JOURNAL de Théorie des Nombres de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Sandra ROZENSZTAJN

Asymptotic values of modular multiplicities for $\mathbf{GL_2}$

Tome 26, n° 2 (2014), p. 465-482.

<http://jtnb.cedram.org/item?id=JTNB_2014__26_2_465_0>

Journal de Théorie des Nombres de Bordeaux **26** (2014), 465–482

Asymptotic values of modular multiplicities for $\mathbf{GL_2}$

par Sandra ROZENSZTAJN

Résumé. *Valeurs asymptotiques de multiplicités modulaires pour* GL_2.
Nous étudions les constituants irréductibles de la réduction modulo p d'une représentation algébrique irréductible V du groupe $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ pour une extension finie K de $\mathbb{Q}_p$. Nous montrons qu'asymptotiquement, la multiplicité de chaque constituant ne dépend que de la dimension de V et du caractère central de sa réduction modulo p. Nous appliquons ce résultat au calcul de la valeur asymptotique de multiplicités qui sont l'objet de la conjecture de Breuil-Mézard.

Abstract. We study the irreducible constituents of the reduction modulo p of irreducible algebraic representations V of the group $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ for K a finite extension of $\mathbb{Q}_p$. We show that asymptotically, the multiplicity of each constituent depends only on the dimension of V and the central character of its reduction modulo p. As an application, we compute the asymptotic value of multiplicities that are the object of the Breuil-Mézard conjecture.

1. Introduction

1.1. Main result. Let p be a prime number and K a finite extension of $\mathbb{Q}_p$ of degree h. We write $H_K = \mathrm{Hom}(K, \overline{\mathbb{Q}}_p)$. The irreducible representations of the algebraic group $G = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ are in bijection with families $(n,m) = (n_\tau, m_\tau)_{\tau\in H_K} \in (\mathbb{Z}_{\geq 0}\times\mathbb{Z})^{H_K}$: they are the representations $V_{n,m} = \otimes_\tau(\mathrm{Symm}^{n_\tau} W_\tau \otimes \det_\tau^{m_\tau})$, where W_τ is the standard representation of GL_2 given by the morphism $\tau : K \to \overline{\mathbb{Q}}_p$ and $\det_\tau = \det W_\tau$. We can consider the semi-simplification $\overline{V}_{n,m}^{ss}$ of the reduction in characteristic p of $V_{n,m}$, this is well-defined and doesn't depend on the choice of a G-stable lattice in $V_{n,m}$ used to define the reduction. Then $\overline{V}_{n,m}^{ss}$ is a representation of the group $\mathrm{GL}_2(\mathbb{F}_q)$ (where q is the cardinality of the residue field of K). In general this representation is not irreducible, and we are interested in the multiplicities $a_\sigma(n,m)$ of the irreducible representations σ of $\mathrm{GL}_2(\mathbb{F}_q)$ that appear as constituents of $\overline{V}_{n,m}^{ss}$. They are well defined and do not depend

Manuscrit reçu le 27 décembre 2012, accepté le 19 février 2013.

on the choice of the lattice in $V_{n,m}$. In particular we want to study their asymptotic value as the coefficients n_τ grow to infinity. We show that the asymptotic value of $a_\sigma(n,m)$ depends only on σ and the dimension and central character of $\overline{V}^{ss}_{n,m}$. This follows from the theorem (corollary 3.2):

Theorem 1.1. *There exists a family $(\mathcal{S}_\alpha)_\alpha$ of elements of the Grothendieck group R of representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in characteristic p, indexed by central characters α of $\mathrm{GL}_2(\mathbb{F}_q)$, such that for any representation W of $\mathrm{GL}_2(\mathbb{F}_q)$ that has a central character, for any representation U of $\mathrm{GL}_2(\mathbb{F}_q)$ that is the reduction in characteristic p of an irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$, the class $[V]$ in R of the representation $V = W \otimes U$ satisfies:*

$$[V] = (\dim V)\mathcal{S}_{\alpha(V)} + O((\dim U)^{1-1/h})$$

where $\alpha(V)$ denotes the central character of V.

1.2. Motivation. The motivation for this study comes from a question about Galois representations that we now explain. We denote the absolute Galois group of K by $\mathrm{Gal}(\overline{K}/K)$. We fix a representation $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$ where V is a 2-dimensional vector space over some finite field k. We denote by F a finite extension of $\mathbb{Q}_p$ with residue field k and by π a uniformizer of F. In all that follows we assume that $\mathrm{End}_{k[\mathrm{Gal}(\overline{K}/K)]}(\overline{\rho}) = k$.

Let $R(\overline{\rho})$ be the universal deformation ring of $\overline{\rho}$ over F. It is an $\mathcal{O}_F$-algebra that parameterizes the liftings of $\overline{\rho}$ in characteristic 0. That is, for each closed point x of $\mathrm{Spec}\, R(\overline{\rho})[1/p]$, there exists a 2-dimensional vector space V_x over some extension of F and a representation $\rho_x : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V_x)$ lifting $\overline{\rho}$.

Let now $v = \{(n_\tau, m_\tau), \tau \in H_K\}$ be a family of pairs of integers indexed by H_K as before, let $\mathtt{t}$ be a Galois type, that is a representation of the inertia subgroup I_K of $\mathrm{Gal}(\overline{K}/K)$ with values in $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ and open kernel, and let ψ be a character $\mathrm{Gal}(\overline{K}/K) \to \mathcal{O}_F^*$ satisfying some compatibility relations with v and $\mathtt{t}$ (details on these relations can be found in Section 2 of [BM12], but they will play no role in this article).

A representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(L)$ for some extension L of F is said to be of type $(v, \mathtt{t}, \psi)$ if ρ is potentially semi-stable with Hodge-Tate weights $(m_\tau, m_\tau + n_\tau + 1)_{\tau \in H_K}$ and the restriction to I_K of the Weil-Deligne representation attached to $D_{pst}(\rho)$ is isomorphic to $\mathtt{t}$ and $\det \rho = \psi$.

Kisin showed in the article [Kis08] (see also [Kis10]) that, after possibly enlarging F, liftings of $\overline{\rho}$ of type $(v, \mathtt{t}, \psi)$ are parameterized by a deformation ring $R^{\psi}_{v,\mathtt{t}}(\overline{\rho})$ that is a quotient of $R(\overline{\rho})$: for any closed point x of $\mathrm{Spec}\, R(\overline{\rho})[1/p]$, the representation ρ_x is of type $(v, \mathtt{t}, \psi)$ if and only if x is a point of the closed subscheme defined by $R^{\psi}_{v,\mathtt{t}}(\overline{\rho})$.

The Breuil-Mézard conjecture ([BM02]) and its refinements ([BM12] and [EG11], [Kis09]), proved by Kisin in [Kis09] in the case $K = \mathbb{Q}_p$ for most

representations $\overline{\rho}$, and also by Paškūnas in [Paš12], is a statement about the ring $R^{\psi}_{v,\mathtt{t}}(\overline{\rho})$. The goal is to give a measure of its complexity using the Hilbert-Samuel multiplicity $e(R^{\psi}_{v,\mathtt{t}}(\overline{\rho})/\pi)$ of its special fiber. The conjecture states that this multiplicity, called Galois multiplicity, equals an automorphic multiplicity that can be written as a sum $\mu_{Aut}(\overline{\rho}, v, \mathtt{t}) = \Sigma_\sigma a_\sigma(v, \mathtt{t})\mu_\sigma(\overline{\rho})$ over all irreducible representations σ in characteristic p of $\mathrm{GL}_2(\mathbb{F}_q)$. The $a_\sigma(v, \mathtt{t})$ are nonnegative integers depending only on σ, v and $\mathtt{t}$ and the $\mu_\sigma(\overline{\rho})$ are nonnegative integers, that we call intrinsinc multiplicities following the terminology of [Dav13]. They depend only on σ and $\overline{\rho}$.

Let us recall the definition of $a_\sigma(v, \mathtt{t})$: Using the theory of types, Henniart (in the appendix of [BM02]) attaches to $\mathtt{t}$ a finite-dimensional representation $\sigma(\mathtt{t})$ of $\mathrm{GL}_2(\mathcal{O}_K)$ with coefficients in $\overline{\mathbb{Q}}_p$. We attach to $v = (n_\tau, m_\tau)$ a representation $\sigma(v)$ that is the algebraic irreducible representation of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ given by $\otimes_{\tau\in H_K}(\mathrm{Symm}^{n_\tau} W_\tau \otimes \det_\tau^{m_\tau})^\tau$. Let $\sigma(v, \mathtt{t}) = \sigma(\mathtt{t}) \otimes \sigma(v)$ and let $T_{v,\mathtt{t}}$ be a lattice in $\sigma(v, \mathtt{t})$ that is preserved by the action of $\mathrm{GL}_2(\mathcal{O}_K)$. We then define $a_\sigma(v, \mathtt{t})$ to be the multiplicity of the irreducible representation σ of $\mathrm{GL}_2(\mathbb{F}_q)$ in the semi-simplification of the reduction to characteristic p of the lattice $T_{v,\mathtt{t}}$, which we denote by $\overline{\sigma(v, \mathtt{t})}$.

Thus when we apply the results of Theorem 1.1 to $U = \sigma(\mathtt{t})$ and $W = \sigma(v)$ we get an asymptotic value of $a_\sigma(v, \mathtt{t})$, and therefore of the automorphic multiplicity (provided we know the intrinsic multiplicities) and so conjecturally, we get the asymptotic value of the Hilbert-Samuel multiplicity of the special fiber of the deformation ring for these Hodge-Tate weights and this Galois type $\mathtt{t}$. For example, we treat in Paragraph 4.3 the case where K is the unramified extension of $\mathbb{Q}_p$ of degree h, and $\overline{\rho}$ is an irreducible representation that is as generic as possible. We obtain the following asymptotic value for the automorphic multiplicity:

$$\mu_{Aut}(\overline{\rho}, v, \mathtt{t}) = \frac{4^h}{p^{2h}-1}\dim\sigma(\mathtt{t})\Big(\prod_{i=0}^{h-1}(n_i+1)\Big) + O\Big(\prod_{i=0}^{h-1}(n_i+1)\Big)^{1-1/h}\Big)$$

for values of $v = (n_i, m_i)_{0\le i\le h-1}$ such that the central character of $\overline{\sigma(v, \mathtt{t})}$ has some fixed value depending on $\overline{\rho}$, and $\mu_{Aut}(\overline{\rho}, v, \mathtt{t}) = 0$ otherwise.

1.3. Plan of the article. The plan of this note is as follows: in Section 2, we recall some results on irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in characteristic p and the multiplicities of these representations in principal series representations. In Section 3 we compute asymptotic values of the multiplicities of the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$, in the reduction in characteristic p of irreducible algebraic representations of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ in

characteristic 0, and we prove Theorem 1.1. In Section 4 we apply this result to the computation of asymptotic values, as Hodge-Tate weights go to infinity, of automorphic multiplicities for some mod p Galois representations.

1.4. Acknowledgements. I would like to thank Christophe Breuil for useful discussions and helpful comments.

2. Representations of $\mathbf{GL_2}(\mathbb{F}_q)$ in characteristic p

2.1. Notations. In the following we fix a prime number p and $q = p^f$. We denote by E an algebraic closure of $\mathbb{F}_q$ and we choose a morphism $\mathbb{F}_q \to E$. This allows us to view $\mathbb{F}_q$ as a subfield of E.

We denote $\mathrm{GL}_2(\mathbb{F}_q)$ by G and the Borel subgroup of upper triangular matrices by B.

2.2. Irreducible representations. We recall the description of the irreducible characteristic p representations of G. In this case everything can be described in a very explicit manner. The proofs of the statements of this Paragraph can be found in Chapter 10 of the book [Bon11] (the results in [Bon11] are for the case of $\mathrm{SL}_2(\mathbb{F}_q)$ but they extend readily to the case of $\mathrm{GL}_2(\mathbb{F}_q)$). The description of the irreductible representations can also be found in Section 1 of [BL94].

We denote by $\mathcal{W}$ the standard representation of G on E^2. For all $n \in \mathbb{Z}_{\geq 0}$ we let $S_n = \mathrm{Symm}^n(\mathcal{W})$. The representation S_n is of dimension $n+1$ and can be identified with the vector space $E[x,y]_n$ of homogeneous polynomials of degree n with the action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ given by $(g \cdot P)(x,y) = P(ax + cy, bx + dy)$. In particular S_0 is the trivial representation of G.

Let (ρ, V) be a finite-dimensional E-representation of G. We denote by $V(i)$ the twist of V by the i-th power of the determinant for an $i \in \mathbb{Z}/(q-1)\mathbb{Z}$. For any integer j we denote by $V^{[j]}$ the representation $\rho \circ F^j$, where F is the Frobenius of $\mathbb{F}_q$. Note that $V^{[f]} = V$.

We now describe the irreducible representations of G (Theorem 10.1.8 of [Bon11]): Let n be an integer written in base p as $\sum_{i \geq 0} n_i p^i$. We denote by L_n the sub-representation of $S_n = E[x,y]_n$ generated by x^n. It is the subspace generated by the monomials $x^m y^{n-m}$ for all integers m that can be written in base p as $\sum_{i \geq 0} m_i p^i$ with $0 \leq m_i \leq n_i$ for all i. As a G-representation, L_n is isomorphic to $\otimes_{i \geq 0} S_{n_i}^{[i]}$.

Proposition 2.1. *The irreducible representations of G are exactly the representations $L_n(m)$ with $0 \leq n \leq q-1$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$.*

Moreover the following property holds (Proposition 10.1.18 and Paragraph 10.2.1 of [Bon11]):

Proposition 2.2. *For any* $0 \leq n \leq q-1$, L_n *is a multiplicity one constituent of* S_n. *The other irreducible constituents in* S_n *are among the representations* $L_i(j)$ *with* $i < n$.

2.3. Frobenius action and θ operators. The Frobenius acts as a permutation on the set $\{L_n(m), 0 \leq n \leq q-1, m \in \mathbb{Z}/(q-1)\mathbb{Z}\}$. In order to describe this permutation we introduce an operator on $\{0, \ldots, q-1\}$ and $\mathbb{Z}/(q-1)\mathbb{Z}$ that we denote by θ. On $\mathbb{Z}/(q-1)\mathbb{Z}$, θ is simply multiplication by p. Let n be an element of $\{0, \ldots, q-1\}$. We write it in base p as $n = \sum_{i=0}^{f-1} a_i p^i$ and we set $\theta n = a_{f-1} + a_0 p + \ldots a_{f-2} p^{f-1}$. These θ operators are compatible with respect to reduction modulo $q-1$. We now have $L_n(m)^{[1]} = L_{\theta n}(\theta m)$.

2.4. The Grothendieck group of representations of G in characteristic p. Let $\mathrm{Rep}_E(G)$ be the Grothendieck group of finite-dimensional representations of G with coefficients in E. We denote by $[V]$ the image in $\mathrm{Rep}_E(G)$ of a representation V of G.

$\mathrm{Rep}_E(G)$ is a ring with the product given by $[V \otimes W] = [V][W]$. The twist by a power of the determinant and the action of Frobenius extend to $\mathrm{Rep}_E(G)$, and we denote $[V(i)]$ by $[V](i)$ and $[V^{[j]}]$ by $[V]^{[j]}$.

We can endow $\mathrm{Rep}_E(G)$ with two natural bases: The first one, which we call basis L, is given by the irreducible representations of G, that is the $[L_n(m)]$ for $0 \leq n \leq q-1$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. Note that tensoring an element of L by a power of the determinant or twisting it by a power of Frobenius transforms this element into another element of L. The second one, which we call basis S, is given by the $[S_n(m)]$ for $0 \leq n \leq q-1$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. That it is indeed a basis follows from Proposition 2.2.

We denote by $[V : L_n(m)]$ the coefficient of $[L_n(m)]$ when we write an element $V \in \mathrm{Rep}_E(G)$ in the basis L.

Finally let $R = \mathrm{Rep}_E(G) \otimes_{\mathbb{Z}} \mathbb{R}$. All the notations that we have just defined extend to R.

2.5. Recurrence formula for tensor products. The following formula holds for the representations $S_n = \mathrm{Symm}^n(\mathcal{W})$ of G:

Proposition 2.3. *For all nonnegative* n, m *we have* $[S_n][S_m] = [S_{n+m}] + [S_{n-1}][S_{m-1}](1)$.

Proof. We have in fact a more precise result which is proved in Glover's article [Glo78]: the following is an exact sequence:

$$0 \to (S_{n-1} \otimes S_{m-1})(1) \to S_n \otimes S_m \to S_{n+m} \to 0,$$

where the second arrow is given by multiplication of polynomials and the first one is given by $u \otimes v \mapsto (xu \otimes yv) - (yu \otimes xv)$. □

2.6. The Dixon invariant. Let $\Theta = xy^q - x^q y$. For all $g \in \mathrm{GL}_2(\mathbb{F}_q)$ we have $g(\Theta) = (\det g)\Theta$. For all $n \in \mathbb{Z}_{\geq 0}$ we can consider the map $\iota : S_n(1) \to S_{n+q+1}$ given by multiplication by Θ. This map is $\mathrm{GL}_2(\mathbb{F}_q)$-equivariant and injective. We let $\mathcal{V}_{n+2} = S_{n+q+1}/\iota S_n(1)$.

2.7. Principal series representations.

2.7.1. *Definition.* Let $\mathcal{F}$ be the space of functions on G with values in $\mathbb{F}_q$. The group G acts on $\mathcal{F}$ by $(gf)(\gamma) = f(\gamma g)$. If χ is a character of B, we let $\mathcal{V}(\chi) = \mathrm{ind}_B^G \chi$. It is the subspace of $\mathcal{F}$ formed by the functions satisfying $f(bg) = \chi(b)f(g)$ for all $b \in B$ and $g \in G$. It is an $\mathbb{F}_q$-vector space of dimension $q+1$. The dual space $\mathcal{V}(\chi)^*$ of $\mathcal{V}(\chi)$ is isomorphic to $\mathcal{V}(\chi^{-1})$.

For $r \in \mathbb{Z}$, we denote by λ_r the character of B with values in $\mathbb{F}_q^*$ given by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto d^r$. This character depends only on r modulo $q-1$. Let us note that all characters of B with values in $\mathbb{F}_q^*$ are of the form $\lambda_r \otimes \det^i$. We will thus restrict our study to the case of the characters λ_r.

2.7.2. *Relation to* $[S_n(r)]$. Let $n \geq 0$ be an integer. We define a G-equivariant morphism $u_n : S_n = E[x,y]_n \to \mathcal{F}$ by $P \mapsto (g \mapsto P((0,1)g))$. Its image is contained in $\mathcal{V}(\lambda_n)$. We have the following result:

Lemma 2.4. *Let $r \in \mathbb{Z}/(q-1)\mathbb{Z}$ and let $n \geq 0$ be an integer congruent to r modulo $q-1$. If $n \leq q$ then u_n is an embedding $S_n \to \mathcal{V}(\lambda_r)$, and it is an isomorphism if $n = q$. If $n \geq q+1$, then u_n induces an isomorphism $\mathcal{V}_{n-q+1} = S_n/\iota S_{n-q-1}(1) \to \mathcal{V}(\lambda_r)$.*

Proof. The kernel of u_n is the set of homogenous polynomials of degree n that are zero on $\mathbb{F}_q^2$, that is, multiples of the Dixon invariant Θ. In particular, u_n is injective when $n < q+1$, and the kernel of u_n is $\iota S_{n-q-1}(1)$ otherwise. Hence when $n \geq q+1$ the morphism u_n induces an embedding of $\mathcal{V}_{n-q+1} = S_n/\iota S_{n-q-1}(1)$ into $\mathcal{V}(\lambda_r)$. As both spaces are of dimension $q+1$, this is an isomorphism. □

In particular:

Lemma 2.5. *For $n \geq 2$, $\mathcal{V}_n$ and $\mathcal{V}(\lambda_n)$ are isomorphic and the isomorphism class of $\mathcal{V}_n$ depends only on the class of n modulo $q-1$.*

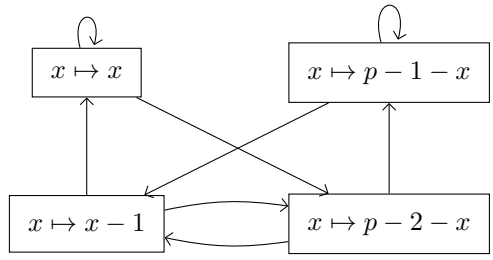
In the following we will use the notation $\mathcal{V}_n$ for $\mathcal{V}(\lambda_n)$ also in the case where n is an element of $\mathbb{Z}/(q-1)\mathbb{Z}$ and not necessarily an integer ≥ 2.

We can then recover a classical result (that we won't use in the sequel) using the fact that S_n^* is isomorphic to $S_n(-n)$ and that S_n and $S_{q-1-n}(n)$ have no common constituent if $n \leq q-1$ (see Proposition 2.7):

Proposition 2.6. *For all $0 \leq n \leq q-1$ there is an exact sequence $0 \to S_n \to \mathcal{V}(\lambda_n) \to S_{q-1-n}(n) \to 0$. It is split when $n = 0$ or $q-1$.*

2.7.3. ***Constituents of principal series representations.*** The constituents of the representations $[\mathcal{V}_r]$ have been computed by Diamond in the article [Dia07]. We give here a reformulation of this description. Consider the following graph:

(2.1)



Let $\mathcal{C}_f$ be the set of closed paths of length f in graph (2.1). Let c be such a path. We denote by c_i the vertex reached at the i-th step of the path. We thus get a list $c_0, \dots, c_{f-1}, c_f = c_0$ of vertices of the graph.

To each path c we attach a family of functions $(\lambda_i^c)_{0 \le i \le f-1}$ by taking for λ_i^c the function written on the vertex c_i. We also define a function ℓ_c, by $\ell_c(x_0, \dots, x_{f-1}) = (\sum_i p^i(x_i - \lambda_i^c(x_i)))/2$ if the vertex c_{f-1} is in the left column of the graph, and $\ell_c(x_0, \dots, x_{f-1}) = (p^f - 1 + \sum_i p^i(x_i - \lambda_i^c(x_i))/2$ if the vertex c_{f-1} is in the right column.

Let $0 \le n < q-1$. We write n in base p as $n = \sum_{i=0}^{f-1} p^i n_i$. Let $\lambda^c(n) = \sum_{i=0}^{f-1} p^i \lambda_i^c(n_i)$, and $\ell_c(n) = \ell_c(n_0, \dots, n_{f-1})$. Observe that $\lambda^c(n) = \lambda^{c'}(n)$ and $\ell_c(n) = \ell_{c'}(n)$ if and only if $c = c'$.

Diamond's result is the following:

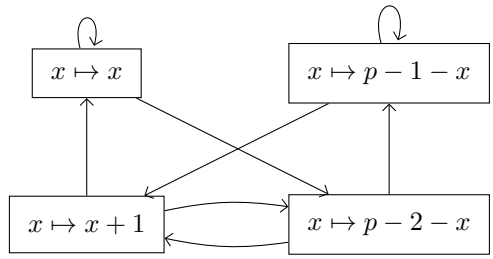
Proposition 2.7.

$$[\mathcal{V}_n(m)] = \sum [L_{\lambda^c(n)}(m + \ell_c(n))]$$

where the sum is over all $c \in \mathcal{C}_f$ *such that* $0 \le \lambda_i^c(n_i) \le p-1$ *for all* i. *In particular, all constituents appear with multiplicity* 1.

2.7.4. ***Representations having*** $[L_n(m)]$ ***as a constituent.*** Let $A(n,m)$ be the set of $(n', m') \in \{0, \dots, q-2\} \times \mathbb{Z}/(q-1)\mathbb{Z}$ such that $L_n(m)$ is a constituent of $\mathcal{V}_{n'}(m')$. We give a description of $A(n,m)$. Consider the following graph:

(2.2)



Let $\mathcal{A}_f$ be the set of closed paths of length f in graph (2.2). Let c be such a path and write c_i for the vertex reached at step i. We attach to c a family of functions $(\mu_i^c)_{0\leq i\leq f-1}$ where μ_i^c is the function written on the vertex c_i.

We say that c is compatible with $n = \sum_i n_i p^i$ if $0 \leq \mu_i^c(n_i) \leq p-1$ for all i. In this case let $\mu^c(n) = \sum \mu_i^c(n_i)p^i$.

Proposition 2.8. *The set $A(n,m)$ is the set of $(\mu^c(n), m - \ell_c(\mu^c(n)))$, for c in the set of elements of $\mathcal{A}_f$ that are compatible with n and such that $\mu^c(n) \neq q-1$. Distinct c's give distinct elements in $A(n,m)$. The cardinality of $A(n,m)$ doesn't depend on m.*

Proof. We denote by the same letter c a path in graph (2.2) and the analogous path in graph (2.1). Then μ^c and λ^c are inverse functions, so that $\lambda^c(n') = n$ if and only if $\mu^c(n) = n'$. Hence the description of $A(n,m)$. In particular, the cardinality of $A(n,m)$ doesn't depend on m. The fact that distinct c's give distinct elements in $A(n,m)$ follows from the fact that $\lambda^c(n) = \lambda^{c'}(n)$ and $\ell_c(n) = \ell_{c'}(n)$ if and only if $c = c'$. □

We denote by $\omega(n)$ the cardinality of $A(n,m)$. Let $0 \leq n \leq q-1$ be written as $\sum_i n_i p^i$ in base p. We denote by r_n the number of coefficients n_i that are equal to $p-1$. Then :

Proposition 2.9. (1) $\omega(0) = 2^f - 1$
(2) *if $n \neq 0$ then* $\omega(n) = 2^{f-r_n}$

Proof. Computing $\omega(n)$ is the same as counting elements $c \in \mathcal{A}_f$ that are compatible with n and such that $\mu^c(n) \neq q-1$.

We have to understand for which cases we have $\mu^c(n) = q-1$. This can happen only if path c never reaches the vertex labelled $x \mapsto p-2-x$, so c is necessarily a constant path staying either on the vertex $x \mapsto x$ or on the vertex $x \mapsto p-1-x$. For such a c we have $\mu^c(n) = q-1$ if and only if $n = 0$ or $n = q-1$.

For each pair of vertices of graph (2.2) there is exactly one path of length 2 from the first point to the second one. Hence the cardinality of $\mathcal{A}_f$ is 2^f: we have 4 choices for the first vertex of path c, 2 choices for each following

step except for step $f-1$ where we have only one possibility that allows us to close the path at step f.

If $r_n < f$ there exist exactly 2^{f-r_n} elements of $\mathcal{A}_f$ that are compatible with n. Indeed we can assume without loss of generality that $n_{f-1} < p-1$. Then for all i such that $n_i = p-1$ the number of possible paths is divided by 2, as the path cannot reach the lower row of graph (2.2) at step i.

If $n = q-1$, the paths compatible with n are those that avoid the lower row of graph 2.2. Hence there are only 2 compatible paths: the constant paths staying either on the vertex labelled $x \mapsto x$ or on the vertex labelled $x \mapsto p-1-x$. The first one gives $\mu^c(q-1) = q-1$. So we get $\omega(q-1) = 1$. □

3. Asymptotic study of representations

3.1. Central characters. Let Z be the center of G. Using the embedding $\mathbb{F}_q \to E$ that we fixed in Paragraph 2.1, we can identify the group of characters of Z to $\mathbb{Z}/(q-1)\mathbb{Z}$ so that $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ corresponds to $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mapsto a^i$.

Let V be a representation of G. If is has a central character, we denote it by $\alpha(V)$.

3.2. The family $(\mathcal{S}_\alpha)$. Let $\mathcal{S} = \frac{1}{q^2-1}\sum_{\chi: B \to \mathbb{F}_q^*} [\mathcal{V}(\chi)] \in R$. Let i be an element of $\mathbb{Z}/(q-1)\mathbb{Z}$ which we see as a character of Z. Let $\mathcal{S}_i = \frac{1}{q^2-1}\sum_{\chi: B \to \mathbb{F}_q^*, \chi_{|Z}=i} [\mathcal{V}(\chi)]$. We have normalized $\mathcal{S}_i$ in such a way that $\dim \mathcal{S}_i = 1$, and by construction $\alpha(\mathcal{S}_i) = i$.

Note that $\mathcal{V}(\chi)^{[j]} = \mathcal{V}(p^j\chi)$, and $\alpha(p^j\chi) = p^j\alpha(\chi)$. Thus $\mathcal{S}_i^{[j]} = \mathcal{S}_{\theta^j i}$. We also have $\mathcal{S}_i(j) = \mathcal{S}_{i+2j}$.

3.3. Norms on R. We endow the $\mathbb{R}$-algebra $R = \mathrm{Rep}_E(G) \otimes_{\mathbb{Z}} \mathbb{R}$ with a norm $\|\cdot\|$ that satisfies the following properties:

(1) it is an algebra norm: that is, for all V and W, we have the inequality $\|V \cdot W\| \leq \|V\|\|W\|$
(2) it is unchanged by twist by the determinant
(3) it is unchanged under the action of Frobenius
(4) for all $V \in R$ and $n \in \{0, \dots, q-1\}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, we have $|[V : L_n(m)]| \leq \|V\|$

Such a norm can be constructed as follows: first endow R with the norm $|\cdot|_{L,\infty}$ defined by $|V|_{L,\infty} = \max_{n,m} |[V : L_n(m)]|$. This norm is unchanged by twist by the determinant and action of Frobenius as these actions permute the elements of the basis L. For all $V \in R$ we set

$$\|V\| = \sup_{W \neq 0} |V \cdot W|_{L,\infty} / |W|_{L,\infty}$$

which is finite as R is a finite dimensional $\mathbb{R}$-vector space and multiplication by V is $\mathbb{R}$-linear. Then $\|\cdot\|$ is a norm on R that clearly satisfies the first three properties. That it also satisfies the last one can be seen by noticing that $\|V\| \geq |V \cdot L_0|_{L,\infty} = |V|_{L,\infty}$.

We also need another norm on R that we denote by $|\cdot|_{S,1}$. An element V of R can be written $V = \sum_{n,m} \lambda_{n,m}[S_n(m)]$ in the basis S. We then set $|V|_{S,1} = \sum_{n,m} |\lambda_{n,m}|$.

As R is a finite-dimension $\mathbb{R}$-vector space, all norms on it are equivalent. Hence there exists a positive real number M such that for all $V \in R$ we have $|V|_{S,1} \leq M\|V\|$.

3.4. Statement of the results. We devote the rest of Section 3 to the proof of the following theorem and its corollary:

Theorem 3.1. *Let $r \geq 1$ be an integer. There exists a constant C_r depending only on r, such that for any representation W of G with a central character, and any representation U of the form $U = S_{k_1}^{[j_1]} \otimes \cdots \otimes S_{k_r}^{[j_r]}$, the representation $V = W \otimes U$ satisfies $[V] = (\dim V)\mathcal{S}_{\alpha(V)} + r_V$ with $\|r_V\| \leq C_r|W|_{S,1}(\dim U)/\min_i(k_i+1)$.*

Corollary 3.2. *There exists a constant C depending only on K such that for any representation W of G with a central character, and any representation U of G that is the reduction in characteristic p of an irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$, the representation $V = W \otimes U$ satisfies $[V] = (\dim V)\mathcal{S}_{\alpha(V)} + r_V$ with $\|r_V\| \leq C\|W\|(\dim U)^{1-1/h}$.*

3.5. Computation of $[S_k]$. Let $k = u(q^2-1) + v(q+1) + w$, where $u = \lfloor k/(q^2-1)\rfloor$, and where $v(q+1)+w$ is the remainder of the euclidean division of k by q^2-1, and w is the remainder of the division of k by $q+1$.

Making use of the equality $[S_{n+q+1}] = [S_n](1) + [\mathcal{V}_{n+2}]$ we get $[S_k] = \sum_{i=1}^{u(q-1)+v}[\mathcal{V}_{k-i(q+1)+2}](i-1) + [S_w](u(q-1)+v)$.

As $[\mathcal{V}_n]$ is $(q-1)$-periodic we have:

$$\sum_{i=u(q-1)+1}^{u(q-1)+v}[\mathcal{V}_{k-i(q+1)+2}](i-1) = \sum_{i=1}^{v}[\mathcal{V}_{k-i(q+1)+2}(i-1)] = \sum_{i=0}^{v-1}[\mathcal{V}_{k-2i}(i)]$$

and

$$\sum_{i=1}^{u(q-1)}[\mathcal{V}_{k-i(q+1)+2}](i-1) = u\sum_{i=1}^{q-1}[\mathcal{V}_{k-i(q+1)+2}(i-1)] = u\sum_{i=0}^{q-2}[\mathcal{V}_{k-2i}(i)]$$

Moreover $[\sum_{i=0}^{q-2} V_{k-2i}(i)] = (q^2-1)\mathcal{S}_{(k \bmod q-1)}$. Let

$$r_k = (u(q^2-1) - k - 1)\mathcal{S}_{(k \bmod q-1)} + \sum_{i=0}^{q-2}[\mathcal{V}_{k-2i}(i)] + [S_w](u(q-1)+v)$$

We define the constant A as follows:

$$A = (q^2+2q)\max\{\|[S_r]\|, 0 \le r < q^2-1, \|\mathcal{S}_i\|, i \in \mathbb{Z}/(q-1)\mathbb{Z}\}$$

As $|u(q^2-1)-k-1| \le q^2$ we get:

Proposition 3.3. $[S_k] = (k+1)\mathcal{S}_{(k \bmod q-1)} + r_k$ *and* $\|r_k\| \le A$.

3.6. Action of Frobenius.

Proposition 3.4. $[S_k]^{[j]} = (k+1)\mathcal{S}_{\theta^j(k \bmod q-1)} + r_k^{[j]}$ *and* $\|r_k^{[j]}\| \le A$.

Proof. From $[S_k] = (k+1)\mathcal{S}_{(k \bmod q-1)} + r_k$ we get the equality $[S_k]^{[j]} = (k+1)\mathcal{S}^{[j]}_{(k \bmod q-1)} + r_k^{[j]}$. We know that $\mathcal{S}_i^{[j]} = \mathcal{S}_{\theta^j i}$, and moreover $\|r_k\| = \|r_k^{[j]}\|$. □

Corollary 3.5. *Let* $t_{j,k}$ *be an integer such that* $2t_{j,k} = \theta^j k - k \bmod q-1$. *Then* $\|[S_k]^{[j]} - [S_k](t_{j,k})\| \le 2A$.

Proof. This follows from Proposition 3.4 as soon as we have proved that such a $t_{j,k}$ exists. When $p=2$, multiplication by 2 is an automorphism of $\mathbb{Z}/(q-1)\mathbb{Z}$, hence $t_{j,k}$ exists (and is unique). When $p>2$, $\theta^j k$ has the same parity as k, as θ is the same as multiplication by p modulo $q-1$, so $\theta^j k - k$ is even which gives the existence of $t_{j,k}$. □

3.7. Product.

Proposition 3.6. *Let* $a, b \in \mathbb{Z}/(q-1)\mathbb{Z}$. *Then* $\mathcal{S}_a\mathcal{S}_b = \mathcal{S}_{a+b}$.

Proof. Let $u, v \to +\infty$ with $u = a \pmod{q-1}$, $v = b \pmod{q-1}$ and $u \le v$.

We have the following equality:

$$\begin{aligned}[S_u][S_v] &= ((u+1)\mathcal{S}_{(u \bmod q-1)} + O(1))((v+1)\mathcal{S}_{(v \bmod q-1)} + O(1)) \\ &= (u+1)(v+1)\mathcal{S}_{(u \bmod q-1)}\mathcal{S}_{(v \bmod q-1)} + O(\max(u,v))\end{aligned}$$

Moreover:

$$[S_u][S_v] = (u+1)(v+1)\mathcal{S}_{(u+v \bmod q-1)} + O(\min(u,v))$$

Indeed it follows from 2.3 that $[S_u][S_v] = [S_{u+v}] + [S_{u-1}][S_{v-1}](1)$, so $[S_u][S_v] = \sum_{i=0}^{u}[S_{u+v-2i}](i)$. Hence:

$$\begin{aligned}[S_u][S_v] &= \sum_{i=0}^{u}((u+v-2i+1)\mathcal{S}_{(u+v \bmod q-1)} + r_{u+v-2i}) \\ &= (u+1)(v+1)\mathcal{S}_{(u+v \bmod q-1)} + \sum_{i=0}^{u} r_{u+v-2i}\end{aligned}$$

Finally $\|[S_u][S_v] - (u+1)(v+1)\mathcal{S}_{(u+v \bmod q-1)}\| \le (\min(u,v)+1)A$.

The result then follows from the comparison of the two formulas for $[S_u][S_v]$. □

3.8. Proof of the theorem. We can now prove Theorem 3.1 and its corollary.

Proof of Theorem 3.1. Using linearity and the invariance of the norm with respect to twisting by the determinant, we need only prove it when $W = S_m$ and $0 \leq m < q$.

Suppose first that $r = 1$. We write k, j for k_1, j_1. Let $V = W \otimes S_k^{[j]}$ as in the statement of the theorem.

Let t be an integer as in corollary 3.5, so that $[S_k^{[j]}] = [S_k](t) + a$ for some a with $\|a\| \leq 2A$. Then $[W \otimes S_k^{[j]}] = [S_m][S_k](t) + [W]a$. By the computations in the proof of Proposition 3.6 we see that

$$[S_m][S_k] = (m+1)(k+1)\mathcal{S}_{(m+k \bmod q-1)} + b$$

for some $b \in R$ with $\|b\| \leq qA$. Finally

$$[W \otimes S_k^{[j]}] = (\dim(W \otimes S_k^{[j]}))\mathcal{S}_{\alpha(W \otimes S_k^{[j]})} + c$$

where $c = b(t) + [W]a$, so that $\|c\| \leq qA + 2A^2$. We get the result with $C = qA + 2A^2$.

Let us now return to the general case. Let $U = S_{k_1}^{[j_1]} \otimes \cdots \otimes S_{k_r}^{[j_r]}$ and $V = W \otimes U$.

We set $V_1 = W \otimes S_{k_1}^{[j_1]}$ and $V_i = S_{k_i}^{[j_i]}$ for $i \geq 2$. Then $[V] = \prod_{i=1}^r [V_i]$.

For all $i \geq 2$ we have $[V_i] = (\dim V_i)\mathcal{S}_{\alpha(V_i)} + c_i$ with $\|c_i\| \leq A$ by Proposition 3.4. For $i = 1$ we have $[V_1] = (\dim V_1)\mathcal{S}_{\alpha(V_1)} + c_1$ with $\|c_1\| \leq C_1 = A(q + 2A)$ by the case $r = 1$ of the theorem. Moreover $\dim V_1 \leq q(k_1 + 1)$, as $\dim W \leq q$.

Let $r_V = [V] - (\dim V)\mathcal{S}_{\alpha(V)}$. We have:

$$[V] = \prod_i ((\dim V_i)\mathcal{S}_{\alpha(V_i)} + c_i) = \sum_{I \subset \{1,\ldots,r\}} (\prod_{i \in I} (\dim V_i)\mathcal{S}_{\alpha(V_i)})(\prod_{i \notin I} c_i).$$

Using Proposition 3.6 we see that $(\dim V)\mathcal{S}_{\alpha(V)} = \prod_i (\dim V_i)\mathcal{S}_{\alpha(V_i)}$, hence:

$$r_V = \sum_{I \subset \{1,\ldots,r\}, I \neq \{1,\ldots,r\}} (\prod_{i \in I} (\dim V_i)\mathcal{S}_{\alpha(V_i)})(\prod_{i \notin I} c_i).$$

We have:

$$\|(\prod_{i \in I} \mathcal{S}_{\alpha(V_i)})(\prod_{i \notin I} c_i)\| \leq \prod_{i \in I} \|\mathcal{S}_{\alpha(V_i)}\| \prod_{i \notin I} \|c_i\| \leq A^r(2A + q)$$

as for each i we have $\|\mathcal{S}_{\alpha(V_i)}\| \leq A$ by definition of A, and $\|c_i\| \leq A$ except for $\|c_1\| \leq A(2A + q)$. Finally:

$$\|r_V\| \leq A^r(2A + q) \sum_{I \subset \{1,\ldots,r\}, I \neq \{1,\ldots,r\}} \prod_{i \in I} \dim V_i.$$

Moreover $\prod_{i\in I}\dim V_i \le q(\dim U)/\min_{1\le i\le r}(k_i+1)$. Hence:

$$\|r_V\| \le q2^r A^r(2A+q)(\dim U)/\min_i(k_i+1).$$

The result follows with $C_r = q2^r A^r(2A+q)$. □

Proof of Corollary 3.2. The irreducible representations of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ are of the form $\otimes_{\tau\in\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}(\mathrm{Symm}^{k_\tau} W_\tau \otimes \det_\tau^{j_\tau})$ for some $k_\tau \in \mathbb{Z}_{\ge 0}$ and $j_\tau \in \mathbb{Z}$. Let U be the reduction in characteristic p of such a representation. Then $[U]$ is of the form $[U] = [S_{k_1}^{[j_1]} \otimes \cdots \otimes S_{k_h}^{[j_h]}(n)]$ where $h = [K:\mathbb{Q}_p]$.

Let I be a nonempty subset of $\{1,\dots,h\}$ and let r be the cardinality of I. We apply Theorem 3.1 to the decomposition $V = W'\otimes U'$ with $W' = W(n)\otimes(\otimes_{i\notin I} S_{k_i}^{[j_i]})$ and $U' = \otimes_{i\in I} S_{k_i}^{[j_i]}$. It gives us $[V] = (\dim V)\mathcal{S}_{\alpha(V)} + r_V$ with $\|r_V\| \le C_r|W'|_{S,1}(\dim U')/(\min_{i\in I} k_i + 1)$.

We have $|W'|_{S,1} \le M\|W'\|$ with M as in Paragraph 3.3. As the norm $\|\cdot\|$ is an algebra norm, we get $\|W'\| \le \|W(n)\|\prod_{i\notin I}\|S_{k_i}^{[j_i]}\|$. Moreover $\|W(n)\| = \|W\|$ by the properties of $\|\cdot\|$.

We apply Proposition 3.4 to each representation $S_{k_i}^{[j_i]}$ and we get an inequality $\|S_{k_i}^{[j_i]}\| \le (k_i+1)\|\mathcal{S}_{(\theta^{j_i}k_i \bmod q-1)}\| + A$, so that $\|S_{k_i}^{[j_i]}\| \le 2A(k_i+1)$ and $|W'|_{S,1} \le M\|W\|\prod_{i\notin I}(k_i+1)$.

From $\dim U = (\dim U')\prod_{i\notin I}(k_i+1)$ we get:

$$\|r_V\| \le MC_r(2A)^{h-r}\|W\|(\dim U)/(\min_{i\in I} k_i + 1).$$

Using the value of C_r that we computed in the proof of Theorem 3.1 we see that:

$$\|r_V\| \le Mq(2A+q)(2A)^h\|W\|(\dim U)/(\min_{i\in I} k_i + 1).$$

The best choice for I is to take $I = \{i_0\}$ where i_0 is such that $k_{i_0} = \max k_i$, so that:

$$\|r_V\| \le Mq(2A+q)(2A)^h\|W\|(\dim U)/(\max_{1\le i\le h} k_i + 1)$$

In order to have a formula that works for all U independently of the values of the k_i's, the best inequality we can use is $\max_i k_i + 1 \ge (\dim U)^{1/h}$, which gives the corollary with $C = Mq(2A+q)(2A)^h = MC_h$. □

3.9. Asymptotic values of multiplicities. Let $a_{n,m}(V) = [V : L_n(m)]$. Our goal is to find the asymptotic value of $a_{n,m}(V)$.

We observe first:

Lemma 3.7. *If V has a central character $\alpha(V)$, then $a_{n,m}(V) = 0$ if $n + 2m \ne \alpha(V) \pmod{q-1}$.*

Moreover Proposition 2.9 implies:

Proposition 3.8. (1) $[\mathcal{S}_i : L_n(m)] = 0$ *if* $n + 2m \neq i \pmod{q-1}$
(2) $[\mathcal{S}_i : L_n(m)] = \omega(n)/(q^2-1)$ *if* $n + 2m = i \pmod{q-1}$.

Finally we get (the value of C_r is given in Paragraph 3.8 and the value of $\omega(n)$ in Proposition 2.9):

Theorem 3.9. *Let $r \geq 1$ be an integer. Let W be a representation of G with a central character. Let $U = S_{k_1}^{[j_1]} \otimes \cdots \otimes S_{k_r}^{[j_r]}$ and $V = W \otimes U$. Then $a_{n,m}(V) = 0$ if $n + 2m \neq \alpha(V) \pmod{q-1}$ and*

$$|a_{n,m}(V) - \omega(n)(\dim V)/(q^2-1)| \leq C_r |W|_{S,1} (\dim U)/\min_i (k_i + 1)$$

if $n + 2m = \alpha(V) \pmod{q-1}$.

Proof. Write $[V] = (\dim V)\mathcal{S}_{\alpha(V)} + r_V$ as in Theorem 3.1. Then

$$|[V : L_n(m)] - (\dim V)[\mathcal{S}_{\alpha(V)} : L_n(m)]| \leq |[r_V : L_n(m)]|.$$

By the conditions we imposed on the norm $\|\cdot\|$ in Paragraph 3.3, we have $|[r_V : L_n(m)]| \leq \|r_V\|$. The result follows. □

Corollary 3.10. *Let W be a representation of G with a central character, let U be a representation of G that is the reduction in characteristic p of an algebraic irreducible representation in characteristic 0 of $\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_2$ and let $V = W \otimes U$. Then*

$$|a_{n,m}(V) - \omega(n)(\dim V)/(q^2-1)| \leq C\|W\|(\dim U)^{1-1/h}$$

if $n + 2m = \alpha(V) \pmod{q-1}$, and $a_{n,m}(V) = 0$ otherwise.

4. Application

4.1. The Breuil-Mézard conjecture. Let us consider again the situation described in Paragraph 1.2 of the introduction. Let $K/\mathbb{Q}_p$ be a finite extension and let $h = [K : \mathbb{Q}_p]$. We denote by κ the residue field of K so that $\kappa = \mathbb{F}_q$ for some $q = p^f$. Let k be a finite field and let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(k)$ be a Galois representation such that $\mathrm{End}_{k[\mathrm{Gal}(\overline{K}/K)]}(\overline{\rho}) = k$.

Fix a Galois type $\mathtt{t}$, so that $\sigma(\mathtt{t})$ is an irreducible finite-dimensional representation of $\mathrm{GL}_2(\mathcal{O}_K)$ with coefficients in $\overline{\mathbb{Q}}_p$. Fix also $v = (n_\tau, m_\tau)_{\tau \in H_K}$ with $n_\tau \in \mathbb{Z}_{\geq 0}$ and $m_\tau \in \mathbb{Z}$. We denote by $\sigma(v)$ the representation $\otimes_\tau (\mathrm{Symm}^{n_\tau} W_\tau \otimes \det_\tau^{m_\tau})$.

Let $\sigma(v, \mathtt{t}) = \sigma(\mathtt{t}) \otimes \sigma(v)$ and let $\overline{\sigma(v,\mathtt{t})}^{ss}$ be the semisimplification of the reduction to characteristic p of $\sigma(v, \mathtt{t})$. It is a finite-dimensional k-vector space with an action of $\mathrm{GL}_2(\mathcal{O}_K)$ that factors through $\mathrm{GL}_2(\kappa)$. Note that we can write $\overline{\sigma(v,\mathtt{t})}^{ss}$ as $(\overline{\sigma(\mathtt{t})} \otimes \otimes_\tau (\mathrm{Symm}^{n_\tau} E^2 \otimes \det_\tau^{m_\tau}))^{ss}$.

In order to compute the automorphic multiplicity, which is defined by $\mu_{Aut}(\overline{\rho}, v, t) = \sum_\sigma \mu_\sigma(\overline{\rho}) a_\sigma(v, \mathtt{t})$, we have to find the multiplicity $a_\sigma(v, \mathtt{t})$

of σ in the semi-simplification of $\overline{\sigma(v,\mathtt{t})}$ for each irreducible representation σ of $\mathrm{GL}_2(\kappa)$.

For a fixed Galois type $\mathtt{t}$ and v going to infinity (that is, with the dimension of $\sigma(v)$ going to infinity) we can use the results of Section 3 in order to have an asymptotic estimate of the values of the $a_\sigma(v,\mathtt{t})$ and therefore of $\mu_{Aut}(\overline{\rho},v,\mathtt{t}) = \Sigma_\sigma \mu_\sigma(\overline{\rho}) a_\sigma(v,\mathtt{t})$, provided we know the intrinsic multiplicities $\mu_\sigma(\overline{\rho})$.

It is conjectured that $\mu_\sigma(\overline{\rho})$ can be nonzero only if σ is an element of the set $D(\overline{\rho})$ of Serre weights defined in the article [BDJ10] in the case where K is unramified, and in [Sch08] and [BLGG] in the general case. In particular, as all elements of $D(\overline{\rho})$ have the same central character, $\mu_{Aut}(\overline{\rho},v,\mathtt{t})$ is expected to be nonzero for only one possible value of the central character $\alpha(\overline{\sigma(v,\mathtt{t})})$.

4.2. Example: the case $K = \mathbb{Q}_p$. In the case $K = \mathbb{Q}_p$ the Breuil-Mézard conjecture is entirely known thanks to results of Kisin ([Kis09]) and Paškūnas ([Paš12]). We will treat as an example the case where $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ is an absolutely irreducible representation. All the results we will use in this example can be found in the article [BM02].

In this case, $v = (a,b) \in (\mathbb{Z}_{\geq 0} \times \mathbb{Z})$ and $\sigma(v) = \mathrm{Symm}^a \overline{\mathbb{Q}}_p^2 \otimes \det^b$. The irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ are the $\sigma_{n,m} = \mathrm{Symm}^n E^2 \otimes \det^m$ for $0 \leq n \leq p-1$ and $m \in \mathbb{Z}/(p-1)\mathbb{Z}$, and $\sigma_{n,m}$ is the representation denoted by $L_n(m)$ in Section 3. We denote $a_{\sigma_{n,m}}(v,\mathtt{t})$ by $a_{n,m}(v,\mathtt{t})$. In this example we take for $\mathtt{t}$ the trivial type.

Let ω be the mod p cyclotomic character and ω_2 the Serre fundamental character of level 2, so that $\omega_2^{p+1} = \omega$.

As $\overline{\rho}$ is irreducible we can write $\overline{\rho}_{|I_p}$ as

$$\begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$$

for some $0 \leq n \leq p-2$ and $m \in \mathbb{Z}/(p-1)\mathbb{Z}$. There are exactly two nonzero intrinsic multiplicities: we have $\mu_{n,m}(\overline{\rho}) = \mu_{p-1-n,n+m}(\overline{\rho}) = 1$.

As we have chosen $\mathtt{t}$ to be the trivial type, the representation $\sigma(\mathtt{t})$ is the Steinberg representation $\mathrm{Symm}^{p-1} \overline{\mathbb{Q}}_p^2$ and is of dimension p. The automorphic multiplicity $\mu_{Aut}(\overline{\rho},v,\mathtt{t})$ can be nonzero only for $v = (a,b)$ such that the central character of $\overline{\sigma(v,\mathtt{t})}$ is that same as the central character of those $\sigma_{n,m}$ for which $\mu_{n,m}(\overline{\rho}) \neq 0$, that is $p + a + 2b = n + 2m \pmod{p-1}$.

Finally we get, as $a \to +\infty$ and $p + a + 2b = n + 2m \pmod{p-1}$:

- $\mu_{Aut}(\overline{\rho},(a,b),\mathtt{t}) = 4p(a+1)/(p^2-1) + O(1)$ if $1 \leq n \leq p-2$ (generic case)
- $\mu_{Aut}(\overline{\rho},(a,b),\mathtt{t}) = 2p(a+1)/(p^2-1) + O(1)$ if $n = 0$.

The constructions of Section 1.2 can be generalized to the case of deformation rings parameterizing representations that have the additional property of being potentially crystalline. These rings differ from the ones previously described only when the Galois type $\mathtt{t}$ is a scalar (otherwise the condition of being potentially crystalline is automatically satisfied). In this case, we need to replace the representation $\sigma(\mathtt{t})$ by another representation which we call $\sigma^{\mathrm{cr}}(\mathtt{t})$. When $\mathtt{t}$ is trivial, $\sigma^{\mathrm{cr}}(\mathtt{t})$ is the trivial representation.

In this case we get, when $\mathtt{t}$ is the trivial type and as $a \to +\infty$ and $a+2b = n+2m \pmod{p-1}$:

- $\mu^{\mathrm{cr}}_{Aut}(\overline{\rho},(a,b),\mathtt{t}) = 4(a+1)/(p^2-1)+O(1)$ if $1 \leq n \leq p-2$.
- $\mu^{\mathrm{cr}}_{Aut}(\overline{\rho},(a,b),\mathtt{t}) = 2(a+1)/(p^2-1)+O(1)$ if $n = 0$.

4.3. Example: K an unramified extension of $\mathbb{Q}_p$, and $\overline{\rho}$ as generic as possible. Let now K be the unramified extension of degree h of $\mathbb{Q}_p$. In this situation we don't know if the Breuil-Mézard conjecture holds and we don't even know which intrinsic multiplicities would make it true. However, thanks to results of Gee and Kisin ([GK12], Theorem A), we have some results about the intrinsic multiplicities in the case where K is unramified. They show that $\mu_\sigma(\overline{\rho})$ is nonzero if and only if σ is an element of the set $D(\overline{\rho})$ of Serre weights defined in the article [BDJ10]. Moreover they show that if σ is what they call Fontaine-Laffaille regular (definition 2.1.8 in [GK12]), then $\mu_\sigma(\overline{\rho}) = 1$ when it is nonzero. In our notations, Fontaine-Laffaille regular weights are representations $L_n(m)$ where the digits of n in base p are all in $\{0,\dots,p-3\}$.

Let us fix a morphism $\mathbb{F}_{p^h} \to E$. We choose a numbering $\{\tau_0,\dots,\tau_{h-1}\}$ of $H_K = \mathrm{Hom}(K,\overline{\mathbb{Q}}_p)$ such that τ_i reduces modulo p to the i-th power of Frobenius. The algebraic irreducible representations of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_2$ are then the $\sigma(v) = \otimes_{i=0}^{h-1}(\mathrm{Symm}^{a_i} W_{\tau_i} \otimes \det_{\tau_i}^{b_i})$, for $v = (a_i,b_i) \in (\mathbb{Z}^h_{\geq 0} \times \mathbb{Z}^h)$. Such a representation reduces in characteristic p to the representation $S_{a_0}(b_0)^{[0]} \otimes S_{a_1}(b_1)^{[1]} \dots \otimes S_{a_{h-1}}(b_{h-1})^{[h-1]}$. Note that this representation has central character $\sum_{i=0}^{h-1} p^i(a_i+2b_i)$.

The irreducible representations in characteristic p of $\mathrm{GL}_2(\mathbb{F}_q)$, $q = p^h$ are those described in Section 2 with $f = h$. We denote $a_{L_n(m)}(v,\mathtt{t})$ by $a_{n,m}(v,\mathtt{t})$ as in Section 3.

Let $\overline{\rho}$ a 2-dimensional irreducible representation of G_K with coefficients in E. After twisting $\overline{\rho}$ by some character we can write:

$$\overline{\rho}_{|I} = \begin{pmatrix} \omega_{2h}^{r_0+1+p(r_1+1)+\dots+p^{h-1}(r_{h-1}+1)} & 0 \\ 0 & \omega_{2h}^{p^h r_0+1+p^{h+1}(r_1+1)+\dots+p^{2h}(r_{h-1}+1)} \end{pmatrix}$$

for some $0 \leq r_0 \leq p-1$ and some $-1 \leq r_i \leq p-2$ for $1 \leq i \leq h-1$ and where ω_{2h} denotes the Serre fundamental character of level $2h$.

In this example we take $\overline{\rho}$ to be as generic as possible, that is we make the following assumption:

$$2 \leq r_0 \leq p-3 \text{ and } 1 \leq r_i \leq p-4 \text{ for } i \geq 1.$$

Note that in this case $\overline{\rho}$ is sufficiently generic in the sense of the article [BP12], Lemma 11.4. As a consequence, we see that $D(\overline{\rho})$ has exactly 2^h elements. Moreover the elements of $D(\overline{\rho})$ can be described using the method of [BP12], Lemma 11.4. Using this description, we see easily that our additional genericity condition ensures that all $\sigma \in D(\overline{\rho})$ are Fontaine-Laffaille regular in the sense of [GK12], and thus have $\mu_\sigma(\overline{\rho}) = 1$. This also ensures that for all $L_n(m) \in D(\overline{\rho})$, we have $\omega(n) = 2^h$, as no digit of n in base p is $p-1$ and $n \neq 0$.

Note that all weights in $D(\overline{\rho})$ have central character $r_0 + 1 + p(r_1+1) + \cdots + p^{h-1}(r_{h-1}+1)$.

Finally, under this hypothesis on the genericity of $\overline{\rho}$, we get, for a fixed Galois type $\mathtt{t}$, (a_i, b_i) such that $\alpha(\overline{\sigma(v,\mathtt{t})}) = \alpha(L_n(m))$ for all $L_n(m)$ in $D(\overline{\rho})$, that is

$$\sum_{i=0}^{h-1} p^i(a_i + 2b_i) + \alpha(\overline{\sigma(\mathtt{t})}) = r_0 + 1 + p(r_1+1) + \cdots + p^{h-1}(r_{h-1}+1),$$

and $\dim \sigma(v) \to +\infty$:

$$\mu_{Aut}(\overline{\rho}, v, \mathtt{t}) = \frac{4^h}{p^{2h}-1} \dim \sigma(\mathtt{t}) \Big(\prod_{i=0}^{h-1}(a_i+1)\Big) + O\Big(\prod_{i=0}^{h-1}(a_i+1)\Big)^{1-1/h}\Big)$$

References

[BDJ10] K. Buzzard, F. Diamond, and F. Jarvis, *On Serre's conjecture for mod ℓ Galois representations over totally real fields*, Duke Math. J., **155**, 1 (2010), 105–161.

[BL94] L. Barthel and R. Livné, *Irreducible modular representations of* GL_2 *of a local field*, Duke Math. J., **75**, 2 (1994), 261–292.

[BLGG] T. Barnet-Lamb, T. Gee, and D. Geraghty, *Serre weights for rank two unitary groups*, to appear in Math Ann.

[BM02] C. Breuil and A. Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J., **115(2)**, (2002), 205–310. With an appendix by Guy Henniart.

[BM12] C. Breuil and A. Mézard, *Multiplicités modulaires raffinées*, Bull. Soc. Math. France **142**, (2014), 127–175.

[Bon11] C. Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$, Algebra and Applications, **13**, (2011), Springer-Verlag London Ltd., London.

[BP12] C. Breuil and V. Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc., **216(1016)**:vi+114, (2012).

[Dia07] F. Diamond, *A correspondence between representations of local Galois groups and Lie-type groups, L-functions and Galois representations*, London Math. Soc. Lecture Note Ser. **320** (2007), 187–206. Cambridge Univ. Press, Cambridge.

[Dav13] A. David, *Calculs de multiplicités dans la conjecture de Breuil-Mézard*, preprint, (2013).

[EG11] M. Emerton and T. Gee, *A geometric perspective on the Breuil-Mézard conjecture* 09, (2011).

[GK12] T. Gee and M. Kisin, *The Breuil-Mézard conjecture for potentially Barsotti-Tate representations*, preprint (2012).

[Glo78] D.J. Glover, *A study of certain modular representations*, J. Algebra, **51(2)**, (1978), 425–475.

[Kis08] M. Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc., **21(2)**, (2008), 513–546.

[Kis09] M. Kisin, *The Fontaine-Mazur conjecture for* GL_2, J. Amer. Math. Soc., **22(3)**, (2009), 641–690.

[Kis10] M. Kisin, *The structure of potentially semi-stable deformation rings*, Proceedings of the International Congress of Mathematicians, New Delhi, Hindustan Book Agency **II**, (2010), 294–311.

[Paš12] V. Paškūnas, *On the Breuil-Mézard conjecture*, preprint (2012).

[Sch08] M.M. Schein, *Weights in Serre's conjecture for Hilbert modular forms, the ramified case*, Israel J. Math., **166**, (2008), 369–391.

Sandra Rozensztajn
UMPA, ENS de Lyon
UMR 5669 du CNRS
46, allée d'Italie
69364 Lyon Cedex 07
France
E-mail: `sandra.rozensztajn@ens-lyon.fr`