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Résumé. Nous donnons une présentation de la construction due à S. Sekiguchi et N. Suwa d'une isogénie cyclique de schémas en groupes affines et lisses qui unifie les isogénies de Kummer et d'Artin-Schreier-Witt. Nous effectuons la construction sur un anneau de base arbitraire. Nous étendons les énoncés de certains résultats de manière à en donner une forme adaptée à une recherche future des modèles des schémas en groupes de racines de l'unité.

Abstract. We present an account of the construction by S. Sekiguchi and N. Suwa of a cyclic isogeny of affine smooth group schemes unifying the Kummer and Artin-Schreier-Witt isogenies. We complete the construction over an arbitrary base ring. We extend the statements of some results in a form adapted to a further investigation of the models of the group schemes of roots of unity.

1. Introduction

Given a prime p and an integer $n \geqslant 1$, consider the problem of describing étale cyclic coverings of order p^n of algebras, or schemes. Over a field of characteristic 0, the Kummer isogeny provides such a covering which is universal on local rings. Over a field of characteristic p, an isogeny with the same virtues is given by the Artin-Schreier-Witt theory. In the end of the nineties, T. Sekiguchi and N. Suwa gave the construction of an isogeny of smooth affine n-dimensional group schemes over a discrete valuation ring of mixed characteristic, putting the Kummer isogeny and the Artin-Schreier-Witt isogeny into a continuous family satisfying a certain universality property. This is presented in the papers [SS1] and [SS3] and we give a more detailed overview in Section 2 below.

The present paper is an account of this construction, with emphasis on some features that we found especially interesting. We have three main goals in writing such an account.

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Our first goal is to generalize their theory in such a way that it can handle as many isogeny kernels as possible. The fact is that the formalism developed by Sekiguchi and Suwa in their series of papers in order to construct a unification of the Kummer and Artin-Schreier theories provides us with a machinery to handle many models of the group scheme of p^n-th roots of unity μ_{p^n}. For this, we need to give some complements to the papers [SS1] and [SS3] and make sure that the proofs of the generalized statements work. The result is Theorem 6.2. Also, since the article [SS3] was never published, we wanted to check thoroughly all the details so as to rely safely on it.

Our second goal is to emphasize the geometric nature of the construction. Indeed, the assumption that the base is a discrete valuation ring is almost useless in [SS1] and [SS3]. With suitable formulations, everything works over an (almost) arbitrary $\mathbb{Z}_{(p)}$-algebra, and the result is a parameterization of a nice family of affine smooth group schemes called *filtered group schemes*, containing plenty of models of μ_{p^n}. The parameter space is a countable union of schemes of finite type over $\mathbb{Z}_{(p)}$, as we prove in Theorem 5.1. We show how to formulate things in this geometric, functorial way.

Our third goal is to propose a hopefully pleasant exposition of the theory, with the idea that this tremendous piece of algebra deserves to be better known. We introduce some terminology for important concepts when we think that it may be enlightening (fundamental morphisms, framed group schemes, Kummer subgroup). We focus on key points rather than lengthy calculations. We emphasize the inductive nature of the intricate constructions with an algorithmic presentation. We do not claim that reading our text is a gentle stroll leading without effort to a transparent understanding of the papers [SS1] and [SS3]. Rather, we hope that having a slightly different viewpoint will help the interested reader to immerse into these papers.

Summary of contents. We first present the main lines of the strategy of Sekiguchi and Suwa to construct some affine smooth group schemes embodying the unification of Kummer and Artin-Schreier-Witt theories. (§1.1-1.3). Our aim is to describe as many isogenies as possible between these groups, and to study their kernels (§1.4). We recall the necessary notions on Witt vectors (§2). We define and classify framed formal groups by a universal object (Theorem 3.2.9). We emphasize that the construction by induction is given by an explicit and computable algorithm (§3). Section 4 is devoted to framed group schemes. In order to obtain algebraic objects we have to truncate the previous formal objects carefully. At last, we consider explicit isogenies between framed group schemes and we obtain the condition to define finite flat Kummer group schemes (§5). In Appendix A, we compare Ext groups of sheaves in the small and big flat sites of a scheme.

Notations. The roman and the greek alphabets do not contain enough symbols for Sekiguchi-Suwa theory. Using the same letters for different objects could not always be avoided. We tried our best to choose good notations, but in some places they remain very heavy. In other places, we changed slightly the notations of Sekiguchi and Suwa. We apologize for the inconvenience.

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Contents

2. Overview of Sekiguchi-Suwa theory

2.1. Unifying Kummer and Artin-Schreier-Witt theories. Fix a discrete valuation ring R with fraction field K of characteristic 0 and residue field k of characteristic $p > 0$. Let W_n be the scheme of Witt vectors of length n and $\mathbb{G}_m$ the multiplicative group scheme. The work of Sekiguchi and Suwa provides an explicit construction of an isogeny $\mathcal{W}_n \longrightarrow \mathcal{V}_n$ of smooth affine n-dimensional group schemes over R with special fibre isomorphic to the Artin-Schreier-Witt isogeny

$$\wp : W_{n,k} \longrightarrow W_{n,k} \ , \ x \mapsto x^p - x,$$

and generic fibre isomorphic to the Kummer-type isogeny

$$\Theta : (\mathbb{G}_{m,K})^n \longrightarrow (\mathbb{G}_{m,K})^n \ , \ (x_1, \dots, x_n) \mapsto (x_1^p, x_2^p x_1^{-1}, \dots, x_n^p x_{n-1}^{-1}),$$

such that any p^n-cyclic finite étale extension of local flat R-algebras is obtained by base change from $\mathcal{W}_n \longrightarrow \mathcal{V}_n$. The isogeny Θ is essentially equivalent to the usual one-dimensional isogeny $x \mapsto x^{p^n}$ for the purposes of Kummer theory, and is of course best-suited to the unification with the Artin-Schreier-Witt theory. In the strategy of Sekiguchi and Suwa to complete this goal, let us single out three steps:

(A) Describe a family of smooth n-dimensional group schemes that are good candidates to be the domain and target of the sought-for isogeny (this is done in Sections 3, 4, 5 of [SS3]). These are called *filtered group schemes*.
(B) Choose suitably the parameters in the previous constructions so as to produce a group scheme $\mathcal{W}_n$ (Section 8 of [SS3]) with a finite flat subgroup scheme $(\mathbb{Z}/p^n\mathbb{Z})_R$. This step requires R to contain the p^n-th roots of unity.
(C) Compute the group $\mathcal{V}_n = \mathcal{W}_n/(\mathbb{Z}/p^n\mathbb{Z})$ and the isogeny $\mathcal{W}_n \to \mathcal{V}_n$ (Section 9 of [SS3]).

We will now present these steps in a little more detail.

2.2. Filtered group schemes. Let us briefly describe Step (A), the description of the family of smooth group schemes relevant to the problem. The groups are constructed with two guiding principles: firstly they are models of $(\mathbb{G}_{m,K})^n$, and secondly they are extensions of a group of the same type in dimension one less by a 1-dimensional group, in the same way as $W_{n,k}$ is an extension of $W_{n-1,k}$ by $\mathbb{G}_{a,k}$. For $n = 1$, the smooth models of $\mathbb{G}_{m,K}$ with connected fibres are known as some group schemes $\mathcal{G}^\lambda = \mathrm{Spec}(R[X, 1/(1 + \lambda X)])$, where $\lambda \in R$ is a parameter (see the papers [WW] and [SOS]). Thus we are led to consider *filtered group schemes of type* $(\lambda_1, \dots, \lambda_n)$ for various n-tuples of elements $\lambda_i \in R$, defined recursively as the extensions of a group $\mathcal{E}$ of type $(\lambda_1, \dots, \lambda_{n-1})$ by the group $\mathcal{G}^{\lambda_n}$. We see that in order to obtain the n-dimensional group schemes, we have to describe the group $\mathrm{Ext}^1(\mathcal{E}, \mathcal{G}^\lambda)$ classifying such extensions. This is easy when λ is invertible i.e. $\mathcal{G}^\lambda \simeq \mathbb{G}_{m,R}$, since one can prove easily by dévissage that $\mathrm{Ext}^1(\mathcal{E}, \mathbb{G}_{m,R}) = 0$. Therefore, in order to understand $\mathrm{Ext}^1(\mathcal{E}, \mathcal{G}^\lambda)$ we must measure the difference between $\mathcal{G}^\lambda$ and $\mathbb{G}_{m,R}$. This is done with an exact sequence of sheaves on the small flat site

$$0 \longrightarrow \mathcal{G}^\lambda \longrightarrow \mathbb{G}_{m,R} \xrightarrow{\rho} i_*\mathbb{G}_{m,R/\lambda} \longrightarrow 0$$

where $i : \mathrm{Spec}(R/\lambda R) \longrightarrow \mathrm{Spec}(R)$ is the closed immersion (we make the convention that all sheaves supported on the empty set are 0, e.g. $i_*\mathbb{G}_{m,R/\lambda} = 0$ if λ is invertible). The long exact sequence for the functor $\mathrm{Hom}(\mathcal{E}, \cdot)$ gives

$$\begin{array}{r} \dots \longrightarrow \mathrm{Hom}(\mathcal{E}, i_*\mathbb{G}_{m,R/\lambda}) \longrightarrow \mathrm{Hom}(\mathcal{E}_{R/\lambda}, \mathbb{G}_{m,R/\lambda}) \longrightarrow \mathrm{Ext}^1(\mathcal{E}, \mathcal{G}^\lambda) \\ \downarrow \quad \\ \dots \longleftarrow \mathrm{Ext}^1(\mathcal{E}, \mathbb{G}_m) \end{array}$$

Moreover we observe that we have an adjunction

$$\mathrm{Hom}(\mathcal{E}, i_*\mathbb{G}_{m,R/\lambda}) \simeq \mathrm{Hom}(i^*\mathcal{E}, \mathbb{G}_{m,R/\lambda})$$

in the small site. Since $\mathcal{E}$ is flat of finite presentation, and hence representable in the small fppf site, then $i^*\mathcal{E}$ is representable by the group $\mathcal{E}_{|R/\lambda}$. Therefore:

$$\mathrm{Hom}(\mathcal{E}, i_*\mathbb{G}_{m,R/\lambda}) \simeq \mathrm{Hom}(\mathcal{E}_{R/\lambda}, \mathbb{G}_{m,R/\lambda}).$$

We want to underline a subtle point here. The groups of homomorphisms and the groups of extensions Ext^1 involved are calculated, a priori, in the small fppf site. But, as showed in the Appendix A, it does not change anything if we calculate the same groups in the big fppf site, which is the case we are interested in.

Finally, since $\mathrm{Ext}^1(\mathcal{E}, \mathbb{G}_{m,R}) = 0$ ([SS4, Example 2.7]), then

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{G}^\lambda) \simeq \mathrm{Hom}(\mathcal{E}_{R/\lambda}, \mathbb{G}_{m,R/\lambda})/\rho_* \mathrm{Hom}(\mathcal{E}, \mathbb{G}_{m,R})$$

(and $\mathrm{Hom}(\mathcal{E}_{R/\lambda}, \mathbb{G}_{m,R/\lambda}) = 0$ if λ is invertible, according to our previous convention).

At this point, the problem becomes to describe the group

$$\mathrm{Hom}(\mathcal{E}_{R/\lambda}, \mathbb{G}_{m,R/\lambda}).$$

Technically, this is one of the key points of Sekiguchi and Suwa's work. This group of homomorphisms is parameterized by a suitable generalization of the classical Artin-Hasse exponential series. It is therefore really in the formal world that the crucial objects live, as formal power series satisfying the important identities. Accordingly, the formal theory (the construction of filtered formal groups) precedes, and is the inspiration for, the algebraic theory (the construction of filtered group schemes). Here, it is worth pointing out that the construction of extensions in the formal case takes a slightly different turn, because no analogue of the exact sequence $0 \to \mathcal{G}^\lambda \to \mathbb{G}_m \to i_*\mathbb{G}_m \to 0$ is available. Instead one considers the composition

$$\partial : \mathrm{Hom}(\hat{\mathcal{E}}, \hat{\mathbb{G}}_m) \longrightarrow H_0^2(\hat{\mathcal{E}}, \hat{\mathcal{G}}^\lambda) \longrightarrow \mathrm{Ext}^1(\hat{\mathcal{E}}, \hat{\mathcal{G}}^\lambda)$$

that associates to a morphism a Hochschild 2-cocycle and then the extension it gives rise to. The point is that in the algebraic case, the map ∂ is obtained as the connecting homorphism of a long exact cohomology sequence which is *not* available in the formal case, while in the formal case the map ∂ is obtained using the *surjective* map $H_0^2(\hat{\mathcal{E}}, \hat{\mathcal{G}}^\lambda) \to \mathrm{Ext}^1(\hat{\mathcal{E}}, \hat{\mathcal{G}}^\lambda)$ which tends to be *zero* in the algebraic case.

2.3. Finite flat subgroup schemes. Let us now make some comments on Steps (B) and (C). Filtered group schemes $\mathcal{E}$ have *filtered subgroup schemes*, obtained by successive extensions of subgroups. We will see that their construction provides natural morphisms $\alpha : \mathcal{E} \to (\mathbb{G}_m)^n$ that are *model maps*, that is to say, isomorphisms on the generic fibre. On the generic fibre, these morphisms provide natural filtered subgroup schemes of $\mathcal{E}_K$ isomorphic to $\mu_{p^n,K}$: one just has to pullback via α the kernel of the Kummer isogeny $\Theta_K : (\mathbb{G}_{m,K})^n \to (\mathbb{G}_{m,K})^n$. By taking the closure in $\mathcal{E}$, one produces interesting candidates to be finite flat models of $\mu_{p^n,K}$. If R contains the p^n-roots of unity, and for suitable choices of the parameters of the extensions, one obtains a filtered group scheme $\mathcal{E} = \mathcal{W}_n$ and a model of $\mu_{p^n,K} \simeq (\mathbb{Z}/p^n\mathbb{Z})_K$ which turns out to be the constant group $(\mathbb{Z}/p^n\mathbb{Z})_R$. Sekiguchi and Suwa specialize to this case and study the quotient isogeny. They prove that these objects realize the unification of the Kummer and Artin-Schreier-Witt exact sequences.

2.4. Our presentation of the theory. Our personal interest does not lie in *one* single model of $\mu_{p^n,K}$ but in *all* possible models one can exhibit (see the article [MRT]). It is therefore very important for us to leave the parameters as free as possible. We call *Kummer subschemes* the subschemes

G obtained by scheme-theoretic closure in the way described in 2.3. Then the framework of Sekiguchi and Suwa allows to characterize when a Kummer subscheme is finite locally free over the base ring R. In fact, the 'good' object is the isogeny $\mathcal{E} \to \mathcal{F} = \mathcal{E}/G$ itself, and we are able to construct isogenies between filtered group schemes, whose kernels are the finite flat models of $\mu_{p^n,K}$ we are interested in.

If we incorporate the various choices of parameters into the definitions, we obtain a notion of *framed group scheme* whose moduli problem is (tautologically) representable by a scheme. This scheme is a nice parameter space for filtered group schemes. It has a formal and an algebraic version. We formulate things with this vocabulary.

Finally, we point out that almost no restriction on the base ring R is necessary. In particular, it need not be a discrete valuation ring, not even an integral domain. The only important point is that the parameters λ_i of the successive extensions should be nonzerodivisors. Thus we work throughout with an arbitrary $\mathbb{Z}_{(p)}$-algebra.

3. Witt vectors

The prime number p is fixed. This section is devoted to generalities on the ring scheme of Witt vectors W. We first recall basic notations concerning W and some of its endomorphisms. Then we define the formal completion of W and study its stability under the endomorphisms defined before. Finally we introduce various objects related to the scheme of Witt vectors over the affine line. As a general rule, we keep the notations of the papers [SS3] and [SS1].

3.1. Witt vectors. We briefly indicate our notations for the ring scheme W of Witt vectors over the integers. The letters X, Y, Z, A denote infinite vectors of indeterminates, with $X = (X_0, X_1, \dots)$, etc.

3.1.1. *Ring scheme structure.* The *scheme of Witt vectors* is

$$W = \mathrm{Spec}(\mathbb{Z}[Z_0, Z_1, \dots]).$$

Its structure is defined using the *Witt polynomials* defined for all integers $r \geqslant 0$ by:

$$\Phi_r(Z) = \Phi_r(Z_0, \dots, Z_r) = Z_0^{p^r} + pZ_1^{p^{r-1}} + \dots + p^r Z_r.$$

The addition and multiplication of the Witt ring scheme are defined respectively, on the function ring level, by the assignments $Z_r \mapsto S_r(X, Y)$ and $Z_r \mapsto P_r(X, Y)$, where

$$S_r(X, Y) = S_r(X_0, \dots, X_r, Y_0, \dots, Y_r),$$
$$P_r(X, Y) = P_r(X_0, \dots, X_r, Y_0, \dots, Y_r)$$

are the unique polynomials with integer coefficients satisfying for all $r \geqslant 0$ the identities:

$$\Phi_r(S_0(X,Y),\dots,S_r(X,Y)) = \Phi_r(X_0,\dots,X_r) + \Phi_r(X_0,\dots,Y_r),$$
$$\Phi_r(P_0(X,Y),\dots,P_r(X,Y)) = \Phi_r(X_0,\dots,X_r)\,\Phi_r(X_0,\dots,Y_r).$$

3.1.2. *Frobenius,Verschiebung,Teichmüller, T map.* The ring scheme endomorphism $F : W \to W$ called *Frobenius* is defined by the assignment $Z_r \mapsto F_r(X)$, where the $F_r(X) = F_r(X_0,\dots,X_{r+1})$ are the unique polynomials satisfying for all $r \geqslant 0$ the identities:

$$\Phi_r(F_0(X),\dots,F_r(X)) = \Phi_{r+1}(X_0,\dots,X_{r+1}).$$

The additive group scheme endomorphism

$$V : W = \mathrm{Spec}(\mathbb{Z}[X_0,X_1,\dots]) \to W = \mathrm{Spec}(\mathbb{Z}[Z_0,Z_1,\dots])$$

called *Verschiebung* is defined by the assignments $Z_0 \mapsto 0$ and $Z_r \mapsto X_{r-1}$ for $r \geqslant 1$. Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[X_0])$ be the affine line over $\mathbb{Z}$. Then the multiplicative morphism $[\,\cdot\,] : \mathbb{A}^1 \to W$ called *Teichmüller representative* is defined by the assignments $Z_0 \mapsto X_0$ and $Z_r \mapsto 0$ if $r \geqslant 1$.

An important role in Sekiguchi-Suwa theory is played by the morphism $T : W \times W \to W$ called (by us) the T *map*, defined by the assignment $Z_r \mapsto T_r(Y,X)$, where the $T_r(Y,X)$ are the unique polynomials satisfying for all $r \geqslant 0$ the identities:

$$\begin{aligned}&\Phi_r(T_0(Y,X),\dots,T_r(Y,X))\\ &\qquad = Y_0^{p^r}\Phi_r(X) + pY_1^{p^{r-1}}\Phi_{r-1}(X) + \cdots + p^rY_r\Phi_0(X).\end{aligned}$$

Existence and uniqueness of the sequence

$$T(Y,X) = (T_0(Y,X),T_1(Y,X),\dots)$$

are granted by Bourbaki [B], § 1, no. 2, Prop. 2, applied to the ring $\mathbb{Z}[Y,X]$ endowed with the endomorphism σ raising each variable to the p-th power. Note that in [SS3] the notation for $T(Y,X)$ is T_YX, a notation that we will also use. The morphism T is additive in the second variable i.e. gives rise to a morphism $T : W \to \mathrm{End}(W,+)$.

Some of these definitions are really more pleasant in terms of functors of points. This is typically the case for the morphisms V, T and $[\,\cdot\,]$. Let us indicate them: given a ring A and Witt vectors $a = (a_0,a_1,\dots)$, $x = (x_0,x_1,\dots) \in W(A)$, we have $V(x) = (0,x_0,x_1,\dots)$, $[x_0] = (x_0,0,0,\dots)$ and $T_ax = \sum_{r\geqslant 0} V^r([a_r]x)$, see [SS3], Lemma 4.2.

3.2. Formal completion. The formal completion of the group scheme of Witt vectors along the zero section is the subfunctor $\hat{W} \subset W$ defined by:

$$\hat{W}(A) \stackrel{\mathrm{df}}{=} \left\{ \begin{array}{c} \boldsymbol{a} = (a_0,a_1,a_2,\dots) \in W(A), \\ a_i \text{ nilpotent for all } i,\ a_i = 0 \text{ for } i \gg 0 \end{array} \right\}.$$

This is the completion in Cartier's sense (see [Ca]); note that in infinite dimension, several reasonable definitions of completion exist (a different one may be found for example in [Ya], example 3.24).

Lemma 3.1. *The formal completion $\hat{W}$ is an ideal of W.*

Proof. We introduce a filtration of $\hat{W}$ by subfunctors $\hat{W}_{M,N}$ ($M, N \geqslant 1$ integers) with

$$\hat{W}_{M,N}(A) = \{\boldsymbol{a} \in W(A),\, a_i = 0 \text{ for } i \geqslant M \text{ and } (a_i)^N = 0 \text{ for } i \geqslant 0\}.$$

It is clear that this filtration is exhaustive. Hence it is enough to prove that for all M, N there exist M', N' such that $\hat{W}_{M,N} + \hat{W}_{M,N} \subset \hat{W}_{M',N'}$ and $W \times \hat{W}_{M,N} \subset \hat{W}_{M',N'}$. The proof in the two cases is very similar, so we will treat only the case of the sum.

Step 1: we may assume that p is invertible in the base ring. Indeed, $\hat{W}_{M,N}$ is a closed subfunctor of $\hat{W}$ which is representable by a finite flat $\mathbb{Z}$-scheme. So if the addition map on $\hat{W}_{M,N}$ factors over $\mathbb{Z}[1/p]$ through some $\hat{W}_{M',N'}$, then by taking scheme-theoretic closures one finds that it factors through $\hat{W}_{M',N'}$ over $\mathbb{Z}$ as well.

Step 2: let (X, Y) be the universal point of $W_{M,N} \times W_{M,N}$, where $X = (X_0, X_1, \dots)$ and $Y = (Y_0, Y_1, \dots)$. Then the coefficients of the sum $S = X + Y$ are nilpotent. This is clear, since for each i the coefficient S_i is a polynomial in the X_j, Y_j.

Step 3: for all $i \geqslant r := M - 1 + \log_p(N)$, we have $\Phi_i(S) = 0$. Indeed, we have

$$\Phi_i(X) = \sum_{j=0}^{i} p^j (X_j)^{p^{i-j}} = \sum_{j=0}^{M-1} p^j (X_j)^{p^{i-j}} = 0$$

since $j \leqslant M-1$ implies that $p^{i-j} \geqslant p^{i-M+1} \geqslant p^{r-M+1} \geqslant N$. Similarly we have $\Phi_i(Y) = 0$ and hence $\Phi_i(S) = \Phi_i(X) + \Phi_i(Y) = 0$.

Step 4: by Step 2, let P be such that $(S_0)^P = \cdots = (S_{r-1})^P = 0$. Then $S_i = 0$ for all $i \geqslant \log_p((P-1)(1 + p + \cdots + p^{r-1}))$. For the weight w such that $w(X_i) = w(Y_i) = p^i$, the element S_i is homogeneous of weight p^i. Since p is invertible, using Step 3 and induction we see that for all $i \geqslant r$, the element S_i is a polynomial in $S_0, \dots, S_{r-1}$. By the choice of P, a monomial $(S_0)^{j_0} \dots (S_{r-1})^{j_{r-1}}$ will be nonzero only if all exponents $j_0, \dots, j_{r-1}$ are less than $P-1$, hence the weight is $j_0 + pj_1 + \cdots + p^{r-1} j_{r-1} \leqslant (P-1)(1 + p + \cdots + p^{r-1})$. We get the claim by contraposition.

Step 5: conclusion. By Step 4, we can take $M' = \log_p((P-1)(1 + p + \cdots + p^{r-1}))$ and the existence of N' is given by Step 2. $\square$

Remark 3.1. In Sections 5 and 6, we try to give a presentation of Sekiguchi-Suwa theory adapted to computations. In particular, in Lemma 5.1 we give an explicit degree of truncation for the Artin-Hasse exponentials that is sufficient to compute filtered group schemes. It is equally desirable to have explicit bounds for the number of nonzero terms of the Witt vectors that appear, but this desire is in fact limited by the difficulty to give a reasonably explicit bound for the number of nonzero coefficients of the *sum* of two Witt vectors, as we saw in the proof of Lemma 3.1.

Lemma 3.3. *The formal completion $\hat{W}$ is stable under F and V.*

Proof. For V there is nothing to say, and for F the strategy of the proof of Lemma 3.1 works almost unchanged. ☐

Lemma 3.4. *Let W^f be the subfunctor of W composed of Witt vectors with finitely many nonzero coefficients. Then T induces a morphism $W^f \times \hat{W} \to \hat{W}$.*

We point out that W^f has no (additive or whatever) structure.

Proof. Using the formulas $T_a x = \sum_{r \geqslant 0} V^r([a_r]x)$ and

$$[a]x = (ax_0, a^p x_1, a^{p^2} x_2, \dots),$$

this is obvious. ☐

3.3. Witt vectors over the affine line. Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[\Lambda])$ be the affine line over the integers, and let $i : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{A}^1$ be the closed immersion of the origin, given by $\Lambda = 0$. In the paper [SOS], the study of the multiplicative group scheme over the affine line leads to introduce a certain group scheme $\mathcal{G}^\Lambda$ (the notation in *loc. cit.* is $\mathcal{G}^{(\lambda)}$). In this section, following Sekiguchi and Suwa (see especially [SS2], sections 1.15, 1.21 and 4.4), we expand the idea behind the introduction of this group scheme, because when we consider a group scheme over $\mathbb{A}^1$ (favourite examples are $\mathbb{G}_m$ or W), the groups of elements *vanishing at the origin* and those *supported at the origin* are especially important. In this way, we introduce a W-module scheme W^Λ. We recall the definition of $\mathcal{G}^\Lambda$ which fits in the same framework. Note that we simplify the notations $F^{(\Lambda)}$, $\mathcal{G}^{(\Lambda)}$, $\alpha^{(\Lambda)}$, $W^{(\Lambda)}$ from the papers [SOS], [SS2], [SS3] to F^Λ, $\mathcal{G}^\Lambda$, α^Λ, W^Λ.

In the proposition below and throughout the paper, what we call the *small flat site* X_{fl} of a scheme X is the category of flat locally finitely presented X-schemes endowed with the topology generated by the families $\{U_i \to U\}_{i \in I}$ such that $\amalg U_i \to U$ is faithfully flat and locally finitely presented. A discussion of the comparison with the big flat site can be found in Appendix A.

Proposition 3.1. *Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[\Lambda])$ be the affine line over the integers, and let $i : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{A}^1$ be the closed immersion given by $\Lambda = 0$. Let $\mathbb{A}^1_{\mathrm{fl}}$ denote the small flat site of $\mathbb{A}^1$.*

(1) *The canonical morphism $\mathbb{G}_m \to i_*\mathbb{G}_m$ fits into an exact sequence*

$$0 \longrightarrow \mathcal{G}^\Lambda \xrightarrow{\alpha^\Lambda} \mathbb{G}_m \longrightarrow i_*\mathbb{G}_m \longrightarrow 0$$

of abelian sheaves on $\mathbb{A}^1_{\mathrm{fl}}$, where $\mathcal{G}^\Lambda$ is a flat commutative group scheme.
(2) *The canonical morphism $W \to i_*W$ fits into an exact sequence*

$$0 \longrightarrow W^\Lambda \xrightarrow{\Lambda} W \longrightarrow i_*W \longrightarrow 0$$

of abelian sheaves on $\mathbb{A}^1_{\mathrm{fl}}$, where W^Λ is a flat W-module scheme. Here, the scheme W^Λ has the same underlying scheme as W and the first map is

$$x = (x_0, x_1, x_2, \dots) \mapsto \Lambda.x := (\Lambda x_0, \Lambda x_1, \Lambda x_2, \dots).$$

An algebra R and an element $\lambda \in R$ define an R-point $\mathrm{Spec}(R) \to \mathbb{A}^1$. The pullbacks of α^Λ and $\Lambda : W^\Lambda \to W$ along this point give a morphism of R-group schemes which we will denote $\alpha^\lambda : \mathcal{G}^\lambda \to \mathbb{G}_m$ and a morphism of R-schemes in W-modules which we will denote $\lambda : W^\lambda \to W$.

Proof. We treat only case (2), since case (1) is similar and even simpler. The scheme W^Λ and the map $\Lambda : W^\Lambda \to W$ are defined in the statement. These fit into an exact sequence, functorial in the flat $\mathbb{Z}[\Lambda]$-algebra R:

$$0 \longrightarrow W^\Lambda(R) \xrightarrow{\Lambda} W(R) \longrightarrow (i_*W)(R) = W(R/\Lambda R) \longrightarrow 0.$$

Thus the map Λ identifies $W^\Lambda(R)$ with the ideal of $W(R)$ of vectors all whose components are multiples of Λ. It follows that for all $u, v \in W^\Lambda(R)$ and $a \in W(R)$, the sum $u + v$ and the product au, computed in $W(R)$, again lie in this ideal. By taking for R the function ring of W^Λ, we see that the universal polynomials giving Witt vector addition and multiplication

$$S_0(\Lambda.u, \Lambda.v), S_1(\Lambda.u, \Lambda.v), S_2(\Lambda.u, \Lambda.v), \dots$$
$$P_0(\Lambda.a, \Lambda.v), P_1(\Lambda.a, \Lambda.v), P_2(\Lambda.a, \Lambda.v), \dots$$

are divisible by Λ, that is $S_i(\Lambda.u, \Lambda.v) = \Lambda S_i'(u, v)$ and $P_i(\Lambda.a, \Lambda.u) = \Lambda P_i'(a, u)$. By flatness, the polynomials S_i' and P_i' are uniquely determined and they define the W-module structure on the scheme W^Λ. $\square$

Remark 3.2. We could also define W^Λ and $\mathcal{G}^\Lambda$ as dilatations of W and $\mathbb{G}_m$ along the respective unit sections of the special fibre $\Lambda = 0$. When the base ring is a discrete valuation ring R, the dilatation of an R-scheme X along a closed subscheme of the special fibre is defined in Chapter 3 of [BLR]. The same construction works in the following more general setting. Consider a base scheme S, a Cartier divisor $S_0 = V(\mathcal{I})$, an S-scheme X, and a closed subscheme Y_0 of $X_0 = X \times_S S_0$. Then there exists a morphism of S-schemes $u : X' \to X$ where X' is an S-scheme without $\mathcal{I}$-torsion such that $u(X_0') \subset Y_0$, and which is universal with these properties. The scheme X' is called the *dilatation of X along Y_0*.

We close the section with a lemma that plays a key role in the development of the theory.

Lemma 3.7. *Let W be the ring scheme of Witt vectors over the affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[\Lambda])$. Then, the additive endomorphism $F^\Lambda := F - [\Lambda^{p-1}] : W \to W$ is faithfully flat.*

Of course, here again, for an algebra R and an element $\lambda \in R$ we obtain a faithfully flat endomorphism $F^\lambda : W_R \to W_R$.

Proof. See [SS1], Prop. 1.6 and Cor. 1.7-1.8, and [SS3], Lemma 4.5. □

4. Formal theory

In Subsection 4.1, we introduce the deformed Artin-Hasse exponentials studied by Sekiguchi and Suwa. These power series satisfy important identities that allow to construct formal filtered group schemes by successive extensions. This is explained in 4.2, with Theorem 4.2 summarizing the main properties of the construction.

4.1. Deformed Artin-Hasse exponentials. In order to describe the homomorphisms from formal filtered group schemes (introduced in Subsection 5.3) to the formal multiplicative group $\hat{\mathbb{G}}_m$, we will need some deformations of Artin-Hasse exponentials. For simplicity, we will call them *deformed exponentials.* In the non-formal case, we will also need some truncations of these series. We introduce all these objects here.

Given indeterminates Λ, U and T, we define a formal power series in T with coefficients in $\mathbb{Q}[\Lambda, U]$ by

$$E_p(U,\Lambda,T) = (1+\Lambda T)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} (1+\Lambda^{p^k}T^{p^k})^{\frac{1}{p^k}\left(\left(\frac{U}{\Lambda}\right)^{p^k} - \left(\frac{U}{\Lambda}\right)^{p^{k-1}}\right)}.$$

It satisfies basic properties such as $E_p(0,\Lambda,T) = 1$ and $E_p(MU, M\Lambda, T) = E_p(U,\Lambda,MT)$, where M is another indeterminate. It is a deformation of the classical Artin-Hasse exponential $E_p(T) = \prod_{k=0}^{\infty} \exp(T^{p^k}/p^k)$ in the sense that $E_p(1,0,T) = E_p(T)$. Given a vector of indeterminates

$$\mathbb{U} = (U_0, U_1, \dots),$$

we define a power series in T with coefficients in $\mathbb{Q}[\Lambda, U_0, U_1, \dots]$ by

$$E_p(\mathbb{U},\Lambda,T) = \prod_{\ell=0}^{\infty} E_p(U_\ell, \Lambda^{p^\ell}, T^{p^\ell}). \tag{4.1}$$

It is proven in [SS1], Cor. 2.5 that the series $E_p(U,\Lambda,T)$ and $E_p(\mathbb{U},\Lambda,T)$ are integral at p, that is, they have their coefficients in $\mathbb{Z}_{(p)}[\Lambda, U]$ and $\mathbb{Z}_{(p)}[\Lambda, U_0, U_1, \dots]$ respectively. It follows that given a $\mathbb{Z}_{(p)}$-algebra A, elements $\lambda, a \in A$ and $\boldsymbol{a} = (a_0, a_1, \dots) \in A^{\mathbb{N}}$, we have specializations

$E_p(a,\lambda,T)$ and $E_p(\boldsymbol{a},\lambda,T)$ which are power series in T with coefficients in A. We usually consider $\boldsymbol{a}$ as a Witt vector, i.e. as an element in $W(A)$. One must however be aware that since $W(A)$ has the extra structure of a ring, this introduces the slight ambiguity that $E_p(\boldsymbol{a},\lambda,T)$ might be interpreted as the result of specializing U to $\boldsymbol{a}$ in the series $E_p(U,\Lambda,T)$, resulting in a series with coefficients in $W(A)$ (note that if A is a $\mathbb{Z}_{(p)}$-algebra then so is $W(A)$). However, in Sekiguchi-Suwa theory the symbol $E_p(\boldsymbol{a},\lambda,T)$ always denotes a specialization of $E_p(\mathbb{U},\Lambda,T)$ so that no confusion can come up.

Now we borrow some terminology from Fourier analysis.

Definition. Let A be a $\mathbb{Z}_{(p)}$-algebra, $\lambda\in A$ an element and $k\geqslant 1$ a prime-to-p integer. A series of the form $E_p(\boldsymbol{a},\lambda,T^k)$ is called a *k-th harmonic* and a 1-st harmonic is also called a *fundamental*. A morphism $\hat{\mathcal{G}}^\lambda\to\hat{\mathbb{G}}_m$ defined by a fundamental is called a *fundamental morphism*.

The significance of this terminology is explained by the following easy lemma, which is stated as Remark 2.10 in [SS1] without proof. We give a proof for the reader's convenience.

Lemma 4.2. *Let A be a $\mathbb{Z}_{(p)}$-algebra and $\lambda\in A$. Then every formal power series $G\in A[[T]]$ such that $G(0)=1$ may be decomposed uniquely as a product of harmonics. More precisely, there exist unique vectors*

$$\boldsymbol{a}_k=(a_{k0},a_{k1},\dots)\in W(A)$$

for all prime-to-p integers k, such that $G(T)=\prod_{p\nmid k}E_p(\boldsymbol{a}_k,\lambda,T^k)$.

Proof. The claim will follow simply from the fact that $E_p(U,\Lambda,T)\equiv 1+UT \mod T^2$. Write $G(T)=1+g_1T+g_2T^2+\dots$ and let $v:\mathbb{N}\setminus\{0\}\to\mathbb{N}$ be the p-adic valuation. We prove by induction on $n\geqslant 1$ that there exist unique elements $b_1,\dots,b_n$ in A such that

$$G(T)E_p(b_1,\lambda^{p^{v(1)}},T)^{-1}E_p(b_2,\lambda^{p^{v(2)}},T^2)^{-1}\dots E_p(b_n,\lambda^{p^{v(n)}},T^n)^{-1} \equiv 1 \mod T^{n+1}.$$

For $n=1$ we have $G(T)\equiv 1+g_1T \mod T^2$ and then it is necessary and sufficient to put $b_1=g_1$. If the claim is proven for $n\geqslant 1$, then we have

$$G(T)\prod_{i=1}^{n}E_p(b_i,\lambda^{v(i)},T^i)^{-1}\equiv 1+c_{n+1}T^{n+1} \mod T^{n+2}$$

for some $c_{n+1}\in A$, and it is necessary and sufficient to put $b_{n+1}=c_{n+1}$. Finally we obtain

$$G(T)=\prod_{i=1}^{\infty}E_p(b_i,\lambda^{p^{v(i)}},T^i)$$

and the claim follows by defining $\boldsymbol{a}_k:=(b_k,b_{kp},b_{kp^2},\dots)$. $\square$

Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}_{(p)}[\Lambda])$ be the affine line over the p-integers. We finally remark that, generalizing what happens for the classical Artin-Hasse exponential (see [SS1], Corollary 2.9.1), the exponential $E_p(\mathbb{U}, \Lambda, T)$ gives a homomorphism

$$W_{\mathbb{A}^1} \longrightarrow \mathbf{\Lambda}_{\mathbb{A}^1},$$

where $\mathbf{\Lambda}_{\mathbb{A}^1} = \mathrm{Spec}(\mathbb{Z}_{(p)}[\Lambda, X_1, \dots, X_n, \dots])$ is the $\mathbb{A}^1$-group scheme whose group of R-points, for any $\mathbb{Z}_{(p)}[\Lambda]$-algebra R, is the abelian multiplicative group $1+TR[[T]]$. The above homomorphism is in fact a closed immersion, and by the above lemma there is an isomorphism

$$\prod_{p\nmid k} W_{\mathbb{A}^1} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathbb{A}^1}.$$

4.2. Construction of framed formal groups. Let R be a $\mathbb{Z}_{(p)}$-algebra and let $\lambda_1, \lambda_2, \dots$ be elements of R.

Definition. A *filtered formal R-group of type* $(\lambda_1, \dots, \lambda_n)$ is a sequence

$$\hat{\mathcal{E}}_0 = 0, \hat{\mathcal{E}}_1, \dots, \hat{\mathcal{E}}_n$$

of affine smooth commutative formal group schemes such that for each $i = 1, \dots, n$ the formal group $\hat{\mathcal{E}}_i$ is an extension of $\hat{\mathcal{E}}_{i-1}$ by $\hat{\mathcal{G}}^{\lambda_i}$.

We now indicate a procedure due to Sekiguchi and Suwa for constructing filtered formal groups. It works under the following:

Assumption 4.1. The elements $\lambda_1, \lambda_2, \dots$ are not zero divisors in R.

The procedure involves some choices which we take into account by introducing notions of *frames* and *framed formal groups*. In this way, the refined procedure becomes universal. We adapt the construction of [SS3] accordingly.

Let W be the R-group scheme of infinite Witt vectors. For each $\lambda \in R$, we have the morphisms of R-group schemes $\alpha^\lambda : \mathcal{G}^\lambda \to \mathbb{G}_m$ and $\lambda : W^\lambda \to W$ introduced in Subsection 3.3. For each integer $n \geqslant 1$, we have a product morphism $\lambda \times \cdots \times \lambda : (W^\lambda)^n \to W^n$ which by abuse we again denote by the symbol λ.

4.2.1. ***Description of the procedure.*** Before we define all the objects more precisely, it may help the reader to have a loose description of the construction. We will define by induction a sequence of quadruples

$$(\boldsymbol{e}^n, D_{n-1}, \hat{\mathcal{E}}_n, U^n)$$

for $n \geqslant 1$, where:

- $\boldsymbol{e}^n = (\boldsymbol{a}^n, \boldsymbol{b}^n)$ is a *frame*, that is, a point of a certain fibred product Fr_{n-1}, a closed subscheme of $W^{n-1} \times (W^\lambda)^{n-1}$. Frames are the parameters of the construction, to be chosen at each step.

- $D_{n-1} : \hat{\mathcal{E}}_{n-1} \to \hat{\mathbb{G}}_m$ is a morphism of formal R-schemes which mod λ_n induces a morphism of formal $(R/\lambda_n R)$-groups.
- $\hat{\mathcal{E}}_n$ is a commutative formal group extension of $\hat{\mathcal{E}}_{n-1}$ by $\hat{\mathcal{G}}^{\lambda_n}$ such that the map $\alpha_{\hat{\mathcal{E}}_n} : \hat{\mathcal{E}}_n \to (\hat{\mathbb{G}}_m)^n$ defined on the points by

  $$(x_1, \dots, x_n) \mapsto (D_0 + \lambda_1 x_1, D_1 + \lambda_2 x_2, \dots, D_{n-1} + \lambda_n x_n)$$

  is a morphism of formal groups, where $D_i = D_i(x_1, \dots, x_i)$ for the natural coordinates $x_1, \dots, x_i$ on $\hat{\mathcal{E}}_i$.
- $U^n : W^n \to W^n$ is a morphism of R-group schemes.

4.2.2. ***Initialization.*** The induction is initialized at $n = 1$. Let $W^0 = 0$ and $\hat{\mathcal{E}}_0 = 0$. We set $\boldsymbol{e}^1 = (0, 0)$, $D_0 : \hat{\mathcal{E}}_0 \to \hat{\mathbb{G}}_m$ equal to 1, $\hat{\mathcal{E}}_1 = \hat{\mathcal{G}}^{\lambda_1}$ and $U^1 = F^{\lambda_1} : W \to W$ (see Lemma 3.7).

4.2.3. ***Induction.*** For the inductive step of the construction, we assume that $(\boldsymbol{e}^i, D_{i-1}, \hat{\mathcal{E}}_i, U^i)$ has been constructed for $1 \leqslant i \leqslant n$ and we explain how to produce $(\boldsymbol{e}^{n+1}, D_n, \hat{\mathcal{E}}_{n+1}, U^{n+1})$. For this, we introduce frames. Let $\lambda \in R$ be a nonzerodivisor and consider the morphism

$$U^n - \lambda : W^n \times (W^\lambda)^n \to W^n$$

taking an element $(\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1}) \in W^n \times (W^\lambda)^n$ to $U^n(\boldsymbol{a}^{n+1}) - \lambda.\boldsymbol{b}^{n+1}$.

Definition. A *λ-frame (relative to $\mathcal{E}_n$)* is an R-point $\boldsymbol{e}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1})$ of the kernel of $U^n - \lambda$. The *scheme of frames of dimension n* is $\mathrm{Fr}_n = \ker(U^n - \lambda)$.

Now the induction goes in four steps A-B-C-D.

A. Choose a λ_{n+1}-frame $\boldsymbol{e}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1}) \in \mathrm{Fr}_n(R)$.

B. The main input of Sekiguchi-Suwa theory lies in the definition and properties of D_n. Let A be an R-algebra. Let us extend the terminology of Definition 4.1 by calling a morphism of formal A-schemes $\hat{\mathcal{E}}_{n,A} \to \hat{\mathbb{G}}_{m,A}$ *fundamental* if it is a product of Artin-Hasse exponentials

$$E_p(\boldsymbol{a}_1^{n+1}, \lambda_1, X_1/D_0)\, E_p(\boldsymbol{a}_2^{n+1}, \lambda_2, X_2/D_1) \dots E_p(\boldsymbol{a}_n^{n+1}, \lambda_n, X_n/D_{n-1})$$

for some n-tuple of Witt vectors $\boldsymbol{a}^{n+1} = (\boldsymbol{a}_1^{n+1}, \dots, \boldsymbol{a}_n^{n+1}) \in W(A)^n$. Then, we have:

Theorem 4.1. *Denote by* FS/R *the category of formal R-schemes and by* FG/R *the category of formal R-groups. Then with the above notation we have:*

(1) *The deformed Artin-Hasse exponentials define a monomorphism of R-group functors*

$$\mathrm{Fund} : W^n \longrightarrow \mathrm{Hom}_{\mathrm{FS}/R}(\hat{\mathcal{E}}_n, \hat{\mathbb{G}}_m)$$

taking an n-tuple of Witt vectors $\boldsymbol{a}^{n+1} = (\boldsymbol{a}_1^{n+1}, \dots, \boldsymbol{a}_n^{n+1}) \in W(A)^n$ to the corresponding fundamental morphism $\prod_{i=1}^n E_p(\boldsymbol{a}_i^{n+1}, \lambda_i, X_i/D_{i-1})$. Here, the group law on the target is induced by the group law of $\hat{\mathbb{G}}_m$.
(2) *The map* Fund *induces an isomorphism of R-group functors*

$$\ker(U^n : W^n \to W^n) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{FG}/R}(\hat{\mathcal{E}}_n, \hat{\mathbb{G}}_m).$$

In particular, any morphism of formal R-groups $\hat{\mathcal{E}}_n \to \hat{\mathbb{G}}_m$ is fundamental.

Proof. Point (1) is [SS1], Corollary 2.9 and point (2) is [SS3], Theorem 5.1. □

It follows from the definition of a frame and from point (2) of the theorem that if we take for $\boldsymbol{a}^{n+1}$ the first component of the frame $\boldsymbol{e}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1})$ chosen in Step A, then $\boldsymbol{a}^{n+1}$ lies in the kernel of U^n modulo λ_{n+1} and the fundamental morphism of formal R-schemes

$$D_n = \prod_{i=1}^n E_p(\boldsymbol{a}_i^{n+1}, \lambda_i, X_i/D_{i-1})$$

induces modulo λ_{n+1} a morphism of formal $(R/\lambda_{n+1}R)$-groups.

C. We now build $\hat{\mathcal{E}}_{n+1}$. Since D_n gives a morphism of formal $(R/\lambda_{n+1}R)$-groups, then the expression $D_n(X)D_n(Y)D_n(X \star Y)^{-1} - 1$ vanishes mod λ_{n+1}, where $X \star Y$ denotes the group law in $\hat{\mathcal{E}}_n$. Since λ_{n+1} is a nonzerodivisor, this implies that

$$H_n(X, Y) = \frac{1}{\lambda_{n+1}} \left(\frac{D_n(X)D_n(Y)}{D_n(X \star Y)} - 1 \right)$$

is well-defined. It is a symmetric 2-cocycle $\hat{\mathcal{E}}_n \times \hat{\mathcal{E}}_n \to \hat{\mathcal{G}}^\lambda$ i.e. an element of the Hochschild cohomology group $\mathrm{H}_0^2(\hat{\mathcal{E}}_n, \hat{\mathcal{G}}^\lambda)$ of symmetric 2-cocycles. From a 2-cocycle we can construct an extension of $\hat{\mathcal{E}}_n$ by $\hat{\mathcal{G}}^\lambda$ in the usual way: this is $\hat{\mathcal{E}}_{n+1}$.

D. Define $U^{n+1} : W^{n+1} \to W^{n+1}$ by the matrix

$$U^{n+1} = \begin{pmatrix} & & & -T_{\boldsymbol{b}_1^{n+1}} \\ & U^n & & \vdots \\ & & & -T_{\boldsymbol{b}_n^{n+1}} \\ 0 & \dots & 0 & F^{\lambda_{n+1}} \end{pmatrix}.$$

With the following definition and theorem, we point out that this construction is universal:

Definition. A *framed formal R-group of type* $(\lambda_1, \dots, \lambda_n)$ is a sequence

$$\hat{\mathcal{E}}_0 = 0, (\hat{\mathcal{E}}_1, \boldsymbol{e}^1), \dots, (\hat{\mathcal{E}}_n, \boldsymbol{e}^n)$$

of pairs composed of an affine smooth commutative formal group scheme and a frame, such that for each $i = 1, \dots, n$ the formal group scheme $\hat{\mathcal{E}}_i$ is the extension of $\hat{\mathcal{E}}_{i-1}$ by $\hat{\mathcal{G}}^{\lambda_i}$ determined by the λ_i-frame $\boldsymbol{e}^i$. We often write $\hat{\mathcal{E}}_n$ as a shortcut for this data.

Theorem 4.2. *Let $\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}_{(p)}[\Lambda_1, \dots, \Lambda_n])$ be affine n-space over $\mathbb{Z}_{(p)}$. Then there exists an affine flat $\mathbb{A}^n$-scheme $\mathscr{S}_n = \mathrm{Spec}(\mathscr{R}_n)$ and a framed formal $\mathscr{R}_n$-group $\hat{\mathscr{E}}_n$ of type $(\Lambda_1, \dots, \Lambda_n)$ with the following universal property : for any $\mathbb{Z}_{(p)}$-algebra R, any nonzerodivisors $\lambda_1, \dots, \lambda_n \in R$ and any framed formal R-group $\hat{\mathcal{E}}_n$ of type $(\lambda_1, \dots, \lambda_n)$, there exists a unique map $\mathscr{R}_n \to R$ taking Λ_i to λ_i such that $\hat{\mathcal{E}}_n \simeq \hat{\mathscr{E}}_n \otimes_{\mathscr{R}_n} R$.*

Proof. The proof is almost tautological, because framed formal groups are more or less by construction pullback of a universal one. Let us however sketch it. What we have to do is to carry out the induction as before, in a universal way. Let $W^0 = 0$ and $\hat{\mathcal{E}}_0 = 0$.

For $n = 1$ we put $\mathscr{R}_1 = \mathbb{Z}_{(p)}[\Lambda_1]$, $\boldsymbol{e}^1 = (0,0)$, $D_0 = 1$, $\hat{\mathscr{E}}_1 = \hat{\mathcal{G}}^{\Lambda_1}$ and $U^1 = F^{\Lambda_1} : W \to W$.

Once $\mathscr{S}_i$, $\boldsymbol{e}^i$, D_{i-1}, $\hat{\mathscr{E}}_i$ and U^i have been constructed for $1 \leqslant i \leqslant n$, we find $\mathscr{S}_{n+1}$, $\boldsymbol{e}^{n+1}$, D_n, $\hat{\mathscr{E}}_{n+1}$ and U^{n+1} as follows. We take as a base ring the ring $R' := \mathscr{R}_n \otimes \mathbb{Z}_{(p)}[\Lambda_{n+1}]$. We define $\mathscr{S}_{n+1}$ as the scheme of frames $\mathrm{Fr}_n = \ker(U^n - \Lambda_{n+1})$, and we set $\boldsymbol{e}^{n+1}$ equal to the universal point of $\mathscr{S}^{n+1}$. Note that since U^n is given by a triangular matrix whose diagonal entries are flat morphisms by Lemma 3.7, it follows immediately that it is a flat morphism. By the definition of $\mathscr{S}_{n+1}$ as the fibred product

$$\begin{tikzcd}
\mathscr{S}_{n+1} \arrow[r] \arrow[d] & (W^{\Lambda_{n+1}})^n \arrow[d, "\Lambda_{n+1}"] \\
W^n \arrow[r, "U^n"] & W^n
\end{tikzcd}$$

we see that it is flat over $(W^{\Lambda_{n+1}})^n$, hence flat over $\mathbb{A}^{n+1}$. It follows that Λ_{n+1} is not a zerodivisor in the function ring $\mathscr{R}_{n+1}$ of $\mathscr{S}_{n+1}$. Now the coefficient $\boldsymbol{a}^{n+1}$ of the frame $\boldsymbol{e}^{n+1}$ determines a fundamental morphism

$$D_n = \prod_{i=1}^{n} E_p(\boldsymbol{a}_i^{n+1}, \Lambda_i, X_i / D_{i-1}),$$

a 2-cocycle

$$H_n(X,Y) = \frac{1}{\Lambda_{n+1}} \left(\frac{D_n(X) D_n(Y)}{D_n(X \star Y)} - 1 \right)$$

and then an extension $\hat{\mathscr{E}}_{n+1}$ in the same way as before. The coefficient $\boldsymbol{b}^{n+1}$ of the frame determines a matrix U^{n+1} by the same formula as in Step D of the induction. Once the construction is over, the verification of the universal property is immediate. $\square$

5. Algebraic theory

In this section, we show how to adapt the formal constructions in order to provide (algebraic) filtered group schemes. This is done by truncating the power series and the Witt vector coefficient in a suitable way. We give some preliminaries on truncations in Subsections 5.1 and 5.2. Then we proceed to construct filtered group schemes in 5.3, with Theorem 5.1 as the final point.

5.1. Truncation of deformed Artin-Hasse exponentials. In order to produce non-formal group schemes, we will need the deformed exponentials to be polynomials. We can achieve this either by letting enough coefficients specialize to nilpotent elements, or by truncating. We know from [SS3], Prop. 2.11 that if $\Lambda, U_0, U_1, \dots$ specialize to nilpotent elements, only finitely many of them nonzero, then $E_p(\mathbb{U}, \Lambda, T)$ specializes to a polynomial. In the following lemma, we give an exact bound for the degree of this polynomial, in terms of bounds on the number of nonzero coefficients and the nilpotency indices.

Lemma 5.1. *Let $L, M, N \geqslant 1$ be integers. Then if we reduce the coefficients of the deformed exponential $E_p(\mathbb{U}, \Lambda, T)$ modulo the ideal generated by*

$$\Lambda^L, (U_0)^N, (U_1)^N, \dots, (U_{M-1})^N, U_M, U_{M+1}, \dots$$

then the series $E_p(\mathbb{U}, \Lambda, T)$ is a polynomial in T of degree at most

$$(N-1)\frac{p^M-1}{p-1} + (L-1).$$

Proof. For each ℓ, we have $E_p(U_\ell, \Lambda^{p^\ell}, T^{p^\ell}) = E_p(U_\ell T^{p^\ell}, \Lambda^{p^\ell} T^{p^\ell}, 1)$. It follows that the latter series is a sum of monomials of the form

$$(U_\ell T^{p^\ell})^i (\Lambda^{p^\ell} T^{p^\ell})^j$$

for varying i, j. Now let us take images in the indicated quotient ring. There, for all $\ell \geqslant M$ we have $U_\ell = 0$ and $E_p(U_\ell, \Lambda^{p^\ell}, T^{p^\ell}) = 1$. It follows that only the first M factors show up in the product defining $E_p(\mathbb{U}, \Lambda, T)$. A typical monomial in this series is obtained by picking a monomial of index i_ℓ, j_ℓ in each factor; the result is the product of

$$(U_0)^{i_0} (U_1)^{i_1} \dots (U_{M-1})^{i_{M-1}} \times T^{i_0 + i_1 p + \dots + i_{M-1} p^{M-1}}$$

by

$$\Lambda^{j_0 + j_1 p + \dots + j_{M-1} p^{M-1}} \times T^{j_0 + j_1 p + \dots + j_{M-1} p^{M-1}}.$$

For this to be nonzero, we must have $i_\ell \leqslant N-1$ for each ℓ and

$$j_0 + j_1 p + \cdots + j_{M-1}p^{M-1} \leqslant L-1$$

for each $(j_0, \ldots, j_{M-1})$. Thereby the T-degree of the monomial is less than

$$(N-1)(1+p+\cdots+p^{M-1}) + (L-1),$$

which is what the lemma claims. $\square$

Definition. Let $L, M, N \geqslant 1$ be integers and let $\tau_{L,M,N}$ be the truncation map of power series in degrees $\geqslant (N-1)\frac{p^M-1}{p-1} + (L-1) + 1$. Then the polynomial

$$E_p^{L,M,N}(\mathbb{U}, \Lambda, T) \overset{\mathrm{df}}{=} \tau_{L,M,N} E_p(\mathbb{U}, \Lambda, T) \in \mathbb{Z}_{(p)}[\Lambda, U_0, U_1, \ldots][T]$$

is called the *truncated (deformed) exponential of level* (L, M, N).

5.2. Truncation of Witt vectors. We will make big use of the functor $\hat{W}$ and its pushforward $i_*\hat{W}$ by the closed immersion $i : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[\Lambda])$. Since $\hat{W}$ is naturally filtered, this leads to consider various truncations of W and $\hat{W}$, over $\mathrm{Spec}(\mathbb{Z})$ and over $\mathbb{A}^1$. In order to define them, we fix integers $M, N \geqslant 1$.

5.2.1. ***Truncation by the length.***
(1) W_M is the $\mathbb{Z}$-subfunctor of W defined by $W_M(A) = \{a \in W(A),\, a_i = 0 \text{ for } i \geqslant M\}$. We emphasize that it is of course *not* a subgroup functor; it should not be confused with the quotient ring of Witt vectors of length N, which will not appear in the present paper.
(2) W_M^Λ is the $\mathbb{A}^1$-subfunctor of W^Λ defined by

$$W_M^\Lambda(A) = \{a \in W^\Lambda(A),\, a_i = 0 \text{ for } i \geqslant M\}.$$

(3) $\hat{W}_M = \hat{W} \cap W_M$ is a $\mathbb{Z}$-subfunctor of $\hat{W}$.

5.2.2. ***Truncation by the nilpotency index.***
(4) $W_{M,N,\Lambda} \subset W_M$ is the $\mathbb{A}^1$-subfunctor defined by

$$W_{M,N,\Lambda}(A) = \{\boldsymbol{a} \in W_M(A),\, (a_i)^N \equiv 0 \mod \Lambda \text{ for all } i\}.$$

(5) $\hat{W}_{M,N} = W_{M,N,0} \subset \hat{W}_M$ is the $\mathbb{Z}$-subfunctor of $\hat{W}_M$ introduced in the proof of Lemma 3.1.

We view all these functors as sheaves over the small flat sites $\mathrm{Spec}(\mathbb{Z})_{\mathrm{fl}}$ and $\mathbb{A}^1_{\mathrm{fl}}$. Then W_M and W_M^Λ are representable by M-dimensional affine spaces over $\mathrm{Spec}(\mathbb{Z})$, $\hat{W}_{M,N}$ is representable by a finite flat $\mathbb{Z}$-scheme, and $W_{M,N,\Lambda}$ is representable by a scheme which is a finite flat N^M-sheeted cover of an M-dimensional affine space over $\mathbb{A}^1$. Of these statements, only the last deserves a comment. The basic observation is that the sheaf F on $\mathbb{A}^1_{\mathrm{fl}}$ defined by $F(A) = \{a \in A,\, a^N \equiv 0 \mod \Lambda\}$ is represented by the scheme

$\mathrm{Spec}(\mathbb{Z}[\Lambda][u,v]/(u^N-\Lambda v))$, and then $W_{M,N,\Lambda}$ is obviously represented by the M-fold product of F.

5.3. Construction of framed group schemes. Here, we build framed group schemes. The precise argument developed in the 2-dimensional case in [SS2] explains how to proceed in higher dimensions.

Let R be a $\mathbb{Z}_{(p)}$-algebra and $\lambda_1, \lambda_2, \dots$ elements of R. Filtered R-group schemes are defined just like their formal analogues in Definition 4.2.

Definition. A *filtered R-group scheme of type* $(\lambda_1, \dots, \lambda_n)$ is a sequence

$$\mathcal{E}_0 = 0, \mathcal{E}_1, \dots, \mathcal{E}_n$$

of affine smooth commutative group schemes such that for each $i = 1, \dots, n$ the group scheme $\mathcal{E}_i$ is an extension of $\mathcal{E}_{i-1}$ by $\mathcal{G}^{\lambda_i}$.

Assumption 5.1. The elements $\lambda_1, \lambda_2, \dots$ are not zero divisors in R, and λ_i is nilpotent modulo λ_{i+1} for each $i \geqslant 1$.

We will see that under this assumption, and provided we make suitable truncations, the procedure described in Subsection 4.2 in the formal case gives filtered group schemes. In order to carry out the construction, we fix positive integers $L_1, L_2, \dots$ such that $(\lambda_i)^{L_i} \in \lambda_{i+1}R$ for all $i \geqslant 1$. We also fix a pair of positive integers (M, N) serving as a truncation level.

5.3.1. ***Description of the procedure.*** Contrary to the formal situation, here the polynomials giving the fundamental morphisms D_i will not be invertible over R but only over R/λ_{i+1}. Because the inductive definition of the D_i requires lifts of the inverses, we have to consider such lifts to be part of the data that we need to produce. So this time, the n-th step of the induction will produce 5-tuples $(\boldsymbol{e}^n, D_{n-1}, D_{n-1}^{-1}, \mathcal{E}_n, U^n)$ where, more precisely:

- $\boldsymbol{e}^n = (\boldsymbol{a}^n, \boldsymbol{b}^n)$ is a *frame*, that is, a point of a certain fibred product Fr_{n-1}, a closed subscheme of the product of $(W_{M,N,\lambda_n})^{n-1}$ by $(W_{M_n}^{\lambda_n})^{n-1}$, where M_n is an integer used below. Frames are the parameters of the construction, to be chosen at each step.
- $D_{n-1}, D_{n-1}^{-1} : \mathcal{E}_{n-1} \to \mathbb{A}^1$ are truncated deformed exponentials, that is morphisms of R-schemes which mod λ_n induce mutually inverse morphisms of $(R/\lambda_n R)$-group schemes $\mathcal{E}_{n-1} \to \mathbb{G}_m$.
- $\mathcal{E}_n$ is a commutative R-group scheme extension of $\mathcal{E}_{n-1}$ by $\mathcal{G}^{\lambda_n}$, with underlying scheme

$$\mathcal{E}_n = \mathrm{Spec}\left(R\left[X_1, \dots, X_n, \frac{1}{D_0 + \lambda_1 X_1}, \dots, \frac{1}{D_{n-1} + \lambda_n X_n}\right]\right),$$

such that the map $\alpha_{\mathcal{E}_n} : \mathcal{E}_n \to (\mathbb{G}_m)^n$ defined on the points by

$$(x_1, \dots, x_n) \mapsto (D_0 + \lambda_1 x_1, D_1 + \lambda_2 x_2, \dots, D_{n-1} + \lambda_n x_n)$$

is a morphism of R-group schemes.

- $U^n : (W_{M,N,\lambda_{n+1}})^n \to (W_{M_n,N_n,\lambda_{n+1}})^n$ is a morphism of R-schemes represented by a square matrix of size n, where M_n, N_n are integers.

5.3.2. *Initialization.* We set $W^0 = (W_{M,N})^0 = 0$ and $\mathcal{E}_0 = 0$. The induction is initialized at $n = 1$ by setting $\boldsymbol{e}^1 = (0,0)$, $D_0 = D_0^{-1} : \mathcal{E}_0 \to \mathbb{G}_m \subset \mathbb{A}^1$ equal to 1, and $\mathcal{E}_1 = \mathcal{G}^{\lambda_1}$. It follows from Lemmas 3.1 and 3.3 that the endomorphism $F^{\lambda_1} : W \otimes (R/\lambda_2) \to W \otimes (R/\lambda_2)$ leaves $\hat{W} \otimes (R/\lambda_2)$ stable, so it maps $\hat{W}_{M,N} \otimes (R/\lambda_2)$ into $\hat{W}_{M_1,N_1} \otimes (R/\lambda_2)$ for some integers M_1, N_1. It follows that the composition of $F^{\lambda_1} : W_{M,N,\lambda_2} \to W$ with the truncation map $\tau_{\geqslant M_1} : W \to W_{M_1}$ factors through W_{M_1,N_1,λ_2}. The result is a morphism $U^1 = F^{\lambda_1} : W_{M,N,\lambda_2} \to W_{M_1,N_1,\lambda_2}$.

5.3.3. *Induction.* For the inductive step of the construction, we assume that

$$(\boldsymbol{e}^i, D_{i-1}, D_{i-1}^{-1}, \mathcal{E}_i, U^i)$$

has been constructed for $1 \leqslant i \leqslant n$ and we explain how to produce $(\boldsymbol{e}^{n+1}, D_n, D_n^{-1}, \mathcal{E}_{n+1}, U^{n+1})$. We do this in four steps A-B-C-D.

A. To start with, we choose $\boldsymbol{e}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1})$ such that $U^n(\boldsymbol{a}^{n+1}) = \lambda_{n+1}.\boldsymbol{b}^{n+1}$. To be more formal, this is a section over R of the *scheme of frames* Fr_n defined as the fibred product of the morphisms

$$U^n : (W_{M,N,\lambda_{n+1}})^n \to (W_{M_n,N_n,\lambda_{n+1}})^n \subset (W_{M_n})^n$$

and

$$\lambda_{n+1} : (W_{M_n}^{\lambda_{n+1}})^n \to (W_{M_n})^n,$$

that is:

$$\mathrm{Fr}_n = (W_{M,N,\lambda_{n+1}})^n \times_{(W_{M_n})^n} (W_{M_n}^{\lambda_{n+1}})^n.$$

The choice of $\boldsymbol{e}^{n+1}$ will determine the other four objects in the 5-tuple.

B. Using the first component $\boldsymbol{a}^{n+1}$ of the frame, we define:

$$\begin{aligned} D_n &= \textstyle\prod_{i=1}^n E_p^{L_i,M_i,N_i}(\boldsymbol{a}_i^{n+1}, \lambda_i, D_{i-1}^{-1}X_i) \\ D_n^{-1} &= \textstyle\prod_{i=1}^n E_p^{L_i,M_i,N_i}(-\boldsymbol{a}_i^{n+1}, \lambda_i, D_{i-1}^{-1}X_i). \end{aligned}$$

Note that this is where the D_i^{-1} are useful, since they are involved in the definition of the D_i. It follows from the choice of the truncations (involved in the choice of $\boldsymbol{a}^{n+1}$ and in the truncated exponentials, see Lemma 5.1 and Definition 5.1) and from Theorem 5.1 of [SS3] (in the case of nilpotent coefficients), that D_n and D_n^{-1} induce morphisms of $(R/\lambda_{n+1}R)$-group schemes $\mathcal{E}_n \to \mathbb{G}_m$ inverse to each other.

C. Now we define $\mathcal{E}_{n+1}$. At this step, the strategy differs from the formal case because the truncated deformed exponentials are not invertible and do not give rise to 2-cocycles like in the formal case. In fact, the Hochschild cohomology group $\mathrm{H}^2_0(\mathcal{E}_n, \mathcal{G}^{\lambda_{n+1}})$ is usually very small. Instead, we use the exact sequence of sheaves on the small flat site of $\mathrm{Spec}(R)$:

$$0 \longrightarrow \mathcal{G}^{\lambda_{n+1}} \longrightarrow \mathbb{G}_m \longrightarrow i_*\mathbb{G}_m \longrightarrow 0$$

where $i : \mathrm{Spec}(R/\lambda_{n+1}R) \hookrightarrow \mathrm{Spec}(R)$ is the closed immersion. There is a connecting homomorphism $\mathrm{Hom}(\mathcal{E}_n, i_*\mathbb{G}_m) \to \mathrm{Ext}^1(\mathcal{E}_n, \mathcal{G}^{\lambda_{n+1}})$. In the groups involved here, the cohomology groups are understood on the small flat site of $\mathrm{Spec}(R)$ but note that the sheaves that appear are restrictions of sheaves on the big flat site. In Appendix A we prove that these groups are canonically isomorphic to the same cohomology groups on the big flat site, see in particular Lemma A.1 and Lemma A.2. This is not trivial, because the obvious candidate to be a morphism from the big flat site to the small flat site is not a morphism of sites, see Corollary A.1. Using the adjunction (i^*, i_*) in the big site, we have an isomorphism:

$$\mathrm{Hom}(\mathcal{E}_n, i_*\mathbb{G}_m) \longrightarrow \mathrm{Hom}(i^*\mathcal{E}_n, \mathbb{G}_m)$$

where now the right-hand group is computed on the big flat site of the scheme $\mathrm{Spec}(R/\lambda_{n+1})$. Notice that because we are on the big flat site, $i^*\mathcal{E}_n$ is simply the sheaf defined by the R/λ_{n+1}-scheme $\mathcal{E}_n \otimes_R R/\lambda_{n+1}$. Therefore the reduction mod λ_{n+1} of D_n defines an element of $\mathrm{Hom}(\mathcal{E}_n, i_*\mathbb{G}_m)$. We can finally define $\mathcal{E}_{n+1}$ as the extension obtained by pulling back the extension $0 \to \mathcal{G}^{\lambda_{n+1}} \to \mathbb{G}_m \to i_*\mathbb{G}_m \to 0$ along D_n. In particular for each flat R-algebra A, we have:

$$\begin{aligned}\mathcal{E}_{n+1}(A) &= \{(v,w) \in \mathcal{E}_n(A) \times A^\times \,,\, D_n(v) \equiv w \mod \lambda_{n+1}\} \\ &= \{(v,w) \in \mathcal{E}_n(A) \times A^\times \,,\, D_n(v) + \lambda_{n+1}x = w \text{ for some } x \in A\} \\ &= \{(v,x) \in \mathcal{E}_n(A) \times A \,,\, D_n(v) + \lambda_{n+1}x \in A^\times\}.\end{aligned}$$

This sheaf is represented by the scheme

$$\mathcal{E}_{n+1} = \mathrm{Spec}\left(R[\mathcal{E}_n][X_n, \frac{1}{D_n + \lambda_{n+1}X_{n+1}}]\right).$$

As far as the group law is concerned, note that by the assumption on D_n there exists a unique function $K = K(X,Y)$ on $\mathcal{E}_n \times \mathcal{E}_n$ such that $D_n(X)D_n(Y) = D_n(X \star Y) + \lambda_{n+1}K(X,Y)$, where $X \star Y$ denotes the group law in $\mathcal{E}_n$. Then it is easy to see that the group law in $\mathcal{E}_{n+1}$ is given on the points by:

$$(v_1, x_1) \star' (v_2, x_2) = \big(v_1 \star v_2, x_1 D_n(v_2) + x_2 D_n(v_1) + \lambda_{n+1}x_1x_2 + K(v_1, v_2)\big).$$

Equivalently, the group law is the only one such that the map

$$\begin{aligned}\alpha_{\mathcal{E}_{n+1}} :\ & \mathcal{E}_{n+1} \to (\mathbb{G}_m)^{n+1} \\ & (x_1, \ldots, x_{n+1}) \mapsto (D_0 + \lambda_1 x_1, D_1 + \lambda_2 x_2, \ldots, D_n + \lambda_{n+1}x_{n+1})\end{aligned}$$

is a morphism of R-group schemes.

D. Finally, using the second component $\boldsymbol{b}^{n+1}$ of the frame, we consider the matrix:

$$U^{n+1} = \begin{pmatrix} & & & -T_{\boldsymbol{b}_1^{n+1}} \\ & U^n & & \vdots \\ & & & -T_{\boldsymbol{b}_n^{n+1}} \\ 0 & \dots & 0 & F^{\lambda_{n+1}} \end{pmatrix}.$$

Let $\lambda \in R$ be a nonzerodivisor such that λ_{n+1} is nilpotent modulo λ. If we reduce modulo λ, then according to Lemmas 3.1, 3.3, 3.4, the endomorphism

$$U^{n+1} \otimes (R/\lambda R) : W^{n+1} \otimes (R/\lambda R) \to W^{n+1} \otimes (R/\lambda R)$$

leaves $\hat{W}^{n+1} \otimes (R/\lambda R)$ stable. It follows that there exist integers M_{n+1}, N_{n+1} such that $U^{n+1} \otimes (R/\lambda R)$ maps $(\hat{W}_{M,N})^{n+1}$ into $(\hat{W}_{M_{n+1},N_{n+1}})^{n+1}$. Therefore the composition of

$$U^{n+1} : (W_{M,N,\lambda})^{n+1} \subset W^{n+1} \to W^{n+1}$$

with the truncation map $W^{n+1} \to (W_{M_{n+1}})^{n+1}$ factors through the functor $(W_{M_{n+1},N_{n+1},\lambda})^{n+1}$. Fixing $\lambda = \lambda_{n+2}$, the result is a morphism of R-schemes

$$U^{n+1} : (W_{M,N,\lambda_{n+2}})^{n+1} \to (W_{M_{n+1},N_{n+1},\lambda_{n+2}})^{n+1}$$

This is the last object in our sought-for 5-tuple.

Remark 5.1. A priori, the integers M_n, N_n depend on the particular frames involved in the matrices U^n. However, considering the universal case (see Theorem 5.1), it is seen that in fact, once (M, N) is fixed then (M_n, N_n) may be chosen uniform, minimal and hence completely determined by M_1, N_1 and n.

Definition. A *framed R-group scheme of type* $(\lambda_1, \dots, \lambda_n)$ is a sequence

$$\mathcal{E}_0 = 0, (\mathcal{E}_1, \boldsymbol{e}^1), \dots, (\mathcal{E}_n, \boldsymbol{e}^n)$$

of pairs composed of an affine smooth commutative group scheme and a frame, such that for each $i = 1, \dots, n$ the group scheme $\mathcal{E}_i$ is the extension of $\mathcal{E}_{i-1}$ by $\mathcal{G}^{\lambda_i}$ determined by the frame $\boldsymbol{e}^i$. We often write $\mathcal{E}_n$ as a shortcut for this data.

In order to state the analogue of Theorem 4.2 in the algebraic context, we must make sure that the coefficients λ_i satisfy Assumption 5.1. This means that for some integer $\nu \geqslant 1$ they are points of the space

$$\mathbb{B}_\nu^n = \operatorname{Spec}\left(\frac{\mathbb{Z}_{(p)}[\Lambda_1, \dots, \Lambda_n, M_2, \dots, M_n]}{\Lambda_1^\nu - M_2\Lambda_2, \dots, \Lambda_{n-1}^\nu - M_n\Lambda_n} \right).$$

This is a finite flat cover of the affine space

$$\mathbb{A}^n = \operatorname{Spec}(\mathbb{Z}_{(p)}[M_2, \dots, M_n, \Lambda_n]).$$

Moreover, there are obvious projections $\mathbb{B}_\nu^{n+1} \to \mathbb{B}_\nu^n$ given by the inclusion of function rings. Below, we denote by Λ the product of the Λ_i.

Theorem 5.1. *Let $\mathbb{B}_\nu^n$ be the finite flat covers of affine space $\mathbb{A}^n$ defined above. Then there exists a sequence indexed by $\nu \geqslant 1$ of affine $\mathbb{B}_\nu^n$-schemes $\mathscr{S}_n^\nu = \operatorname{Spec}(\mathscr{R}_n^\nu)$ of finite type, without Λ-torsion, and framed $\mathscr{R}_n^\nu$-group schemes $\mathscr{E}_n^\nu$ of type $(\Lambda_1, \dots, \Lambda_n)$ with the following universal property : for any $\mathbb{Z}_{(p)}$-algebra R, any nonzerodivisors $\lambda_1, \dots, \lambda_n \in R$ such that λ_i is nilpotent modulo λ_{i+1} for each i, and any framed R-group scheme $\mathcal{E}_n$ of type $(\lambda_1, \dots, \lambda_n)$, there exists ν and a unique map $\mathscr{R}_n^\nu \to R$ taking Λ_i to λ_i such that $\mathcal{E}_n \simeq \mathscr{E}_n^\nu \otimes_{\mathscr{R}_n^\nu} R$.*

Proof. For a fixed $\nu \geqslant 1$, we first give $\mathscr{S}_n^\nu \to \mathbb{B}_\nu^n$ and $\mathscr{E}_n^\nu \to \mathscr{S}_n^\nu$. The construction is by induction on n and follows the proof of Theorem 4.2, with minor differences which we indicate. The main difference is that in the present case, the function ring of the schemes of frames in dimension n is bound to play the role of the coefficient ring in dimensions $\geqslant n+1$ and so needs to be free of Λ-torsion. Thus we have to kill torsion in the adequate fibred product.

We set $L = M = N = \nu$. In order to keep the notation light, we will sometimes omit the symbol ν in the indices and exponents. We initialize by $\mathscr{E}_0 = 0$, $\mathscr{S}_1 = \mathbb{B}^1$, $\boldsymbol{e}^1 = (0,0)$, $D_0 = D_0^{-1} = 1$, $\mathscr{E}_1 = \mathcal{G}^{\Lambda_1}$ over $\mathscr{S}_1$, and $U^1 = F^{\Lambda_1} : W_{M,N,\Lambda_2} \to W_{M_1,N_1,\Lambda_2}$ is the morphism of $\mathbb{B}^2$-schemes constructed like in 5.3.2. Now assuming that for $1 \leqslant i \leqslant n$ we have objects $\mathscr{S}_i, \boldsymbol{e}^i, D_{i-1}, D_{i-1}^{-1}, \mathscr{E}_i, U^i$, here is how to construct

$$\mathscr{S}_{n+1}, \boldsymbol{e}^{n+1}, D_n, D_n^{-1}, \mathscr{E}_{n+1}, U^{n+1}.$$

Consider the morphisms of $\mathscr{S}_n \times_{\mathbb{B}^n} \mathbb{B}^{n+1}$-schemes

$$U^n : (W_{\nu,\nu,\Lambda_{n+1}})^n \to (W_{M_n,N_n,\Lambda_{n+1}})^n \subset (W_{M_n})^n$$

and

$$\Lambda_{n+1} : (W_{M_n}^{\Lambda_{n+1}})^n \to (W_{M_n})^n.$$

Call $\mathscr{S}_{n+1}$ the closed subscheme of the fibred product of U^n and Λ_{n+1} defined by the ideal of Λ-torsion, where $\Lambda = \Lambda_1 \dots \Lambda_{n+1}$. Let $\boldsymbol{e}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1})$ be the universal point of $\mathscr{S}_{n+1}$. Then $D_n, D_n^{-1}, \mathscr{E}_{n+1}, U^{n+1}$ are constructed as in steps B, C, D of 5.3.3 and we do not repeat the details.

If $\mathcal{E}_n$ is a framed group scheme of type $(\lambda_1, \dots, \lambda_n)$ over a ring R, then there exists L such that $(\lambda_i)^L \in \lambda_{i+1}R$. Moreover $\mathcal{E}_n$ is described by Witt vectors with a number of nonzero coefficients bounded by some M and nilpotency indices bounded by some N. For $\nu = \max(L, M, N)$ it is clear that $\mathcal{E}_n$ is uniquely a pullback of $\mathscr{E}_n^\nu$. This proves the universality property of the statement of the theorem. $\square$

Proposition 5.1. *Let R be a $\mathbb{Z}_{(p)}$-algebra which is a unique factorization domain. Then, any filtered group scheme is induced by a framed group scheme.*

Proof. By induction, it is enough to prove that given a filtered group scheme $\mathcal{E}$ of some type $(\lambda_1, \dots, \lambda_n)$ and a nonzero element $\lambda \in R$, any extension of $\mathcal{E}$ by $\mathcal{G}^\lambda$ may be defined by a frame. Consider the long exact sequence

$$\dots \longrightarrow \mathrm{Hom}(\mathcal{E}_{R/\lambda}, \mathbb{G}_{m,R/\lambda}) \xrightarrow{\partial} \mathrm{Ext}^1(\mathcal{E}, \mathcal{G}^\lambda) \longrightarrow \mathrm{Ext}^1(\mathcal{E}, \mathbb{G}_m) \longrightarrow \dots$$

derived from the exact sequence (1) in Proposition 3.1. It is enough to prove that the connecting homomorphism ∂ is surjective. But since R is a unique factorization domain, this follows from the fact that $\mathrm{Ext}^1(\mathcal{E}, \mathbb{G}_m) = 0$, proven as in the proof of Theorem 3.2 of [S] and Proposition 3.1 of [SS3]. $\square$

6. Kummer subschemes

Let R be a $\mathbb{Z}_{(p)}$-algebra and let $(\lambda_1, \dots, \lambda_n)$ be as in Assumption 5.1. We call λ the product of the λ_i and we write $K = R[1/\lambda]$. For an R-scheme X, it will be a convenient abuse of terminology to call the restriction X_K the *generic fibre* of X. Let $\mathcal{E}$ be a framed group scheme of type $(\lambda_1, \dots, \lambda_n)$. By construction $\mathcal{E}$ comes with a map $\alpha_{\mathcal{E}} : \mathcal{E} \to (\mathbb{G}_m)^n$ which is an isomorphism over K. Let $\Theta^n : (\mathbb{G}_m)^n \to (\mathbb{G}_m)^n$ be the morphism defined by

$$\Theta^n(t_1, \dots, t_n) = (t_1^p, t_2^p t_1^{-1}, \dots, t_n^p t_{n-1}^{-1}).$$

The kernel of Θ^n is a subgroup isomorphic to $\mu_{p^n,R}$ which we call the *Kummer μ_{p^n} of $\mathbb{G}_m^n$*. Via the map α, we can see the Kummer $\mu_{p^n,K}$ as a closed subgroup scheme of $\mathcal{E}_K$. We define the *Kummer subscheme* as the scheme-theoretic closure of $\mu_{p^n,K}$ in $\mathcal{E}$. Note that in general the multiplication of G_K need not extend to G. The main question we want to address in this section is : when is the Kummer subscheme G finite locally free over $\mathrm{Spec}(R)$? When this happens, then the multiplication extends and accordingly, we shall prefer to call G the *Kummer subgroup*. In order to study this question, we first study the one-dimensional case in 6.1. Then, we consider extensions and we sketch the usual inductive procedure producing isogenies between filtered group schemes, in 6.2.

Before we start, let us make a couple of easy remarks. First, note that G is the smallest closed subscheme of $\mathcal{E}$ with generic fibre isomorphic to $\mu_{p^n,K}$. It is also the only closed subscheme of $\mathcal{E}$ without λ-torsion with generic fibre isomorphic to $\mu_{p^n,K}$. In particular, if there exists a closed subscheme of $\mathcal{E}$ which is finite locally free over R and has generic fibre isomorphic to $\mu_{p^n,K}$, then this subscheme is equal to G.

6.1. Dimension 1. If λ^{p-1} divides p in R, then the polynomial $\lambda^{-p}((\lambda x+1)^p-1)$ has coefficients in R and the morphism $\psi:\mathcal{G}^\lambda\to\mathcal{G}^{\lambda^p}$ defined by $\psi(x)=\lambda^{-p}((\lambda x+1)^p-1)$ is an isogeny. Following the notation in [To], we put $G_{\lambda,1}=\ker(\psi)$.

Lemma 6.1. *Let $\lambda\in R$ be a nonzerodivisor and $\mathcal{E}=\mathcal{G}^\lambda$.*
(1) *The Kummer subscheme G is finite locally free over R if and only if λ^{p-1} divides p in R.*
(2) *If G is finite locally free, its ideal sheaf in $\mathcal{O}_\mathcal{E}$ is generated by the polynomial $\lambda^{-p}((\lambda x+1)^p-1)$, and the quotient $\mathcal{E}\to\mathcal{E}/G$ is isomorphic to the isogeny $\psi:\mathcal{G}^\lambda\to\mathcal{G}^{\lambda^p}$.*

Proof. We begin with a couple of remarks. Let us introduce the polynomial $P=(\lambda x+1)^p-1$. If $s:=\max\{t\leqslant p,\ \lambda^{t-1}\text{ divides }p\}$, there exists $u\in R$ such that $p=u\lambda^{s-1}$. Then we can write $P=\lambda^sQ$ where:

$$Q=\lambda^{p-s}x^p+\sum_{i=1}^{p-1}\left\{{p\atop i}\right\}u\lambda^{i-1}x^i$$

with $\binom{p}{i}=\left\{{p\atop i}\right\}p$ for $1\leqslant i\leqslant p-1$; and Q is not divisible by λ. The ideal of G is:

$$I=\{F\in R[x,(\lambda x+1)^{-1}],\ \exists n\geqslant 0,\ \exists F'\in R[x,(\lambda x+1)^{-1}],\lambda^nF=QF'\}.$$

Note that because $\lambda x+1$ is invertible modulo P and also modulo Q, we may always choose $F'\in R[x]$ above. Now let $R[x]\to R[x,(\lambda x+1)^{-1}]$ be the natural inclusion and let J be the preimage of I. We have $J=\{F\in R[x],\ \exists n\geqslant 0,\ \exists F'\in R[x],\lambda^nF=QF'\}$ and it is clear that the natural map $R[x]/J\to R[x,(\lambda x+1)^{-1}]/I$ is an isomorphism. We now prove (1) and (2).

(1) We have to prove that the algebra $R[x]/J$ is finite locally free over R if and only if λ^{p-1} divides p. If λ^{p-1} divides p, that is if $s=p$, then Q is monic and we claim that $J=(Q)$. Consider $F\in J$ and n,F' such that $\lambda^nF=QF'$. We assume n is minimal, i.e. λ does not divide F'. If $n>0$ then $QF'\equiv 0\mod\lambda$ hence $F'\equiv 0\mod\lambda$ since Q is monic hence a nonzerodivisor. This is a contradiction, so $n=0$ and $F\in(Q)$. Thus $J=(Q)$ and $R[x]/J$ is finite free over R.

Conversely, assume that $R[x]/J$ is finite locally free. We will prove that Q is monic and generates J. It is enough to prove these properties locally over $\mathrm{Spec}(R)$, hence we may assume that $R[x]/J$ is finite free over R. Then there is a monic polynomial G that generates J, see Eisenbud [Ei], Prop. 4.1. From the fact that $Q\in(G)$ and $\lambda^nG\in(Q)$ we see that $\deg(G)=\deg(Q)=p$. Writing $\lambda^nG=QF'$ we see that $F'=\lambda^{n-p+s}$ so that $Q=\lambda^{p-s}G$. Since λ does not divide Q this is possible only if $s=p$, that is λ^{p-1} divides p.

(2) The isogeny $\psi:\mathcal{E}=\mathcal{G}^\lambda\to\mathcal{G}^{\lambda^p}$ is G-invariant and induces a morphism $\mathcal{E}/G\to\mathcal{G}^{\lambda^p}$ which is finite flat of degree 1, hence an isomorphism. ☐

If $\mathcal{E}$ is an n-dimensional framed group scheme, then what we have just proved gives some one-dimensional necessary conditions for the Kummer subscheme G to be finite locally free over R, as we shall now see. Indeed if G is finite locally free over R, then the quotient $\mathcal{F} = \mathcal{E}/G$ is a smooth affine n-dimensional R-group scheme and the quotient map $\nu : \mathcal{E} \to \mathcal{F}$ is an isogeny (smoothness follows from [EGA], Chap. 0 (préliminaires), 17.3.3.(i)). Consider the subgroup $\mathcal{G}^{\lambda_n} \subset \mathcal{E}$, its scheme-theoretic image $\mathcal{G}$ under ν and the restriction $\nu' : \mathcal{G}^{\lambda_n} \to \mathcal{G}$ of ν. In the fibre over any point $s \in \mathrm{Spec}(R)$, the scheme $\mathcal{G}_s$ is the quotient of $\mathcal{G}_s^{\lambda_n}$ by the equivalence relation induced by G_s, that is, it is the quotient of $\mathcal{G}_s^{\lambda_n}$ by the stabilizer

$$H = \{g \in G_s,\ g(\mathcal{G}_s^{\lambda_n}) \subset \mathcal{G}_s^{\lambda_n}\}.$$

In particular $\mathcal{G}_s$ is a quotient of a smooth $k(s)$-group scheme by a finite flat subgroup scheme, hence it is a smooth $k(s)$-group scheme and the map $\mathcal{G}_s^{\lambda_n} \to \mathcal{G}_s$ is flat. By the criterion of flatness in fibres, it follows that ν' is flat and that $\mathcal{G}$ is smooth. Then the kernel $H_n = \ker(\nu')$ is flat of degree p, with generic fibre equal to the Kummer $\mu_{p,K}$ inside G. Moreover $\mathcal{G}$ is isomorphic to $\mathcal{G}^{\lambda_n^p}$ and ν' is isomorphic to the isogeny $\mathcal{G}^{\lambda_n} \to \mathcal{G}^{\lambda_n^p}$, by Lemma 6.1. Set $G_{n-1} = G/H_n$ and $\mathcal{F}_{n-1} = \mathcal{F}/\mathcal{G}^{\lambda_n^p}$. Then we have exact sequences

$$0 \longrightarrow H_n \longrightarrow G \longrightarrow G_{n-1} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{G}^{\lambda_n^p} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{n-1} \longrightarrow 0.$$

By induction we see immediately that G and $\mathcal{F}$ have filtrations $G_0 = 0, G_1, \dots, G_n = G$ and $\mathcal{F}_0 = 0, \mathcal{F}_1, \dots, \mathcal{F}_n = \mathcal{F}$ where $G_i \subset \mathcal{F}_i$ is finite locally free of rank p^i and $\mathcal{F}_i/\mathcal{F}_{i-1} \simeq \mathcal{G}^{\lambda_i^p}$. In particular $\mathcal{F}$ is a filtered group scheme of type $(\lambda_1^p, \dots, \lambda_n^p)$ and G is a successive extensions of the groups $G_{\lambda_1,1}, \dots, G_{\lambda_n,1}$. Another consequence of our discussion is that the scheme-theoretic closure of $\mu_{p,K}$ inside $\mathcal{G}^{\lambda_n}$ is H_n and in particular is finite locally free over R. Similarly, by induction the scheme-theoretic closure of $\mu_{p,K}$ inside $\mathcal{G}^{\lambda_i}$ is equal to the kernel of $G_i \to G_{i-1}$ and is finite locally free. By Lemma 6.1, this proves that the following reinforcement of Assumption 5.1 is satisfied.

Assumption 6.1. For each $i \geqslant 1$ we have: λ_i is not a zero divisor in R, λ_i is nilpotent modulo λ_{i+1}, and λ_i^{p-1} divides p.

6.2. Construction of Kummer group schemes. From now on, we work under Assumption 6.1. Because filtered group schemes are defined by successive extensions, the condition that the Kummer subscheme be finite locally free is also naturally expressed at each extension step. Assume that we have a filtered group scheme $\mathcal{E}_n$ of dimension n with finite locally free Kummer subgroup G_n. Then, there is a quotient morphism $\Psi_n : \mathcal{E}_n \to \mathcal{F}_n = \mathcal{E}_n/G_n$ and for each $\lambda \in R$ a pullback

$$(\Psi^n)^* : \mathrm{Hom}_{R/\lambda R-\mathrm{Gr}}(\mathcal{F}_n, \mathbb{G}_m) \to \mathrm{Hom}_{R/\lambda R-\mathrm{Gr}}(\mathcal{E}_n, \mathbb{G}_m).$$

If $\mathcal{E}_{n+1}$ is an extension of $\mathcal{E}_n$ by $\mathcal{G}^\lambda$ determined by a frame $\boldsymbol{e}^{n+1}$, then we shall see that the condition for the Kummer subscheme G_{n+1} to be finite locally free is expressed in terms of $(\Psi^n)^*$ and $\boldsymbol{e}^{n+1}$. This will be integrated in an inductive construction where we build at the same time the group schemes $\mathcal{E}_n, \mathcal{F}_n$ and the isogeny between them, by making compatible choices of frames. We explain how to do this, along the same lines as before but giving a little less detail.

We start with a well-known fact.

Lemma 6.2. *If $p \geqslant 3$, then in the Witt ring $W(\mathbb{Z})$ we have*

$$p = (p, 1 - p^{p-1}, \epsilon_2 p^{p-1}, \epsilon_3 p^{p-1}, \epsilon_4 p^{p-1}, \dots)$$

where $\epsilon_2, \epsilon_3, \epsilon_4, \dots$ are principal p-adic units. If $p = 2$, then in $W(\mathbb{Z})$ we have

$$2 = (2, -1, \epsilon_2 2^2, \epsilon_3 2^3, \epsilon_4 2^5, \dots, \epsilon_n 2^{2^{n-2}+1}, \dots)$$

where $\epsilon_2, \epsilon_3, \epsilon_4, \dots$ are 2-adic units.

Proof. We start by proving that for $i \geqslant 1$ we have:

$$(1 - p^{p-1})^{p^i} = \begin{cases} 1 - p^{i+p-1} + \frac{p^i-1}{2} p^{i+2(p-1)} + \dots & \text{if } p > 2, \\ 1 & \text{if } p = 2. \end{cases}$$

If $p = 2$ this is clear and we assume $p \geqslant 3$. Now

$$(1 - p^{p-1})^{p^i} = 1 - x_1 + x_2 - \dots + (-1)^{p^i} x_{p^i}$$

where $x_j = \binom{p^i}{j} p^{j(p-1)}$ has valuation $v(x_j) = i - v(j) + j(p-1)$ whenever $1 \leqslant j \leqslant p^i$. Let us write $j = up^a$ with $u \geqslant 1$ prime to p and $a \geqslant 0$. Then $v(x_j) = i - a + up^a(p-1)$ which is increasing both as a function of a and as a function of u. If $j \geqslant 2$, then either $u \geqslant 2$ or $a \geqslant 1$. In the first case we have $v(x_j) \geqslant i + 2(p-1)$ and we have equality for $j = 2$. In the second case we have $v(x_j) \geqslant i - 1 + p(p-1) > i + 2(p-1)$ since $p \geqslant 3$. The claim follows. Now we come to the statement of the lemma itself. The proof for $p = 2$ is similar and we focus on the case $p \geqslant 3$. The Witt vector $p = (a_0, a_1, a_2, \dots)$ is determined by the equalities

$$a_0^{p^n} + p a_1^{p^{n-1}} + \dots + p^{n-1} a_{n-1}^p + p^n a_n = p$$

for all $n \geqslant 0$. In particular $a_0 = p$ and $a_1 = 1 - p^{p-1}$. By the computation of the p-adic first terms of $(1 - p^{p-1})^{p^i}$ which we started with, if $n \geqslant 2$ we have

$$\frac{p - pa_1^{p^{n-1}}}{p^n} = \frac{p^{n-1+p-1} + \dots}{p^{n-1}} = p^{p-1} + \dots$$

For $n \geqslant 2$, by induction using the equality

$$a_n = \frac{p - pa_1^{p^{n-1}}}{p^n} - p^{-n}(a_0^{p^n} + p^2 a_2^{p^{n-2}} + \dots + p^{n-1}a_{n-1}^p),$$

we see that the p-adic leading term of a_n is p^{p-1}. $\square$

Corollary 6.1. *Let $\mathcal{O} = \mathbb{Z}[C, \Lambda]/(p - C\Lambda^{p-1})$ and let $c, \lambda \in \mathcal{O}$ be the images of C, Λ. There exists a unique $d = (d_0, d_1, d_2, \dots) = (c, 1 - p^{p-1}, d_2, \dots)$ in $W^{\lambda^p}(\mathcal{O})$ such that*

$$p[\lambda] = \lambda^p(d) = (\lambda^p d_0, \lambda^p d_1, \lambda^p d_2, \dots).$$

Proof. If $p \geqslant 3$, then from the lemma we deduce

$$p[\lambda] = (c\lambda^p, (1 - p^{p-1})\lambda^p, \epsilon_2 p^{p-1}\lambda^{p^2}, \epsilon_3 p^{p-1}\lambda^{p^3}, \dots).$$

The coefficients of this vector are divisible by λ^p, thus $d_0, d_1, d_2, \dots$ exist. They are unique since λ is not a zero divisor in $\mathcal{O}$. If $p = 2$, the proof works similarly. $\square$

Thus for any $\mathbb{Z}$-algebra R' and any elements $c', \lambda' \in R$ satisfying $p = c'\lambda'^{p-1}$ there is a well-determined $d' \in W^{\lambda'^p}(R')$ such that

$$p[\lambda'] = (\lambda'^p d'_0, \lambda'^p d'_1, \lambda'^p d'_2, \dots).$$

In particular, our choice of elements $\lambda_i \in R$ satisfying Assumption 6.1 determines elements $d_i = (d_{i0}, d_{i1}, \dots) \in W^{\lambda_i^p}(R)$ such that

$$p[\lambda_i] = (\lambda_i^p d_{i0}, \lambda_i^p d_{i1}, \lambda_i^p d_{i2}, \dots).$$

These are the elements denoted $p\widetilde{\lambda_i}/\lambda_i^p$ in [SS3] and $p[\lambda_i]/\lambda_i^p$ in [MRT].

6.2.1. ***Description of the procedure.*** As in 5.3, we fix positive integers L_i such that $(\lambda_i)^{L_i} \in \lambda_{i+1}R$ for all $i \geqslant 1$ and positive integers M, N. The n-th step of the induction produces data:

- $\boldsymbol{h}^n = (\boldsymbol{a}^n, \boldsymbol{b}^n, \boldsymbol{u}^n, \boldsymbol{v}^n, \boldsymbol{z}^n)$ is a *big frame* including two frames of definition $\boldsymbol{e}^n = (\boldsymbol{a}^n, \boldsymbol{b}^n)$ and $\boldsymbol{f}^n = (\boldsymbol{u}^n, \boldsymbol{v}^n)$ of filtered group schemes and a compatibility between them given by $\boldsymbol{z}^n$,
- $(\boldsymbol{e}^n, D_{n-1}^{-1}, D_{n-1}, \mathcal{E}_n, U^n)$ is a framed group scheme of type
$$(\lambda_1, \dots, \lambda_{n+1}),$$
- $(\boldsymbol{f}^n, E_{n-1}^{-1}, E_{n-1}, \mathcal{F}_n, \overline{U}^n)$ is a framed group scheme of type
$$(\lambda_1^p, \dots, \lambda_{n+1}^p),$$
- $\Psi^n : \mathcal{E}_n \to \mathcal{F}_n$ is an isogeny commuting with the morphism Θ^n,

- $\Upsilon^n : (W_{M,N,\lambda_{n+1}})^n \to (W_{M,N,\lambda_{n+1}})^n$ is a matrix of operators (made precise below) describing $(\Psi^n)^*$.

The condition that Ψ^n commutes with Θ^n involves implicitly the maps $\alpha_{\mathcal{E}_n} : \mathcal{E}_n \to (\mathbb{G}_m)^n$ and $\beta_{\mathcal{F}_n} : \mathcal{F}_n \to (\mathbb{G}_m)^n$ provided by the construction of framed group schemes, and may be pictured by the commutative diagramme:

$$\begin{tikzcd} \mathcal{E}_n \arrow[r, "\Psi^n"] \arrow[d, "\alpha_{\mathcal{E}_n}"'] & \mathcal{F}_n \arrow[d, "\beta_{\mathcal{F}_n}"] \\ (\mathbb{G}_m)^n \arrow[r, "\Theta^n"] & (\mathbb{G}_m)^n. \end{tikzcd}$$

Since $\beta_{\mathcal{F}_n}$ is an isomorphism on the generic fibre, there is in any case a rational map $\mathcal{E}_n \dashrightarrow \mathcal{F}_n$. The morphism Ψ^n is determined as the unique morphism extending this rational map. In fact, the choice of the big frame $\boldsymbol{h}^n$ will guarantee that Ψ^n exists and we may as well remove it from the list above; we included it for clarity of the picture.

6.2.2. ***Initialization.*** We set $W^0 = (W_{M,N})^0 = 0$, $\mathcal{E}_0 = \mathcal{F}_0 = 0$ and

- $\boldsymbol{h}^1 = (0,0,0,0,0)$,
- $D_0 = D_0^{-1} = 1$, $\mathcal{E}_1 = \mathcal{G}^{\lambda_1}$,
- $E_0 = E_0^{-1} = 1$, $\mathcal{F}_1 = \mathcal{G}^{\lambda_1^p}$,
- $U^1 = F^{\lambda_1} : W_{M,N,\lambda_2} \to W_{M_1,N_1,\lambda_2}$,
- $\overline{U}^1 = F^{\lambda_1^p} : W_{M,N,\lambda_2^p} \to W_{M_1,N_1,\lambda_2^p}$,
- $\Upsilon^1 = T_{d_1} : W_{M,N,\lambda_2^p} \to W_{M,N,\lambda_2}$,

where M_1, N_1 are suitable integers whose existence comes from Lemmas 3.1 and 3.3.

6.2.3. ***Induction.*** As usual, we assume that objects in dimension i have been constructed for $1 \leqslant i \leqslant n$ and we explain how to produce $\boldsymbol{h}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}^{n+1}, \boldsymbol{z}^{n+1})$ and the related data.

A. In order to define the big scheme of frames, first we introduce an $n+1$-dimensional vector $\boldsymbol{c}^{n+1} = (\boldsymbol{a}^n, [\lambda_n])$. We recall that Assumption 6.1 is supposed to be satisfied. The fundamental ingredient of the induction is given by the following result.

Theorem 6.1. *Let $\mathcal{E}_{n+1}, \mathcal{F}_{n+1}$ be framed group schemes of types*

$$(\lambda_1, \ldots, \lambda_{n+1}) \quad \textit{and} \quad (\lambda_1^p, \ldots, \lambda_{n+1}^p).$$

Let $(\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1})$ and $(\boldsymbol{u}^{n+1}, \boldsymbol{v}^{n+1})$ be the defining frames. Assume that the Kummer subscheme $G_n \subset \mathcal{E}_n$ is finite locally free and that the rational map $\mathcal{E}_n \dashrightarrow \mathcal{F}_n$ extends to an isogeny with kernel G_n. Then, the following conditions are equivalent:

(1) *the Kummer subscheme* $G_{n+1} \subset \mathcal{E}_{n+1}$ *is finite locally free and the rational map* $\mathcal{E}_{n+1} \dashrightarrow \mathcal{F}_{n+1}$ *extends to an isogeny with kernel* G_{n+1}*,*
(2) *there exists* $\boldsymbol{z}^{n+1} \in (W^{\lambda^p_{n+1}})^n(R)$ *such that* $p\boldsymbol{a}^{n+1} - \boldsymbol{c}^{n+1} - \Upsilon^n \boldsymbol{u}^{n+1} = \lambda^p_{n+1}(\boldsymbol{z}^{n+1})$*.*

Proof. This is proven in [MRT], Theorem 7.1.1 in the case where the ring R is a discrete valuation ring with uniformizer π, the element λ_i being replaced by π^{l_i}. The proof uses general power series computations and it is clear while reading it that it works for an arbitrary $\mathbb{Z}_{(p)}$-algebra R satisfying our assumptions. ☐

Given this theorem, we can choose a big frame

$$\boldsymbol{h}^{n+1} = (\boldsymbol{a}^{n+1}, \boldsymbol{b}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}^{n+1}, \boldsymbol{z}^{n+1})$$

living in a big scheme of frames whose heavy but obvious definition we omit.

B. Using the components $\boldsymbol{a}^{n+1}$ and $\boldsymbol{u}^{n+1}$ of the frame, we define:

$$\begin{aligned}
D_n &= \textstyle\prod_{i=1}^n E_p^{L_i,M_i,N_i}(\boldsymbol{a}_i^{n+1}, \lambda_i, D_{i-1}^{-1} X_i) \\
D_n^{-1} &= \textstyle\prod_{i=1}^n E_p^{L_i,M_i,N_i}(-\boldsymbol{a}_i^{n+1}, \lambda_i, D_{i-1}^{-1} X_i) \\
E_n &= \textstyle\prod_{i=1}^n E_p^{L_i,M_i,N_i}(\boldsymbol{u}_i^{n+1}, \lambda_i^p, E_{i-1}^{-1} Y_i) \\
E_n^{-1} &= \textstyle\prod_{i=1}^n E_p^{L_i,M_i,N_i}(-\boldsymbol{u}_i^{n+1}, \lambda_i^p, E_{i-1}^{-1} Y_i).
\end{aligned}$$

C. At this step, we define $\mathcal{E}_{n+1}$ and $\mathcal{F}_{n+1}$ in the same way as in 5.3.3, Step C.

D. At this step, we define morphisms

$$U^{n+1}, \overline{U}^{n+1} : (W_{M,N,\lambda_{n+2}})^{n+1} \to (W_{M_{n+1},N_{n+1},\lambda_{n+2}})^{n+1}$$

like in 5.3.3, Step D (here $\overline{U}^{n+1}$ is attached to the group scheme $\mathcal{F}_{n+1}$ in the same way as U^{n+1} is attached to the group scheme $\mathcal{E}_{n+1}$), and the operator $\Upsilon^{n+1} : (W_{M,N,\lambda_{n+2}})^{n+1} \to (W_{M,N,\lambda_{n+2}})^{n+1}$ by the matrix

$$\Upsilon^{n+1} = \begin{pmatrix} & & & -T_{\boldsymbol{z}_1^{n+1}} \\ & \Upsilon^n & & \vdots \\ & & & -T_{\boldsymbol{z}_n^{n+1}} \\ 0 & \dots & 0 & T_{d_{n+1}} \end{pmatrix}.$$

This concludes the inductive construction.

Theorem 6.2. *Let* $\mathbb{B}^n_\nu$ *be the finite flat covers of affine space* $\mathbb{A}^n$ *defined in 5.1. There exists a sequence indexed by* $\nu \geqslant 1$ *of affine* $\mathbb{B}^n_\nu$*-schemes* $\mathscr{K}^\nu_n = \mathrm{Spec}(\mathscr{M}^\nu_n)$ *of finite type, without* Λ*-torsion, framed* $\mathscr{M}^\nu_n$*-group schemes* $\mathscr{E}^\nu_n$ *of type* $(\Lambda_1, \dots, \Lambda_n)$ *and* $\mathscr{F}^\nu_n$ *of type* $(\Lambda^p_1, \dots, \Lambda^p_n)$*, and an isogeny* $\mathscr{E}^\nu_n \to \mathscr{F}^\nu_n$

with finite locally free kernel $\mathscr{G}_n^\nu$ compatible with the maps to $(\mathbb{G}_m)^n$. This isogeny is universal in the same sense as in 5.1.

Proof. Omitted. □

The family $\mathscr{G}_n = (\mathscr{G}_n^\nu)_{\nu \geqslant 1}$ is a finite flat group scheme over the ind-scheme $(\mathscr{K}_n^\nu)_{\nu \geqslant 1}$. We call it the *universal Kummer group scheme.*

Example. Let $R = \mathbb{Z}_{(p)}[\zeta_n]$ be the extension of the localization of the ring of integers at p obtained by adjoining a primitive p^n-th root of unity. In sections 8 and 9 of [SS3], the authors provide an explicit isogeny $\mathcal{W}_n \to \mathcal{V}_n$ between filtered group schemes over R, with kernel isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})_R$. Setting $\lambda := \zeta_1 - 1$, the group $\mathcal{W}$ is of type $(\lambda, \dots, \lambda)$ and the group $\mathcal{V}_n$ is of type $(\lambda^p, \dots, \lambda^p)$. It would be very interesting to find explicit values for the integers L_i, M_i, N_i of the construction procedure, but this seems quite difficult (see the earlier remarks 3.1 and 5.1).

We conclude with a remark on the operator Υ^n. By construction, it represents the pullback $(\Psi^n)^*$, which implies that modulo λ_{n+1} it maps the subspace $\ker(\overline{U}^n)$ into the subspace $\ker(U^n)$. In fact, we can do better: it is possible to include in the induction the construction of a matrix Ω^n such that $U^n\,\Upsilon^n = \Omega^n\,\overline{U}^n$. This is a reflection of the fact that among the morphisms from a filtered group to $\mathbb{G}_m$, not only the *group morphisms* (represented by $\ker(U^n)$) but also the *fundamental morphisms* (represented by the ambient W^n) are meaningful. On the diagonal, the entries of the matrix Ω^n should be operators T'_{d_i} (see below) satisfying $F^{\lambda_i} \circ T_{d_i} = T'_{d_i} \circ F^{\lambda_i^p}$. In fact, these matrices are defined by $\Omega^1 = T'_{d_1}$ and

$$\Omega^{n+1} = \begin{pmatrix} & & & * \\ & \Omega^n & & \vdots \\ & & & * \\ 0 & \dots & 0 & T'_{d_{n+1}} \end{pmatrix}.$$

We do not want to go into the full details of the construction of Ω^n. We simply note that the essential task is to define the diagonal entries T'_{d_i}. We end the paper with the proof of existence and unicity of these endomorphisms.

Lemma 6.3. *Let $\mathcal{O} = \mathbb{Z}[C, \Lambda]/(p - C\Lambda^{p-1})$ and let $c, \lambda \in \mathcal{O}$ be the images of C, Λ. Let $d = (c, 1 - p^{p-1}, \dots)$ be the unique vector such that $p[\lambda] = (\lambda^p d_0, \lambda^p d_1, \dots)$, as in Corollary 6.1. Then there exists a unique endomorphism $T'_d : W \to W$ such that $F^\lambda \circ T_d = T'_d \circ F^{\lambda^p}$ as endomorphisms of the $\mathcal{O}$-group scheme W.*

Proof. Since F^{λ^p} is an epimorphism, then T'_d is unique and we only have to prove that it exists. Let $\Phi : W \to (\mathbb{G}_a)^{\mathbb{N}}$ be the Witt morphism of $\mathcal{O}$-ring schemes. Given that the schemes $\mathrm{Spec}(\mathcal{O})$ and W have no p-torsion, the morphism Φ is a monomorphism and it is enough to look for $T'_d : W \to W$ such that $\Phi \circ F^\lambda \circ T_d = \Phi \circ T'_d \circ F^{\lambda^p}$. Let f and t_d be the endomorphisms of $(\mathbb{G}_a)^\mathbb{N}$ such that $\Phi \circ F = f \circ \Phi$ and $\Phi \circ T_d = t_d \circ \Phi$. They are defined by:

- $f(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$,
- $t_d(x_0, x_1, x_2, \dots) = (y_0, y_1, y_2, \dots)$ with $y_n = d_0^{p^n} x_n + p d_1^{p^{n-1}} x_{n-1} + \dots + p^n d_n x_0$.

We first construct $t'_d : (\mathbb{G}_a)^\mathbb{N} \to (\mathbb{G}_a)^\mathbb{N}$ such that

$$\big(f - \Phi([\lambda^{p-1}]) \operatorname{Id}\big) \circ t_d = t'_d \circ \big(f - \Phi([\lambda^{(p-1)p}]) \operatorname{Id}\big).$$

Let $y = (y_0, y_1, y_2, \dots)$ be a vector of indeterminates (a point of $(\mathbb{G}_a)^\mathbb{N}$) and let us write

$$\big([f - \Phi([\lambda^{p-1}])] \circ t_d\big)(y) = (\alpha_0, \alpha_1, \alpha_2, \dots),$$

$$\Big[f - \Phi([\lambda^{(p-1)p}])\Big](y) = (\beta_0, \beta_1, \beta_2, \dots).$$

Given that $\Phi([a]) = (a, a^p, a^{p^2}, \dots)$ and that addition and multiplication in $(\mathbb{G}_a)^\mathbb{N}$ are componentwise, we compute:

$$\begin{aligned}\alpha_n =& \big(d_0^{p^{n+1}} y_{n+1} + p d_1^{p^n} y_n + \dots + p^n d_n^p y_1 + p^{n+1} d_{n+1} y_0\big) \\ & - \lambda^{p^n(p-1)} \big(d_0^{p^n} y_n + p d_1^{p^{n-1}} y_{n-1} + \dots + p^{n-1} d_{n-1}^p y_1 + p^n d_n y_0\big)\end{aligned}$$

and $\beta_n = y_{n+1} - \lambda^{p^{n+1}(p-1)} y_n$ for all $n \geqslant 0$. The existence of t'_d means that α_n is a polynomial with coefficients in $\mathcal{O}$ in the variables $\beta_0, \beta_1, \beta_2, \dots$ for each n. Since the α_n and β_n are linear in y, this in turn means that we get $\alpha_n = 0$ under the specializations

$$y_1 = \lambda^{p(p-1)} y_0 \ , \ y_2 = \lambda^{p^2(p-1)} y_1 \ , \dots , \ y_{i+1} = \lambda^{p^{i+1}(p-1)} y_i \ , \dots$$

This amounts to $y_i = \lambda^{p(p^i-1)} y_0$ for each i. Now

$$\alpha_n \Big(y_0, \lambda^{p(p-1)} y_0, \lambda^{p(p^2-1)} y_0, \lambda^{p(p^3-1)} y_0, \dots \Big)$$

is equal to y_0 times

$$\begin{aligned}&\big(d_0^{p^{n+1}} \lambda^{p(p^{n+1}-1)} + p d_1^{p^n} \lambda^{p(p^n-1)} + \dots + p^n d_n^p \lambda^{p(p-1)} + p^{n+1} d_{n+1}\big) \\ &- \lambda^{p^n(p-1)} \big(d_0^{p^n} \lambda^{p(p^n-1)} + p d_1^{p^{n-1}} \lambda^{p(p^{n-1}-1)} + \dots + p^{n-1} d_{n-1}^p \lambda^{p(p-1)} + p^n d_n\big).\end{aligned}$$

If we recall that $p\lambda^{p^i} = \lambda^{p^{i+1}} d_0^{p^i} + p\lambda^{p^i} d_1^{p^{i-1}} + \dots + p^i \lambda^p d_i$ for all i by definition of d, then we indeed find that this quantity vanishes. This proves the existence of t'_d as required. In order to find a morphism T'_d such that $\Phi \circ T'_d = t'_d \circ \Phi$, we use Bourbaki [B], § 1, no. 2, Prop. 2, applied to $t'_d \circ \Phi$, viewed as a sequence of elements in the ring $H^0(W, \mathcal{O}_W) = \mathcal{O}[Z_0, Z_1, \dots]$ endowed with the endomorphism raising each variable to the p-th power. □

Appendix A. Comparison of Ext groups in the small and the big sites

For more details on the basic facts concerning sites, we refer to [SGA4-1], Exp. IV, especially § 3 and § 4 but note that our notations may differ from *loc. cit.* For a site $\mathcal{C}$ we denote by $\mathcal{C}^\wedge$, $\mathcal{C}^\sim$, $\mathrm{Ab}(\mathcal{C})$ the corresponding categories of presheaves, sheaves and abelian sheaves, respectively. Let $\mathcal{C},\mathcal{D}$ be sites. Recall that a functor $u:\mathcal{C}\to\mathcal{D}$ is *continuous* if the pullback $u^p:\mathcal{D}^\wedge\to\mathcal{C}^\wedge$ maps sheaves to sheaves and hence induces a functor $u^s:\mathcal{D}^\sim\to\mathcal{C}^\sim$. In this case, by the general theory u^s has a left adjoint $u_s:\mathcal{C}^\sim\to\mathcal{D}^\sim$. A *morphism of sites* $f:\mathcal{D}\to\mathcal{C}$ is by definition a continuous functor $u:\mathcal{C}\to\mathcal{D}$ such that u_s is exact. We write $f^*=u_s$ and $f_*=u^s$.

Let X be a scheme. The *small flat site* X_{fl} is the category Fppf/X of flat locally finitely presented X-schemes endowed with the topology generated by the families $\{U_i\to U\}_{i\in I}$ such that $\amalg U_i\to U$ is faithfully flat and locally finitely presented. The *big flat site* X_{FL} is the category of X-schemes Sch/X with the same topology as above. The inclusion functor $u:\mathrm{Fppf}/X\to\mathrm{Sch}/X$ is continuous and the functor on sheaves u^s is exact. Let us write $f^*=u_s$ and $f_*=u^s$.

We wish to study homomorphisms and extensions between sheaves defined by flat commutative locally finitely presented X-group schemes. It is in fact better to work with algebraic spaces because they enjoy better descent properties, so that we obtain statements that not only are slightly more general but more significantly are easier to apply, even for schemes. Any X-group algebraic space G defines sheaves on X_{FL} and X_{fl}, and it is clear that the image under f_* (i.e. the restriction) of the sheaf on the big site is canonically isomorphic to the sheaf on the small site. Thus it will be notationally convenient to denote the sheaf on the big site by G and the sheaf on the small site by f_*G, thereby systematically identifying f_* of the former with the latter. For these sheaves, we have the following.

Lemma A.1. *Let G be a flat commutative locally finitely presented group algebraic space over X. Then, we have an isomorphism of functors in $\mathcal{H}\in\mathrm{Ab}(X_{\mathrm{FL}})$:*

$$f_*:\mathrm{Hom}_{\mathrm{Ab}(X_{\mathrm{FL}})}(G,\mathcal{H})\to\mathrm{Hom}_{\mathrm{Ab}(X_{\mathrm{fl}})}(f_*G,f_*\mathcal{H}).$$

In particular, the adjunction morphism $f^*f_*G\to G$ is an isomorphism.

Proof. Write G as the quotient of a scheme by an étale equivalence relation of schemes $s,t:R\rightrightarrows U$. Then U,R are flat and locally of finite presentation, hence objects of the underlying category of the site X_{fl}. Applying Yoneda's lemma first in the big site and then in the small site, we get a functorial bijection:

$$\mathrm{Hom}_{X_{\mathrm{FL}}^\sim}(U,\mathcal{H})=\mathcal{H}(U)=(f_*\mathcal{H})(U)=\mathrm{Hom}_{X_{\mathrm{fl}}^\sim}(f_*U,f_*\mathcal{H})$$

and similarly for R. These fit into a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{X_{\mathrm{FL}}^{\sim}}(U,\mathcal{H}) \arrow[r, "\sim"] \arrow[d, shift right, "s^*,t^*"'] \arrow[d, shift left] & \mathrm{Hom}_{X_{\mathrm{fl}}^{\sim}}(f_*U, f_*\mathcal{H}) \arrow[d, shift left, "s^*,t^*"] \arrow[d, shift right] \\ \mathrm{Hom}_{X_{\mathrm{FL}}^{\sim}}(R,\mathcal{H}) \arrow[r, "\sim"] & \mathrm{Hom}_{X_{\mathrm{fl}}^{\sim}}(f_*R, f_*\mathcal{H}) \end{tikzcd}$$

and we obtain an induced bijection between the equalizers

$$\mathrm{Hom}_{X_{\mathrm{FL}}^{\sim}}(G,\mathcal{H}) = \mathrm{Hom}_{X_{\mathrm{fl}}^{\sim}}(f_*G, f_*\mathcal{H}).$$

We have been dealing here with morphisms of sheaves of *sets*. The axiom for a morphism $G \to \mathcal{H}$ to be a morphism of *groups* gives a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{X_{\mathrm{FL}}^{\sim}}(G,\mathcal{H}) \arrow[r, "\sim"] \arrow[d, shift right] \arrow[d, shift left] & \mathrm{Hom}_{X_{\mathrm{fl}}^{\sim}}(f_*G, f_*\mathcal{H}) \arrow[d, shift left] \arrow[d, shift right] \\ \mathrm{Hom}_{X_{\mathrm{FL}}^{\sim}}(G\times_X G,\mathcal{H}) \arrow[r, "\sim"] & \mathrm{Hom}_{X_{\mathrm{fl}}^{\sim}}(f_*G\times f_*G, f_*\mathcal{H}) \end{tikzcd}$$

where the pairs of vertical arrows are induced by the multiplications of G and $\mathcal{H}$, and we used the fact that $f_*(G\times_X G) = f_*G \times f_*G$. We obtain an induced bijection

$$\mathrm{Hom}_{\mathrm{Ab}(X_{\mathrm{FL}})}(G,\mathcal{H}) = \mathrm{Hom}_{\mathrm{Ab}(X_{\mathrm{fl}})}(f_*G, f_*\mathcal{H}).$$

By the adjunction (f^*, f_*) this gives functorial isomorphisms

$$\mathrm{Hom}_{\mathrm{Ab}(X_{\mathrm{FL}})}(G,\mathcal{H}) = \mathrm{Hom}_{\mathrm{Ab}(X_{\mathrm{FL}})}(f^*f_*G, \mathcal{H})$$

so by Yoneda's lemma the adjunction $f^*f_*G \to G$ is an isomorphism. ☐

In particular, the lemma proves that the functor f^* from the category of flat locally finitely presented commutative group schemes over X to the category of representable abelian sheaves over X_{fl} is an equivalence of categories.

For the purposes of the present article, we need to have a similar result for the first derived functor of Hom, namely, we want an isomorphism $\mathrm{Ext}^1_{\mathrm{Ab}(X_{\mathrm{FL}})}(G,\mathcal{H}) \to \mathrm{Ext}^1_{\mathrm{Ab}(X_{\mathrm{fl}})}(f_*G, f_*\mathcal{H})$. More generally, consider the functors $T^i = \mathrm{Ext}^i_{\mathrm{Ab}(X_{\mathrm{FL}})}(\mathcal{G}, -)$ and $U^i = \mathrm{Ext}^i_{\mathrm{Ab}(X_{\mathrm{fl}})}(f_*\mathcal{G}, f_*(-))$ defined on the category of abelian sheaves $\mathrm{Ab}(X_{\mathrm{FL}})$. The sequence $\{T^i\}$ is a universal cohomological δ-functor. Using the fact that f_* is exact, one sees that $\{U^i\}$ has long exact cohomology sequences and hence is a cohomological δ-functor. If U^i did vanish on injectives for $i > 0$ then the sequence $\{U^i\}$ also would be a universal cohomological δ-functor and the isomorphism $f_* : T^0 \to U^0$ of Lemma A.1 would extend to an isomorphism $T^i \to U^i$, giving in particular the result we need for $i = 1$. However, we do not know if U^i vanishes on injectives for $i > 0$. For example the required vanishing on injectives for U^i would follow rather easily if f_* had its left adjoint f^* exact;

for then f_* would take injectives to injectives. However this is unfortunately not always the case:

Corollary A.1. *Let $X = \mathrm{Spec}(R)$ where R is a discrete valuation ring, and $X_\lambda = \mathrm{Spec}(R/\lambda R)$ where λ lies in the maximal ideal of R. Consider the X-group scheme $\mathcal{G}^\lambda = \mathrm{Spec}(R[T, 1/(1+\lambda T)])$ with multiplication law given by $(t_1, t_2) \mapsto t_1 + t_2 + \lambda t_1 t_2$. Then the image under f^* of the exact sequence on the small flat site*

$$0 \longrightarrow f_*\mathcal{G}^\lambda \longrightarrow f_*\mathbb{G}_{m,X} \longrightarrow f_* i_* \mathbb{G}_{m,X_\lambda} \longrightarrow 0$$

described in 3.1 is not exact on the left. In particular f^* is not exact and $f : X_{\mathrm{FL}} \to X_{\mathrm{fl}}$ is not a morphism of sites.

Proof. Using the adjunction $f^* f_* \to \mathrm{id}$ which according to Lemma A.1 is an isomorphism on the category of flat finitely presented group schemes, we see that the sequence $0 \to f^* f_* \mathcal{G}^\lambda \to f^* f_* \mathbb{G}_{m,X}$ is simply the sequence of sheaves on the big site $0 \to \mathcal{G}^\lambda \to \mathbb{G}_{m,X}$. This sequence is not exact, as one sees by evaluating on an X-scheme which has λ-torsion. $\square$

As we will see below, one crucial point here is that the cokernel of $f_*\mathcal{G}^\lambda \to f_*\mathbb{G}_{m,X}$ is not representable.

We shall nevertheless prove that when *both* sheaves G and $\mathcal{H}$ are representable, we have an isomorphism for the Ext^1's. The proof of this has been suggested to us by Jilong Tong.

Lemma A.2. *Let G, H be flat locally finitely presented commutative X-group algebraic spaces. Then the map*

$$f_* : \mathrm{Ext}^1_{\mathrm{Ab}(X_{\mathrm{FL}})}(G, H) \longrightarrow \mathrm{Ext}^1_{\mathrm{Ab}(X_{\mathrm{fl}})}(f_*G, f_*H)$$

is an isomorphism.

Proof. We show that f^* gives an inverse to f_*. Start from an extension $0 \to f_*H \to \mathcal{E} \to f_*G \to 0$ in the small flat site. Then $\mathcal{E}$ is representable by a flat locally finitely presented group algebraic space; this is a standard fact recalled in Lemma A.3 below. Applying f^* which is right exact and using the adjunction isomorphism (Lemma A.1), we obtain an exact sequence $H \to f^*\mathcal{E} \to G \to 0$ of sheaves on the big site. Since f_*H is the kernel of $\mathcal{E} \to f_*G$, we may write $H \simeq \mathcal{E} \times_G 0$ as group algebraic spaces, where 0 is the trivial group scheme and $0 \to G$ is the unit section. This isomorphism remains valid when we view both sides as sheaves on the big sites. This proves that H is the kernel of the map of sheaves $f^*\mathcal{E} \to G$, that is, the sequence $0 \to H \to f^*\mathcal{E} \to G \to 0$ is exact on the left also. In this way we have defined a map $f^* : \mathrm{Ext}^1_{\mathrm{Ab}(X_{\mathrm{fl}})}(f_*G, f_*H) \to \mathrm{Ext}^1_{\mathrm{Ab}(X_{\mathrm{FL}})}(G, H)$. Using the fact that the adjunctions for representable group sheaves are isomorphisms (Lemma A.1), it is immediate that f^* is an inverse for f_*. $\square$

We finish with the lemma that was used in the proof. Here, it is more convenient to denote by G, H, the sheaves on the small site defined by G and H.

Lemma A.3. *With G and H as above, any extension in the small flat site of G by H is representable by a flat locally finitely presented group algebraic space.*

Proof. This result is well-known in the big site and in fact the usual proof works in the small site. Let us write the details. Let $0 \to H \to \mathcal{E} \to G \to 0$ be such an extension. Since $\pi : \mathcal{E} \to G$ is a surjection of sheaves, there exists a faithfully flat morphism $q : Y \to G$ and an element $p \in \mathcal{E}(Y)$, viewed as a morphism $Y \to \mathcal{E}$, lifting q:

$$\begin{tikzcd} & & & Y \arrow[dl, "p"'] \arrow[d, "q"] & \\ 0 \arrow[r] & H \arrow[r] & \mathcal{E} \arrow[r, "\pi"] & G \arrow[r] & 0. \end{tikzcd}$$

Then we have a morphism $Y \times_G \mathcal{E} \to Y \times_X H$ given on the sections by $(a, b) \mapsto (a, b - p(a))$. It is easy to see that this is an isomorphism. This means that $\mathcal{E} \to G$ is an H-torsor in the small flat site. The result follows, because by Artin's theorem all torsors under a flat locally finitely presented group algebraic space are again flat locally finitely presentated algebraic spaces. □

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