real number x is *Q-normal of order k* if for all blocks B of length k,

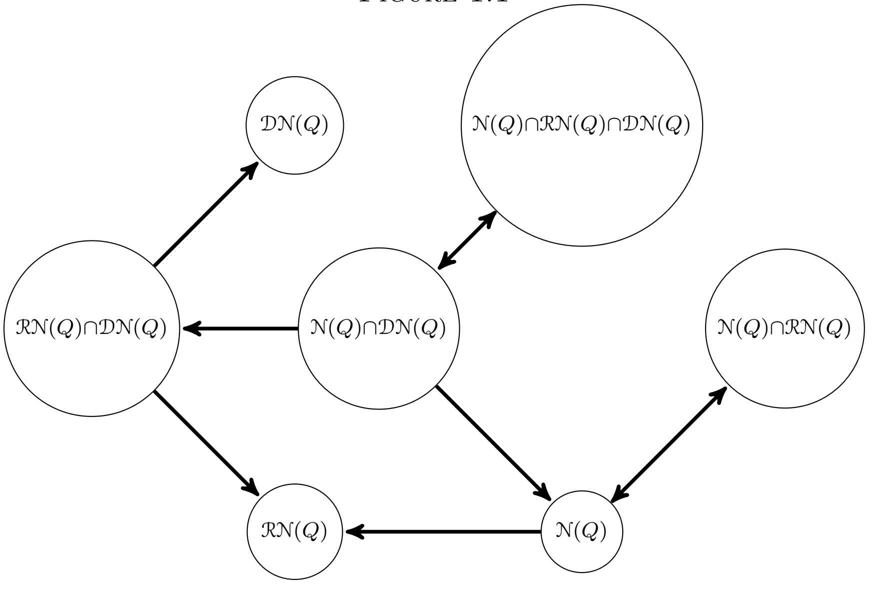
$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = 1.$$

We let $\mathcal{N}_k(Q)$ be the set of numbers that are Q-normal of order k. A real number x is *Q-normal* if $x \in \mathcal{N}(Q) := \bigcap_{k=1}^{\infty} \mathcal{N}_k(Q)$ and x is *simply Q-normal* if it is Q-normal of order 1. Additionally, x is *Q-ratio normal of*

[2]Uniqueness can be proven in the same way as for the b-ary expansions.

[3]For the remainder of this paper, we will assume the convention that the empty sum is equal to 0 and the empty product is equal to 1.

Figure 1.1



order k (here we write $x \in \mathcal{RN}_k(Q)$) if for all blocks B_1 and B_2 of length k

$$\lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = 1.$$

We say that x is *Q-ratio normal* if $x \in \mathcal{RN}(Q) := \bigcap_{k=1}^{\infty} \mathcal{RN}_k(Q)$. A real number x is *Q-distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Let $\mathcal{DN}(Q)$ be the set of Q-distribution normal numbers.

It is easy to show that for every basic sequence Q, the set of Q-distribution normal numbers has full Lebesgue measure. For Q that are infinite in limit, it has been shown that the set of all real numbers x that are Q-normal of order k has full Lebesgue measure if and only if Q is k-divergent [12]. Early work in this direction has been done by A. Rényi [17], T. Šalát [22], and F. Schweiger [20]. Therefore if Q is infinite in limit, then the set of all real numbers x that are Q-normal has full Lebesgue measure if and only if Q is fully divergent. We will show that $\mathcal{RN}_1(Q)$ is a set of zero measure if Q is infinite in limit and 1-convergent. This will follow immediately from a result of P. Erdős and A. Rényi [3].

Note that in base b, where $q_n = b$ for all n, the corresponding notions of Q-normality, Q-ratio normality, and Q-distribution normality are equivalent. This equivalence is fundamental in the study of normality in base b. It is surprising that this equivalence breaks down in the more general context of Q-Cantor series for general Q.

We refer to the directed graph in Figure 1 for the complete containment relationships between these notions when Q is infinite in limit and fully divergent. The vertices are labeled with all possible intersections of one, two, or three choices of the sets $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$. The set labeled on vertex A is a subset of the set labeled on vertex B if and only if there is a directed path from vertex A to vertex B. For example, $\mathcal{N}(Q) \cap \mathcal{DN}(Q) \subseteq \mathcal{RN}(Q)$, so all real numbers that are Q-normal and Q-distribution normal are also Q-ratio normal. These relations are fully explored and examples are given in [10].

It is usually most difficult to establish a lack of a containment relationship. The first non-trivial result in this direction was in [1] where a basic sequence Q and a real number x is constructed where $x \in \mathcal{N}(Q) \backslash \mathcal{DN}(Q)$.[4] By far the most difficult of these to establish is the existence of a basic sequence Q where $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q) \neq \emptyset$. This case is considered in [10] and requires more sophisticated methods. Other related examples may be found in [11],[13], and [10].

It should be noted that for every Q that is fully divergent infinite and infinite in limit, the sets $\mathcal{RN}(Q) \backslash \mathcal{N}(Q)$, $\mathcal{DN}(Q) \backslash \mathcal{RN}(Q)$, and $\mathcal{N}(Q) \backslash \mathcal{DN}(Q)$ are non-empty. It is likely that $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q)$ is also always non-empty. In this paper, we will be concerned with the Hausdorff dimension of sets of this form.

Definition. Let $P = (p_n)$ and $Q = (q_n)$ be basic sequences. We say that $P \sim_s Q$ if

$$q_n = \prod_{j=1}^{s} p_{s(n-1)+j}.$$

The main result of this paper is the following theorem, which concerns the Hausdorff dimension of countable intersections of sets of the form $\mathcal{DN}(Q) \backslash \mathcal{RN}_1(Q)$:

Theorem 1.1. *Suppose that $(Q_j)_{j=1}^{\infty}$ is a sequence of basic sequences that are infinite in limit. Then*

$$dim_H \left(\bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j) \backslash \mathcal{RN}_1(Q_j) \right) = 1$$

if either of the following conditions hold.

(1) *For all j, the basic sequence Q_j is 1-convergent.*
(2) *The basic sequence Q_1 is 1-divergent and there exists some basic sequence $S = (s_n)$ with*

$$Q_1 \sim_{s_1} Q_2 \sim_{s_2} Q_3 \sim_{s_3} Q_4 \cdots .$$

[4] This real number x satisfies a much stronger condition than not being Q-distribution normal: $T_{Q,n}(x) \to 0$.

Corollary 1.1. *Suppose that $(Q_j)_{j=1}^{\infty}$ is a sequence of basic sequences that are infinite in limit. Then*

$$dim_H\left(\bigcap_{j=1}^{\infty}\mathcal{DN}(Q_j)\backslash\mathcal{RN}(Q_j)\right)=dim_H\left(\bigcap_{j=1}^{\infty}\mathcal{DN}(Q_j)\backslash\mathcal{N}(Q_j)\right)=1,$$

under the same conditions as Theorem 1.1. Additionally, for any Q that is infinite in limit,

$$dim_H(\mathcal{DN}(Q)\backslash\mathcal{RN}(Q))=dim_H(\mathcal{DN}(Q)\backslash\mathcal{N}(Q))=1.$$

Proof. This is immediate as $\mathcal{N}(Q)\subseteq\mathcal{RN}(Q)\subseteq\mathcal{RN}_1(Q)$ for every basic sequence Q that is infinite in limit. □

We note the following fundamental fact about Q-distribution normal numbers that follows directly from a theorem of T. Šalát [23]:[5]

Theorem 1.2. *Suppose that $Q=(q_n)$ is a basic sequence and $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{q_n}=0$. Then $x=E_0.E_1E_2\cdots$ w.r.t. Q is Q-distribution normal if and only if (E_n/q_n) is uniformly distributed mod 1.*

The first part of Theorem 1.1 is trivial: we show in this case that the sets $\mathcal{DN}(Q_j)\backslash\mathcal{RN}_1(Q_j)$ are of full measure. Part (2) will be more difficult to establish. We will provide an explicit construction of a Cantor set $\Theta_{Q,S}\subsetneq\bigcap_{j=1}^{\infty}\mathcal{DN}(Q_j)\backslash\mathcal{RN}_1(Q_j)$ with $\dim_{\mathrm{H}}(\Theta_{Q,S})=1$ by refining the methods used in [13]. Moreover, this construction will give us explicit examples of members of $\bigcap_{j=1}^{\infty}\mathcal{DN}(Q_j)\backslash\mathcal{RN}_1(Q_j)$ for any collection of basic sequences (Q_j) that are infinite in limit with $Q_1\sim_{s_1}Q_2\sim_{s_2}Q_3\sim_{s_3}Q_4\cdots$. To see that the second part of Theorem 1.1 would not immediately follow if we were to prove that $\dim_{\mathrm{H}}(\mathcal{DN}(Q)\backslash\mathcal{RN}_1(Q))=1$, consider two basic sequences $P=(p_n)$ and $Q=(q_n)$ given by

$$(p_1,p_2,p_3,\cdots)=(2,2,4,4,4,4,6,6,6,6,6,6,8,8,8,8,8,8,8,8,\cdots);$$
$$(q_1,q_2,q_3,\cdots)=(4,16,16,36,36,36,64,64,64,64,\cdots).$$

Define the sequences (E_n) and (F_n) by

$$(E_1,E_2,E_3,\cdots)=(0,1,0,2,1,3,0,3,1,4,2,5,0,4,1,5,2,6,3,7,\cdots);$$
$$(F_1,F_2,F_3,\cdots)=(0,0,8,0,12,24,0,16,32,48,\cdots).$$

Let $x=\sum_{n=1}^{\infty}\frac{E_n}{p_1\cdots p_n}$ and $y=\sum_{n=1}^{\infty}\frac{F_n}{q_1\cdots q_n}$. Clearly, $x\in\mathcal{DN}(P)$ and $y\in\mathcal{DN}(Q)$ by Theorem 1.2. However, $y=0.00002000204000204060\cdots$ w.r.t.

[5]The original theorem of T. Šalát says: Given a basic sequence Q and a real number x with Q-Cantor series expansion $x=\lfloor x\rfloor+\sum_{n=1}^{\infty}\frac{E_n}{q_1q_2\dots q_n}$, if $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{q_n}=0$ then x is Q-distribution normal iff $E_n=\lfloor\theta_nq_n\rfloor$ for some uniformly distributed sequence (θ_n). N. Korobov [7] proved this theorem under the stronger condition that Q is infinite in limit. For this paper, we will only need to consider the case where Q is infinite in limit.

P, so $y \notin \mathcal{DN}(P)$. Furthermore, note that

$$x = 0.2273(10)(17)4(13)(22)(31)\cdots \text{ w.r.t. } Q.$$

So $T_{Q,n}(x) < 1/2$ for all n and $x \notin \mathcal{DN}(Q)$. Thus, we have demonstrated an example of two basic sequences P and Q with $P \sim_2 Q$ where $\mathcal{DN}(P)\backslash\mathcal{DN}(Q) \neq \emptyset$, $\mathcal{DN}(Q)\backslash\mathcal{DN}(P) \neq \emptyset$, and $\mathcal{DN}(P) \neq \mathcal{DN}(Q)$. It should be noted that these examples are in sharp contrast with a well known theorem of W. M. Schmidt [18]:

Theorem 1.3. *We write $r \sim s$ if there exist integers n, m with $r^n = s^m$. If $r \sim s$, then any number normal to base r is normal to base s. If $r \nsim s$, then the set of numbers which are normal to base r but not even simply normal to base s has the power of the continuum.*

While there is no reason to expect uncountable intersections to preserve Hausdorff dimension, it is not immediately clear that there are not numbers that are Q-distribution normal for every basic sequence Q that is infinite in limit. If this were the case then it might be possible that Theorem 1.1 could be extended to arbitrary uncountable intersections.

Theorem 1.4. *There is an uncountable family of basic sequences $(Q_j)_{j\in J}$ that are infinite in limit such that*

$$\bigcap_{j\in J} \mathcal{DN}(Q_j)\backslash\mathcal{RN}_1(Q_j) = \emptyset.$$

Theorem 1.4 can be proven with only a trivial modification of the proof of Theorem 1.1.4 in the dissertation of P. Laffer [9]. P. Laffer's Theorem 1.1.4 shows that no number is Q-distribution normal for all basic sequences Q. It should be noted that every irrational number is Q-distribution normal for uncountably many basic sequences Q and not Q-distribution normal for uncountably many basic sequences Q. P. Laffer [9] also provides further refinements of these statements.

Moreover, we will also show the following for Q that are infinite in limit:

(1) The sets $\mathcal{DN}(Q)^{\mathsf{c}}$ and $\mathcal{RN}_2(Q)^{\mathsf{c}}$ are α-winning sets (in the sense of Schmidt's game) for every α in $(0, 1/2)$.
(2) $\mathcal{DN}(Q)$ and $\mathcal{RN}_1(Q)$ are sets of the first category.

2. Properties of $\mathcal{RN}_k(Q)$, and $\mathcal{DN}(Q)$

2.1. Winning sets. In [19], W. Schmidt proposed the following game between two players: Alice and Bob. Let $\alpha \in (0,1)$, $\beta \in (0,1)$, $S \subseteq \mathbb{R}$, and let $\rho(I)$ denote the radius of a set I. Bob first picks any closed interval $B_1 \subsetneq \mathbb{R}$. Then Alice picks a closed interval $A_1 \subsetneq B_1$ such that $\rho(A_1) = \alpha\rho(B_1)$. Bob then picks a closed interval $B_2 \subsetneq A_1$ with $\rho(B_2) = \beta\rho(A_1)$. After this, Alice picks a closed interval $A_2 \subsetneq B_2$ such that $\rho(A_2) = \alpha\rho(B_2)$, and so on. We say that the set S is (α, β)-*winning* if Alice can play so that

$$\bigcap_{n=1}^{\infty} B_n \subsetneq S. \tag{2.1}$$

The set S is (α, β)-*losing* if it is not (α, β)-winning. S is α-*winning* if it is (α, β)-winning for all $0 < \beta < 1$. Winning sets satisfy the following properties:

(1) If S is an α-winning set, then the Hausdorff dimension of S is 1.
(2) The intersection of countably many α-winning sets is α-winning.
(3) Bi-Lipshitz homeomorphisms of $\mathbb{R}$ preserve winning sets.

We write windim S to be the supremum of all α such that S is α-winning. N. G. Moshchevitin [14] proved

Theorem 2.1. *Let (t_n) be a sequence of positive numbers and*

$$\forall \epsilon > 0\ \exists N_0\ \forall n \geq N_0 : \frac{t_{n+1}}{t_n} \geq 1 + \frac{1}{n^{\epsilon}}.$$

Then for every number $\delta > 0$ the set

$$\mathcal{A}_\delta = \left\{ x \in \mathbb{R} : \exists c(x) > 0\ \forall n \in \mathbb{N}\ \|t_n x\| > \frac{c(x)}{n^{\delta}} \right\}$$

is an α-winning set for all α in $(0, 1/2)$. Thus, windim $\mathcal{A}_\delta = 1/2$.

Corollary 2.1. *For every basic sequence Q, windim $\mathcal{DN}(Q)^{\mathsf{c}} = 1/2$. Moreover, $\mathcal{DN}(Q) \backslash \mathcal{RN}_1(Q)$ is not an α-winning set for any α.*

Proof. Let $t_n = q_1 q_2 \cdots q_n$. Clearly, for all $\delta > 0$, $\mathcal{A}_\delta \subsetneq \mathcal{DN}(Q)^{\mathsf{c}}$. Thus, windim $\mathcal{DN}(Q)^{\mathsf{c}} = 1/2$. But $\mathcal{DN}(Q)^{\mathsf{c}} \cap \mathcal{DN}(Q) = \emptyset$ and the property of being α-winning is preserved by countable intersections, so $\mathcal{DN}(Q)$ and $\mathcal{DN}(Q) \backslash \mathcal{RN}_1(Q)$ are not α-winning sets for any α. □

Lemma 2.1. *If Q is infinite in limit, $x \in \mathcal{RN}_2(Q)$, and t is a non-negative integer, then*

$$\lim_{n \to \infty} N_n^Q((t), x) = \infty.$$

Proof. Since Q is infinite in limit and $x \in \mathcal{RN}_2(Q)$, for all $i, j \geq 0$, we have

$$\lim_{n \to \infty} \frac{N_n^Q((t,i), x)}{N_n^Q((t,j), x)} = 1.$$

So, for all j there is an n such that $N_n^Q((t,j), x) \geq 1$. Since there are infinitely many choices for j, the lemma follows. □

Let $\mathcal{FZ}(Q)$ be the set of real numbers whose Q-Cantor series expansion contains at most finitely many copies of the digit 0.

Corollary 2.2. *If Q is infinite in limit, then $\mathcal{FZ}(Q) \subsetneq \mathcal{RN}_2(Q)^{\mathsf{c}}$.*

Theorem 2.2. *If Q is infinite in limit, then windim $\mathcal{RN}_2(Q)^{\mathsf{c}} = 1/2$. Moreover, if Q is 1-divergent, then windim $\mathcal{N}_1(Q)^{\mathsf{c}} = 1/2$.*

Proof. We note that windim $\mathcal{FZ}(Q) = 1/2$ by Theorem 2.1. The first conclusion follows directly from this and Corollary 2.2. If Q is 1-divergent and $x \in \mathcal{N}_1(Q)$, then every digit occurs infinitely often in the Q-Cantor series expansion of x. So, $\mathcal{FZ}(Q) \subsetneq \mathcal{N}_1(Q)^{\mathsf{c}}$ and windim $\mathcal{N}_1(Q)^{\mathsf{c}} = 1/2$. □

It should be noted that Theorem 2.2 is in some ways stronger than the corresponding result for b-ary expansions. The original proof due to W. Schmidt that the set of numbers not normal in base b is 1/2-winning heavily uses the fact that a real number x is normal in base b if and only if x is simply normal in base b^k for all k. In fact, the set of numbers not normal of order 2 in base b is not an α-winning set for any α. The reasoning used in the proof of Theorem 2.2 and in the preceeding lemmas only works because Q is infinite in limit.

2.2. $\mathcal{DN}(Q)$, $\mathcal{RN}_k(Q)$, and $\mathcal{N}_k(Q)$ are sets of the first category. Given a sequence $Z = (z_1, \ldots, z_n)$ in $\mathbb{R}$ and $0 < \gamma \leq 1$, we define

$$A_n([0,\gamma), z) := |\{i; 1 \leq i \leq n \text{ and } \{z_i\} \in [0,\gamma)\}|.$$

Theorem 2.3. *For any basic sequence Q, the set $\mathcal{DN}(Q)$ is of the first category.*

Proof. We define

$$G_m = \bigcap_{n=m}^{\infty} \left\{ x \in \mathbb{R} : \frac{A_n([0,1/2), T_{Q,n-1}(x))}{n} < 2/3 \right\} \tag{2.2}$$

and put $G = \bigcup_{m=1}^{\infty} G_m$. Clearly, $\mathcal{DN}(Q) \subsetneq G$ and each of the sets G_m is nowhere dense, so $\mathcal{DN}(Q)$ is of the first category. □

We also note the following, which is proven similarly to Theorem 2.3.

Theorem 2.4. *For any basic sequence Q and positive integer k, the set $\mathcal{RN}_k(Q)$ is of the first category. Since $\mathcal{N}_k(Q) \subsetneq \mathcal{RN}_k(Q)$, $\mathcal{N}_k(Q)$ is also of the first category.*[6]

3. Proof of Theorem 1.1

Suppose that Q is a basic sequence and $x = E_0.E_1E_2\cdots$ w.r.t. Q. We let $S(x)$ be the set of all positive integers which occur at least once in the sequence (E_n). P. Erdős and A. Rényi [3] proved the following theorem.

Theorem 3.1. *If Q is infinite in limit and 1-convergent, then the density of $S(x)$ is with probability 1 equal to 0.*

[6] $\mathcal{N}_k(Q)$ could be empty. See Proposition 5.1 in [12]. It is proven in [10] that $\mathcal{N}_k(Q) \subsetneq \mathcal{RN}_k(Q)$ for all Q that are infinite in limit.

Corollary 3.1. *If Q is infinite in limit and 1-convergent, then*

$$\lambda(\mathcal{RN}_1(Q)) = 0.$$

The first part of Theorem 1.1 follows immediately from Corollary 3.1 as the sets $\mathcal{DN}(Q)\backslash\mathcal{RN}_1(Q)$ have full measure when Q is infinite in limit and 1-convergent. The remainder of this paper will be devoted to proving the second part of Theorem 1.1.

3.1. Construction of $\Theta_{Q,S}$. For the rest of this section, we fix basic sequences $Q = (q_n)$ and $S = (s_n)$. We let $Q_1 = Q$ and define basic sequences $Q_j = (q_{j,n})$ by

$$Q_1 \sim_{s_1} Q_2 \sim_{s_2} Q_3 \sim_{s_3} Q_4 \cdots.$$

We will define the following notation. Let $S_j = \prod_{k=1}^{j-1} s_j$ and set $\nu_j = \min\left\{t \in \mathbb{Z} : q_m \geq S_j^{2j} \text{ for } m \geq t\right\}$. Put $l_1 = s_1\nu_2$ and

$$l_j = \frac{\left(\sum_{k=1}^{j-1} S_k l_k\right) \cdot (2js_j\nu_{j+1} - 1)}{S_j}.$$

Given $l_1, l_2, \cdots, l_j$, define $L_j = \sum_{k=1}^{j} S_k l_k$. Thus, we may write

$$l_j = \frac{L_{j-1} \cdot (2js_j\nu_{j+1} - 1)}{S_j}.$$

Let $\mathcal{U} = \{(j, b, c) \in \mathbb{N}^3 : b \leq l_j, c \leq S_j\}$. Put

$$\phi(j, b, c) = L_{j-1} + (b - 1)S_j + c.$$

Note that $\phi : \mathbb{N}^3 \to \mathbb{N}$ is a bijection. Define

$$(i(n), b(n), c(n)) = \phi^{-1}(n)$$

and put $a(n) = S_{i(n)}$. Let

$$\mathcal{F} = \left\{ \left(F_{(j,b,c)}\right)_{(j,b,c)\in\mathcal{U}} \subseteq \mathbb{N}^3 \middle| \frac{F_{(j,b,c)}}{q_{\phi(j,b,c)}} \in V_{j,b,c} \right\},$$

where

$$V_{j,b,c} = \begin{cases} \left[\frac{1}{q_{\phi(j,b,c)}}, \frac{2}{q_{\phi(j,b,c)}}\right) & \text{if } j = 1 \\ \left[\frac{c}{a(\phi(j,b,c))} + \frac{1}{a(\phi(j,b,c))^2}, \frac{c}{a(\phi(j,b,c))} + \frac{2}{a(\phi(j,b,c))^2}\right] & \text{if } j > 1 \end{cases}.$$

Given $F \in \mathcal{F}$, we set $E_{F,n} = F_{\phi^{-1}(n)}$, $E_F = (E_{F,n})_{n=1}^{\infty}$, and put

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \cdots q_n}.$$

We set $\Theta_{Q,S} = \{x_F : F \in \mathcal{F}\}$. It will be proven that $\Theta_{Q,S}$ is non-empty, has full Hausdorff dimension, and

$$\Theta_{Q,S} \subsetneq \bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j) \backslash \mathcal{RN}_1(Q_j).$$

3.2. Distribution normality of members of $\Theta_{Q,S}$.

Let $\omega(n) = \#\{E_{F,n} : F \in \mathcal{F}\}$.

Lemma 3.1. *$\omega(n) = 1$ if $i(n) = 1$ and*

$$\omega(n) \geq q_n^{1-1/i(n)} > 2$$

if $i(n) \geq 2$.

Proof. By construction, $E_{F,n} = 1$ when $i(n) = 1$. If $i(n) \geq 2$, then

$$\frac{E_{F,n}}{q_n} \in \left[\frac{c(n)}{a(n)} + \frac{1}{a(n)^2}, \frac{c(n)}{a(n)} + \frac{2}{a(n)^2}\right],$$

which has length $1/a(n)^2$. Thus,

$$\omega(n) = 1 + \lfloor q_n \cdot (1/a(n)^2) \rfloor \geq \frac{q_n}{a(n)^2}.$$

By construction, $q_n \geq a(n)^{2i(n)}$, thus $q_n^{1/i(n)} \geq a(n)^2$. So $\frac{q_n^{1/i(n)}}{a(n)^2} \geq 1$ and $\frac{q_n}{a(n)^2} \geq q_n^{1-1/i(n)}$. Additionally, $n \geq \nu_2$, so $q_n \geq s_1^{2\cdot 2} \geq 16$. Thus, $q_n^{1-1/i(n)} \geq 4 > 2$. $\square$

Lemma 3.1 guarantees that $\Theta_{Q,S}$ is non-empty, but will also be critical in determining $\dim_{\mathrm{H}}(\Theta_{Q,S})$. If $x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \cdots q_n}$, then the Q_j-Cantor series expansion of x_F is

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,j,n}}{q_{j,1} q_{j,2} \cdots q_{j,n}},$$

where

$$E_{F,j,n} = \sum_{v=1}^{S_j} \left(E_{F,S_j\cdot(n-1)+v} \cdot \prod_{w=v+1}^{S_j} q_{S_j\cdot(n-1)+w} \right) \text{ and } q_{j,n} = \prod_{w=1}^{S_j} q_{S_j\cdot(n-1)+w}.$$

Lemma 3.2. *For all $j, n \geq 1$*

$$0 \leq \frac{E_{F,j,n}}{q_{j,n}} - \frac{E_{F,S_j\cdot(n-1)+1}}{q_{S_j\cdot(n-1)+1}} < \frac{S_j}{q_{S_j\cdot(n-1)+1}};$$

$$\lim_{n\to\infty} \frac{E_{F,j,n}}{q_{j,n}} - \frac{E_{F,S_j\cdot(n-1)+1}}{q_{S_j\cdot(n-1)+1}} = 0.$$

Proof.

$$
\begin{aligned}
0 \le & \frac{E_{F,j,n}}{q_{j,n}} - \frac{E_{F,S_j\cdot(n-1)+1}}{q_{S_j\cdot(n-1)+1}} = \frac{\sum_{v=2}^{S_j}\left(E_{F,S_j\cdot(n-1)+v}\cdot\prod_{w=v+1}^{S_j} q_{S_j\cdot(n-1)+w}\right)}{\prod_{w=1}^{S_j} q_{S_j\cdot(n-1)+w}} \\
\le & \sum_{v=2}^{S_j} \frac{(q_{S_j\cdot(n-1)+v}-1)\prod_{w=v+1}^{S_j} q_{S_j\cdot(n-1)+w}}{\prod_{w=1}^{S_j} q_{S_j\cdot(n-1)+w}} < \sum_{v=2}^{S_j} \frac{\prod_{w=v}^{S_j} q_{S_j\cdot(n-1)+w}}{\prod_{w=1}^{S_j} q_{S_j\cdot(n-1)+w}} \\
= & \sum_{v=2}^{S_j} \frac{1}{\prod_{w=1}^{v-1} q_{S_j\cdot(n-1)+w}} = \frac{1}{q_{S_j\cdot(n-1)+1}} \cdot \sum_{v=2}^{S_j} \frac{1}{\prod_{w=2}^{v-1} q_{S_j\cdot(n-1)+w}} \\
\le & \frac{S_j}{q_{S_j\cdot(n-1)+1}} \to 0, \text{ as } q_{S_j\cdot(n-1)+1} \to \infty.
\end{aligned}
$$

□

Lemma 3.2 suggests the key observation that the Q_j-distribution normality of a member of $\Theta_{Q,S}$ is determined entirely by its digits (E_n) in base Q, where $n \equiv 1 \pmod{S_j}$. Thus, we prove the following.

Lemma 3.3. *For all $j \ge 1$, S_{j+1} divides L_j.*

Proof. We prove this by induction. The base case holds as $L_1 = l_1 = s_1\nu_2$. Assume that $S_j | L_{j-1}$. Then

$$L_j = L_{j-1} + S_j \cdot \frac{L_{j-1}\cdot(2js_j\nu_{j+1}-1)}{S_j} = 2jL_{j-1}\nu_{j+1}s_j \tag{3.1}$$

$$= \left(2j\nu_{j+1}\cdot\frac{L_{j-1}}{S_j}\right)S_js_j = \left(2j\nu_{j+1}\cdot\frac{L_{j-1}}{S_j}\right)S_{j+1}. \tag{3.2}$$

□

Lemma 3.4. *For all $j \ge 1$, l_j is an integer, $l_j \ge js_j$, and $L_j \ge \nu_{j+1} - 1$.*

Proof. $l_1 = s_1\nu_2$ is an integer. To show that l_j is an integer for $j \ge 2$, we write

$$l_j = \frac{L_{j-1}}{S_j}\cdot(2js_j\nu_{j+1}-1), \tag{3.3}$$

which is an integer by Lemma 3.3. Since $\nu_{j+1} \ge 1$, $2\nu_{j+1}js_j - 1 \ge js_j$. Thus, by (3.3), $l_j \ge js_j$. The last assertion follows directly from (3.1). □

Definition. For a finite sequence $z = (z_1, \ldots, z_n)$, we define the *star discrepancy* $D_n^* = D_n^*(z_1, \ldots, z_n)$ as

$$\sup_{0<\gamma\le 1}\left|\frac{A([0,\gamma), z)}{n} - \gamma\right|.$$

Given an infinite sequence $w = (w_1, w_2, \ldots)$, we define

$$D_n^*(w) = D_n^*(w_1, w_2, \ldots, w_n).$$

For convenience, set $D^*(z_1, \ldots, z_n) = D_n^*(z_1, \ldots, z_n)$.

Theorem 3.2. *The sequence $w = (w_1, w_2, \ldots)$ is uniformly distributed mod 1 if and only if $\lim_{n\to\infty} D_n^*(w) = 0$.*

We will make use of the following definition from [8]:

Definition. For $0 \le \delta < 1$ and $\epsilon > 0$, a finite sequence $x_1 < x_2 < \cdots < x_N$ in $[0, 1)$ is called an almost-arithmetic progression-(δ, ϵ) if there exists an η, $0 < \eta \le \epsilon$, such that the following conditions are satisfied:

$$0 \le x_1 \le \eta + \delta\eta; \tag{3.4}$$

$$\eta - \delta\eta \le x_{n+1} - x_n \le \eta + \delta\eta \text{ for } 1 \le n \le N - 1; \tag{3.5}$$

$$1 - \eta - \delta\eta \le x_N < 1. \tag{3.6}$$

Almost arithmetic progressions were introduced by P. O'Neil in [16]. He proved that a sequence (x_n) of real numbers in $[0, 1)$ is uniformly distributed mod 1 if and only if the following holds: for any three positive real numbers δ, ϵ, and ϵ', there exists a positive integer N such that for all $n > N$, the initial segment $x_1, x_2, \ldots, x_n$ can be decomposed into an almost-arithmetic progression-(δ, ϵ) with at most N_0 elements left over, where $N_0 < \epsilon' N$. We will use the following theorem from [15]:

Theorem 3.3. *Let $x_1 < x_2 < \cdots < x_N$ be an almost arithmetic progression-(δ, ϵ) and let η be the positive real number corresponding to the sequence according to Definition 3.2. Then*

$$D_N^* \le \frac{1}{N} + \frac{\delta}{1 + \sqrt{1 - \delta^2}} \text{ for } \delta > 0 \text{ and } D_N^* \le \min\left(\eta, \frac{1}{N}\right) \text{ for } \delta = 0.$$

Corollary 3.2. *Let $x_1 < x_2 < \cdots < x_N$ be an almost arithmetic progression-(δ, ϵ) and let η be the positive real number corresponding to the sequence according to Definition 3.2. Then $D_N^* \le \frac{1}{N} + \delta$.*

For $j < k$ set

$$Y_{F,j,k,b} = \left(\frac{E_{F,\phi(j,b,1+S_j n)}}{q_{\phi(j,b,1+S_j n)}}\right)_{n=1}^{S_k/S_j - 1}$$

and let $D_{F,j,k,b}^* = D^*(Y_{F,j,k,b})$. Put

$$Y_{F,j} = Y_{F,j,j+1,1} Y_{F,j,j+1,2} \ldots Y_{F,j,j+1,l_1} Y_{F,j,j+2,1} \times Y_{F,j,j+2,2} \ldots Y_{F,j,j+2,l_2} Y_{F,j,j+3,1}.$$

If we prove that $Y_{F,j}$ is uniformly distributed mod 1, it will immediately follow that $x_F \in \mathcal{DN}(Q_j)$.

Lemma 3.5. *If $F \in \mathcal{F}$ and $j < k$, then $Y_{F,j,k,b}$ is an almost arithmetic progression-$\left(\frac{1}{S_j S_k}, \frac{S_j}{S_k}\right)$. Thus,*

$$D^*_{F,j,k,b} \leq |Y_{F,j,k,b}| + \frac{1}{S_j S_k} = \frac{S_j}{S_k} + \frac{1}{S_j S_k} \leq 2\frac{S_j}{S_k}. \tag{3.7}$$

Proof. We verify only (3.5) as (3.4) and (3.6) may be verified similarly. Note that

$$\frac{E_{F,\phi(k,b,1+S_j n)}}{q_{\phi(k,b,1+S_j n)}} \in \left[\frac{1+S_j n}{S_k} + \frac{1}{S_k^2}, \frac{1+S_j n}{S_k} + \frac{2}{S_k^2}\right];$$

$$\frac{E_{F,\phi(k,b,1+S_j(n+1))}}{q_{\phi(k,b,1+S_j(n+1))}} \in \left[\frac{1+S_j(n+1)}{S_k} + \frac{1}{S_k^2}, \frac{1+S_j(n+1)}{S_k} + \frac{2}{S_k^2}\right].$$

Therefore,

$$\begin{aligned}\frac{E_{F,\phi(k,b,1+S_j(n+1))}}{q_{\phi(k,b,1+S_j(n+1))}} - \frac{E_{F,\phi(k,b,1+S_j n)}}{q_{\phi(k,b,1+S_j n)}} &\leq \left(\frac{1+S_j(n+1)}{S_k} + \frac{2}{S_k^2}\right) \\ &\qquad - \left(\frac{1+S_j n}{S_k} + \frac{1}{S_k^2}\right) \\ &= \frac{S_j}{S_k} + \frac{1}{S_k^2}.\end{aligned}$$

Similarly, it may be shown that

$$\frac{E_{F,\phi(k,b,1+S_j(n+1))}}{q_{\phi(k,b,1+S_j(n+1))}} - \frac{E_{F,\phi(k,b,1+S_j n)}}{q_{\phi(k,b,1+S_j n)}} \geq \frac{S_j}{S_k} - \frac{1}{S_k^2}.$$

Thus, with $\eta = \epsilon$, we have $\eta - \delta\eta \leq \frac{E_{F,\phi(k,b,1+S_j(n+1))}}{q_{\phi(k,b,1+S_j(n+1))}} - \frac{E_{F,\phi(k,b,1+S_j n)}}{q_{\phi(k,b,1+S_j n)}} \leq \eta + \delta\eta$. □

We will need the following corollary of Theorem 2.6 in Chapter 2 of [8].

Corollary 3.3. *If t is a positive integer and for $1 \leq j \leq t$, z_j is a finite sequence in $\mathbb{R}$ with star discrepancy at most ϵ_j, then*

$$D^*\left(z_1^{l_1} \cdots z_t^{l_t}\right) \leq \frac{\sum_{j=1}^t l_j |z_j| \epsilon_j}{\sum_{j=1}^t l_j |z_j|}.$$

For any given positive integer n and $j < i(n)$, we can write $n = \frac{L_{i(n)-1}}{S_j} + m_j(n)$, where $m_j(n)$ can be uniquely written in the form

$$m_j(n) = \alpha_j(n)\frac{S_{i(n)}}{S_j} + \beta_j(n),$$

with $0 \le \alpha_j(n) \le l_{i(n)}$ and $0 \le \beta_j(n) < \frac{S_{i(n)}}{S_j}$. For $j < t$, define

$$f_{j,t}(w,z) = \frac{L_j/S_j + \sum_{k=j+1}^{t-1} 2l_k + 2w + z}{L_j/S_j + \sum_{k=j+1}^{t-1} l_k \cdot \frac{S_k}{S_j} + \frac{S_t}{S_j} w + z};$$

$$\bar{\epsilon}_{j,t} = \frac{L_j/S_j + \sum_{k=j+1}^{t-1} 2l_k + S_t/S_j}{L_j/S_j + \sum_{k=j+1}^{t-1} l_k \cdot \frac{S_k}{S_j} + S_t/S_j}.$$

The following lemma is proven similarly to Lemma 11 in [1].

Lemma 3.6. *If $1 \le j < t$ and $(w,z) \in \{0,\cdots,l_t\} \times \{0,\cdots,S_t/S_j\}$, then*

$$f_{j,t}(w,z) < f_{j,t}(0, S_t/S_j) = \bar{\epsilon}_{j,t}.$$

Lemma 3.7. *Suppose that $j < i(n)$. Then*

$$D_n^*(Y_{F,j}) \le f_{j,i(n)}(\alpha_j(n), \beta_j(n)) < \bar{\epsilon}_{j,i(n)}.$$

Proof. This follows from Lemma 3.5, Corollary 3.3, and Lemma 3.6. □

We will need the following basic lemma.

Lemma 3.8. *Let L be a real number and $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be two sequences of positive real numbers such that*

$$\sum_{n=1}^{\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = L.$$

Lemma 3.9. *The limit $\lim_{t\to\infty} \bar{\epsilon}_{j,t}$ is equal to 0.*

Proof. For $1 \le k \le j$, put $a_k = b_k = \frac{L_j}{jS_j}$. For $k > j$, set $a_k = 2l_k + \frac{S_{k+1}-S_k}{S_j}$ and $b_k = l_k \cdot \frac{S_k}{S_j} + \frac{S_{k+1}-S_k}{S_j}$. Clearly, $\bar{\epsilon}_{j,t} = \frac{a_1+\cdots+a_{t-1}}{b_1+\cdots+b_{t-1}}$ for $t > j$. Then

$$\frac{a_k}{b_k} = \frac{2l_kS_j + S_{k+1} - S_k}{l_kS_k + S_{k+1} - S_k} \le \frac{l_kS_j}{l_kS_k} + \frac{S_{k+1} - S_k}{l_kS_k}$$
$$= \frac{1}{s_js_{j+1}\cdots s_{k-1}} + \frac{s_k - 1}{l_k} < \frac{1}{s_js_{j+1}\cdots s_{k-1}} + \frac{1}{k} \to 0,$$

by Lemma 3.4. Thus, the conclusion follows directly from Lemma 3.8. □

Theorem 3.4. *If $F \in \mathcal{F}$, then $x_F \in \bigcap_{j=1}^\infty \mathcal{DN}(Q_j)$.*

Proof. Let $j \ge 1$ and $F \in \mathcal{F}$. By Lemma 3.7 and Lemma 3.9, $Y_{F,j}$ is uniformly distributed mod 1. Thus, by Lemma 3.2, $x_F \in \mathcal{DN}(Q_j)$. □

Theorem 3.5. *If $F \in \mathcal{F}$, then $x_F \notin \bigcup_{j=1}^\infty \mathcal{RN}_1(Q_j)$.*

Proof. By construction, $E_{F,n} \neq 0$ for all natural numbers n and $F \in \mathcal{F}$. Note that $E_{F,j,n}$ can only be equal to 0 if

$$\sum_{v=1}^{S_j} \left(E_{F,S_j\cdot(n-1)+v} \cdot \prod_{w=v+1}^{S_j} q_{S_j\cdot(n-1)+w} \right) = 0.$$

But this is impossible as $E_{F,S_j\cdot(n-1)+v} \neq 0$ for all v. Thus, $E_{F,j,n} \neq 0$ for all j and n, so $x_F \notin \bigcup_{j=1}^{\infty} \mathcal{RN}_1(Q_j)$. □

Corollary 3.4. *We have the following containment*

$$\Theta_{Q,S} \subsetneq \bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j) \backslash \mathcal{RN}_1(Q_j).$$

We note the following theorem that is proven similarly to Theorem 3.8 and Theorem 3.10 in [13].

Theorem 3.6. *The set $\Theta_{Q,S}$ is perfect and nowhere dense.*

3.3. Hausdorff dimension of $\boldsymbol{\Theta_{Q,S}}$. We will make use of the following general construction found in [4]. Suppose that $[0,1] = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \ldots$ is a decreasing sequence of sets, with each I_k a union of a finite number of disjoint closed intervals (called k^{th} *level basic intervals*). Then we will consider the set $\cap_{k=0}^{\infty} I_k$. We will construct a set $\Theta'_{Q,S}$ that may be written in this form such that $\dim_{\mathrm{H}}(\Theta_{Q,S}) = \dim_{\mathrm{H}}\left(\Theta'_{Q,S}\right)$.

Given a block of digits $B = (b_1, b_2, \ldots, b_s)$ and a positive integer n, define

$$\mathcal{S}_{Q,B} = \{x = 0.E_1E_2\ldots \text{ w.r.t } Q : E_1 = b_1, \ldots, E_t = b_s\}.$$

Let P_n be the set of all possible values of $E_n(x)$ for $x \in \Theta_{Q,S}$. Put $J_0 = [0,1)$ and

$$J_k = \bigcup_{B \in \prod_{n=1}^{k} P_n} \mathcal{S}_{Q,B}.$$

Then $J_k \subsetneq J_{k-1}$ for all $k \geq 0$ and $\Theta_{Q,S} = \cap_{k=0}^{\infty} J_k$, which gives the following.

Proposition 3.1. *The set $\Theta_{Q,S}$ can be written in the form $\cap_{k=0}^{\infty} J_k$, where each J_k is the union of a finite number of disjoint half-open intervals.*

We now set $I_k = \overline{J_k}$ for all $k \geq 0$ and put $\Theta'_{Q,S} = \cap_{k=0}^{\infty} I_k$. Since each set J_k consists of only a finite number of intervals, the set $I_k \backslash J_k$ is finite.

Lemma 3.10. *For all Q and S, we have $\dim_H(\Theta_{Q,S}) = \dim_H\left(\Theta'_{Q,S}\right)$.*

Proof. The lemma follows as $\Theta'_{Q,S} \backslash \Theta_{Q,S}$ is a countable set. □

We need the following key theorem from [4].

Theorem 3.7. *Suppose that each $(k-1)^{th}$ level interval of I_{k-1} contains at least m_k k^{th} level intervals ($k = 1, 2, \ldots$) which are separated by gaps of at least ϵ_k, where $0 \le \epsilon_{k+1} < \epsilon_k$ for each k. Then*

$$dim_H\left(\bigcap_{k=0}^{\infty} I_k\right) \ge \liminf_{k\to\infty} \frac{\log(m_1 m_2 \cdots m_{k-1})}{-\log(m_k \epsilon_k)}.$$

Theorem 3.8. *Suppose that Q is infinite in limit and*

$$\log q_k = o\left(\sum_{n=1}^{k-1} \log q_n\right). \tag{3.8}$$

Then $dim_H(\Theta_{Q,S}) = 1$.

Proof. We wish to better describe the k^{th} level basic intervals J_k in order to apply Theorem 3.7. We note that when $a(k) > 1$, each k^{th} level basic interval is contained in

$$\left[\sum_{n=1}^{k-1} \frac{E_n}{q_1 \cdots q_n} + \frac{1}{q_1 \cdots q_{k-1}} \cdot \left(\frac{c(k)}{a(k)} + \frac{1}{a(k)^2}\right), \right. \\ \left. \sum_{n=1}^{k-1} \frac{E_n}{q_1 \cdots q_n} + \frac{1}{q_1 \cdots q_{k-1}} \cdot \left(\frac{c(k)}{a(k)} + \frac{2}{a(k)^2}\right)\right] \tag{3.9}$$

for some $(E_1, E_2, \cdots, E_{k-1}) \in \prod_{n=1}^{k-1} P_n$. Thus, by Lemma 3.1, there are at least $q_{k-1}^{1-1/i(k-1)}$ k^{th} level basic intervals contained in each $(k-1)^{th}$ level basic interval. By (3.9), they are separated by gaps of length at least

$$\begin{aligned}
&\left(\sum_{n=1}^{k-2} \frac{E_n}{q_1 \cdots q_n} + \frac{E_{k-1}+1}{q_1 \cdots q_{k-1}} + \frac{1}{q_1 \cdots q_{k-1}} \cdot \left(\frac{c(k)}{a(k)} + \frac{1}{a(k)^2}\right)\right) \\
&- \left(\sum_{n=1}^{k-1} \frac{E_n}{q_1 \cdots q_n} + \frac{1}{q_1 \cdots q_{k-1}} \cdot \left(\frac{c(k)}{a(k)} + \frac{1}{a(k)^2}\right)\right) \\
=&\frac{(E_{k-1}+1) - E_{k-1}}{q_1 \cdots q_{k-1}} - \frac{1}{q_1 \cdots q_{k-1}} \frac{1}{a(k)^2} = \frac{1 - 1/a(k)^2}{q_1 \cdots q_{k-1}}.
\end{aligned}$$

Thus, we may apply Theorem 3.7 with $m_k = q_{k-1}^{1-1/i(k-1)}$ and $\epsilon_k = \frac{1-1/a(k)^2}{q_1\cdots q_{k-1}}$. But $1 - 1/a(k) \to 1$, so

$$\begin{aligned}\dim_{\mathrm{H}}\Theta_{Q,S} &\geq \liminf_{k\to\infty} \frac{\log \prod_{n=L_1}^{k-1} q_n^{1-1/i(n)}}{-\log\left(\left(q_k^{1-1/i(k)}+1\right)\cdot \frac{1}{q_1 q_2\cdots q_{k-1}}\right)} \\ &= \liminf_{k\to\infty} \frac{\sum_{n=1}^{k-1}\left(1-\frac{1}{i(n)}\right)\log q_n}{\sum_{n=1}^{k-1}\log q_n - \left(1-\frac{1}{i(k)}\right)\log q_k} \\ &= \liminf_{k\to\infty} \frac{\sum_{n=1}^{k-1}\left(1-\frac{1}{i(n)}\right)\log q_n}{\sum_{n=1}^{k-1}\log q_n} = 1\end{aligned}$$

by Lemma 3.8 and (3.8) since

$$\lim_{k\to\infty} \frac{\left(1-\frac{1}{i(k-1)}\right)\log q_{k-1}}{\log q_{k-1}} = \lim_{k\to\infty}\left(1-\frac{1}{i(k)}\right) = 1.$$

Thus, $\dim_{\mathrm{H}}\Theta_{Q,S} = 1$. □

Clearly, every 1-convergent basic sequence satisfies (3.8). So, part (2) of Theorem 1.1 follows by Corollary 3.4 and Theorem 3.8.

4. Further Remarks

We observed after Lemma 3.2 that it was key to be able to approximate $\frac{E_n}{q_n}$ for $n \equiv 1 \pmod{S_j}$. Part (2) of Theorem 1.1 can be extended to a larger intersection of sets of the form $\mathcal{DN}(Q)\backslash\mathcal{RN}_1(Q)$ by estimating $\frac{E_n}{q_n}$ for $n \equiv r \pmod{S_j}$, $r = 0, 1, \cdots, S_j - 1$. Given $Q = Q_1 \sim_{s_1} Q_2 \sim_{s_2} Q_3 \cdots$, define $Q_{j,k} = (q_{j,k,n})$ by

$$q_{j,k,n} = \begin{cases} \prod_{j=1}^{k} q_j & \text{if } n = 1 \\ \prod_{j=1}^{s} q_{s(n-1)+j+k} & \text{if } n > 1 \end{cases},$$

so $Q_j = Q_{j,0}$. With only small modifications of the preceeding proofs, we may conclude that

$$\Theta_{Q,S} \subsetneq \bigcap_{j=1}^{\infty}\bigcap_{k=0}^{S_j-1} \mathcal{DN}(Q_{j,k})\backslash\mathcal{RN}_1(Q_{j,k})$$

and

$$\dim_{\mathrm{H}}\left(\bigcap_{j=1}^{\infty}\bigcap_{k=0}^{S_j-1} \mathcal{DN}(Q_{j,k})\backslash\mathcal{RN}_1(Q_{j,k})\right) = 1.$$

The techniques introduced in this paper are unlikely to settle the following questions. For an arbitrary countable collection of infinite in limit basic sequences (Q_j), is it true that

$$\dim_{\mathrm{H}}\left(\bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j)\backslash\mathcal{RN}_1(Q_j)\right) = 1?$$

A more difficult problem would be to construct an explicit example of a member of $\bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j)\backslash\mathcal{RN}_1(Q_j)$. The problem gets much harder if we loosen the restriction that Q_j is infinite in limit. In fact, it is still an open problem to construct an explicit example of a member of $\mathcal{DN}(Q)$ for an arbitrary Q. See [9] for more information.

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