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## The distribution of second $p$ -class groups on coclass graphs

par DANIEL C. MAYER

*Dedicated to the memory of Emil Artin*

RÉSUMÉ. Des concepts et des stratégies généraux sont développés pour identifier le type d'isomorphisme du deuxième  $p$ -groupe de classes  $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ , c'est donc le groupe de Galois du deuxième  $p$ -corps de classes de Hilbert  $\mathbb{F}_p^2(K)$ , d'un corps de nombres  $K$ , pour un nombre premier  $p$ . Le type d'isomorphisme détermine la position de  $G$  sur un des graphes de coclasses  $\mathcal{G}(p, r)$ ,  $r \geq 0$ , dans le sens de Eick, Leedham-Green et Newman. Il est démontré que, pour des types spéciaux du corps de base  $K$  et de son  $p$ -groupe de classes  $\text{Cl}_p(K)$ , la position de  $G$  est limitée à certaines branches d'arbres de coclasses admissibles par des règles de sélection. Une compréhension plus profonde, en particulier la densité de population de sommets particuliers au niveau des graphes de coclasses, est acquise en calculant la distribution effective des deuxièmes  $p$ -groupes de classes  $G$  pour diverses séries de corps de nombres  $K$  ayant des  $p$ -groupes de classes  $\text{Cl}_p(K)$  de type fixe et  $p \in \{2, 3, 5, 7\}$ .

ABSTRACT. General concepts and strategies are developed for identifying the isomorphism type of the second  $p$ -class group  $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$ , that is the Galois group of the second Hilbert  $p$ -class field  $\mathbb{F}_p^2(K)$ , of a number field  $K$ , for a prime  $p$ . The isomorphism type determines the position of  $G$  on one of the coclass graphs  $\mathcal{G}(p, r)$ ,  $r \geq 0$ , in the sense of Eick, Leedham-Green, and Newman. It is shown that, for special types of the base field  $K$  and of its  $p$ -class group  $\text{Cl}_p(K)$ , the position of  $G$  is restricted to certain admissible branches of coclass trees by selection rules. Deeper insight, in particular, the density of population of individual vertices on coclass graphs, is gained by computing the actual distribution of second  $p$ -class groups  $G$  for various series of number fields  $K$  having  $p$ -class groups  $\text{Cl}_p(K)$  of fixed type and  $p \in \{2, 3, 5, 7\}$ .

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*Mots clefs.*  $p$ -class groups,  $p$ -class field tower, principalization of  $p$ -classes, quadratic fields, cubic fields, dihedral fields, metabelian  $p$ -groups, coclass graphs.

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### 1. Introduction

Let  $p$  denote a prime and let  $K$  be an algebraic number field. By the Hilbert  $p$ -class field  $F_p^1(K)$  of  $K$  we understand the maximal abelian unramified  $p$ -extension of  $K$ . The Hilbert  $p$ -class field tower, briefly  $p$ -tower, of  $K$ ,  $F_p^0(K) \leq F_p^1(K) \leq F_p^2(K) \leq \dots$ , is defined recursively by  $F_p^0(K) = K$  and  $F_p^n(K) = F_p^1(F_p^{n-1}(K))$ , for  $n \geq 1$ . According to the uniqueness theorem of class field theory, all members of the  $p$ -tower are Galois extensions of  $K$ , and their union  $F_p^\infty(K) = \cup_{n=0}^\infty F_p^n(K)$  is the maximal unramified pro- $p$  extension of  $K$ . Let  $Cl_p(K)$  be the  $p$ -class group of  $K$ , that is the Sylow  $p$ -subgroup of the class group  $Cl(K)$ . If  $Cl_p(K) = 1$  is trivial, then  $F_p^1(K) = K$  and the  $p$ -tower of  $K$  has length  $\ell_p(K) = 0$ . If  $Cl_p(K) \neq 1$  and  $F_p^{n-1}(K) < F_p^n(K) = F_p^{n+1}(K)$ , for some  $n \geq 1$ , the  $p$ -tower is finite of length  $\ell_p(K) = n$ . Otherwise the  $p$ -tower of  $K$  is infinite and  $F_p^\infty(K)$  is an infinite Galois extension of  $K$ , having a pro- $p$  group  $G_p^\infty(K) = \text{Gal}(F_p^\infty(K)|K)$  as Galois group, endowed with the Krull topology. For each  $n \geq 1$ , the finite quotient  $G_p^n(K) = \text{Gal}(F_p^n(K)|K) = G_p^\infty(K) / (G_p^\infty(K))^{(n)}$  of the  $p$ -tower group  $G_p^\infty(K)$  by the closed subgroup  $(G_p^\infty(K))^{(n)} = \text{Gal}(F_p^\infty(K)|F_p^n(K))$  is called the  $n$ th  $p$ -class group of  $K$ , in analogy to  $G_p^1(K) = \text{Gal}(F_p^1(K)|K)$ , which is isomorphic to the (first)  $p$ -class group  $Cl_p(K)$  of  $K$ , by Artin's reciprocity law. All these higher  $p$ -class groups  $G_p^n(K)$  of  $K$  with  $n \geq 2$  are usually non-abelian and share two essential common invariants, as the following theorem shows. The germs of these general concepts are contained in Artin's famous papers [2, 3].

**Theorem 1.1.** *Suppose that  $n \geq 2$  and  $G = G_p^n(K)$ . For any subgroup  $H \leq G$  which contains the commutator subgroup  $G'$  of  $G$ , there exists a unique intermediate field  $K \leq L \leq F_p^1(K)$  such that  $H = \text{Gal}(F_p^n(K)|L)$ , and the following statements hold.*

- (1) *The abelianization  $H/H'$  is isomorphic to the  $p$ -class group  $Cl_p(L)$ . In particular,*

$$G/G' \simeq Cl_p(K) \text{ and } G'/G'' \simeq Cl_p(F_p^1(K)).$$

- (2) *The kernel  $\ker(T_{G,H})$  of the transfer  $T_{G,H} : G/G' \rightarrow H/H'$  is isomorphic to the  $p$ -principalization kernel of  $K$  in  $L$ , that is, the kernel  $\ker(j_{L|K})$  of the natural class extension homomorphism  $j_{L|K} : Cl_p(K) \rightarrow Cl_p(L)$ . In particular,*

$$\ker(T_{G,G}) \simeq \ker(j_{K|K}) = 1$$

and

$$\ker(T_{G,G'}) = G/G' \simeq \ker(j_{F_p^1(K)|K}) = Cl_p(K).$$

*Proof.* Since  $H$  contains  $G'$ ,  $H$  is a normal subgroup of  $G = \text{Gal}(\mathbb{F}_p^n(K)|K)$ . The intermediate field  $K \leq L \leq \mathbb{F}_p^1(K)$  of degree  $[L : K] = (G : H)$ , such that  $H = \text{Gal}(\mathbb{F}_p^n(K)|L)$ , is determined uniquely as the fixed field  $\text{Fix}(H)$  within  $\mathbb{F}_p^n(K)$ , by the Galois correspondence. From the viewpoint of class field theory, the norm class group  $N_{L|K}(\text{Cl}_p(L))$  of  $L|K$  is isomorphic to  $H/G' = \text{Gal}(\mathbb{F}_p^n(K)|L)/\text{Gal}(\mathbb{F}_p^n(K)|\mathbb{F}_p^1(K)) \simeq \text{Gal}(\mathbb{F}_p^1(K)|L)$  and thus of index  $[L : K]$  in  $\text{Cl}_p(K) \simeq \text{Gal}(\mathbb{F}_p^1(K)|K)$ , as required.

(1) We have

$$H/H' = \text{Gal}(\mathbb{F}_p^n(K)|L)/\text{Gal}(\mathbb{F}_p^n(K)|\mathbb{F}_p^1(L)) \simeq \text{Gal}(\mathbb{F}_p^1(L)|L) \simeq \text{Cl}_p(L),$$

by the Galois correspondence and Artin's reciprocity law [2].

(2) The isomorphism  $\ker(T_{G,H}) \simeq \ker(j_{L|K})$  is a consequence of the commutativity of the diagram in Table 1.1, which was proved by Artin [3] and investigated in more detail by Miyake [65]. The special case  $\ker(T_{G,G'}) = G/G'$  is the principal ideal theorem [38]. □

TABLE 1.1. Class extension homomorphism  $j_{L|K}$  and transfer  $T_{G,H}$

		$j_{L K}$		
	$\text{Cl}_p(K)$	$\longrightarrow$	$\text{Cl}_p(L)$	
Artin isomorphism	$\updownarrow$	$///$	$\updownarrow$	Artin isomorphism
	$G/G'$	$\longrightarrow$	$H/H'$	
		$T_{G,H}$		

Since each finite quotient  $G = G_p^n(K)$  of the  $p$ -tower group  $G_p^\infty(K)$  of  $K$ , which is isomorphic to the inverse limit  $\varprojlim_{n \geq 1} G_p^n(K)$ , behaves in the same manner with respect to the kernels  $\ker(T_{G,H})$  and targets  $H/H'$  of the transfers  $T_{G,H} : G/G' \rightarrow H/H'$ , for  $G' \leq H \leq G$ , we define two invariants  $\tau(K) = \tau(G)$  and  $\varkappa(K) = \varkappa(G)$  either of the entire  $p$ -tower of  $K$  or of the individual  $n$ th  $p$ -class group  $G$ .

**Definition 1.1.** Let  $p$  be a prime and  $K$  be a number field.

- (1) The family  $\tau(K) = (\text{Cl}_p(L))_{K \leq L \leq \mathbb{F}_p^1(K)}$  of  $p$ -class groups of all intermediate fields  $L$  between  $K$  and  $\mathbb{F}_p^1(K)$  is called *transfer target type*, briefly TTT, of the  $p$ -tower of  $K$ .
- (2) The family  $\varkappa(K) = (\ker(j_{L|K}))_{K \leq L \leq \mathbb{F}_p^1(K)}$  of  $p$ -principalization kernels of  $K$  in all intermediate fields  $L$  between  $K$  and  $\mathbb{F}_p^1(K)$  is called *transfer kernel type*, briefly TKT, of the  $p$ -tower of  $K$ .

In general, third and higher  $p$ -class groups  $G_p^n(K)$  of  $K$ ,  $n \geq 3$ , are non-metabelian with rather complex structure. For this reason, we focus our investigation on the *second*  $p$ -class group  $G = G_p^2(K) = \text{Gal}(\mathbb{F}_p^2(K)|K)$  which is *metabelian* with commutator subgroup

$$G' = \text{Gal}(\mathbb{F}_p^2(K)|\mathbb{F}_p^1(K)) \simeq \text{Cl}_p(\mathbb{F}_p^1(K)).$$

It is the simplest group admitting the calculation of the TTT,  $\tau(K) = (H/H')_{G' \leq H \leq G}$ , and the TKT,  $\varkappa(K) = (\ker(T_{G,H}))_{G' \leq H \leq G}$ , of  $K$  by means of the transfers from  $G$  to the subgroups  $H \leq G$  containing  $G'$ .

**1.1. Identifying  $G_p^2(K)$  via  $\varkappa(K)$  and  $\tau(K)$ .** First, we illustrate that second  $p$ -class groups  $G_p^2(K)$  of number fields  $K$  can frequently but not always be identified uniquely by means of TKT and TTT.

For  $p = 5$ , we apply our recently calculated TKTs of § 3.5 to prove the following Theorem. It gives criteria for second 5-class groups of number fields in terms of TKTs which were unknown up to now.

**Theorem 1.2.** *Let  $K$  be an arbitrary number field with 5-class group  $\text{Cl}_5(K)$  of type  $(5, 5)$ . In the following four cases, the second 5-class group  $G_5^2(K)$  of  $K$  is determined uniquely by the TKT and TTT of  $K$ .*

- (1)  $\varkappa(K) = (1, 2, 3, 4, 5, 6)$  (*identity*),  $\tau(K) = ((5, 5, 5)^6) \implies G_5^2(K) \simeq \langle 3125, 14 \rangle$ .
- (2)  $\varkappa(K) = (1, 2, 5, 3, 6, 4)$  (*4-cycle*),  $\tau(K) = ((5, 25)^4, (5, 5, 5)^2) \implies G_5^2(K) \simeq \langle 3125, 11 \rangle$ .
- (3)  $\varkappa(K) = (5, 1, 2, 6, 4, 3)$  (*6-cycle*),  $\tau(K) = ((5, 25)^6) \implies G_5^2(K) \simeq \langle 3125, 12 \rangle$ .
- (4)  $\varkappa(K) = (3, 1, 2, 5, 6, 4)$  (*two 3-cycles*),  $\tau(K) = ((5, 25)^6) \implies G_5^2(K) \simeq \langle 3125, 9 \rangle$ .

*In one case, there are two possibilities for  $G_5^2(K)$ .*

$\varkappa(K) = (6, 1, 2, 4, 3, 5)$  (*5-cycle*),  $\tau(K) = ((5, 25)^5, (5, 5, 5)) \implies$  either  $G_5^2(K) \simeq \langle 3125, 8 \rangle$  or  $G_5^2(K) \simeq \langle 3125, 13 \rangle$ .

*The powers in TTTs denote iteration and the 5-groups are identified by their numbers in the SmallGroups library [14].*

*Proof.* In section § 3.5, we prove that the metabelianization  $G_5^2(K) = \text{Gal}(\mathbb{F}_5^2(K)|K) = G/G''$  of the 5-tower group  $G = G_5^\infty(K)$  of any algebraic number field  $K$  with 5-class group  $\text{Cl}_5(K)$  of type  $(5, 5)$ , having one of the five pairs of TKT and TTT in Theorem 1.2, is one of the six terminal top vertices  $\langle 3125, 8 \dots 9 \rangle$  and  $\langle 3125, 11 \dots 14 \rangle$  of the coclass graph  $\mathcal{G}(5, 2)$  in Figure 3.8. □

**Remark.** We conjecture that the TKT alone suffices for the characterization of  $G_5^2(K)$  in Theorem 1.2.

**Example 1.1.** Discriminants  $D$  with smallest absolute values of complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  having one of the five pairs of TKT and TTT in Theorem 1.2 are given by  $-89751, -37363, -11199, -17944, -12451$ , in the same order. They were computed by means of MAGMA [56]. According to section § 3.5.3, the vertices  $\langle 3125, 8 \rangle$  and  $\langle 3125, 13 \rangle$  of coclass graph  $\mathcal{G}(5, 2)$  in Figure 3.8, corresponding to the last case of Theorem 1.2, are populated by 167 occurrences (17.4%) of 959 complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  with  $-2\,270\,831 \leq D < 0$  and  $\text{Cl}_5(K)$  of type  $(5, 5)$ . This shows that even the last case alone occurs with rather high density.

For  $p = 3$ , we use four TKTs which occurred repeatedly in the literature [72, 45, 25]. These TKTs define infinite sequences, in fact periodic coclass families (§ 2), of possible groups  $G_3^2(K)$ , and neither Heider and Schmithals [45] nor Brink and Gold [24, 25] have been aware that the TTT is able to identify a unique member of the sequences, as we proved in [58, 60].

**Theorem 1.3.** *Let  $K$  be an arbitrary number field with 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$ . In the following four cases, the second 3-class group  $G_3^2(K)$  of  $K$  is either determined uniquely or up to the sign of the relational exponent  $\gamma$  by the TKT and the parametrized TTT of  $K$ , for each integer  $j \geq 2$ .*

- (1)  $\varkappa(K) = (1, 3, 1, 3), \tau(K) = ((3^j, 3^{j+1}), (3, 9)^2, (3, 3, 3)) \implies G_3^2(K) \simeq G_0^{2j+2, 2j+3}(1, -1, 1, 1).$
- (2)  $\varkappa(K) = (2, 3, 1, 3), \tau(K) = ((3^j, 3^{j+1}), (3, 9)^2, (3, 3, 3)) \implies G_3^2(K) \simeq G_0^{2j+2, 2j+3}(0, -1, \pm 1, 1).$
- (3)  $\varkappa(K) = (1, 2, 3, 1), \tau(K) = ((3^j, 3^{j+1}), (3, 9)^3) \implies G_3^2(K) \simeq G_0^{2j+2, 2j+3}(1, 0, -1, 1).$
- (4)  $\varkappa(K) = (2, 2, 3, 1), \tau(K) = ((3^j, 3^{j+1}), (3, 9)^3) \implies G_3^2(K) \simeq G_0^{2j+2, 2j+3}(0, 0, \pm 1, 1).$

The groups in the first two cases are located on the coclass tree  $\mathcal{T}(\langle 243, 6 \rangle)$  in Fig. 3.6, the groups in the last two cases on the coclass tree  $\mathcal{T}(\langle 243, 8 \rangle)$  in Fig. 3.7. 3-groups of order  $3^n$  and index  $m$  of nilpotency are identified by their parametrized presentations given in the form  $G_p^{m,n}(\alpha, \beta, \gamma, \delta)$  in § 3.3.3.

*Proof.* We proved that the second derived quotient  $G_3^2(K) = G/G''$  of the 3-tower group  $G = G_3^\infty(K)$  of any algebraic number field  $K$  with 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$ , transfer kernel type E.6,  $\varkappa(K) = (1, 3, 1, 3)$ , resp. E.14,  $\varkappa(K) = (2, 3, 1, 3)$  [59, Tbl. 6, p. 492], and parametrized transfer target type  $\tau(K) = ((3^j, 3^{j+1}), (3, 9)^2, (3, 3, 3))$ ,  $j \geq 2$ , is isomorphic to the unique group  $G_0^{2j+2, 2j+3}(1, -1, 1, 1)$ , resp. to one of the two groups  $G_0^{2j+2, 2j+3}(0, -1, \pm 1, 1)$ , of the coclass tree  $\mathcal{T}(\langle 243, 6 \rangle)$  in Figure 3.6 [60, Thm. 4.4, Tbl. 8].

Similarly, we proved for transfer kernel type E.8,  $\varkappa(K) = (1, 2, 3, 1)$ , resp. E.9,  $\varkappa(K) = (2, 2, 3, 1)$ , and parametrized transfer target type  $\tau(K) = ((3^j, 3^{j+1}), (3, 9)^3)$ ,  $j \geq 2$ , that  $G_3^2(K)$  is isomorphic to the unique group  $G_0^{2j+2, 2j+3}(1, 0, -1, 1)$ , resp. to one of the two groups  $G_0^{2j+2, 2j+3}(0, 0, \pm 1, 1)$ , of the coclass tree  $\mathcal{T}(\langle 243, 8 \rangle)$  in Figure 3.7.  $\square$

**Example 1.2.** By [58, Thm. 5.2, p. 492], the complex quadratic field  $K = \mathbb{Q}(\sqrt{-9748})$  is a number field having the TKT and TTT of the last case in Theorem 1.3, actually with smallest absolute discriminant. It was first mentioned by Scholz and Taussky [72, p. 25]. Among the 93 complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  with discriminants  $-6 \cdot 10^4 < D < 0$  and  $\text{Cl}_3(K)$  of type  $(3, 3)$ , there are 11 cases (12%) having the TKT and TTT of the last case in Theorem 1.3. So even the last case alone occurs quite frequently.

In contrast, we can also prove that certain metabelian  $p$ -groups are excluded as second  $p$ -class groups for special base fields. The following negative result for  $p = 3$  gives an exact justification for a particular instance of our *weak leaf conjecture*, Cnj. 3.1.

**Theorem 1.4.** *The 3-group  $\langle 243, 4 \rangle$ , resp.  $\langle 243, 9 \rangle$ , cannot occur as second 3-class group  $G_3^2(K)$  for a complex quadratic field  $K = \mathbb{Q}(\sqrt{D})$ ,  $D < 0$ , whose TKT and TTT are given by  $\varkappa(K) = (4, 4, 4, 3)$  and  $\tau(K) = ((3, 9), (3, 3, 3)^3)$ , resp.  $\varkappa(K) = (2, 1, 4, 3)$  and  $\tau(K) = ((3, 9)^4)$ .*

*Proof.* The assumption that one of the groups  $g \in \{\langle 243, 4 \rangle, \langle 243, 9 \rangle\}$  were the second 3-class group  $g = G_3^2(K)$  of a complex quadratic field  $K$  implies two contradictory consequences. On the one hand, both groups  $g$  are of class  $\text{cl}(g) = 3$ , whence the fourth lower central  $\gamma_4(g) = 1$  is trivial. According to Heider and Schmithals [45, p. 20], any number field  $K$  whose second  $p$ -class group  $g = G_p^2(K)$  has trivial  $\gamma_4(g) = 1$  possesses a  $p$ -tower of length  $\ell_p(K) = 2$ . On the other hand, since  $K$  is complex quadratic, its 3-tower group  $G$  must be a Schur  $\sigma$ -group [74], [50, p. 58]. The Schur multiplier  $H_2(g, \mathbb{Z})$  of both groups  $g$  is non-trivial of order 3, as can be verified by means of GAP [40]. Hence they cannot be Schur  $\sigma$ -groups [20, p. 6]. Therefore, the 3-tower of  $K$  cannot stop at the second stage,  $G \neq G_3^2(K)$ , and  $G$  must be a non-metabelian group of derived length at least 3, that is, the 3-tower of  $K$  has length  $\ell_3(K) \geq 3$ .  $\square$

**1.2. Length of  $p$ -towers.** As the following Theorems 1.5–1.6 show, the length  $\ell_p(K)$  of the  $p$ -tower of  $K$  can either be determined exactly or at least be estimated by a lower bound, once the second  $p$ -class group  $G_p^2(K)$  of  $K$  and its properties are known in sufficient detail.

A criterion for 3-towers of exact length 2 was proved in three independent ways by Scholz and Taussky [72], by Heider and Schmithals [45], and by

Brink and Gold [24, 25]. With our new methods, we can give a short proof of this criterion.

**Theorem 1.5.** *Let  $K$  be an arbitrary number field with 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$ . In the following two cases, the second 3-class group  $G_3^2(K)$  of  $K$  and the TTT of  $K$  are determined uniquely by the TKT of  $K$ .*

- (1)  $\varkappa(K) = (2, 2, 4, 1) \implies G_3^2(K) \simeq \langle 243, 5 \rangle, \tau(K) = ((3, 9)^3, (3, 3, 3))$ .
- (2)  $\varkappa(K) = (4, 2, 2, 4) \implies G_3^2(K) \simeq \langle 243, 7 \rangle, \tau(K) = ((3, 9)^2, (3, 3, 3)^2)$ .

*In both cases, the 3-class field tower of  $K$  has exact length  $\ell_3(K) = 2$ .*

*Proof.* The metabelianization  $G_3^2(K) \simeq G/G''$  of the 3-tower group  $G = G_3^\infty(K)$  of any algebraic number field  $K$  with 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$  having transfer kernel type D.10,  $\varkappa(K) = (2, 2, 4, 1)$ , resp. D.5,  $\varkappa(K) = (4, 2, 2, 4)$  [59, Tbl. 6, p. 492], is isomorphic to the terminal top vertex  $\langle 243, 5 \rangle$ , resp.  $\langle 243, 7 \rangle$ , of the sporadic part  $\mathcal{G}_0(3, 2)$  of the coclass graph  $\mathcal{G}(3, 2)$  in Figure 3.5, according to Nebelung [66, Thm. 6.14, p. 208]. In [60, Thm. 4.2, Tbl. 4] it is shown that the corresponding transfer target type is given by  $\tau(K) = ((3, 9)^3, (3, 3, 3))$ , resp.  $\tau(K) = ((3, 9)^2, (3, 3, 3)^2)$ .

According to the proof of [20, Thm. 4.2, p. 14],  $\langle 243, 5 \rangle$  and  $\langle 243, 7 \rangle$  are Schur  $\sigma$ -groups [1, 26].

However, independently from  $K$  being complex quadratic or not, when the second derived quotient  $G/G'' \simeq G_3^2(K)$  of  $G$  is a Schur  $\sigma$ -group, then the 3-tower group  $G$  of  $K$  must be isomorphic to it,  $G \simeq G_3^2(K)$ , by the argument given in [22, Lem. 4.10]. Consequently, the 3-tower of  $K$  stops at the second stage and has length  $\ell_3(K) = 2$ . □

**Remark.** We point out that the figure in [20, p. 10] is not a coclass graph in our sense (§ 2.2), since it contains vertices of four different coclass graphs  $\mathcal{G}(3, r)$ ,  $1 \leq r \leq 4$ , partially connected by edges of depth 2. The top level of this figure, where  $\langle 243, 5 \rangle$  and  $\langle 243, 7 \rangle$  are emphasized by surrounding circles, coincides with the top vertices of our Figure 3.5.

**Example 1.3.** Discriminants  $D$  with smallest absolute values of complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  having one of the two TKTs in Theorem 1.5 are given by  $-4027, -12131$ , in the same order. The first was communicated in [72, p. 22] (see also [11, Cor. 3.3, p. 164]), the second in [45, p. 19]. Corresponding minimal discriminants of real quadratic fields are 422573, 631769 [58, Tbl. 4, p. 498]. Among the 2020 complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  with discriminants  $-10^6 < D < 0$  and  $\text{Cl}_3(K)$  of type  $(3, 3)$ , there are 936 cases (46.3%) having one of the two pairs of TKT and TTT in Theorem 1.5 [58, Tbl. 3, p. 497]. So these types of fields are definitely among the high-champs with respect to density of population.



In the next Theorem, the second 3-class group  $G_3^2(K)$  is not at all determined by the TKT  $\varkappa(K)$  alone. Furthermore, we must restrict ourselves to an estimate of the 3-tower length  $\ell_3(K) \geq 3$ .

**Theorem 1.6.** *Let  $K = \mathbb{Q}(\sqrt{D})$ ,  $D < 0$ , be a complex quadratic field with 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$ . In the following two cases, the second 3-class group  $G_3^2(K)$  of  $K$  is determined uniquely by the TKT and the TTT of  $K$ .*

- (1)  $\varkappa(K) = (4, 4, 4, 3)$ ,  $\tau(K) = ((3, 9), (3, 3, 3)^3) \implies G_3^2(K) \simeq \langle 729, 45 \rangle$ .
- (2)  $\varkappa(K) = (2, 1, 4, 3)$ ,  $\tau(K) = ((3, 9)^4) \implies G_3^2(K) \simeq \langle 729, 57 \rangle$ .

*In both cases,  $K$  has a 3-class field tower of length  $\ell_3(K) \geq 3$ .*

*Proof.* We proved that the second derived quotient  $G_3^2(K) = G/G''$  of the 3-tower group  $G = G_3^\infty(K)$  of a complex quadratic field  $K$  with 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$ , transfer kernel type H.4,  $\varkappa(K) = (4, 4, 4, 3)$ , resp. G.19,  $\varkappa(K) = (2, 1, 4, 3)$  [59, Tbl. 6, p. 492], and transfer target type  $\tau(K) = ((3, 9), (3, 3, 3)^3)$ , resp.  $\tau(K) = ((3, 9)^4)$  [60, Thm. 4.3, Tbl. 6], is isomorphic to the unique vertex  $\langle 729, 45 \rangle$ , resp.  $\langle 729, 57 \rangle$ , of the sporadic part  $\mathcal{G}_0(3, 2)$  of coclass graph  $\mathcal{G}(3, 2)$  in Figure 3.5. For an arbitrary number field  $K$ , several other candidates for  $G_3^2(K)$  are possible. However, for a complex quadratic field  $K$ ,  $\langle 243, 45 \rangle$ , resp.  $\langle 243, 57 \rangle$ , are discouraged by Theorem 1.4, and the siblings  $\langle 729, 44 \rangle$  and  $\langle 729, 46 \dots 47 \rangle$ , resp.  $\langle 729, 56 \rangle$ , of  $\langle 729, 45 \rangle$ , resp.  $\langle 729, 57 \rangle$ , do not admit the mandatory automorphism of order 2 acting as inversion on the abelianization.

Since  $K$  is complex quadratic, its 3-tower group  $G$  must be a Schur  $\sigma$ -group [74], [50, p. 58]. However, neither  $\langle 729, 45 \rangle$  nor  $\langle 729, 57 \rangle$  is a Schur  $\sigma$ -group [20, p. 6], because the Schur multiplier is non-trivial of type  $(3, 3)$ , as can be verified with the aid of GAP [40]. Therefore, the 3-tower of  $K$  cannot stop at the second stage,  $G \neq G_3^2(K)$ , and  $G$  must be a non-metabelian group of derived length at least 3, that is, the 3-tower has length  $\ell_3(K) \geq 3$ .  $\square$

**Example 1.4.** Discriminants  $D$  with smallest absolute values of complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  having one of the two pairs of TKT and TTT in Theorem 1.6 are given by  $-3896$ ,  $-12067$ , in the same order. They were communicated in [45, p. 19] (see also [11, p. 165]). Among the 2020 complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  with discriminants  $-10^6 < D < 0$  and  $\text{Cl}_3(K)$  of type  $(3, 3)$ , there are 391 cases (19.4%) having one of the two pairs of TKT and TTT in Theorem 1.6 [58, Tbl. 3, p. 497]. This shows that fields with 3-towers of at least three stages occur quite frequently.

Note that the proofs of the preceding Theorems 1.5, and 1.6 are very brief. This is the beginning of powerful new methods of research concerning the maximal unramified pro- $p$  extensions of number fields by joining the class theory of finite  $p$ -groups and suitable generalizations of Schur  $\sigma$ -groups

[21]. We optimistically expect further prolific impact of these new foundations on the investigation of  $p$ -towers and  $p$ -principalization, although Artin called the capitulation problem ‘hopeless’.

**1.3. Overview.** In §§ 3.2 and 3.4, we analyze number fields  $K$  with  $p$ -class group  $\text{Cl}_p(K)$  of type  $(p, p)$ . Based on [58], we prove that the  $p$ -class numbers  $h_p(L_i) = |\text{Cl}_p(L_i)|$  of *two distinguished* intermediate fields  $L_i$ ,  $1 \leq i \leq 2$ , lying strictly between  $K$  and  $F_p^1(K)$ , and the  $p$ -class number  $h_p(F_p^1(K))$  of the Hilbert  $p$ -class field of  $K$ , that is, the orders of three special members of the TTT  $\tau(K)$ , contain sufficient information for determining the order  $|G| = 3^n$ , class  $c = \text{cl}(G)$ , coclass  $r = \text{cc}(G)$ , and the so-called *defect of commutativity*  $k = k(G)$  of the second  $p$ -class group  $G = G_p^2(K)$  of  $K$ . These invariants are related by the equation  $n = \text{cl}(G) + \text{cc}(G)$  and restrict  $G$  to the *finite* subset of groups of equal order  $3^n$  of the coclass graph  $\mathcal{G}(p, r)$ . If the TKT  $\varkappa(K)$  is known additionally, the position of  $G$  can be restricted further, either to a branch  $\mathcal{B}$  of a coclass tree  $\mathcal{T}$ , forming a subgraph of  $\mathcal{G}(p, r)$ , or even to a unique isomorphism type of metabelian  $p$ -groups.

Group theoretic foundations concerning coclass graphs and their mainlines, parametrized presentations, polarization, and defect are provided in preliminary sections §§ 2, 3.1, and 3.3.

In [60], it was shown for number fields  $K$  of type  $(p, p)$ , that the abelian type invariants of the  $p$ -class groups  $\text{Cl}_p(L_i)$  of *all* intermediate fields  $K < L_i < F_p^1(K)$ ,  $1 \leq i \leq p + 1$ , that is, the structures of the *first layer* of the TTT  $\tau(K)$ , usually determine the TKT  $\varkappa(K)$ , at least in the case that  $G_p^2(K)$  is one of the most densely populated metabelian  $p$ -groups.

The density of population of a metabelian  $p$ -group  $G$  by second  $p$ -class groups  $G_p^2(K)$  of certain base fields  $K$  can be calculated explicitly from a purely group theoretic probability measure by non-abelian generalizations of the Cohen-Lenstra-Martinet asymptotic, as developed recently by Boston, Bush, Hajir [20], and also by Bembom [12], resp. Boy [23], under supervision by Mihailescu, resp. Malle. The heuristic is in good accordance with our extensive computational results for quadratic base fields in [58]. Our results have in fact actually been used in [12, pp. 5, 126]. Further, they eliminate all incomplete IPADs (index- $p$  abelianization data), which coincide with the first layer of our TTTs, and correct the frequencies given in [20, Tbl. 1–2, pp. 17–18], which are uniformly slightly too low.

It is to be expected that similar strategies, exploiting the interplay between TTT and TKT, but now extended to the *higher layers* of these invariants, can be used to identify the isomorphism type of the second  $p$ -class group  $G = G_p^2(K)$  of number fields  $K$  with more complicated  $p$ -class group  $\text{Cl}_p(K)$ , for example of type  $(p^2, p)$  or  $(p, p, p)$ . Extensions in this direction

will be presented in subsequent papers, e. g. [61]. An outlook is given in section § 4.

## 2. Visualizing finite $p$ -groups on coclass graphs

**2.1. Periodic patterns.** An important purpose of this paper is to emphasize that coclass graphs  $\mathcal{G}(p, r)$  are particularly well suited for visualizing periodic properties [30, 33] of infinite sequences of finite  $p$ -groups  $G$ , such as parametrized power-commutator presentations [15, 66], automorphism groups  $\text{Aut}(G)$ , Schur multipliers  $H_2(G, \mathbb{Z}_p)$  and other cohomology groups of  $G$ , transfer kernel types  $\varkappa(G)$  [59], transfer target types  $\tau(G)$  [60], and defect of commutativity  $k(G)$  expressed by the depth  $\text{dp}(G)$  (Corollaries 3.1.1 and 3.10.1). In number theoretic applications, selection rules for second  $p$ -class groups  $G = G_p^2(K)$  of special base fields  $K$  [58] are additional periodic properties. Computational results on the density of distribution of second  $p$ -class groups can also be represented very clearly on coclass graphs.

**2.2. Coclass graphs.** For a given prime  $p$ , Leedham-Green and Newman [53] have defined the structure of a directed graph  $\mathcal{G}(p)$  on the set of all isomorphism classes of finite  $p$ -groups. Two vertices are connected by a directed edge  $H \rightarrow G$  if  $G$  is isomorphic to the last lower central quotient  $H/\gamma_c(H)$  of  $H$ , where  $c$  denotes the nilpotency class  $\text{cl}(H)$  of  $H$ .

If the condition  $|H| = p|G|$  is imposed on the edges,  $\mathcal{G}(p)$  is partitioned into countably many disjoint subgraphs  $\mathcal{G}(p, r)$ ,  $r \geq 0$ , called *coclass graphs* of  $p$ -groups  $G$  of coclass  $r = \text{cc}(G) = n - \text{cl}(G)$ , where  $|G| = p^n$  [52, p. 155, 166]. A coclass graph  $\mathcal{G}(p, r)$  is a forest of finitely many coclass trees  $\mathcal{T}_i$ , each with a single infinite mainline having a pro- $p$  group of coclass  $r$  as its inverse limit, and additionally contains a set  $\mathcal{G}_0(p, r)$  of finitely many sporadic groups outside of coclass trees,  $\mathcal{G}(p, r) = (\cup_i \mathcal{T}_i) \cup \mathcal{G}_0(p, r)$ .

The terminology concerning the structure of coclass graphs  $\mathcal{G}(p, r)$  with a prime  $p \geq 2$  and an integer  $r \geq 0$  must be recalled briefly. We adopt the most recent view of coclass graphs, which is given by Eick and Leedham-Green [33], and by Dietrich, Eick, Feichtenschlager [27, p. 46].

- The *coclass*  $\text{cc}(G)$  of a finite  $p$ -group  $G$  of order  $|G| = p^n$  and nilpotency class  $\text{cl}(G)$  is defined by  $n = \text{cl}(G) + \text{cc}(G)$ .
- By a *vertex* of the coclass graph  $\mathcal{G}(p, r)$  we understand the isomorphism class of a finite  $p$ -group  $G$  of coclass  $\text{cc}(G) = r$ .
- The vertex  $H$  is an *immediate descendant* of the vertex  $G$ , if  $G$  is isomorphic to the last lower central quotient  $H/\gamma_c(H)$  of  $H$ , where  $c = \text{cl}(H)$  denotes the nilpotency class of  $H$ , and  $\gamma_c(H)$  is cyclic of order  $p$ , that is,  $\text{cl}(H) = 1 + \text{cl}(G)$  and  $|H| = p|G|$ . In this case,  $H$  and  $G$  are connected by a *directed edge*  $H \rightarrow G$  of the coclass graph and  $G$  is called the *parent*  $G = \pi(H)$  of  $H$ .

- A *capable vertex* has at least one immediate descendant, whereas a *terminal vertex* has no immediate descendants.
- The vertex  $G_m$  is a *descendant* of the vertex  $G_0$ , if there is a *path*  $(G_j \rightarrow G_{j-1})_{m \geq j \geq 1}$  of directed edges from  $G_m$  to  $G_0$ . In particular, the vertex  $G_0$  is descendant of itself, with empty path.
- The *tree*  $\mathcal{T}(G)$  with root  $G$  consists of all descendants of the vertex  $G$ .
- A *coclass tree* is a maximal rooted tree containing exactly one infinite path.
- The *mainline*  $(M_{j+1} \rightarrow M_j)_{j \geq n}$  of a coclass tree  $\mathcal{T}(M_n)$  with root  $M_n$  of order  $|M_n| = p^n$  is its unique maximal infinite path. The projective limit  $S = \varprojlim_{j \geq n} M_j$ , is an infinite pro- $p$  group, whose finite quotients by closed subgroups return the mainline vertices  $M_j$ .
- For  $i \geq n$ , the *branch*  $\mathcal{B}(M_i)$  of a coclass tree  $\mathcal{T}(M_n)$  with tree root  $M_n$  and mainline  $(M_{j+1} \rightarrow M_j)_{j \geq n}$  is the difference set  $\mathcal{T}(M_i) \setminus \mathcal{T}(M_{i+1})$ . The branch  $\mathcal{B}(M_i)$  is briefly denoted by  $\mathcal{B}_i$  and we assume that the order of the branch root  $M_i$  is  $|M_i| = p^i$ .
- The *depth*  $\text{dp}(G) = m - j$  of a vertex  $G$  of order  $|G| = p^m$  on a branch  $\mathcal{B}(M_j)$  of a coclass tree is its distance from the branch root  $M_j$  of order  $|M_j| = p^j$  on the mainline. For  $d \geq 1$ ,  $\mathcal{B}_d(M_j)$  denotes the *pruned branch of bounded depth  $d$*  with root  $M_j$ .
- The *periodic sequence*  $\mathcal{S}(G)$  of a vertex  $G \in \mathcal{B}_d(M_i)$  of order  $|G| = p^m$ ,  $i \leq m \leq i + d$ , on a coclass tree of  $\mathcal{G}(p, r)$ , where  $M_i$  denotes the vertex of order  $p^i$  on the mainline and  $i$  is sufficiently large so that periodicity has set in already [33], is the infinite sequence  $(G_{m+j\ell})_{j \geq 0}$  of vertices defined recursively by  $G_m = G$  and  $G_{m+j\ell} = \varphi_{i+(j-1)\ell}(G_{m+(j-1)\ell})$ , for  $j \geq 1$ , using the periodicity isomorphisms [33, Thm. 27–28, p. 286] of graphs  $\varphi_{i+(j-1)\ell} : \mathcal{B}_d(M_{i+(j-1)\ell}) \rightarrow \mathcal{B}_d(M_{i+j\ell})$  with period length  $\ell$ , which is a divisor of  $p^{r+1}(p-1)$ .

### 3. $p$ -Groups with single layered abelianization of type $(p, p)$

**3.1. Metabelian  $p$ -groups  $G$  of coclass  $\text{cc}(G) = 1$ .** For an arbitrary prime  $p \geq 2$ , let  $G$  be a metabelian  $p$ -group of order  $|G| = p^n$  and nilpotency class  $\text{cl}(G) = n - 1$ , where  $n \geq 3$ . In the terminology of Blackburn [15] and Miech [64],  $G$  is of maximal class, that is, of coclass  $\text{cc}(G) = 1$ , whence the commutator factor group  $G/G'$  of  $G$  is of type  $(p, p)$ . The converse is only true for  $p = 2$ : A 2-group  $G$  with  $G/G' \simeq (2, 2)$  is of coclass 1, a fact which is usually attributed to Taussky [75]. The lower central series of  $G$  is defined recursively by  $\gamma_1(G) = G$  and  $\gamma_j(G) = [\gamma_{j-1}(G), G]$  for  $j \geq 2$ . Nilpotency of  $G$  is expressed by  $\gamma_{n-1}(G) > \gamma_n(G) = 1$ .

**3.1.1. Polarization and defect.** The *two-step centralizer*  $\chi_2(G) = \{g \in G \mid [g, u] \in \gamma_4(G) \text{ for all } u \in \gamma_2(G)\}$  of the two-step factor group  $\gamma_2(G)/\gamma_4(G)$ , which can also be defined by

$$\chi_2(G)/\gamma_4(G) = \text{Centralizer}_{G/\gamma_4(G)}(\gamma_2(G)/\gamma_4(G)),$$

is the largest subgroup of  $G$  such that  $[\chi_2(G), \gamma_2(G)] \leq \gamma_4(G)$ . It is characteristic, contains the commutator subgroup  $\gamma_2(G)$ , and coincides with  $G$  if and only if  $n = 3$ . For  $n \geq 4$ ,  $\chi_2(G)$  is one of the maximal subgroups  $(H_i)_{1 \leq i \leq p+1}$  of  $G$  and causes a *polarization* among them, which will be standardized in Definition 3.1. Let the isomorphism invariant  $k = k(G)$  of  $G$  be defined by

$$[\chi_2(G), \gamma_2(G)] = \gamma_{n-k}(G),$$

where  $k = 0$  for  $n = 3$ ,  $0 \leq k \leq n-4$  for  $n \geq 4$ , and  $0 \leq k \leq \min\{n-4, p-2\}$  for  $n \geq p+1$ , according to Miech [64, p. 331].  $k(G)$  provides a measure for the deviation from the maximal degree of commutativity  $[\chi_2(G), \gamma_2(G)] = 1$  and will be called *defect of commutativity* of  $G$ .

**3.1.2. Parametrized presentation.** Suppose that generators of  $G = \langle x, y \rangle$  are selected such that  $x \in G \setminus \chi_2(G)$ , if  $n \geq 4$ , and  $y \in \chi_2(G) \setminus \gamma_2(G)$ , and define the main commutator by  $s_2 = [y, x] \in \gamma_2(G)$  and the higher commutators by  $s_j = [s_{j-1}, x] = s_{j-1}^{x-1} \in \gamma_j(G)$  for  $j \geq 3$ . We use identifiers  $s_j$  to emphasize those elements of  $G$  for which addition of symbolic exponents  $f_1, f_2$  in the group ring  $\mathbb{Z}[G]$  is commutative,  $s_j^{f_1+f_2} = s_j^{f_2+f_1}$ . Nilpotency of  $G$  is expressed by  $s_n = 1$  and a *power-commutator presentation* of  $G$  with generators  $x, y, s_2, \dots, s_{n-1}$  is given as follows. There are two relations for  $p$ th powers of the generators  $x$  and  $y$  of  $G$ ,

(3.1)

$$x^p = s_{n-1}^w \quad \text{and} \quad y^p \prod_{\ell=2}^p s_\ell^{\binom{p}{\ell}} = s_{n-1}^z \quad \text{with exponents } 0 \leq w, z \leq p-1,$$

according to Miech [64, p. 332, Thm. 2, (3)]. Blackburn uses the notation  $\delta = w$  and  $\gamma = z$  for these relational exponents [15, p. 84, (36), (37)].

Additionally, the group  $G$  satisfies relations for  $p$ th powers of the higher commutators,

$$s_{j+1}^p \prod_{\ell=2}^p s_{j+\ell}^{\binom{p}{\ell}} = 1 \quad \text{for } 1 \leq j \leq n-2,$$

and the commutator relation of Miech [64, p. 332, Thm. 2, (2)], containing the defect  $k = k(G)$ ,

$$(3.2) \quad [y, s_2] = \prod_{\ell=1}^k s_{n-\ell}^{a(n-\ell)} \in [\chi_2(G), \gamma_2(G)] = \gamma_{n-k}(G),$$

with exponents  $0 \leq a(n - \ell) \leq p - 1$  for  $1 \leq \ell \leq k$ , and  $a(n - k) > 0$ , if  $k \geq 1$ . Blackburn restricts his investigations to  $k \leq 2$  and uses the notation  $\beta = a(n - 1)$  and  $\alpha = a(n - 2)$  [15, p. 82, (33)].

By  $G_a^n(z, w)$  we denote the representative of an isomorphism class of metabelian  $p$ -groups  $G$  of coclass  $\text{cc}(G) = 1$  and order  $|G| = p^n$ , which satisfies the relations (3.1) and (3.2) with a fixed system of exponents  $a = (a(n - k), \dots, a(n - 1)), w$ , and  $z$ . We have  $a = 0$  if and only if  $k = 0$ .

**3.1.3. A distinguished maximal subgroup.** Since the maximal normal subgroups  $H_i$ ,  $1 \leq i \leq p + 1$ , of  $G$  contain the commutator subgroup  $G'$  as a normal subgroup of index  $p$ , they are of the shape  $H_i = \langle g_i, G' \rangle$  with suitable generators  $g_i$ , and we can arrange them in a fixed order.

**Definition 3.1.** The *polarization* or *natural order* of the maximal subgroups  $(H_i)_{1 \leq i \leq p+1}$  of  $G$  is given by the *distinguished first generator*  $g_1 = y \in \chi_2(G)$  and the other generators  $g_i = xy^{i-2} \notin \chi_2(G)$  for  $2 \leq i \leq p + 1$ , provided that  $|G| \geq p^4$ . Then, in particular  $\chi_2(G) = H_1 = \langle y, G' \rangle$ .

**3.1.4. Parents of CF groups.** Together with group counts in Blackburn's theorems [15, p. 88, Thm. 4.1–4.3], Theorem 3.1 describes the structure of the *metabelian skeleton* of the unique coclass tree  $\mathcal{T}(C_p \times C_p)$  [28, § 1, p. 851] of the coclass graph  $\mathcal{G}(p, 1)$  with an arbitrary prime  $p \geq 2$ . The graph consists of all isomorphism classes of CF groups (with cyclic factors) [4, § 4, p. 264] of coclass 1.

**Theorem 3.1.** *Let  $p \geq 2$  be an arbitrary prime, and  $G$  be a metabelian  $p$ -group of coclass  $\text{cc}(G) = 1$  having defect of commutativity  $k = k(G)$ , such that  $G \simeq G_a^n(z, w)$  with parameters  $n \geq 3$ ,  $a = (a(n - k), \dots, a(n - 1))$ ,  $0 \leq a(n - k), \dots, a(n - 1), w, z < p$ , where  $a(n - k) > 0$ , if  $k \geq 1$ , that is,  $G$  is of order  $|G| = p^n$  and nilpotency class  $\text{cl}(G) = n - 1$ . Then the parent  $\pi(G)$  of  $G$  on the coclass tree  $\mathcal{T}(C_p \times C_p)$  is given by*

$$\pi(G) \simeq \begin{cases} C_p \times C_p, & \text{if } n = 3 \text{ (and thus } k = 0), \\ G_0^{n-1}(0, 0), & \text{if } n \geq 4, k = 0, \\ G_0^{n-1}(0, 0), & \text{if } n \geq 5, k = 1 \text{ (and thus } p \geq 3), \\ G_{\tilde{a}}^{n-1}(0, 0), & \text{where } \tilde{a} = (a(n - k), \dots, a(n - 2)), \\ & \text{if } n \geq 6, k \geq 2 \text{ (and thus } p \geq 5). \end{cases}$$

**Remark.** The various cases of Theorem 3.1 can be described as follows.

- (1) In the first case,  $n = 3$ , where  $G \simeq G_0^3(0, w)$  with  $0 \leq w \leq 1$  is an extra-special  $p$ -group of order  $p^3$ , the parent  $\pi(G)$  is the abelian

root  $C_p \times C_p$  of the tree  $\mathcal{T}(C_p \times C_p)$ , which can formally be viewed as  $G_0^2(0, 0)$ .

- (2) In the second and third case of a group  $G$  of defect  $k \leq 1$ , the parent  $\pi(G)$  is a mainline group.
- (3) In the last case of a group  $G$  of higher defect  $k \geq 2$ , which can occur only for  $p \geq 5$ , the parent  $\pi(G)$  lies outside of the mainline and the defect  $\tilde{k}$  and family  $\tilde{a}$  of relational exponents of  $\pi(G)$  are given by  $\tilde{k} = k - 1$  and  $\tilde{a} = (\tilde{a}((n - 1) - (k - 1)), \dots, \tilde{a}((n - 1) - 1)) = (a(n - k), \dots, a(n - 2))$ , where  $\tilde{a}((n - 1) - (k - 1)) > 0$ . We point out that the parent is always characterized by parameters  $\tilde{z} = 0$  and  $\tilde{w} = 0$ .

*Proof.* For  $n = 3$ ,  $G$  is an extra special  $p$ -group of nilpotency class  $\text{cl}(G) = n - 1 = 2$  having the commutator subgroup  $\gamma_2(G)$  as its last (non-trivial) lower central  $\gamma_{n-1}(G)$ . In this special case, the definition of the parent  $\pi(G) = G/\gamma_{n-1}(G)$  of  $G$  yields the abelianization  $\pi(G) = G/\gamma_2(G)$  of type  $(p, p)$ , which is isomorphic to the root  $C_p \times C_p$  of  $\mathcal{T}(C_p \times C_p)$ .

For  $n \geq 4$ ,  $G$  can be assumed to be isomorphic to a group  $G \simeq G_a^n(z, w)$  with pc-presentation consisting of the relations (3.1) and (3.2) for the two generators  $x, y$ ,

$$x^p = s_{n-1}^w, \quad y^p \prod_{\ell=2}^p s_{\ell}^{\binom{p}{\ell}} = s_{n-1}^z, \quad [y, s_2] = \prod_{\ell=1}^k s_{n-\ell}^{a(n-\ell)}.$$

Since the parent  $\pi(G) = G/\gamma_{n-1}(G)$  of  $G$  is defined as the last lower central quotient, we denote the left coset of an element  $g \in G$  with respect to  $\gamma_{n-1}(G)$  by  $\bar{g} = g \cdot \gamma_{n-1}(G)$  and we obtain  $\bar{s}_{n-1} = 1$ , because  $\gamma_{n-1}(G) = \langle s_{n-1} \rangle$ . Therefore, the nilpotency class of the parent is  $\text{cl}(\pi(G)) = \text{cl}(G) - 1 = n - 2$  and a pc-presentation of  $\pi(G)$  is given by

$$\bar{x}^p = \bar{s}_{n-1}^w = 1, \quad \bar{y}^p \prod_{j=2}^p \bar{s}_j^{\binom{p}{j}} = \bar{s}_{n-1}^z = 1, \quad [\bar{y}, \bar{s}_2] = \prod_{\ell=1}^k \bar{s}_{n-\ell}^{a(n-\ell)},$$

where the last product equals 1, if  $k \leq 1$ , and  $\prod_{\ell=2}^k \bar{s}_{n-\ell}^{a(n-\ell)} \neq 1$ , if  $k \geq 2$ , because  $\bar{s}_{n-k}^{a(n-k)} \neq 1$ . Since the order of the parent is  $|\pi(G)| = |G| : |\gamma_{n-1}(G)| = p^n : p = p^{n-1}$ , the coclass remains the same  $\text{cc}(\pi(G)) = n - 1 - \text{cl}(\pi(G)) = n - 1 - (n - 2) = 1 = \text{cc}(G)$ .  $\square$

The following principle, that the kernel  $\varkappa(1)$  of the transfer from  $G$  to the first distinguished maximal subgroup  $H_1 = \chi_2(G)$  decides about the relation between depth  $\text{dp}(G)$  and defect  $k = k(G)$  of  $G$ , will turn out to be crucial for metabelian  $p$ -groups  $G$  of coclass  $\text{cc}(G) \geq 2$ , too.

**Corollary 3.1.1.** *For a metabelian  $p$ -group  $G$  of coclass  $\text{cc}(G) = 1$  with defect of commutativity  $k = k(G)$ , the depth  $\text{dp}(G)$  of  $G$  on the coclass tree  $\mathcal{T}(C_p \times C_p)$  of  $\mathcal{G}(p, 1)$  is given by*

$$\text{dp}(G) = \begin{cases} k + 1, & \text{if } \varkappa(1) \neq 0, \\ k, & \text{if } \varkappa(1) = 0, \end{cases}$$

*with respect to the natural order of the maximal subgroups of  $G$ .*

*Proof.* Theorem 3.1 shows that  $(G_0^n(0, 0))_{n \geq 2}$  is the mainline of the coclass tree  $\mathcal{T}(C_p \times C_p)$ , consisting of all groups  $G$  of depth  $\text{dp}(G) = 0$  and defect  $k = 0$ , because each of these vertices occurs as a parent and possesses infinitely many descendants, whereas the groups  $G_a^n(0, 0)$  with  $a \neq 0, k \geq 1$  can only have finitely many descendants, due to the bound  $k \leq p - 2$  by Miech [64]. Since the defect of any group  $G_a^n(z, w)$  with parameter  $a = 0$  is given by  $k = 0$ , all the other groups  $G = G_0^n(z, w), (z, w) \neq (0, 0)$ , which contain  $H_1$  as an abelian maximal subgroup, must be located as terminal vertices at depth  $\text{dp}(G) = 1 = k + 1$ , because they never occur as a parent. On the other hand, the third and fourth case of Theorem 3.1 show that the relation between the defects of parent  $\pi(G)$  and immediate descendant  $G$  is given by  $\tilde{k} = k - 1$  for any group  $G = G_a^n(z, w), a \neq 0$ , with positive defect  $k \geq 1$ , whence the depth, being the number of steps required to reach the mainline by successive construction of parents,  $(G, \pi(G), \pi^2(G), \dots)$ , is given by  $\text{dp}(G) = k$ . Finally, the groups  $G = G_0^n(z, w), (z, w) \neq (0, 0)$ , containing the abelian maximal subgroup  $H_1$ , are characterized uniquely by a partial transfer  $\varkappa(1) \neq 0$  to the distinguished maximal subgroup  $H_1$ , according to [59, Thm. 2.5–2.6].  $\square$

We conjecture that the following property of mainline groups of  $\mathcal{G}(p, 1)$  might be true for mainline groups on any coclass tree of  $\mathcal{G}(p, r), r \geq 1$ .

**Corollary 3.1.2.** *Mainline groups of  $\mathcal{G}(p, 1)$ , that is, groups of depth  $\text{dp}(G) = 0$ , must have a total transfer  $\varkappa(1) = 0$  to the distinguished maximal subgroup  $H_1 = \chi_2(G)$ . The converse is only true for  $p = 2$ : A 2-group  $G \in \mathcal{T}(C_2 \times C_2)$  having  $\varkappa(1) = 0$  is mainline.*

*Proof.* The statement for  $p \geq 2$  is an immediate consequence of Corollary 3.1.1 and it only remains to prove the converse for  $p = 2$ . This, however, is contained in [59, Thm. 2.6, p. 481].  $\square$

Concerning the transfer kernel type  $\varkappa(G)$  of a  $p$ -group  $G$  of coclass 1 we can state:

**Corollary 3.1.3.** *The transfer kernel types of groups on the unique coclass tree  $\mathcal{T}(C_p \times C_p)$  of coclass graph  $\mathcal{G}(p, 1)$  are given by the following rules.*



- (1) The root  $C_p \times C_p$  is of TKT a.1 ( $0^{p+1}$ ) for any prime  $p \geq 2$ . The extra-special group  $G_0^3(0, 1)$  is of TKT A.1 ( $1^{p+1}$ ) for odd  $p \geq 3$ , and of TKT Q.5 (123) for  $p = 2$ . In the sequel, these exceptions are excluded.
- (2) Mainline groups are of TKT a.1 ( $0^{p+1}$ ) for odd  $p \geq 3$ , and of TKT d.8 (032) for  $p = 2$ .
- (3) Groups of depth 1 and defect 0 are of TKT either a.2 ( $1, 0^p$ ) or a.3 ( $2, 0^p$ ) for  $p \geq 3$ , and of TKT either Q.6 (132) or S.4 (232) for  $p = 2$ .
- (4) Groups of positive defect  $1 \leq k \leq p - 2$  are exclusively of TKT a.1 ( $0^{p+1}$ ).

*Proof.* This is a result of combining Theorem 3.1 with Theorems 2.5 and 2.6 in [59]. □

**3.2. Second  $p$ -class groups  $G = G_p^2(K)$  of coclass  $\text{cc}(G) = 1$ .**

**3.2.1. Weak transfer target type  $\tau_0(G)$  expressed by  $p$ -class numbers.** The group theoretic information on the second  $p$ -class group  $G = G_p^2(K)$ , that is, order, class, coclass, and defect, is contained in the  $p$ -class numbers of the distinguished extension  $L_1$  and of the Hilbert  $p$ -class field  $F_p^1(K)$ . Additionally, the principalization  $\kappa(1)$  of  $K$  in the distinguished extension  $L_1$  determines the connection between defect and depth of  $G$ .

**Theorem 3.2.** *Let  $K$  be an arbitrary number field with  $p$ -class group  $\text{Cl}_p(K)$  of type  $(p, p)$ . Suppose that the second  $p$ -class group  $G = \text{Gal}(F_p^2(K)|K)$  is abelian or metabelian of coclass  $\text{cc}(G) = 1$  with defect  $k = k(G)$ , order  $|G| = p^n$ , and class  $\text{cl}(G) = n - 1$ , where  $n \geq 2$ . With respect to the natural order among the maximal subgroups of  $G$ , the weak transfer target type  $\tau_0(G)$  of  $G$ , that is, the family of  $p$ -class numbers of the multiplet  $(L_1, \dots, L_{p+1})$  of unramified cyclic extension fields of  $K$  of relative prime degree  $p \geq 2$  is given for the first layer by*

$$\tau_0(G) = (h_p(L_1), h_p(L_2), \dots, h_p(L_{p+1})) = \begin{cases} (\overbrace{p \dots p}^{p+1 \text{ times}}), & \text{if } n = 2, \\ (p^{\text{cl}(G)-k}, \overbrace{p^2, \dots, p^2}^{p \text{ times}}), & \text{if } n \geq 3, \end{cases}$$

where defect  $k$  and depth  $\text{dp}(G)$  of  $G$  are related by

$$k = \begin{cases} \text{dp}(G) - 1, & \text{if } \varkappa(1) \neq 0, \\ \text{dp}(G), & \text{if } \varkappa(1) = 0, \end{cases}$$

and for the single member of the second layer by

$$h_p(F_p^1(K)) = p^{\text{cl}(G)-1}.$$

*Proof.* The statement is a succinct version of [58, Thm. 3.2], expressed by concepts more closely related to the position of  $G$  on the coclass graph  $\mathcal{G}(p, 1)$  and to the transfer kernel type  $\varkappa(G)$  of  $G$ , using Corollary 3.1.1.  $\square$

**Remark.** Whereas  $h_p(L_2), \dots, h_p(L_{p+1})$  only indicate that  $\text{cc}(G) = 1$ , the  $p$ -class number  $h_p(\mathbb{F}_p^1(K))$  of the Hilbert  $p$ -class field of  $K$  determines the order  $p^n$ ,  $n = \text{cl}(G) + 1$ , and class of  $G$ , and the distinguished  $h_p(L_1)$  gives the defect  $k$  of  $G$ .

With respect to the mainline  $(M_j)_{j \geq 2}$  of the coclass tree  $\mathcal{T}(C_p \times C_p)$ , the order  $|M_i| = 3^i$  of the branch root  $M_i$  of  $G$  is given by  $i = n - \text{dp}(G) = \text{cl}(G) + 1 - \text{dp}(G)$ , where

$$\text{dp}(G) = \begin{cases} k, & \text{if } \varkappa(1) = 0, \\ k + 1, & \text{if } \varkappa(1) \neq 0. \end{cases}$$

**3.2.2. The complete coclass graph  $\mathcal{G}(2, 1)$ .** We start our investigation of special cases by showing that the distribution of second 2-class groups  $G_2^2(K)$  of complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D < 0$ , with  $\text{Cl}_2(K) \simeq (2, 2)$  on  $\mathcal{G}(2, 1)$  is not restricted by selection rules. This distribution will only be given qualitatively, without exact counts.

**Theorem 3.3.** *The diagram in Figure 3.1 visualizes the complete coclass graph  $\mathcal{G}(2, 1)$  up to order  $2^8 = 256$ . It is periodic with length 1. The first period consists of branch  $\mathcal{B}_3$ , whereas branch  $\mathcal{B}_2$  is irregular and forms the pre-period.*

*Proof.*  $\mathcal{G}(2, 1)$  begins with two abelian groups of order  $2^2$ , the isolated cyclic group  $C_4$ , having different abelianization, and Klein’s four group  $V_4$ , that is the bicyclic root  $C_2 \times C_2$  of the unique coclass tree  $\mathcal{T}(C_2 \times C_2)$ .

As immediate descendants of the root,  $\mathcal{G}(2, 1)$  contains the capable mainline group  $D(8)$  and the terminal group  $Q(8)$ , both of order  $2^3$ .

Applying Blackburn’s results [15] on counts of metabelian  $p$ -groups of maximal class and order  $p^n$  with  $n \geq 4$ , to the special case  $p = 2$ , we only need to consider metabelian groups containing an abelian maximal subgroup, characterized by defect  $k = 0$ . They consist of the capable mainline group  $D(2^n) = G_0^n(0, 0)$ , the terminal group  $Q(2^n) = G_0^n(0, 1)$ , and the terminal group  $S(2^n) = G_0^n(1, 0)$ , which is expressed by specialization of [15, p. 88, Thm. 4.3] to  $p = 2$ . The count is independent from  $n$ , yielding the constant number  $2 + (n - 2, p - 1) = 3$ .

$\square$

We recall from [59] that the transfer kernel types  $\varkappa(G)$  for  $p$ -groups of coclass  $\text{cc}(G) = 1$  are exceptional in the case  $p = 2$ , compared to the uniform standard case of odd primes  $p \geq 3$ .

**Theorem 3.4.** *Table 3.1 gives the transfer kernel type  $\varkappa(G)$  of all non-isolated vertices  $G$ , having abelianization  $G/G' \simeq (2, 2)$ , on the coclass graph  $\mathcal{G}(2, 1)$ . The 2-groups  $G$  are identified by their Blackburn invariants  $|G| = 2^n$  and  $a, w, z$  as exponents in the relations (3.1) and (3.2). The graph information gives the depth  $\text{dp}(G)$  and the location of each 2-group  $G$  with respect to the unique coclass tree  $\mathcal{T}(C_2 \times C_2)$  of  $\mathcal{G}(2, 1)$ .*

*The mainline, consisting of the dihedral 2-groups  $D(2^n) = G_0^n(0, 0)$  including the abelian root  $C_2 \times C_2 = D(4) = G_0^2(0, 0)$ , is characterized by the total transfer  $\varkappa(1) = 0$  to the distinguished maximal subgroup  $H_1 = \chi_2(G)$ . Total transfers  $\varkappa(i) = 0$  are counted by  $\nu(G)$ .*

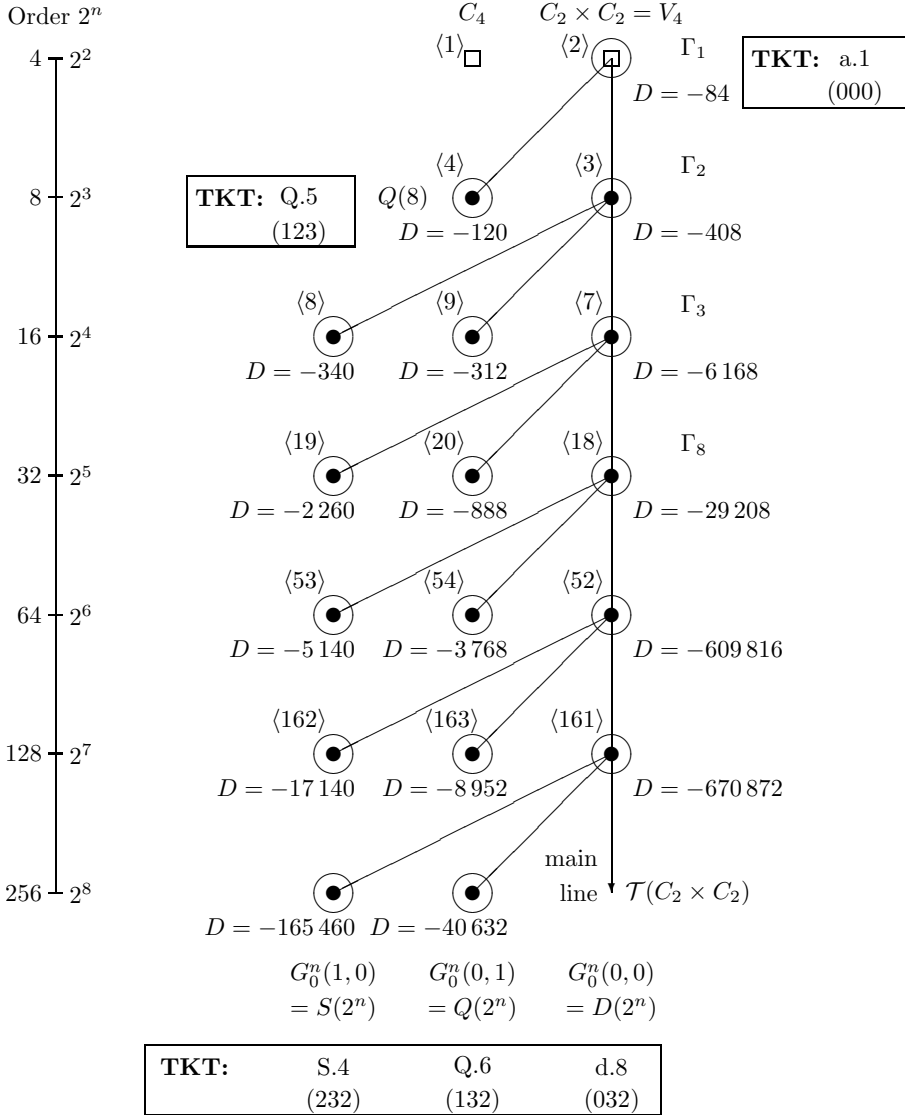
TABLE 3.1.  $\varkappa(G), \nu(G)$  in dependence on non-isolated  $G \in \mathcal{G}(2, 1)$

2-group $G_a^n(z, w)$ of coclass 1							graph information		transfer kernels		
$G$	$\text{cl}(G)$	$n$	$a$	$z$	$w$	$k$	$\text{dp}(G)$	tree position	TKT	$\varkappa(G)$	$\nu(G)$
$C_2 \times C_2$	1	2	0	0	0	0	0	root	a.1	(000)	3
$Q(8)$	2	3	0	0	1	0	1	pre-period	Q.5	(123)	0
$D(2^n)$	$\geq 2$	$\geq 3$	0	0	0	0	0	mainline	d.8	(032)	1
$Q(2^n)$	$\geq 3$	$\geq 4$	0	0	1	0	1	periodic sequence	Q.6	(132)	0
$S(2^n)$	$\geq 3$	$\geq 4$	0	1	0	0	1	periodic sequence	S.4	(232)	0

*Proof.* See [59, Thm. 2.6, Tbl. 2–3] for the technique of determining kernels of transfers and the definition of transfer kernel types as orbits of integer triplets  $[0, 3]^3$  under the action of the symmetric group of degree 3.  $\square$

The statements of Theorem 3.4 can be visualized very conveniently by the diagram of a finite part of the coclass graph  $\mathcal{G}(2, 1)$ , which is shown in Figure 3.1. It contains the isolated vertex  $C_4$ , the root  $C_2 \times C_2$  and branches  $\mathcal{B}_j, 2 \leq j \leq 7$ , of the coclass tree  $\mathcal{T}(C_2 \times C_2)$ . Branch  $\mathcal{B}_2$  consists of two initial exceptions, the elementary abelian bicyclic 2-group  $C_2 \times C_2$  with TKT a.1,  $\varkappa = (000)$ , and the quaternion group  $Q(8)$  with TKT Q.5,  $\varkappa = (123)$ . Periodicity of length  $\ell = 1$  sets in with branch  $\mathcal{B}_3$  which consists of the starting vertices of three periodic sequences,  $\mathcal{S}(D(8))$ , the mainline of dihedral groups,  $\mathcal{S}(Q(16))$ , the sequence of *generalized quaternion* groups, and  $\mathcal{S}(S(16))$ , the sequence of *semi-dihedral* groups. Transfer kernel types (TKT) in the bottom rectangle concern all vertices in the periodic sequence located vertically above. Large contour squares  $\square$  denote abelian groups and big full discs  $\bullet$  denote metabelian groups with defect  $k = 0$ . A number in angles gives the identifier of a group in the SmallGroups Library [14]. The symbols  $\Gamma_s$  denote isoclinism families given by Hall and Senior [43]. The population of each vertex is indicated by a surrounding circle labelled by the discriminant  $D < 0$  of a suitable complex quadratic field  $K = \mathbb{Q}(\sqrt{D})$  of type  $(2, 2)$ . There are no selection rules for  $p = 2$  and the numerical results

FIGURE 3.1. Population of  $\mathcal{G}(2, 1)$  by groups  $G_2^2(K)$  of  $K = \mathbb{Q}(\sqrt{D})$ ,  $D < 0$



suggest the conjecture that the tree  $\mathcal{T}(V_4)$  is covered entirely by second 2-class groups  $G_2^2(K)$  of complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D < 0$ . Ground states are due to Kisilevsky [49, p. 277–278]. All excited states have been determined with the aid of Theorem 3.2. See [58, § 9]. Here we refrain from giving the exact distribution up to some bound for  $|D|$  and we do not claim that the given examples have minimal absolute discriminants.

**3.2.3. Selection Rule for quadratic base fields.** Let  $K = \mathbb{Q}(\sqrt{D})$  be a quadratic number field with discriminant  $D$  and  $p$ -class group  $\text{Cl}_p(K)$  of type  $(p, p)$ , where  $p \geq 3$  denotes an odd prime. Then the  $p + 1$  unramified cyclic extension fields  $(L_1, \dots, L_{p+1})$  of  $K$  of relative prime degree  $p$  have dihedral absolute Galois groups  $\text{Gal}(L_i|K)$  of degree  $2p$ , according to [58, Prop. 4.1].

**Theorem 3.5.** *Let  $G = \text{Gal}(\mathbb{F}_p^2(K)|K)$  be the second  $p$ -class group of  $K$ . If  $G \in \mathcal{G}(p, 1)$ , then  $K$  must be real quadratic,  $D > 0$ , and, with respect to the natural order of the maximal subgroups of  $G$ , the family of  $p$ -class numbers of the non-Galois subfields  $K_i$  of  $L_i$  is given by*

$$(\mathfrak{h}_p(K_1), \mathfrak{h}_p(K_2), \dots, \mathfrak{h}_p(K_{p+1})) = (p^{\frac{\text{cl}(G) - \text{dp}(G)}{2}}, \overbrace{p, \dots, p}^{p \text{ times}}),$$

where depth and defect  $k$  of  $G$  are related via the first component of the TKT  $\varkappa(K)$  by

$$\text{dp}(G) = \begin{cases} k, & \text{if } \varkappa(1) = 0, \\ k + 1, & \text{if } \varkappa(1) \neq 0. \end{cases}$$

Consequently, the order  $|M_i| = p^i$  of the branch root  $M_i$  of  $G \in \mathcal{B}(M_i)$  on the unique coclass tree of  $\mathcal{G}(p, 1)$  with mainline  $(M_j)_{j \geq 2}$  must have odd exponent

$$i = n - \text{dp}(G) = \text{cl}(G) + 1 - \text{dp}(G) \equiv 1 \pmod{2}.$$

**Remark.** Whereas  $\mathfrak{h}_p(K_2), \dots, \mathfrak{h}_p(K_{p+1})$  do not give any information, the distinguished  $p$ -class number  $\mathfrak{h}_p(K_1)$  enforces the congruence  $\text{cl}(G) - \text{dp}(G) \equiv 0 \pmod{2}$ .

*Proof.* The statement is a compact version of [58, Thm. 4.1], expressed by the depth  $\text{dp}(G)$ , and thus more closely related to the position of  $G$  on the coclass graph  $\mathcal{G}(p, 1)$  and to the transfer kernel type  $\varkappa(G)$  of  $G$ , using Corollary 3.1.1. □

**Theorem 3.6.** *Table 3.2 gives the transfer kernel type (TKT)  $\varkappa(G)$  of all non-isolated metabelian vertices  $G$  on the coclass graph  $\mathcal{G}(p, 1)$ , for odd  $p \geq 3$ . The  $p$ -groups  $G$  are identified by their Blackburn-Miech invariants  $|G| = p^n$  and  $a, w, z$  as exponents in the relations (3.1) and (3.2). The graph information gives the depth  $\text{dp}(G)$  and the location of each  $p$ -group  $G$  with respect to the unique coclass tree  $\mathcal{T}(C_p \times C_p)$  of  $\mathcal{G}(p, 1)$ .*

*The mainline, consisting of the  $p$ -groups  $G_0^n(0, 0)$  including the abelian root  $C_p \times C_p = G_0^2(0, 0)$ , and all groups of positive defect  $k \geq 1$  are characterized by the total transfer  $\varkappa(1) = 0$  to the distinguished maximal subgroup  $H_1 = \chi_2(G)$ .*

TABLE 3.2.  $\varkappa(G)$ ,  $\nu(G)$  in dependence on non-isolated  $G \in \mathcal{G}(p, 1)$  for  $p \geq 3$

$p$ -Group $G_a^n(z, w)$ of Coclass 1							graph information		transfer kernels		
$G$	$\text{cl}(G)$	$n$	$a$	$z$	$w$	$k$	$\text{dp}(G)$	tree position	TKT	$\varkappa(G)$	$\nu(G)$
$C_p \times C_p$	1	2				0	0	root	a.1	$\overbrace{(0 \dots 0)}^{p+1 \text{ times}}$	$p+1$
$G_0^3(0, 1)$	2	3	0	0	1	0	1	pre-period	A.1	$\overbrace{(1 \dots 1)}^{p+1 \text{ times}}$	0
$G_0^n(0, 0)$	$\geq 2$	$\geq 3$	0	0	0	0	0	mainline	a.1	$\overbrace{(0 \dots 0)}^{p+1 \text{ times}}$	$p+1$
$G_0^n(0, 1)$	$\geq 3$	$\geq 4$	0	0	1	0	1	periodic sequences	a.2	$\overbrace{(1 0 \dots 0)}^{p \text{ times}}$	$p$
$G_0^n(z, 0)$	$\geq 3$	$\geq 4$	0	$\neq 0$	0	0	1	periodic sequences	a.3	$\overbrace{(2 0 \dots 0)}^{p \text{ times}}$	$p$
$G_a^n(z, w)$	$\geq 4$	$\geq 5$	$\neq 0$			$\geq 1$	$\geq 1$	periodic sequences	a.1	$\overbrace{(0 \dots 0)}^{p+1 \text{ times}}$	$p+1$

*Proof.* See [59, Thm. 2.5, Tab. 1]. □

**3.2.4. The complete coclass graph  $\mathcal{G}(3, 1)$ .** This section and the following sections §§ 3.2.5–3.2.8 will show, that second  $p$ -class groups  $G_p^2(K)$  of real quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D > 0$ , with  $\text{Cl}_p(K) \simeq (p, p)$  are only distributed on odd branches of the metabelian skeleton of  $\mathcal{G}(p, 1)$ , for an odd prime  $p \geq 3$ , in contrast to the complete population of the coclass graph  $\mathcal{G}(2, 1)$ . The effect is due to the number theoretic selection rule in Theorem 3.5. The quantitative distribution for  $p \in \{3, 5, 7\}$  reveals a dominant population of ground states and decreasing frequency of hits of excited states.

**Theorem 3.7.** *The diagram in Figure 3.2 visualizes the complete coclass graph  $\mathcal{G}(3, 1)$  up to order  $3^8 = 6\,561$ . It is periodic with length 2. The period consists of branches  $\mathcal{B}_j$  with  $4 \leq j \leq 5$ , whereas branches  $\mathcal{B}_j$  with  $2 \leq j \leq 3$  are irregular and form the pre-period.*

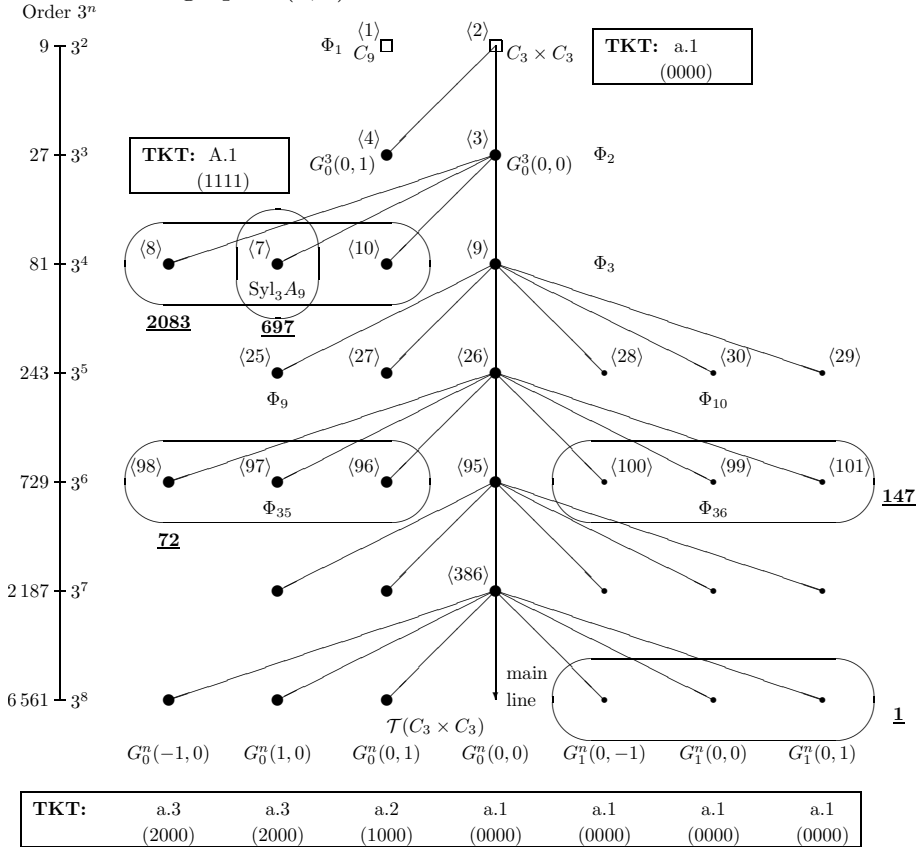
*Proof.* The top of  $\mathcal{G}(3, 1)$  consists of two abelian groups of order  $3^2$ , the isolated cyclic group  $C_9$  and the bicyclic root  $C_3 \times C_3$  of the unique coclass tree  $\mathcal{T}(C_3 \times C_3)$ .

Immediate descendants of the root are the two well-known extra special groups, the capable mainline group  $G_0^3(0, 0)$  of exponent 3 and the terminal group  $G_0^3(0, 1)$  of exponent 9, both of order  $3^3$ .

Blackburn’s results [15] on counting metabelian  $p$ -groups of maximal class and order  $p^n$  with  $n \geq 4$  can now be applied to the special case  $p = 3$ , which is entirely metabelian [16, p. 26, Thm. 6].

We start with metabelian groups containing an abelian maximal subgroup, which are characterized by defect  $k = 0$ . They consist of the capable

FIGURE 3.2. Root  $C_3 \times C_3$  and branches  $\mathcal{B}_j$ ,  $2 \leq j \leq 7$ , of coclass graph  $\mathcal{G}(3, 1)$



mainline group  $G_0^n(0, 0)$ , the terminal group  $G_0^n(0, 1)$  and  $(n - 2, p - 1)$  terminal groups of the form  $G_0^n(z, 0)$ . Specialization of [15, p. 88, Thm. 4.3] to  $p = 3$  in dependence on  $n$  yields their number

$$2 + (n - 2, p - 1) = \begin{cases} 2 + 1 = 3 & \text{for } 5 \leq n \equiv 1 \pmod{2}, \\ 2 + 2 = 4 & \text{for } 4 \leq n \equiv 0 \pmod{2}. \end{cases}$$

Further, the number of metabelian groups with defect  $k = 1$ , which are terminal and of the form  $G_1^n(z, w)$  with  $a = (a(n - 1)) = (1)$ , is given, independently from  $n \geq 5$ , by 3 [15, p. 88, Thm. 4.2]. □

Vertices  $G$  of coclass graph  $\mathcal{G}(3, 1)$  in Figure 3.2 are classified according to their defect  $k(G)$  by using different symbols:

- (1) large contour squares  $\square$  denote abelian groups,

- (2) big full discs  $\bullet$  denote metabelian groups with abelian maximal subgroup and  $k = 0$ ,
- (3) small full discs  $\bullet$  denote metabelian groups with defect  $k = 1$ .

The actual distribution of the 2576 second 3-class groups  $G_3^2(K)$  of real quadratic number fields  $K = \mathbb{Q}(\sqrt{D})$  of type (3, 3) with discriminant  $0 < D < 10^7$  is represented by underlined boldface counters of hits of vertices surrounded by the adjacent oval. See [58, § 6, Tbl. 2] and [60, § 6, Tbl. 6.1]. The results verify the selection rule, Theorem 3.5, for groups  $G_3^2(\mathbb{Q}(\sqrt{D}))$ ,  $D > 0$ , and underpin the *weak leaf conjecture* 3.1 that mainline vertices are forbidden for second 3-class groups of quadratic fields. A remarkably different behavior is revealed by certain biquadratic fields to be presented in a forthcoming paper.

**Conjecture 3.1.** A vertex  $G$  on the metabelian skeleton  $\mathcal{M}(p, r)$  of a coclass graph  $\mathcal{G}(p, r)$ , with an odd prime  $p \geq 3$  and  $r \geq 1$ , cannot be realized as second  $p$ -class group  $G_p^2(K)$  of a quadratic field  $K = \mathbb{Q}(\sqrt{D})$ , if it possesses a metabelian immediate descendant  $H$  having the same transfer kernel type  $\varkappa(H) = \varkappa(G)$  and a higher defect of commutativity  $k(H) > k(G)$ .

**3.2.5. Separating TKTs on  $\mathcal{G}(p, 1)$  via first TTT.** For increasing odd primes  $p \geq 5$ , the structure  $\tau(1)$  of the  $p$ -class group  $\text{Cl}_p(L_1)$  of the distinguished first unramified extension  $L_1$  of degree  $p$  of an arbitrary base field  $K$  with second  $p$ -class group  $G = G_p^2(K)$  of coclass  $\text{cc}(G) = 1$  admits the separation of more and more excited states of the TKTs a.2, with fixed point  $\varkappa(1) = 1$ , and a.3, without fixed point  $\varkappa(1) \in \{2, \dots, p + 1\}$ . Further, the order of the exceptional  $p$ -group  $G \simeq \text{Syl}_p A_{p^2}$  becomes increasingly larger.

**Theorem 3.8.** Let  $p \geq 3$  be an odd prime and  $G \in \mathcal{G}(p, 1)$  a  $p$ -group of order  $|G| = p^n$ ,  $n \geq 4$ , depth  $\text{dp}(G) = 1$ , and defect  $k(G) = 0$ .

- (1) The exceptional case of TKT a.3\*, having an elementary abelian first TTT  $\tau(1)$  of elevated  $p$ -rank  $r_p = p$ , occurs if and only if  $n = p + 1$  and  $G \simeq G_0^{p+1}(1, 0) = \text{Syl}_p A_{p^2}$
- (2) The regular cases of the TKTs a.2 and a.3, having a first TTT  $\tau(1)$  of usual  $p$ -rank  $r_p \leq p - 1$ , can be separated by the structure  $\tau(1)$  of the distinguished first maximal subgroup  $H_1 = \chi_2(G)$  if and only if  $n \leq p$ . In this case,

- (a)  $G$  is of TKT a.2 if and only if  $\tau(1) = (\overbrace{p \dots p}^{n-1 \text{ times}})$  is elementary abelian of rank  $r_p = n - 1 \leq p - 1$ ,



- (b)  $G$  is of TKT a.3 if and only if  $\tau(1) = (p^2, \overbrace{p \dots p}^{n-3 \text{ times}})$  is of rank  $r_p = n - 2 \leq p - 2$ , neither nearly homocyclic nor elementary abelian.

*Proof.* All groups  $G \in \mathcal{G}(p, 1)$  of order  $|G| \geq p^n$ ,  $n \geq 4$ , depth  $\text{dp}(G) = 1$ , and defect  $k(G) = 0$  are metabelian and contain the abelian distinguished maximal subgroup  $A = H_1 = \chi_2(G)$ , having  $A' = 1$ . Thus, all statements are a consequence of [45, Thm. 7, p. 11], where  $G$  is of TKT a.2 if and only if  $G \simeq G_0^n(0, 1)$ , and  $G$  is of TKT a.3 if and only if  $G \simeq G_0^n(z, 0)$ ,  $z \notin \{0, 1\}$ .  $\square$

Table 3.3 displays the possibilities for the first TTT  $\tau(1)$  in dependence on the ground state (GS) and excited states (ES) of TKTs, as stated in Theorem 3.8 for the smallest odd primes  $p \in \{3, 5, 7\}$ . Here, we assume a quadratic base field  $K = \mathbb{Q}(\sqrt{D})$ , taking into account the selection rule, Theorem 3.5, for odd branches.

TABLE 3.3. Separating TKT a.2, a.3 and a.3\* on  $\mathcal{G}(p, 1)$

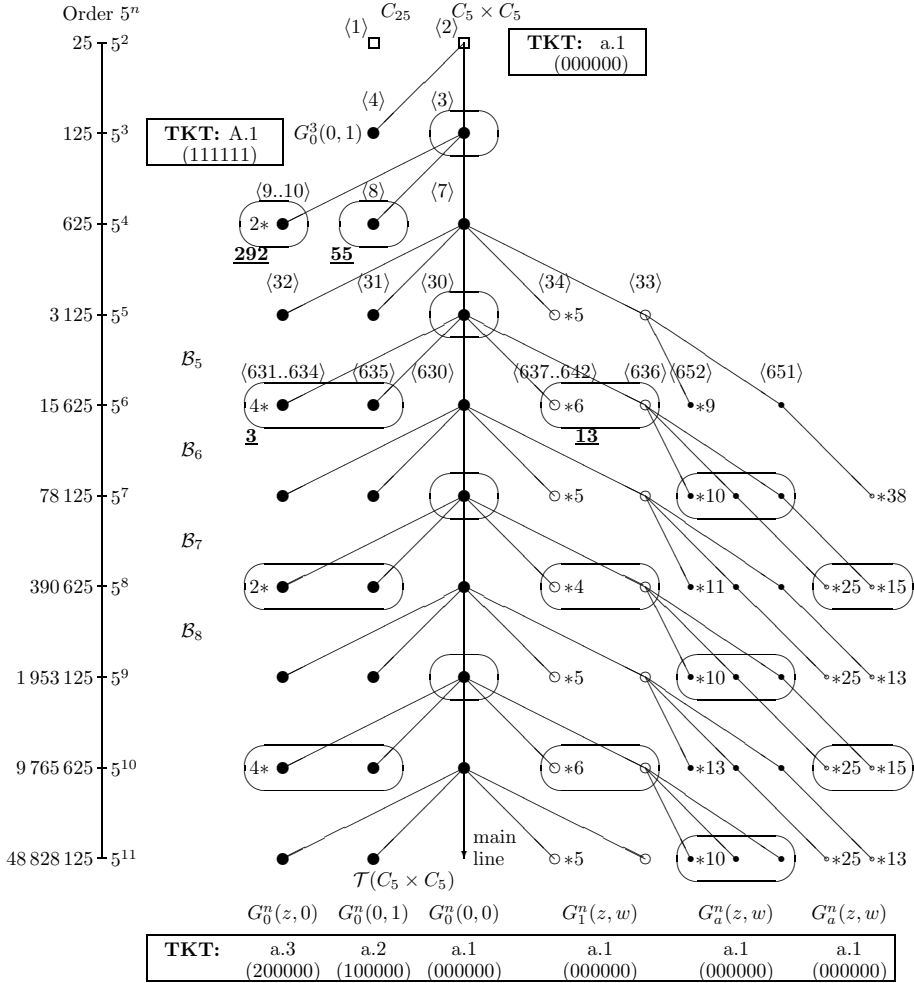
$p$	state	branch of $\mathcal{T}(C_p \times C_p)$	First TTT, $\tau(1)$ , for TKT		
			a.2	a.3	a.3*
3	GS	$\mathcal{B}_3$	$(3^2, 3)$	$(3^2, 3)$	$(3, 3, 3)$
	ES 1	$\mathcal{B}_5$	$(3^3, 3^2)$	$(3^3, 3^2)$	—
	ES 2	$\mathcal{B}_7$	$(3^4, 3^3)$	$(3^4, 3^3)$	—
5	GS	$\mathcal{B}_3$	$(5, 5, 5)$	$(5^2, 5)$	—
	ES 1	$\mathcal{B}_5$	$(5^2, 5, 5, 5)$	$(5^2, 5, 5, 5)$	$(5, 5, 5, 5, 5)$
	ES 2	$\mathcal{B}_7$	$(5^2, 5^2, 5^2, 5)$	$(5^2, 5^2, 5^2, 5)$	—
7	GS	$\mathcal{B}_3$	$(7, 7, 7)$	$(7^2, 7)$	—
	ES 1	$\mathcal{B}_5$	$(7, 7, 7, 7, 7)$	$(7^2, 7, 7, 7)$	—
	ES 2	$\mathcal{B}_7$	$(7^2, 7, 7, 7, 7, 7)$	$(7^2, 7, 7, 7, 7, 7)$	$(7, 7, 7, 7, 7, 7, 7)$

**3.2.6. Metabelian 5-groups  $G$  of coclass  $\text{cc}(G) = 1$ .**

**Theorem 3.9.** *The diagram in Figure 3.3 visualizes the metabelian skeleton  $\mathcal{M}(5, 1)$  of coclass graph  $\mathcal{G}(5, 1)$  up to order  $5^{11} = 48\,828\,125$ . This subgraph of  $\mathcal{G}(5, 1)$  is periodic with length 4. The period consists of the branches  $\mathcal{B}_j$  with  $5 \leq j \leq 8$ , whereas the branches  $\mathcal{B}_j$  with  $2 \leq j \leq 4$  are irregular and form the pre-period.*

Vertices of coclass graph  $\mathcal{G}(5, 1)$  in Figure 3.3 are classified according to their defect  $k$  by using different symbols:

FIGURE 3.3. All metabelian 5-groups of order up to  $5^{11}$  on  $\mathcal{G}(5, 1)$



- (1) large contour squares  $\square$  represent abelian groups,
- (2) big full discs  $\bullet$  represent metabelian groups with defect  $k = 0$ ,
- (3) big contour circles  $\circ$  represent metabelian groups with  $k = 1$ ,
- (4) small full discs  $\bullet$  represent metabelian groups with  $k = 2$ ,
- (5) small contour circles  $\circ$  represent metabelian groups with  $k = 3$ .

The symbol  $n*$  adjacent to a vertex denotes the multiplicity of a batch of  $n$  immediate descendants sharing a common parent. The selection rule, Theorem 3.5, for second 5-class groups  $G_5^2(K)$  of real quadratic number fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D > 0$ , is indicated by ovals surrounding admissible vertices.

The actual distribution of the 377 second 5-class groups  $G_5^2(\mathbb{Q}(\sqrt{D}))$  with discriminant  $0 < D \leq 26\,695\,193$ , discussed in section 3.2.7, is represented by underlined boldface counters of hits of vertices in the adjacent oval. The 13 cases of TKT a.1,  $\varkappa = (000000)$ , underpin the weak leaf conjecture 3.1.

*Proof.*  $\mathcal{G}(5, 1)$  starts with two abelian groups of order  $5^2$ , the isolated cyclic group  $C_{25}$  and the bicyclic root  $C_5 \times C_5$  of the unique coclass tree  $\mathcal{T}(C_5 \times C_5)$ .

As immediate descendants of the root,  $\mathcal{G}(5, 1)$  contains the two well-known extra special groups, the capable mainline group  $G_0^3(0, 0)$  of exponent 5 and the terminal group  $G_0^3(0, 1)$  of exponent 25, both of order  $5^3$ .

Now we use Blackburn’s results [15] on counting metabelian  $p$ -groups of maximal class and order  $p^n$  with  $n \geq 4$ , for the special case  $p = 5$ .

First, we consider the metabelian groups containing an abelian maximal subgroup, which are characterized by the defect  $k = 0$ . They consist of the capable mainline group  $G_0^n(0, 0)$ , the terminal group  $G_0^n(0, 1)$  and  $(n - 2, p - 1)$  terminal groups of the form  $G_0^n(z, 0)$ . Specialization of [15, p. 88, Thm. 4.3] for  $p = 5$  in dependence on  $n$  yields their number

$$2 + (n - 2, p - 1) = \begin{cases} 2 + 1 = 3 & \text{for } 5 \leq n \equiv 1 \pmod{2}, \\ 2 + 2 = 4 & \text{for } 4 \leq n \equiv 0 \pmod{4}, \\ 2 + 4 = 6 & \text{for } 6 \leq n \equiv 2 \pmod{4}. \end{cases}$$

Next, the number of metabelian groups with defect  $k = 1$ , which contain exactly one capable group  $G_1^n(0, 0)$  with  $a = (a(n - 1)) = (1)$  for every  $n \geq 5$ , is given by [15, p. 88, Thm. 4.2]:

$$1 + (2n - 6, p - 1) + (n - 2, p - 1) = \begin{cases} 1 + 4 + 1 = 6 & \text{for } 5 \leq n \equiv 1 \pmod{2}, \\ 1 + 2 + 2 = 5 & \text{for } 8 \leq n \equiv 0 \pmod{4}, \\ 1 + 2 + 4 = 7 & \text{for } 6 \leq n \equiv 2 \pmod{4}. \end{cases}$$

Finally, the number of metabelian groups with defect  $k = 2$ , containing exactly two capable groups  $G_a^n(0, 0)$  with  $a = (1, \pm 1)$  for every  $n \geq 7$  [51, § 3, ramification level], but only one capable group  $G_{(1, -1)}^n(0, 0)$  for  $n = 6$ , is given by [15, p. 88, Thm. 4.1]:

$$p + (2n - 7, p - 1) + (n - 2, p - 1) = 5 + 1 + 4 = 10 \text{ for } n = 6,$$

$$2p + (2n - 7, p - 1) + (n - 2, p - 1) = \begin{cases} 10 + 1 + 1 = 12 & \text{for } 7 \leq n \equiv 1 \pmod{2}, \\ 10 + 1 + 2 = 13 & \text{for } 8 \leq n \equiv 0 \pmod{4}, \\ 10 + 1 + 4 = 15 & \text{for } 10 \leq n \equiv 2 \pmod{4}. \end{cases}$$

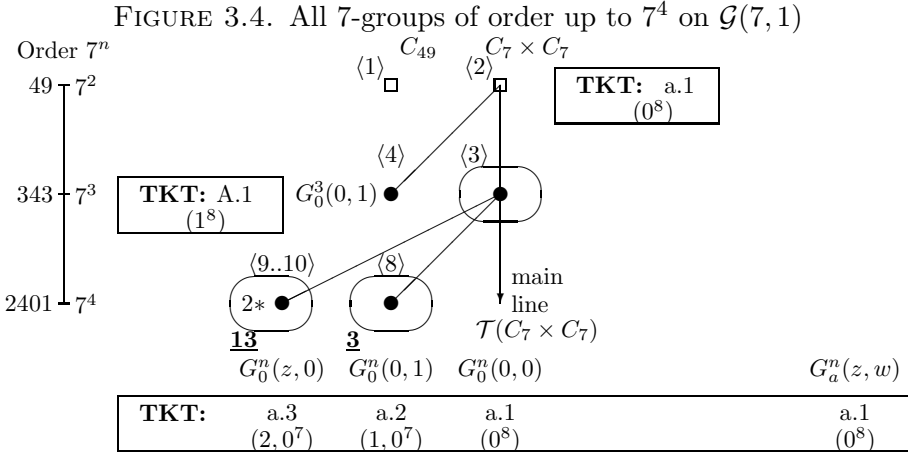
Since Blackburn restricts his investigations to defects  $k \leq 2$ , we need a supplementary count of metabelian groups with defect  $k = 3$ . The results for the head of the 4 virtually periodic branches  $\mathcal{B}_j$  with  $14 \leq j \leq 17$  given by Dietrich, Eick, Feichtenschlager [27, Fig. 7–10, p. 57–60] and by Dietrich [29, Fig. 4–5, p. 1086] are accumulated counts of metabelian and non-metabelian groups, whereas the collars and tails entirely consist of non-metabelian groups. According to private communications by H. Dietrich, one of the two capable groups  $G_{(1, \pm 1)}^{n-1}(0, 0)$  at depth two has always 25 metabelian descendants, which are all terminal, independently from  $n \geq 8$ , and the other has 15, resp. 13, metabelian descendants, which are all terminal, for even  $n \geq 8$ , resp. odd  $n \geq 9$ . The count of metabelian groups with defect  $k = 3$  is also given by Miech [64, Thm. 6–7, p. 336–337].  $\square$

**3.2.7. Distribution of  $G_5^2(K)$  on  $\mathcal{G}(5, 1)$ .** In Table 3.4, we list the 8 variants of second 5-class groups  $G_5^2(K)$  for the 377 real quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  of type  $(5, 5)$  with discriminant  $0 < D \leq 26\,695\,193$ , mainly on the coclass graph  $\mathcal{G}(5, 1)$ , but modestly also on  $\mathcal{G}(5, 2)$ .  $\tau(0)$  denotes the 5-class group of  $F_5^1(K)$ . Schur  $\sigma$ -groups are starred.

TABLE 3.4. 8 variants of  $G = G_5^2(K)$  for 377  $K = \mathbb{Q}(\sqrt{D})$ ,  $0 < D \leq 26\,695\,193$

$D$	$\tau(K); \tau(0)$	Type	$\varkappa(K)$	$G$	$\text{cc}(G)$	#	%
244 641	$(5, 5^2), (5, 5)^5; (5, 5)$	a.3	$(2, 0^5)$	$\langle 625, 9 10 \rangle$	1	292	77.5
1 167 541	$(5, 5, 5), (5, 5)^5; (5, 5)$	a.2	$(1, 0^5)$	$\langle 625, 8 \rangle$	1	55	14.6
1 129 841	$(5, 5, 5, 5), (5, 5)^5; (5, 5, 5, 5)$	a.1	$(0^6)$	$\langle 15625, 637 \dots 642 \rangle$	1	13	3.4
3 812 377	$(5, 5, 5, 5^2), (5, 5)^5; (5, 5, 5, 5)$	a.2.3 $\uparrow$	$(?, 0^5)$	$\langle 15625, 631 \dots 635 \rangle$	1	3	
4 954 652	$(5, 5^2)^6; (5, 5, 5)$		$(B^6)$	$\langle 3125, 9 10 12 \rangle$	2	7	1.9
10 486 805	$(5, 5, 5)^2, (5, 5^2)^4; (5, 5, 5)$		$(A^2, B^4)$	$\langle 3125, 7 11 \rangle$	2	2	
18 070 649	$(5, 5, 5), (5, 5^2)^5; (5, 5, 5)$		$(A, B^5)$	$\langle 3125, 8 13 \rangle^*$	2	1	
7 306 081	$(5, 5, 5, 5), (5, 5, 5), (5, 5^2)^4$		$(0, A, B^4)$	$\langle 3125, 4 \rangle$	2	4	1

There occur 13 cases (3.4%) of TKT a.1,  $\varkappa = (000000)$ , starting with  $D = 1\,129\,841$ , 3 cases of the first excited state of TKT a.2  $\uparrow$ ,  $\varkappa = (100000)$ , or a.3  $\uparrow$ ,  $\varkappa = (200000)$ , for  $D \in \{3\,812\,377, 19\,621\,905, 21\,281\,673\}$ , and 55 cases (14.6%) of the ground state of TKT a.2,  $\varkappa = (100000)$ , starting with  $D = 1\,167\,541$ . The remaining 292 cases (77.5%) of the ground state of TKT a.3,  $\varkappa = (200000)$ , starting with  $D = 244\,641$ , are clearly dominating. The TKTs were identified by means of Theorem 3.8, taking into account the selection rule for quadratic base fields as given in Table 3.3. The distribution of the corresponding second 5-class groups  $G_5^2(K)$  on the coclass graph  $\mathcal{G}(5, 1)$ , resp.  $\mathcal{G}(5, 2)$ , is shown in Figure 3.3, resp. 3.8. See also section § 3.5.3.



**3.2.8. Metabelian 7-groups  $G$  of coclass  $\text{cc}(G) = 1$ .** Figure 3.4 visualizes the lowest range of the distribution of second 7-class groups  $G_7^2(K)$  for the 17 real quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  of type (7, 7) with discriminant  $0 < D < 10^7$  on the coclass graph  $\mathcal{G}(7, 1)$ . With the aid of MAGMA [56], we found 13 cases, 76%, of TKT a.3,  $\varkappa = (20000000)$ , starting with  $D = 1633285$ , and 3 occurrences, 18%, of TKT a.2,  $\varkappa = (10000000)$ , for  $D \in \{2713121, 6872024, 9659661\}$ . These two TKTs can be separated by means of Theorem 3.8. There were no cases of excited states, but for the single discriminant  $D = 6986985$ ,  $G_7^2(K)$  is a top vertex of  $\mathcal{G}(7, 2)$  without total 7-principalization and of Taussky’s coarse transfer kernel type  $\kappa = (BBBBBBBB)$  [76]. Table 3.5 shows the corresponding TTT  $\tau(K) = (\text{Cl}_7(L_i))_{1 \leq i \leq 8}$  using power notation for repetitions and including  $\tau(0) = \text{Cl}_7(\mathbb{F}_7^1(K))$ , separated by a semicolon. 7-groups  $G_a^n(z, w)$  of positive defect  $k \geq 1$  appear in higher branches and are invisible in Figure 3.4.

TABLE 3.5. 3 variants of  $G = G_7^2(K)$  for 17  $K = \mathbb{Q}(\sqrt{D})$ ,  $0 < D < 10^7$

$D$	$\tau(K); \tau(0)$	Type	$\varkappa(K)$	$G$	$\text{cc}(G)$	#	%
1633285	$(7, 7^2), (7, 7)^7; (7, 7)$	a.3	$(2, 0^7)$	$\langle 2401, 9 10 \rangle$	1	13	76
2713121	$(7, 7, 7), (7, 7)^7; (7, 7)$	a.2	$(1, 0^7)$	$\langle 2401, 8 \rangle$	1	3	18
6986985	$(7, 7^2)^8; (7, 7, 7)$		$(B^8)$	$\langle 16807, 10 14 \dots 16 \rangle$	2	1	6

In the next section, we proceed to  $p$ -groups  $G$  of coclass  $\text{cc}(G) \geq 2$ .

**3.3. Metabelian 3-groups  $G$  of coclass  $\text{cc}(G) \geq 2$  with  $G/G' \simeq (3, 3)$ .**

**3.3.1. Non-CF groups.** In contrast to CF groups of coclass 1, metabelian 3-groups  $G$  of coclass  $\text{cc}(G) \geq 2$  with abelianization  $G/G'$  of type  $(3, 3)$  must have at least one bicyclic factor  $\gamma_3(G)/\gamma_4(G)$  [66], and are therefore called *non-CF groups*. They are characterized by an isomorphism *invariant*  $e = e(G)$ , defined by  $e + 1 = \min\{3 \leq j \leq m \mid 1 \leq |\gamma_j(G)/\gamma_{j+1}(G)| \leq 3\}$ . This invariant  $2 \leq e \leq m - 1$  indicates the first cyclic factor  $\gamma_{e+1}(G)/\gamma_{e+2}(G)$  of the lower central series of  $G$ , except  $\gamma_2(G)/\gamma_3(G)$ , which is always cyclic. We can calculate  $e$  from order  $|G| = 3^n$  and nilpotency class  $\text{cl}(G) = m - 1$ , resp. index  $m$  of nilpotency, of  $G$  by the formula  $e = n - m + 2$ . Since the coclass of  $G$  is given by  $\text{cc}(G) = n - \text{cl}(G) = n - m + 1$ , we have the relation  $e = \text{cc}(G) + 1$ . CF groups are characterized by  $e = 2$  and non-CF groups by  $e \geq 3$ .

**3.3.2. Bipolarization and defect.** For a group  $G$  of coclass  $\text{cc}(G) \geq 2$  we need a generalization of the group  $\chi_2(G)$ . Denoting by  $m$  the index of nilpotency of  $G$ , we let  $\chi_j(G)$  with  $2 \leq j \leq m - 1$  be the centralizers of two-step factor groups  $\gamma_j(G)/\gamma_{j+2}(G)$  of the lower central series, that is, the biggest subgroups of  $G$  with the property  $[\chi_j(G), \gamma_j(G)] \leq \gamma_{j+2}(G)$ . They form an ascending chain of characteristic subgroups of  $G$ ,  $\gamma_2(G) \leq \chi_2(G) \leq \dots \leq \chi_{m-2}(G) < \chi_{m-1}(G) = G$ , which contain the commutator subgroup  $\gamma_2(G)$ , and  $\chi_j(G)$  coincides with  $G$  if and only if  $j \geq m - 1$ . We characterize the smallest *two-step centralizer* different from the commutator subgroup by an isomorphism *invariant*  $s = s(G) = \min\{2 \leq j \leq m - 1 \mid \chi_j(G) > \gamma_2(G)\}$ . Again, CF groups are characterized by  $s = 2$  and non-CF groups by  $s \geq 3$ .

Now we can generalize the *defect of commutativity*  $k = k(G)$  to any metabelian 3-group  $G$  with  $G/G'$  of type  $(3, 3)$  by defining  $0 \leq k \leq 1$  such that  $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ .

The following assumptions for a metabelian 3-group  $G$  of coclass  $\text{cc}(G) \geq 2$  with abelianization  $G/\gamma_2(G)$  of type  $(3, 3)$  can always be satisfied, according to Nebelung [66, Thm. 3.1.11, p. 57, and Thm. 3.4.5, p. 94].

Let  $G$  be a metabelian 3-group of coclass  $\text{cc}(G) \geq 2$  with abelianisation  $G/\gamma_2(G)$  of type  $(3, 3)$ . Assume that  $G$  has order  $|G| = 3^n$ , class  $\text{cl}(G) = m - 1$ , and invariant  $e = n - m + 2 \geq 3$ , where  $4 \leq m < n \leq 2m - 3$ . Let generators of  $G = \langle x, y \rangle$  be selected such that the bicyclic factor  $\gamma_3(G)/\gamma_4(G)$  is generated by their third powers,  $\gamma_3(G) = \langle y^3, x^3, \gamma_4(G) \rangle$ , and that  $x \in G \setminus \chi_s(G)$ , if  $s < m - 1$ , and  $y \in \chi_s(G) \setminus \gamma_2(G)$ . This causes a *bipolarization* among the four maximal subgroups  $H_1, \dots, H_4$  of  $G$ , which will be standardized in Definition 3.2.

**3.3.3. Parametrized presentation.** Let the main commutator of  $G$  be declared by  $s_2 = t_2 = [y, x] \in \gamma_2(G)$  and higher commutators recursively by  $s_j = [s_{j-1}, x], t_j = [t_{j-1}, y] \in \gamma_j(G)$  for  $j \geq 3$ . Starting with the powers  $\sigma_3 = y^3, \tau_3 = x^3 \in \gamma_3(G)$ , which generate  $\gamma_3(G)$  modulo  $\gamma_4(G)$ , let  $\sigma_j = [\sigma_{j-1}, x], \tau_j = [\tau_{j-1}, y] \in \gamma_j(G)$  for  $j \geq 4$ . Nilpotency of  $G$  is expressed by  $\sigma_{m-1} = 1$  and  $\tau_{e+2} = 1$ . According to Nebelung [66], the group  $G$  satisfies the following relations with certain exponents  $-1 \leq \alpha, \beta, \gamma, \delta, \rho \leq 1$  as parameters.

$$(3.3) \quad \begin{aligned} s_2^3 &= \sigma_4 \sigma_{m-1}^{-\rho\beta} \tau_4^{-1}, & s_3 \sigma_3 \sigma_4 &= \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^\gamma \tau_e^\delta, \\ t_3^{-1} \tau_3 \tau_4 &= \sigma_{m-2}^{\rho\delta} \sigma_{m-1}^\alpha \tau_e^\beta, & \tau_{e+1} &= \sigma_{m-1}^{-\rho}. \end{aligned}$$

By  $G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$  we denote the representative of an isomorphism class of metabelian 3-groups  $G$ , having  $G/G'$  of type (3, 3), of coclass  $\text{cc}(G) = n - m + 1 \geq 2$ , class  $\text{cl}(G) = m - 1$ , and order  $|G| = 3^n$ , which satisfies the relations (3.3) with a fixed system of exponents  $(\alpha, \beta, \gamma, \delta, \rho)$ . We have  $\rho = 0$  if and only if  $k = 0$ .

**3.3.4. Two distinguished maximal subgroups.** The maximal normal subgroups  $H_1, \dots, H_4$  of  $G$  contain the commutator subgroup  $G'$  as a normal subgroup of index 3 and are thus of the shape  $H_i = \langle g_i, G' \rangle$  with suitable generators  $g_i$ . We want to arrange them in a fixed order.

**Definition 3.2.** The *bipolarization* or *natural order* of the maximal subgroups  $(H_i)_{1 \leq i \leq 4}$  of  $G$  is given by the *distinguished first generator*  $g_1 = y \in \chi_s(G)$ , the *distinguished second generator*  $g_2 = x \notin \chi_s(G)$ , both satisfying  $y^3, x^3 \in \gamma_3(G) \setminus \gamma_4(G)$ , and the other generators  $g_i = xy^{i-2} \notin \chi_s(G)$  for  $3 \leq i \leq 4$ , provided that  $s < m - 1$ . Then, in particular,  $\chi_s(G) = H_1 = \langle y, G' \rangle$ .

**3.3.5. Parents of core and interface groups.**

**Definition 3.3.** For an arbitrary prime  $p$ , let  $G$  be a finite  $p$ -group of nilpotency class  $c = \text{cl}(G)$ . We call  $G$  a *core group*, resp. an *interface group*, if its last lower central  $\gamma_c(G)$  is of order  $p^d$  with  $d = 1$ , resp.  $d \geq 2$ .

If  $G$  is of order  $p^n$ , the last lower central quotient  $Q = G/\gamma_c(G)$  of  $G$  is of order  $|Q| = |G|/p^d = p^{n-d}$  and of class  $\text{cl}(Q) = \text{cl}(G) - 1$ . Therefore, the coclass of  $Q$  is given by

$$\text{cc}(Q) = n - d - \text{cl}(Q) = n - d - \text{cl}(G) + 1 = \text{cc}(G) - (d - 1).$$

Consequently, the last lower central quotient  $Q$  of a core group  $G$  is of the same coclass as  $G$ , whereas the last lower central quotient  $Q$  of an interface group  $G$  is of lower coclass than  $G$ . Obviously, a CF group must necessarily be a core group.

Now we apply these new concepts to the case  $p = 3$  and investigate the parent  $\pi(G)$  of a metabelian 3-group  $G$  with  $G/G'$  of type  $(3, 3)$ . Since the invariant  $e = e(G) = \text{cc}(G) + 1$  indicates the first cyclic quotient  $\gamma_{e+1}(G)/\gamma_{e+2}(G)$ ,  $G$  is an interface group if and only if  $e = \text{cl}(G) = m - 1$ , where  $m$  denotes the index of nilpotency of  $G$ . This maximal possible value of  $e$  enforces a special relation between order  $|G| = 3^n$  and class  $\text{cl}(G) = m - 1$  of  $G$ ,

$$n = e + m - 2 = m - 1 + m - 2 = 2m - 3.$$

Together with group counts in Nebelung's theorem [66, p. 178, Thm. 5.1.16], the following two theorems describe the structure of the *metabelian skeleton* of those subgraphs of the coclass graphs  $\mathcal{G}(3, r)$ ,  $r \geq 2$ , which are formed by isomorphism classes of metabelian 3-groups  $G$  having abelianization  $G/G' \simeq (3, 3)$ . This restriction concerns both, the coclass trees  $\mathcal{T}$  and the sporadic part  $\mathcal{G}_0(3, r)$  of each coclass graph  $\mathcal{G}(3, r)$ . We distinguish *core* groups and *interface* groups and begin with the former.

**Theorem 3.10.** *Let  $G$  be a metabelian 3-group of coclass  $r = \text{cc}(G) \geq 2$  with  $G/G' \simeq (3, 3)$ , such that  $G \simeq G_p^{m,n}(\alpha, \beta, \gamma, \delta) \in \mathcal{G}(3, r)$  with parameters  $-1 \leq \alpha, \beta, \gamma, \delta, \rho \leq 1$ , that is,  $G$  is of order  $|G| = 3^n$ , class  $\text{cl}(G) = m - 1$ ,  $4 \leq m < n \leq 2m - 3$ , coclass  $2 \leq r = n - m + 1 \leq m - 2$ , and invariant  $3 \leq e = n - m + 2 \leq m - 1$ . Assume additionally that  $G$  is a core group with cyclic last lower central  $\gamma_{m-1}(G)$  of order 3, thus having  $5 \leq m < n \leq 2m - 4$  and  $e \leq m - 2$ . Then the parent  $\pi(G)$  of  $G$  is generally given by  $\pi(G) \simeq G_0^{m-1, n-1}(\rho\delta, \beta, \rho\beta, \delta) \in \mathcal{G}(3, r)$ , and in particular,*

$$\pi(G) \simeq \begin{cases} G_0^{m-1, n-1}(0, \beta, 0, \delta), & \text{if } \rho = 0, \\ G_0^{m-1, n-1}(\rho\delta, \beta, \rho\beta, \delta), & \text{if } \rho = \pm 1, (\beta, \delta) \neq (0, 0), \\ G_0^{m-1, n-1}(0, 0, 0, 0), & \text{if } \rho = \pm 1, (\beta, \delta) = (0, 0). \end{cases}$$

**Remark.** The various cases of Theorem 3.10 can be described as follows.

- (1) If  $G$  is a group with parameter  $\rho = 0$ , or equivalently with defect  $k = 0$ , then the parent  $\pi(G) \simeq G_0^{m-1, n-1}(0, \beta, 0, \delta)$  is a mainline group on one of the coclass trees, since these groups are characterized uniquely by  $\alpha = 0, \gamma = 0, \rho = 0$  [67]. A summary is given in Table 3.6.
- (2) However, if  $G$  is a group with  $\rho = \pm 1$  or equivalently  $k = 1$ , then the parent  $\pi(G)$  has defect  $\tilde{k} = 0 = k - 1$  but lies outside of any mainline, either on a branch of a coclass tree  $\mathcal{T}$  or on the sporadic part  $\mathcal{G}_0(3, r)$ .
- (3) The only exception is the very special case that  $G$  with  $\rho = \pm 1$  has the parameters  $\beta = 0$  and  $\delta = 0$ . According to [67], this uniquely characterizes groups  $G$  of transfer kernel type b.10,  $\varkappa = (0043)$ , outside of mainlines, having mainline parent  $\pi(G)$  of the same TKT.



Table 3.6 summarizes parametrized power-commutator presentations  $G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$  with parameters  $\alpha = \gamma = 0$ ,  $-1 \leq \beta, \delta \leq 1$ ,  $\rho = 0$ ,  $4 \leq m < n \leq 2m - 3$ ,  $n = r + m - 1$ , and transfer kernel types  $\varkappa$  of all metabelian mainline groups on coclass trees  $\mathcal{T}$  of the coclass graphs  $\mathcal{G}(3, r)$  with given coclass  $r \geq 2$ . In any case, the metabelian root  $G$  of a tree  $\mathcal{T} = \mathcal{T}(G)$  is given by the top vertex  $G = G_0^{r+2, 2r+1}(0, \beta, 0, \delta)$ , for which  $e = m - 1$ ,  $r = m - 2$ , and thus  $n = 2r + 1 = 2(r + 2) - 3 = 2m - 3$ . For the sake of comparison, the mainline of  $\mathcal{G}(3, 1)$  is also included. Total transfers  $\varkappa(i) = 0$ ,  $1 \leq i \leq 4$ , are counted by the invariant  $\nu(G)$ , cfr. [58, Dfn. 4.2, p. 488].

TABLE 3.6. Metabelian mainline groups of  $\mathcal{G}(3, r)$ ,  $r \geq 1$ , sharing the total transfer kernel  $\varkappa(1) = 0$

3-group of order $3^n$				transfer kernels		
$G$	cc( $G$ )	$m \geq r + 2$	$n \geq 2r + 1$	TKT	$\varkappa(G)$	$\nu(G)$
$G_0^n(0, 0)$	$r = 1$	$\geq 3$	$\geq 3$	a.1	(0000)	4
$G_0^{m,n}(0, 0, 0, 0)$	$r \geq 2$	$\geq 4$	$\geq 5$	b.10	(0043)	2
$G_0^{m,n}(0, -1, 0, 1)$	$r = 2$	$\geq 4$	$\geq 5$	c.18	(0313)	1
$G_0^{m,n}(0, 0, 0, 1)$	$r = 2$	$\geq 4$	$\geq 5$	c.21	(0231)	1
$G_0^{m,n}(0, 1, 0, 1)$	$r \geq 3$	$\geq 5$	$\geq 7$	d*.19	(0443)	1
$G_0^{m,n}(0, -1, 0, 1)$	$r \geq 4$ even	$\geq 6$	$\geq 9$	d*.19	(0343)	1
$G_0^{m,n}(0, 0, 0, 1)$	$r \geq 3$	$\geq 5$	$\geq 7$	d*.23	(0243)	1
$G_0^{m,n}(0, 1, 0, 0)$	$r \geq 3$	$\geq 5$	$\geq 7$	d*.25	(0143)	1
$G_0^{m,n}(0, -1, 0, 0)$	$r \geq 4$ even	$\geq 6$	$\geq 9$	d*.25	(0143)	1

*Proof.* The assumption  $5 \leq m < n \leq 2m - 4$ , and thus  $e = n - m + 2 \leq m - 2$ , ensures that  $G$  is not a top vertex of the coclass graph  $\mathcal{G}(3, r)$ . Therefore, the last lower central  $\gamma_{m-1}(G) = \langle \sigma_{m-1} \rangle$  of  $G$  is cyclic of order 3. Since  $G \simeq G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$ ,  $G$  is defined by the relations (3.3),

$$s_3\sigma_3\sigma_4 = \sigma_{m-2}^{\rho\beta}\sigma_{m-1}^\gamma\bar{\tau}_e^\delta, \quad t_3\tau_3^{-1}\tau_4^{-1} = \sigma_{m-2}^{-\rho\delta}\sigma_{m-1}^{-\alpha}\tau_e^{-\beta}, \quad \tau_{e+1} = \sigma_{m-1}^{-\rho},$$

and the relations for the parent  $\pi(G) = G/\gamma_{m-1}(G)$  of  $G$  are

$$\bar{s}_3\bar{\sigma}_3\bar{\sigma}_4 = \bar{\sigma}_{m-2}^{\rho\beta}\bar{\sigma}_{m-1}^\gamma\bar{\tau}_e^\delta, \quad \bar{t}_3\bar{\tau}_3^{-1}\bar{\tau}_4^{-1} = \bar{\sigma}_{m-2}^{-\rho\delta}\bar{\sigma}_{m-1}^{-\alpha}\bar{\tau}_e^{-\beta}, \quad \bar{\tau}_{e+1} = \bar{\sigma}_{m-1}^{-\rho},$$

where the left coset of an element  $g \in G$  with respect to  $\gamma_{m-1}(G)$  is denoted by  $\bar{g} = g \cdot \gamma_{m-1}(G)$ . In particular, we have  $\bar{\sigma}_{m-1} = 1$ . Since the order of the parent is  $|\pi(G)| = |G| : |\gamma_{m-1}(G)| = 3^n : 3 = 3^{n-1}$  and the nilpotency class is  $\text{cl}(\pi(G)) = \text{cl}(G) - 1 = m - 2$ , the coclass  $r$  and the invariant  $e$  remain the same, and we can view the relations as

$$\bar{s}_3\bar{\sigma}_3\bar{\sigma}_4 = \bar{\sigma}_{m-3}^0\bar{\sigma}_{m-2}^{\rho\beta}\bar{\tau}_e^\delta, \quad \bar{t}_3\bar{\tau}_3^{-1}\bar{\tau}_4^{-1} = \bar{\sigma}_{m-3}^0\bar{\sigma}_{m-2}^{-\rho\delta}\bar{\tau}_e^{-\beta}, \quad \bar{\tau}_{e+1} = 1.$$

Consequently,  $\pi(G) \simeq G_0^{m-1, n-1}(\rho\delta, \beta, \rho\beta, \delta)$ , that is  $\pi(G) \simeq G_0^{m-1, n-1}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$  with  $\tilde{\alpha} = \rho\delta$ ,  $\tilde{\gamma} = \rho\beta$ , but  $\tilde{\beta} = \beta$ ,  $\tilde{\delta} = \delta$  remain unchanged.  $\square$

The following principle, that the kernel  $\varkappa(1)$  of the transfer from  $G$  to the first distinguished maximal subgroup  $H_1 = \chi_s(G)$  decides about the relation between depth  $\text{dp}(G)$  and defect  $k = k(G)$  of  $G$ , is already known from metabelian  $p$ -groups  $G$  of coclass  $\text{cc}(G) = 1$ .

**Corollary 3.10.1.** *For a metabelian 3-group  $G$  of coclass  $r = \text{cc}(G) \geq 2$  having abelianization  $G/G' \simeq (3, 3)$  and defect of commutativity  $k = k(G)$ , which does not belong to the sporadic part  $\mathcal{G}_0(3, r)$ , the depth  $\text{dp}(G)$  of  $G$  on its coclass tree  $\mathcal{T}$ , as a subset of  $\mathcal{G}(3, r)$ , is given by*

$$\text{dp}(G) = \begin{cases} k + 1, & \text{if } \varkappa(1) \neq 0, \\ k, & \text{if } \varkappa(1) = 0, \end{cases}$$

with respect to the natural order of the maximal subgroups of  $G$ .

*Proof.* This follows immediately from Theorem 3.10 and the remark thereafter: The system of all groups  $G_0^{m, n}(0, \beta, 0, \delta)$  with arbitrary  $4 \leq m < n \leq 2m - 3$ ,  $-1 \leq \beta, \delta \leq 1$ , but  $\alpha = \gamma = \rho = 0$ , consists of all mainline groups on coclass trees  $\mathcal{T}$  of  $\mathcal{G}(3, r)$ ,  $r = n - m + 1$ , that is, of all groups  $G$  with depth  $\text{dp}(G) = 0 = k$  equal to the defect  $k$ . According to Table 3.6, all these mainline groups have a total transfer  $\varkappa(1) = 0$  to the first distinguished maximal subgroup  $H_1$ .

Since the defect of a group  $G_\rho^{m, n}(\alpha, \beta, \gamma, \delta)$  with parameter  $\rho = 0$  is  $k = 0$ , all the other groups  $G = G_0^{m, n}(\alpha, \beta, \gamma, \delta)$ ,  $(\alpha, \gamma) \neq (0, 0)$ , with defect  $k = 0$  must be located at depth  $\text{dp}(G) = 1 = k + 1$  on a coclass tree  $\mathcal{T}$  or as a top vertex on the sporadic part  $\mathcal{G}_0(3, r)$  of  $\mathcal{G}(3, r)$ . According to [66, Thm. 6.14, pp. 208 ff], supplemented by [59, Thm. 3.3], all these groups have a partial transfer  $\varkappa(1) \neq 0$  to the first distinguished maximal subgroup  $H_1$ .

On the other hand, Theorem 3.10 shows that the relation between the defects  $\tilde{k}$  of parent  $\pi(G)$  and  $k$  of descendant  $G$  is given by  $\tilde{k} = 0 = k - 1$  for any group  $G = G_\rho^{m, n}(\alpha, \beta, \gamma, \delta)$ ,  $\rho = \mp 1$ , with positive defect  $k = 1$ , whence the depth, that is the number of steps required to reach the mainline by successive construction of parents, is given by

$$\text{dp}(G) = \begin{cases} 2 = k + 1, & \text{if } \varkappa(1) \neq 0, \\ 1 = k, & \text{if } \varkappa(1) = 0. \end{cases}$$

The groups  $G$  with positive defect  $k = 1$  are characterized by a partial transfer  $\varkappa(1) \neq 0$  to the first distinguished maximal subgroup  $H_1$ , according to [66, Thm. 6.14, pp. 208 ff]. The only exception are the groups  $G$  with parameters  $\beta = 0$  and  $\delta = 0$ , that is, those with transfer kernel type b.10,  $\varkappa = (0043)$ ,  $\varkappa(1) = 0$ , outside of mainlines.  $\square$

We conjecture that the following property of mainline groups of  $\mathcal{G}(3, r)$  might be true for mainline groups on any coclass tree of  $\mathcal{G}(p, r)$ ,  $p \geq 3$  prime,  $r \geq 1$ .

**Corollary 3.10.2.** *Mainline groups on a coclass tree  $\mathcal{T}$  of  $\mathcal{G}(3, r)$ ,  $r \geq 1$ , that is, groups of depth  $\text{dp}(G) = 0$ , must have a total transfer  $\varkappa(1) = 0$  to the distinguished maximal subgroup  $H_1 = \chi_s(G)$ .*

*Proof.* See Table 3.6. □

Only the groups of TKT a.1,  $\varkappa = (0000)$ , and b.10,  $\varkappa = (0043)$ , outside of mainlines, prohibit that the converse of Corollary 3.10.2 is also true.

Top vertices  $G$  on coclass trees  $\mathcal{T}$  and on the sporadic part  $\mathcal{G}_0(3, r)$  of a coclass graph  $\mathcal{G}(3, r)$  are groups of minimal class within their coclass  $r \geq 2$ . They are BF groups with bicyclic factors, except  $\gamma_2(G)/\gamma_3(G)$ , in particular having a bicyclic last lower central  $\gamma_{m-1}(G)$  of type  $(3, 3)$ , and consequently they do not possess a parent  $\pi(G)$  on the same coclass graph. They form the *interface* between the coclass graphs  $\mathcal{G}(3, r)$  and  $\mathcal{G}(3, r - 1)$ . We call the last lower central quotient  $G/\gamma_{m-1}(G)$  of  $G$  the *generalized parent*  $\tilde{\pi}(G)$  of  $G$  but we point out that there is no directed edge of depth 1 from  $G$  to  $\tilde{\pi}(G)$ . However, in the complete graph  $\mathcal{G}(3)$  of all finite 3-groups as defined by Newman and O'Brien [69, p. 131], there is a directed edge of depth 2 from  $G$  to  $\tilde{\pi}(G)$ . This supergraph  $\mathcal{G}(3)$  is the disjoint union of all coclass graphs  $\mathcal{G}(3, r)$ ,  $r \geq 0$ .

**Theorem 3.11.** *Let  $G$  be a metabelian 3-group of coclass  $r = \text{cc}(G) \geq 2$  with  $G/G' \simeq (3, 3)$ , such that  $|G| = 3^n$ ,  $\text{cl}(G) = m - 1$ ,  $4 \leq m < n = 2m - 3$ ,  $r = n - m + 1 = m - 2$ ,  $e = n - m + 2 = m - 1$ , and consequently  $k = 0$ , that is,  $G$  is an interface group with bicyclic last lower central  $\gamma_{m-1}(G)$  of type  $(3, 3)$ . Then the generalized parent  $\tilde{\pi}(G) \in \mathcal{G}(3, r - 1)$  of  $G \in \mathcal{G}(3, r)$  is given by*

$$\tilde{\pi}(G) \simeq \begin{cases} G_0^3(0, 0) \in \mathcal{G}(3, 1), & \text{if } m = 4 \\ & \text{(and thus } n = 5, r = 2), \\ G_0^{m-1, n-2}(0, 0, 0, 0) \in \mathcal{G}(3, r - 1), & \text{if } m \geq 5 \\ & \text{(and thus } n \geq 7, r \geq 3). \end{cases}$$

*Proof.* First, we consider the very special transition from second maximal to maximal class. The assumption  $m = 4$  implies  $n = 2m - 3 = 5$ . The last lower central  $\gamma_3(G) = \langle \sigma_3, \tau_3 \rangle$  is bicyclic of order  $3^2$ , and the generalized parent  $\tilde{\pi}(G) = G/\gamma_3(G)$  is of order  $3^3$ , of nilpotency class 2 and of coclass 1. The group  $G$  of type  $G \simeq G_0^{4,5}(\alpha, \beta, \gamma, \delta)$  satisfies the following special form of Nebelung's relations (3.3),

$$s_2^3 = 1, \quad s_3\sigma_3 = \sigma_3^\gamma\tau_3^\delta, \quad t_3\tau_3^{-1} = \sigma_3^{-\alpha}\tau_3^{-\beta},$$

and since  $\bar{\sigma}_3 = 1$  and  $\bar{\tau}_3 = 1$ , the relations for the generalized parent  $\tilde{\pi}(G)$  can be written as Blackburn's relations (3.1) and (3.2),

$$\bar{x}^3 = \bar{\tau}_3 = 1, \quad \bar{y}^3 \bar{s}_2^3 \bar{s}_3 = \bar{\sigma}_3 \cdot 1 \cdot \bar{\sigma}_3^{\gamma-1} \bar{\tau}_3^\delta = 1, \quad [\bar{s}_2, \bar{y}] = \bar{t}_3 = \bar{\sigma}_3^{-\alpha} \bar{\tau}_3^{1-\beta} = 1,$$

which imply that  $\tilde{\pi}(G) \simeq G_0^3(0, 0)$ .

Now, let  $m \geq 5$ . Since  $e = m - 1$ , and the last lower central  $\gamma_{m-1}(G) = \langle \sigma_{m-1}, \tau_{m-1} \rangle$  is bicyclic of type (3, 3), the order of the generalized parent is  $|\tilde{\pi}(G)| = |G| : |\gamma_{m-1}(G)| = 3^n : 3^2 = 3^{n-2}$ , the nilpotency class is  $\text{cl}(\tilde{\pi}(G)) = \text{cl}(G) - 1 = m - 2$ , and the coclass  $\tilde{r} = n - 2 - (m - 2) = 2m - 3 - m = m - 3 = r - 1$  and the invariant  $\tilde{e} = \tilde{r} + 1 = r = e - 1$  decrease by 1. Since  $k = 0, \rho = 0$ , the group  $G$  of type  $G \simeq G_0^{m,n}(\alpha, \beta, \gamma, \delta)$ , is defined by a special form of the relations (3.3),

$$s_3 \sigma_3 \sigma_4 = \sigma_{m-1}^\gamma \tau_e^\delta, \quad t_3 \tau_3^{-1} \tau_4^{-1} = \sigma_{m-1}^{-\alpha} \tau_e^{-\beta}, \quad \tau_{e+1} = 1.$$

Since  $\bar{\sigma}_{m-1} = 1$  and  $\bar{\tau}_{m-1} = 1$ , the relations for the generalized parent  $\tilde{\pi}(G)$  are

$$\bar{s}_3 \bar{\sigma}_3 \bar{\sigma}_4 = \bar{\sigma}_{m-1}^\gamma \bar{\tau}_{m-1}^\delta = 1, \quad \bar{t}_3 \bar{\tau}_3^{-1} \bar{\tau}_4^{-1} = \bar{\sigma}_{m-1}^{-\alpha} \bar{\tau}_{m-1}^{-\beta} = 1, \quad \bar{\tau}_e = \bar{\tau}_{m-1} = 1,$$

and therefore we have  $\tilde{\pi}(G) \simeq G_0^{m-1, n-2}(0, 0, 0, 0)$ . □

### 3.4. Second 3-class groups $G = G_3^2(K)$ of coclass $\text{cc}(G) \geq 2$ with $G/G' \simeq (3, 3)$ .

**3.4.1. Weak transfer target type  $\tau_0(G)$  expressed by 3-class numbers.** The group theoretic information on the second 3-class group  $G = G_3^2(K)$ , that is, its class, coclass, and defect, is contained in the 3-class numbers of the two distinguished extensions  $L_1, L_2$  and of the Hilbert 3-class field  $F_3^1(K)$ . Additionally, the principalization  $\kappa(1)$  of  $K$  in the first distinguished extension  $L_1$  determines the connection between defect and depth of  $G$ .

**Theorem 3.12.** *Let  $K$  be an arbitrary number field having 3-class group  $\text{Cl}_3(K)$  of type (3, 3). Suppose the second 3-class group  $G = \text{Gal}(F_3^2(K)|K)$  of  $K$  is of coclass  $\text{cc}(G) \geq 2$  with defect  $k = k(G)$ , order  $|G| = 3^n$ , and class  $\text{cl}(G) = m - 1$ , where  $4 \leq m < n \leq 2m - 3$ . With respect to the natural order of the maximal subgroups  $(H_i)_{1 \leq i \leq 4}$  of  $G$ , fixed in Definition 3.2, the weak transfer target type  $\tau_0(G) = (|H_i/H_i'|)_{1 \leq i \leq 4}$ , that is the family of 3-class numbers of the quadruplet  $(L_1, \dots, L_4)$  of unramified cyclic cubic extension fields of  $K$ , forming the first layer, is given by*

$$\tau_0(G) = (h_3(L_1), \dots, h_3(L_4)) = (3^{\text{cl}(G)-k}, 3^{\text{cc}(G)+1}, 3^3, 3^3),$$

where, in the case of a non-sporadic group  $G$  on some coclass tree, defect  $k$  and depth  $\text{dp}(G)$  are related by

$$k = \begin{cases} \text{dp}(G) - 1, & \text{if } \varkappa(1) \neq 0, \\ \text{dp}(G), & \text{if } \varkappa(1) = 0. \end{cases}$$

For the second layer, consisting of the Hilbert 3-class field  $F_3^1(K)$  only, the 3-class number is given by

$$h_3(F_3^1(K)) = 3^{\text{cl}(G) + \text{cc}(G) - 2}.$$

*Proof.* The statement is a succinct version of [58, Thm. 3.4], expressed by concepts more closely related to the position of  $G$  on the coclass graphs  $\mathcal{G}(3, r)$ ,  $r \geq 2$ , and to the transfer kernel type  $\varkappa(G)$  of  $G$ , using Corollary 3.10.1.  $\square$

**Remark.** Whereas  $h_3(L_3)$  and  $h_3(L_4)$  only indicate that  $\text{cc}(G) \geq 2$ , the second distinguished  $h_3(L_2)$  gives the precise coclass  $r$  of  $G$ ,  $h_3(F_3^1(K))$  determines the order  $3^n$ ,  $n = \text{cl}(G) + \text{cc}(G)$ , and class of  $G$ , and the first distinguished  $h_3(L_1)$  yields the defect  $k$  of  $G$ .

With respect to the mainline  $(M_j)_{j \geq 2r+1}$  of the coclass tree  $\mathcal{T}$  containing  $G$ , the order  $|M_i| = 3^i$  of the branch root  $M_i$  of a non-sporadic group  $G$  is given by  $i = n - \text{dp}(G) = \text{cl}(G) + \text{cc}(G) - \text{dp}(G)$ , where

$$\text{dp}(G) = \begin{cases} k, & \text{if } \varkappa(1) = 0, \\ k + 1, & \text{if } \varkappa(1) \neq 0. \end{cases}$$

**3.4.2. Selection Rules for quadratic base fields.** Let  $K = \mathbb{Q}(\sqrt{D})$  be a quadratic number field with discriminant  $D$  and 3-class group  $\text{Cl}_3(K)$  of type  $(3, 3)$ . Then the 4 unramified cyclic cubic extension fields  $(L_1, \dots, L_4)$  of  $K$  have dihedral absolute Galois groups  $\text{Gal}(L_i|K)$  of degree 6, according to [58, Prop. 4.1]. Consequently each sextic field  $L_i$  contains a cubic subfield  $K_i$ , whose invariants can be computed easier than those of  $L_i$  and are also sufficient to determine complete information on the group  $G = G_3^2(K)$ .

**Theorem 3.13.** *Let  $G = \text{Gal}(F_3^2(K)|K)$  be the second 3-class group of  $K = \mathbb{Q}(\sqrt{D})$ . If  $G \in \mathcal{G}(3, r)$  for some coclass  $r \geq 2$ , then the family of 3-class numbers of the non-Galois subfields  $K_i$  of  $L_i$ , with respect to the natural order fixed in Definition 3.2 is given by*

$$(h_3(K_1), \dots, h_3(K_4)) = \begin{cases} (3^{\frac{\text{cl}(G) - (k+1)}{2}}, 3^{\frac{\text{cc}(G)}{2}}, 3, 3) & \text{for sporadic } G, \\ & (\text{where always } \varkappa(2) \neq 0,) \\ (3^{\frac{\text{cl}(G) - \text{dp}(G)}{2}}, 3^{\frac{\text{cc}(G)+1}{2}}, 3, 3) & \text{otherwise, if } \varkappa(2) = 0, \\ (3^{\frac{\text{cl}(G) - \text{dp}(G)}{2}}, 3^{\frac{\text{cc}(G)}{2}}, 3, 3) & \text{otherwise, if } \varkappa(2) \neq 0. \end{cases}$$

The order  $|M_i| = 3^i$  of the root  $M_i$  for a non-sporadic group  $G$  on branch  $\mathcal{B}(M_i)$  of some coclass tree is given by

$$i = \text{cl}(G) + \text{cc}(G) - \text{dp}(G) \equiv \begin{cases} 1 \pmod{2}, & \text{if } \varkappa(2) = 0, \\ 0 \pmod{2}, & \text{if } \varkappa(2) \neq 0. \end{cases}$$

**Remark.** While  $h_3(K_3)$  and  $h_3(K_4)$  do not provide any information, the second distinguished  $h_3(K_2)$  indicates the coclass of  $G$  and enforces the parity

$$\text{cc}(G) \equiv \begin{cases} 1 \pmod{2}, & \text{if } \varkappa(2) = 0, \\ 0 \pmod{2}, & \text{if } \varkappa(2) \neq 0, \end{cases}$$

in dependence on the principalization  $\kappa(2)$  of  $K$  in the second distinguished extension  $L_2$ , and the first distinguished  $h_3(K_1)$  demands  $\text{cl}(G) - \text{dp}(G) \equiv 0 \pmod{2}$ , for non-sporadic  $G$ .

*Proof.* See [58, Thm. 4.2.]. For the branch root order  $|M_i| = 3^i$  of a non-sporadic vertex  $G$  of order  $|G| = 3^n$ , we use the relations  $n = \text{cl}(G) + \text{cl}(G)$  and  $i = n - \text{dp}(G)$ , the depth being the number of successive steps on the path between  $G$  and  $M_i$ , each decreasing order and class by 1 and keeping the coclass constant. □

**3.4.3. Identifying densely populated vertices by fast algorithms.** The top vertices  $G$  on  $\mathcal{G}(3, 2)$  with  $G/G'$  of type  $(3, 3)$  in Figure 3.5 can be identified by the fast algorithm given in [60, § 5.2–5.3], using the TTT and the counter  $\varepsilon$  of  $\tau(i) = (3, 3, 3)$ ,  $1 \leq i \leq 4$ , in Table 3.7.

TABLE 3.7. TTT and  $\varepsilon$  of the top vertices of type  $(3, 3)$  on  $\mathcal{G}(3, 2)$

Id of 3-group $G$	isoclinism family	TKT	$\varkappa$	Hilbert TTT $\tau(0)$	TTT $\tau(G)$				$\varepsilon$
					$\tau(1)$	$\tau(2)$	$\tau(3)$	$\tau(4)$	
$\langle 243, 5 \rangle$	$\Phi_6$	D.10	(2241)	(3, 3, 3)	(9, 3)	(9, 3)	(3, 3, 3)	(9, 3)	1
$\langle 243, 7 \rangle$	$\Phi_6$	D.5	(4224)	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	(9, 3)	2
$\langle 243, 9 \rangle$	$\Phi_6$	G.19	(2143)	(3, 3, 3)	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0
$\langle 729, 57 \rangle$	$\Phi_{43}$	G.19	(2143)	(3, 3, 3, 3)	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0
$\langle 243, 4 \rangle$	$\Phi_6$	H.4	(4443)	(3, 3, 3)	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	3
$\langle 729, 45 \rangle$	$\Phi_{42}$	H.4	(4443)	(9, 3, 3)	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	3
$\langle 243, 3 \rangle$	$\Phi_6$	b.10	(0043)	(3, 3, 3)	(9, 3)	(9, 3)	(3, 3, 3)	(3, 3, 3)	2
$\langle 243, 6 \rangle$	$\Phi_6$	c.18	(0313)	(3, 3, 3)	(9, 3)	(9, 3)	(3, 3, 3)	(9, 3)	1
$\langle 729, 49 \rangle$	$\Phi_{23}$	c.18	(0313)	(9, 3, 3)	(9, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1
$\langle 243, 8 \rangle$	$\Phi_6$	c.21	(0231)	(3, 3, 3)	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0
$\langle 729, 54 \rangle$	$\Phi_{23}$	c.21	(0231)	(9, 3, 3)	(9, 9)	(9, 3)	(9, 3)	(9, 3)	0

For quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ , the following metabelian 3-groups  $G \in \Phi_6$  cannot be realized as second 3-class groups  $G_3^2(K)$ .

- $\langle 243, 9 \rangle, \langle 243, 4 \rangle$ , according to Theorem 1.4, if  $K$  is complex.
- $\langle 243, 6 \rangle, \langle 243, 8 \rangle$ , due to the selection rule for branches in Theorem 3.13.
- $\langle 243, 3 \rangle$ , according to the remark after Theorem 3.13, since  $\varkappa(2) = 0$  enforces odd coclass.

**3.4.4. Top vertices of type (3, 3) on  $\mathcal{G}(3, 2)$ .** Figure 3.5 shows the interface between the coclass graphs  $\mathcal{G}(3, 1)$  and  $\mathcal{G}(3, 2)$ . The extra special group  $G_0^3(0, 0)$  of order 27 and exponent 3, which is the second member of the unique mainline of  $\mathcal{G}(3, 1)$ , is the generalized parent  $\tilde{\pi}(G)$  of all top vertices  $G$  of  $\mathcal{G}(3, 2)$ . We point out that the connecting edges of depth 2 neither belong to  $\mathcal{G}(3, 1)$  nor to  $\mathcal{G}(3, 2)$ . The metabelian skeleton of this graph is also shown in [66, p. 189 ff] and the complete graph, including the non-metabelian leaves, was first drawn in [4, Tbl. 1–2, pp. 265–266] and [5, Fig. 4.6–4.7, p. 74]. Among the non-CF groups  $G$  with abelianization  $G/G'$  of type (3, 3) at the top of coclass graph  $\mathcal{G}(3, 2)$ , which form the stem of isoclinism family  $\Phi_6$ , we have, from the left to the right:

- two terminal vertices  $\langle 243, 5 \rangle, \langle 243, 7 \rangle$ , the only Schur  $\sigma$ -groups of order  $3^5$  [20, Thm. 4.2, p. 14],
- two roots  $\langle 243, 9 \rangle$  and  $\langle 243, 4 \rangle$  of finite trees of depth 2, whose metabelian descendants belong to the stem of the isoclinism families  $\Phi_{43}$  and  $\Phi_{42}$ ,
- a root  $\langle 243, 3 \rangle$  of an infinite tree, which is not a coclass tree,
- and two roots  $\langle 243, 6 \rangle, \langle 243, 8 \rangle$  of coclass trees, shown in detail in Figures 3.6 and 3.7.

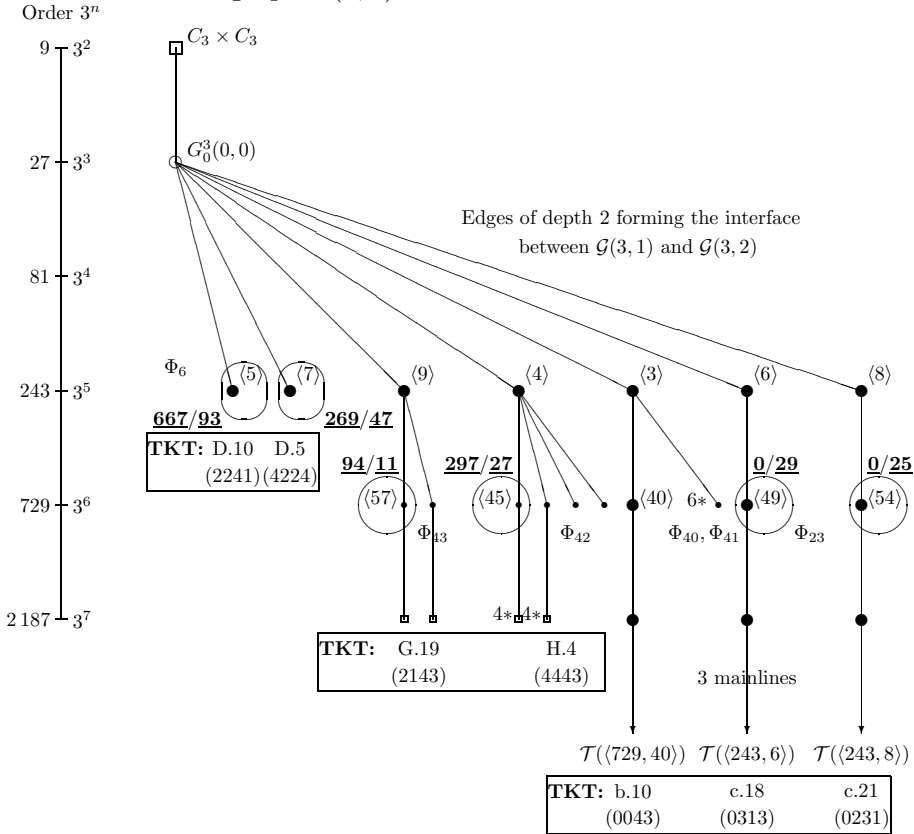
The sporadic part  $\mathcal{G}_0(3, 2)$  of  $\mathcal{G}(3, 2)$  consists of the terminal vertices  $\langle 243, 5 \rangle$  and  $\langle 243, 7 \rangle$ , the finite trees  $\mathcal{T}(\langle 243, 9 \rangle)$  and  $\mathcal{T}(\langle 243, 4 \rangle)$ , and a certain finite subset of the difference  $\mathcal{T}(\langle 243, 3 \rangle) \setminus \mathcal{T}(\langle 729, 40 \rangle)$ .

The vertices of the coclass graph  $\mathcal{G}(3, 2)$  in Figure 3.5 are classified by using different symbols:

- (1) a large contour square  $\square$  represents an abelian group,
- (2) a big contour circle  $\circ$  represents a metabelian group containing an abelian maximal subgroup, all other metabelian groups do not possess abelian subgroups,
- (3) big full discs  $\bullet$  represent metabelian groups with bicyclic centre of type (3, 3) and defect  $k = 0$ ,
- (4) small full discs  $\bullet$  represent metabelian groups with cyclic centre of order 3 and defect  $k = 1$ ,
- (5) small contour squares  $\square$  represent non-metabelian groups.

Groups of particular importance are labelled by a number in angles. This is the identifier in the SmallGroups library [14] of GAP [40] and MAGMA [56], where we omit the order, which is given on the left hand scale.

FIGURE 3.5. Sporadic groups and roots of coclass trees on the coclass graph  $\mathcal{G}(3, 2)$



The actual distribution of the 2020, resp. 2576, second 3-class groups  $G_3^2(K)$  of complex, resp. real, quadratic number fields  $K = \mathbb{Q}(\sqrt{D})$  of type  $(3, 3)$  with discriminant  $-10^6 < D < 10^7$  is represented by underlined bold-face counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval.

It is illuminating to compare these frequencies, which we have computed in [58, § 6, Tbl. 3–5] and [60, § 6, Tbl. 6.3–6.5 and 6.7] with the non-abelian generalization of the asymptotic Cohen-Lenstra-Martinet probability, which is given for complex quadratic fields by [20, p. 18, Tbl. 2] with respect to all 3190 discriminants of 3-rank 2 instead to the 2020 discriminants of type  $(3, 3)$  in the range  $-10^6 < D < 0$ . In three cases of Table 3.8, the actual percentage exceeds the conjectural asymptotic probability. In the first case the excess is significant. A possible interpretation is that the population of vertices of higher order will become more probable in



ranges of considerably bigger absolute values of discriminants so that the percentage of hits of the low order vertices in the table will decrease.

TABLE 3.8. Asymptotic probability for densely populated vertices of  $\mathcal{G}(3, 2)$

Id of $G$	frequency	percentage	probability
$\langle 243, 5 \rangle$	667	20.91	17.56
$\langle 243, 7 \rangle$	269	8.43	8.78
$\langle 729, 57 \rangle$	94	2.95	2.19
$\langle 729, 45 \rangle$	297	9.31	8.78

Identification of the vertex  $\langle 729, 57 \rangle$ , resp.  $\langle 729, 45 \rangle$ , among two, resp. four, closely related vertices in isoclinism family  $\Phi_{43}$ , resp.  $\Phi_{42}$ , was possible by means of the following *Artin criterion* for second  $p$ -class groups  $G$  of quadratic base fields, which can be verified by testing for a suitable automorphism  $\sigma \in \text{Aut}(G)$  of order 2 acting as inversion on  $G/G'$ .

**Theorem 3.14.** *Let  $p \geq 3$  be an odd prime. The second  $p$ -class group  $G = G_p^2(K)$  of a quadratic field  $K = \mathbb{Q}(\sqrt{D})$  admits an extension by the cyclic group  $C_2$ ,  $1 \rightarrow G \rightarrow H \rightarrow C_2 \rightarrow 1$ , such that  $H' \simeq G$ .*

*Proof.* See the letter of E. Artin to H. Hasse from November 19, 1928 [37]. □

**3.4.5. Coclass trees of type (3, 3) and (9, 3) on  $\mathcal{G}(3, 2)$ .**

**Definition 3.4.** Let  $p \geq 3$  be an odd prime. A rooted subtree  $\mathcal{T}$  of a coclass graph  $\mathcal{G}(p, r)$ ,  $r \geq 1$ , is called *forbidden*, if none of its vertices  $G \in \mathcal{T}$  can be realized as the second  $p$ -class group  $G_p^2(K)$  of a quadratic field  $K = \mathbb{Q}(\sqrt{D})$ . Otherwise  $\mathcal{T}$  is called *admissible*.

**Theorem 3.15.** *Let  $\mathcal{T}$  be a coclass tree of a coclass graph  $\mathcal{G}(3, r)$ ,  $r \geq 1$ , such that all its mainline groups  $M$  are metabelian with abelianization of type (3, 3).*

- (1) *The unique tree of  $\mathcal{G}(3, 1)$  and the trees of  $\mathcal{G}(3, 2)$ , whose mainline groups  $M$  are of transfer kernel type either c.18,  $\varkappa = (0313)$ , or c.21,  $\varkappa = (0231)$ , are admissible.*
- (2) *If all mainline groups  $M$  are of transfer kernel type b.10,  $\varkappa = (0043)$ , with distinguished second member  $\varkappa(2) = 0$ , then  $\mathcal{T}$  is forbidden if and only if the coclass  $r \geq 2$  is even.*
- (3) *If all mainline groups  $M$  are of transfer kernel type either d\*.23,  $\varkappa = (0243)$ , or d\*.19,  $\varkappa = (0443)$ , or d\*.25,  $\varkappa = (0143)$ , with distinguished second member  $\varkappa(2) \neq 0$ , then  $\mathcal{T}$  is forbidden if and only if the coclass  $r \geq 3$  is odd.*

Referring to [66, p. 189 ff] we point out the following details.

- (1) There is a periodic pattern of period length 2 of rooted subtrees with metabelian mainlines of type (3, 3) among the coclass graphs  $\mathcal{G}(p, r)$ , setting in with  $r = 3$ . The roots of the trees with fixed coclass  $r$  are of order  $3^{2r+1}$ . The metabelian skeletons of the trees with common mainline TKT and coclass of the same parity are isomorphic as graphs. The same is true for the metabelian skeletons of sporadic groups with coclass of the same parity.
- (2) For odd coclass  $r \geq 3$ , there are 4 trees with mainlines of transfer kernel types b.10,  $\varkappa = (0043)$ , d\*.23,  $\varkappa = (0243)$ , d\*.19,  $\varkappa = (0443)$ , and d\*.25,  $\varkappa = (0143)$ .
- (3) For even coclass  $r \geq 4$ , there are 6 trees with mainlines of transfer kernel types b.10,  $\varkappa = (0043)$ , and d\*.23,  $\varkappa = (0243)$ , each occurring only once, and on the other hand d\*.19,  $\varkappa = (0443)$ , and d\*.25,  $\varkappa = (0143)$ , each occurring in two instances, isomorphic as graphs.

*Proof.* Theorem 3.15 is an immediate consequence of Theorem 3.13, due to the second distinguished member  $\varkappa(2)$  of the TKT. See the diagrams on the pages without numbers, following [66, p. 189]. These diagrams were constructed by means of the lists of representatives  $G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$  for isomorphism classes, given in the appendix [67] of Nebelung’s thesis. The connection with the transfer kernel types  $\varkappa$  is established in [66, Thm. 6.14, pp. 208 ff].  $\square$

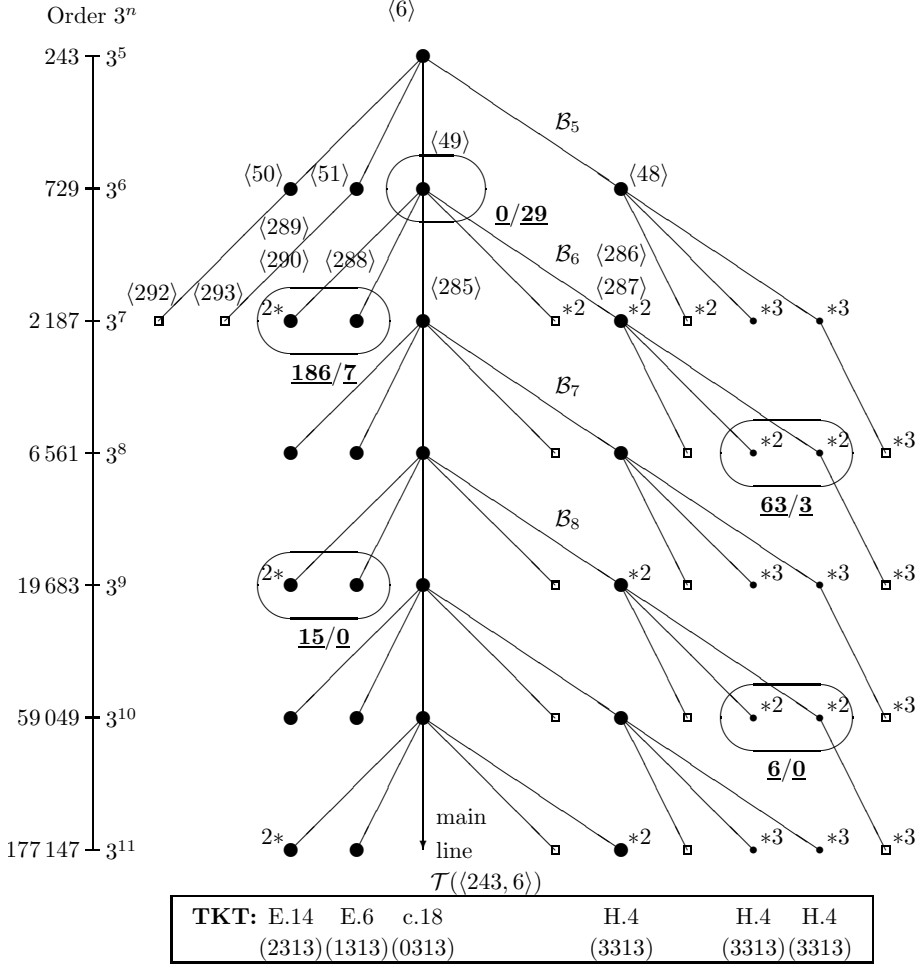
TABLE 3.9. Infinite trees of types (3, 3) or (9, 3) with metabelian mainline on  $\mathcal{G}(3, 2)$

SmallGroups Id of root $G$	$G/G'$	TKT resp. pTKT	$\varkappa$	$\varepsilon$	$\eta \in$	$(f, g, h)$	population
$\langle 729, 40 \rangle$	(3, 3)	b.10	(0043)	2	{1, 2}	(0, 1, 0)	forbidden
$\langle 243, 6 \rangle$	(3, 3)	c.18	(0313)	1	{0, 1}	(0, 1, 2)	admissible
$\langle 243, 8 \rangle$	(3, 3)	c.21	(0231)	0	{2, 3}	(1, 1, 2)	admissible
$\langle 243, 17 \rangle$	(9, 3)	a.1	(000; 0)	2	{3, 4}	(0, 0, 1)	admissible
$\langle 243, 15 \rangle$	(9, 3)	a.1	(000; 0)	1	{3, 4}	(1, 0, 0)	admissible

Aside from the single forbidden coclass tree with metabelian mainline of transfer kernel type b.10,  $\varkappa = (0043)$ , there exist 4 admissible coclass trees with metabelian mainline on  $\mathcal{G}(3, 2)$  which are populated quite densely by second 3-class groups  $G_3^2(K)$  of quadratic fields  $K$  with 3-class group of type (3, 3), resp. (9, 3). They can be characterized by the number of members of the TTT with 3-rank bigger than 2,  $\varepsilon = \#\{1 \leq i \leq 4 \mid r_3(H_i/H_i') \geq 3\}$ , or by the number of members of the TKT, resp. punctured TKT (pTKT), having Taussky’s type A,  $\eta = \#\{1 \leq i \leq 4 \mid \kappa(i) = A\}$ , as shown in Table 3.9. Groups along their mainlines arise as quotients of infinite pro-3

groups of coclass 2 having a non-trivial centre, whose pro-3 presentations are defined by suitable triplets  $(f, g, h)$  of relational exponents in [34, Thm. 4.1].

FIGURE 3.6. Distribution of  $G_3^2(K)$  on the coclass 2 tree  $\mathcal{T}(\langle 243, 6 \rangle)$



**Theorem 3.16.** *The structure of the complete coclass tree  $\mathcal{T}(\langle 243, 6 \rangle)$  as part of the coclass graph  $\mathcal{G}(3, 2)$ , restricted to 3-groups  $G$  with abelianization  $G/G' \simeq (3, 3)$ , is globally characterized by the tree invariant  $\varepsilon(G) = 1$  and given up to order  $3^{11} = 177147$  by Figure 3.6. The branches are of depth 3 and periodic of length 2. The pre-period consists of  $\mathcal{B}_5, \mathcal{B}_6$ , the primitive period of  $\mathcal{B}_7, \mathcal{B}_8$*

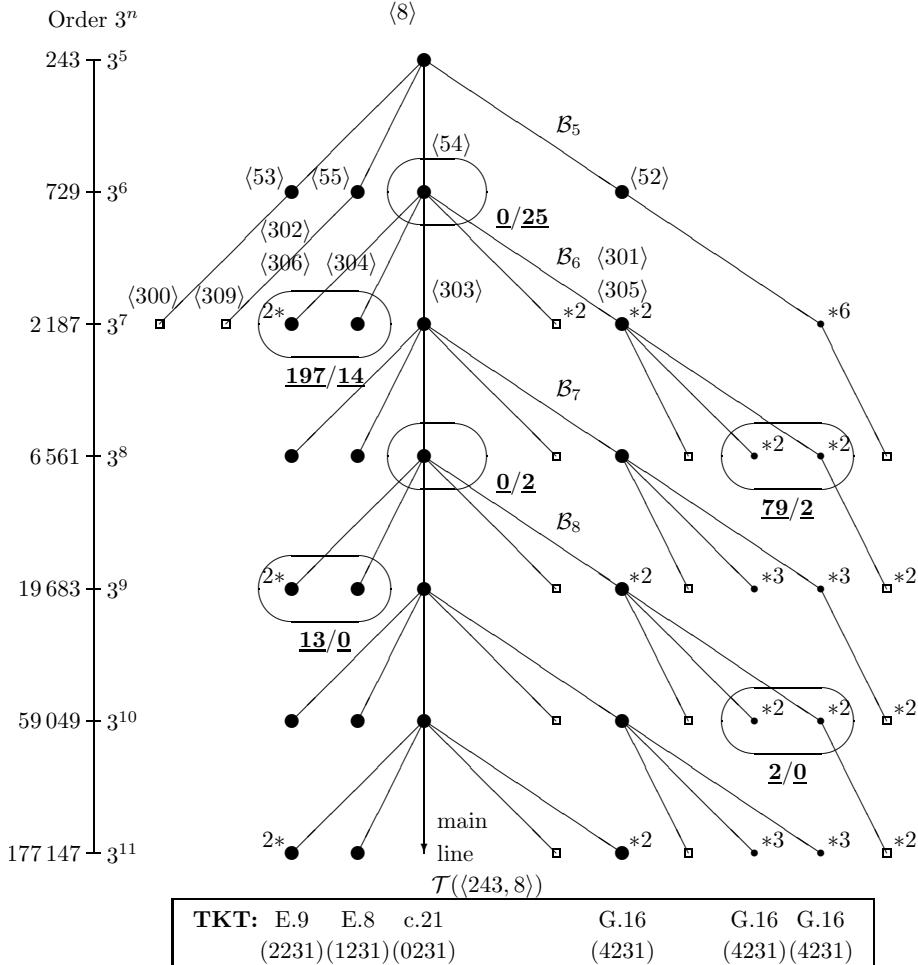
In Figure 3.6, we have  $G_3^2(\mathbb{Q}(\sqrt{D})) \in \mathcal{T}(\langle 243, 6 \rangle)$  for 270 (13.4%) of the 2020 discriminants  $-10^6 < D < 0$  and for 39 (1.5%) of the 2576 discriminants  $0 < D < 10^7$ , investigated in [58, § 6], [60, § 6].

Since the TKT c.18,  $\varkappa = (0313)$ , of the mainline is *total* with  $\varkappa(1) = 0$ , there only occur  $G_3^2(K)$  of *real* quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D > 0$ , on the mainline.

Due to the *Selection Rule* in Theorem 3.13, the  $G_3^2(K)$  are distributed on *even branches* only, since the second distinguished transfer kernel  $\varkappa(2) \neq 0$ .

Underpinning the weak leaf conjecture, there is no actual hit of the vertices at depth 1 with TKT H.4,  $\varkappa = (3313)$ , e. g., of  $\langle 2187, 286 \dots 287 \rangle$ .

FIGURE 3.7. Distribution of  $G_3^2(K)$  on the coclass 2 tree  $\mathcal{T}(\langle 243, 8 \rangle)$



**Theorem 3.17.** *The structure of the complete coclass tree  $\mathcal{T}(\langle 243, 8 \rangle)$  as part of the coclass graph  $\mathcal{G}(3, 2)$ , restricted to 3-groups  $G$  with abelianization  $G/G' \simeq (3, 3)$ , is globally characterized by  $\varepsilon(G) = 0$  and given up to order  $3^{11} = 177\,147$  by Figure 3.7. The branches are of depth 3 and periodic of length 2. The pre-period consists of  $\mathcal{B}_5, \mathcal{B}_6$ , the primitive period of  $\mathcal{B}_7, \mathcal{B}_8$*

In Figure 3.7, we have  $G_3^2(\mathbb{Q}(\sqrt{D})) \in \mathcal{T}(\langle 243, 8 \rangle)$  for 291 (14.4%) of the 2020 discriminants  $-10^6 < D < 0$  and for 43 (1.7%) of the 2576 discriminants  $0 < D < 10^7$ , investigated in [58, § 6], [60, § 6].

Since the TKT c.21,  $\varkappa = (0231)$ , of the mainline is *total* with  $\varkappa(1) = 0$ , there only occur  $G_3^2(K)$  of *real* quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D > 0$ , on the mainline.

Due to the *Selection Rule* in Theorem 3.13, the  $G_3^2(K)$  are distributed on *even branches* only, since the second distinguished transfer kernel  $\varkappa(2) \neq 0$ .

Underpinning the weak leaf conjecture, there is no actual hit of the vertices at depth 1 with TKT G.16,  $\varkappa = (4231)$ , e. g., of  $\langle 2187, 301 \rangle$ ,  $\langle 2187, 305 \rangle$ .

The vertices of the coclass trees in both Figures 3.6 and 3.7 are classified by using different symbols:

- (1) big full discs  $\bullet$  represent metabelian groups with bicyclic centre of type  $(3, 3)$  and defect  $k = 0$ ,
- (2) small full discs  $\bullet$  represent metabelian groups with cyclic centre of order 3 and defect  $k = 1$ ,
- (3) small contour squares  $\square$  represent non-metabelian groups.

A number adjacent to a vertex denotes the multiplicity of a batch of immediate descendants sharing a common parent. The groups of particular importance are labelled by a number in angles, which is the identifier in the SmallGroups library [14] of GAP [40] and MAGMA [56], where the order is omitted since it is given on the left hand scale.

The actual distribution of the 2020, resp. 2576, second 3-class groups  $G_3^2(K)$  of complex, resp. real, quadratic number fields  $K = \mathbb{Q}(\sqrt{D})$  of type  $(3, 3)$  with discriminant  $-10^6 < D < 10^7$  is represented by underlined bold-face counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval. See [58, § 6, Tbl. 3–5] and [60, § 6, Tbl. 6.5–6.8].

The realization of mainline vertices with TKT c.18 and c.21 as  $G_3^2(K)$  is no violation of the weak leaf conjecture 3.1, since these vertices do not possess metabelian immediate descendants of the same TKT and of higher defect of commutativity.

*Proof.* (Proof of Theorems 3.16 and 3.17) The metabelian skeletons were drawn in [66, p. 189 ff], the complete trees were given in [5, p. 76, Fig. 4.8 and p. 123, Fig. 6.1], and  $\varepsilon$  was determined in [60, Cor. 4.4.1].  $\square$

When we had completed our extensive investigation of second 3-class groups  $G_3^2(K)$  of all 4596 quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  of type (3, 3) in the range  $-10^6 < D < 10^7$ , we wondered whether the distribution of second 3-class groups  $G_3^2(K)$  for other sequences of base fields  $K$  of type (3, 3) shows similarities or differences.

Since fields of degree 4 are still within the reach of numerical computations, we were able to analyze bicyclic biquadratic fields, either of Gauss-Dirichlet-Hilbert type  $K = \mathbb{Q}(\sqrt{-1}, \sqrt{d})$  [46] or of Eisenstein-Scholz-Reichardt type  $K = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$  [71, 70]. Whereas the former reveal strong similarities to quadratic fields, the latter exhibit a completely different behavior, as will be shown in a forthcoming paper.

**3.5. Stem of isoclinism family  $\Phi_6$ .** In this section, we provide group theoretic foundations for determining second 5-class groups  $G_5^2(K)$  of co-class  $\text{cc}(G) \geq 2$  for quadratic and quartic number fields  $K$  of type (5, 5). The stem of Hall's isoclinism family  $\Phi_6$  is the key for a deeper understanding of the 5-principalization of these base fields in their six unramified cyclic quintic extensions  $L_1, \dots, L_6$ , which has partially but not completely been investigated by Heider and Schmithals [45] and by Bembom [12].

The stem groups  $G$  of Hall's isoclinism family  $\Phi_6$  [44, p. 139] are  $p$ -groups of order  $|G| = p^5$  with odd prime  $p$ , nilpotency class  $\text{cl}(G) = 3$ , and coclass  $\text{cc}(G) = 2$ . They were discovered in 1898 by Bagnera [10, pp. 182–183], and were constructed as extensions of  $C_p^3$  by  $C_p^2$ , for  $p \geq 5$ , in 1926 by Schreier [73, pp. 341–345]. Bagnera also pointed out that these groups do not have an analog for  $p = 2$ .

Every stem group of isoclinism family  $\Phi_6$  is a 2-generator group  $G = \langle x, y \rangle$  with main commutator  $s_2 = [y, x]$  in  $\gamma_2(G)$  and higher commutators  $s_3 = [s_2, x]$ ,  $t_3 = [s_2, y]$  in  $\gamma_3(G)$ , satisfying the power relations  $s_2^p = s_3^p = t_3^p = 1$ . The lower central series of  $G$  is given by

$$\gamma_2(G) = \langle s_2, s_3, t_3 \rangle \text{ of type } (p, p, p), \quad \gamma_3(G) = \langle s_3, t_3 \rangle \text{ of type } (p, p),$$

$$\gamma_4(G) = 1,$$

and the center by  $\zeta_1(G) = \gamma_3(G)$ . The central quotient  $G/\zeta_1(G)$  is of type  $\Phi_2(1^3) \simeq G_0^3(0, 0)$ , the extra special  $p$ -group of order  $p^3$  and exponent  $p$ , and the abelianization  $G/\gamma_2(G)$  is of type  $(p, p)$ . Therefore, the lower central structure of these groups uniformly consists of two bicyclic factors, the head  $G/\gamma_2(G)$ , and the tail  $\gamma_3(G)/\gamma_4(G)$ , separated by the cyclic factor  $\gamma_2(G)/\gamma_3(G)$ .

For any stem group  $G$  in  $\Phi_6$ , there exists a nice 1-to-1 correspondence between the two bicyclic factors, the head and the tail, by taking the derived subgroups.

**Lemma 3.1.** *The maximal normal subgroups  $H_i$  of  $G$  contain the commutator subgroup  $G' = \gamma_2(G)$  and are given by*

$$H_i = \langle g_i, G' \rangle \text{ with generators } g_1 = y \text{ and } g_i = xy^{i-2} \text{ for } 2 \leq i \leq p + 1.$$

*Their derived subgroups  $H'_i = (G')^{g_i-1}$  are given by*

$$H'_1 = \langle t_3 \rangle \text{ and } H'_i = \langle s_3 t_3^{i-2} \rangle \text{ for } 2 \leq i \leq p + 1.$$

As a consequence of Lemma 3.1, we only have trivial two-step centralizers  $G' = \chi_2(G) < \chi_3(G) = G$  and the invariants  $e(G)$  and  $s(G)$  of section § 3.3 take the same value  $e = s = 3$ .

Individual relations for isomorphism classes by James [47, pp. 620–621] are given in Table 3.10, where  $\nu$  denotes the smallest positive quadratic non-residue modulo  $p$  and  $g$  denotes the smallest positive primitive root modulo  $p$ .

TABLE 3.10. Relations for the stem groups of  $\Phi_6$

stem group	$x^p$	$y^p$	parameters
$\Phi_6(221)_a$	$s_3$	$t_3$	
$\Phi_6(221)_{b_r}$	$s_3$	$t_3^k$	$1 \leq r \leq \frac{p-1}{2}, k = g^r$
$\Phi_6(221)_{c_r}$	$s_3^r t_3^r$	$s_3^{-\frac{r}{4}}$	$r \in \{1, \nu\}$
$\Phi_6(221)_{d_0}$	$t_3^\nu$	$s_3$	
$\Phi_6(221)_{d_r}$	$s_3 t_3$	$s_3^k$	$1 \leq r \leq \frac{p-1}{2}, k = \frac{g^{2r+1}-1}{4}$
$\Phi_6(21^3)_a$	1	$t_3$	$p \geq 5$
$\Phi_6(21^3)_{b_r}$	$t_3^r$	1	$r \in \{1, \nu\}, p \geq 5$
$\Phi_6(1^5)$	1	1	

These presentations for 7 isomorphism classes of 3-groups, resp. 12 isomorphism classes of 5-groups, among the stem of  $\Phi_6$  are now used to calculate the kernels of all transfers  $T_i : G/G' \rightarrow H_i/H'_i, 1 \leq i \leq p + 1$ , whose images are given for  $p = 5$  in very convenient form by Lemma 3.2, since the expressions for *inner transfers* are well-behaved  $p$ th powers. *Outer transfers* always map to  $p$ th powers, anyway.

**Lemma 3.2.** *For any  $1 \leq i \leq 6$ , the image of an arbitrary element  $gG' \in G/G'$  with representation  $x^j y^\ell G', 0 \leq j, \ell \leq 4$ , under the transfer  $T_i$  is given by  $T_i(x^j y^\ell G') = x^{pj} y^{p\ell} H'_i$ .*

In Table 3.11, TKTs  $\varkappa$  of 3-groups in the notation of [59, § 3.3] were determined by Nebelung [66, p. 208, Thm. 6.14] already, using different presentations in equation (3.3), section § 3.3.3. For 5-groups the TKTs are given here for the first time. The only exception is the group  $\langle 3125, 14 \rangle$ , which was discussed in the well-known paper by Taussky [76, p. 436, Thm.2]

TABLE 3.11. TKT of corresponding  $p$ -groups in  $\Phi_6$  for  $p \in \{3, 5\}$

$p = 3$				$p = 5$			
3-group	TKT	$\varkappa$	5-group	$\eta$	$\varkappa$	property	
$\langle 243, 7 \rangle$	$G_0^{4,5}(1, 1, -1, 1)$	D.5	$\langle 3125, 14 \rangle$	$\Phi_6(221)_a$	6	(123456)	identity
$\langle 243, 4 \rangle$	$G_0^{4,5}(1, 1, 1, 1)$	H.4	$\langle 3125, 11 \rangle$	$\Phi_6(221)_{b_1}$	2	(125364)	4-cycle
	no analog		$\langle 3125, 7 \rangle$	$\Phi_6(221)_{b_2}$	2	(126543)	two transpos.
$\langle 243, 8 \rangle$	$G_0^{4,5}(0, 0, 0, 1)$	c.21	$\langle 3125, 8 \rangle$	$\Phi_6(221)_{c_1}$	1	(612435)	5-cycle
$\langle 243, 5 \rangle$	$G_0^{4,5}(0, 0, -1, 1)$	D.10	$\langle 3125, 13 \rangle$	$\Phi_6(221)_{c_2}$	1	(612435)	5-cycle
$\langle 243, 9 \rangle$	$G_0^{4,5}(0, -1, -1, 0)$	G.19	$\langle 3125, 10 \rangle$	$\Phi_6(221)_{d_0}$	0	(214365)	three transpos.
$\langle 243, 6 \rangle$	$G_0^{4,5}(0, -1, 0, 1)$	c.18	$\langle 3125, 12 \rangle$	$\Phi_6(221)_{d_1}$	0	(512643)	6-cycle
	no analog		$\langle 3125, 9 \rangle$	$\Phi_6(221)_{d_2}$	0	(312564)	two 3-cycles
	no analog		$\langle 3125, 4 \rangle$	$\Phi_6(21^3)_a$	2	(022222)	nr1.const.with fp.
	no analog		$\langle 3125, 5 \rangle$	$\Phi_6(21^3)_{b_1}$	1	(011111)	nearly constant
	no analog		$\langle 3125, 6 \rangle$	$\Phi_6(21^3)_{b_2}$	1	(011111)	nearly constant
$\langle 243, 3 \rangle$	$G_0^{4,5}(0, 0, 0, 0)$	b.10	$\langle 3125, 3 \rangle$	$\Phi_6(1^5)$	6	(000000)	constant

as an example to show that the coarse TKT  $\kappa = (\text{AAAAAA})$  can occur for  $p = 5$ , and also for primes  $p \geq 7$ . A convenient partial characterization is provided by counters of fixed point transfer kernels, resp. abelianizations of type  $(5, 5, 5)$ ,  $\eta = \#\{1 \leq i \leq 6 \mid \kappa(i) = \text{A}\} = \#\{1 \leq i \leq 6 \mid \tau(i) = (5, 5, 5)\}$ , which must coincide, according to [45, Thm. 7, p. 11].

The correspondence between  $p = 3$  and  $p = 5$  is due to the formally identical power-commutator presentation. However, it is partially rather shallow, since corresponding groups can have different properties with respect to their role on the coclass graphs  $\mathcal{G}(3, 2)$  and  $\mathcal{G}(5, 2)$ . For example, the 3-groups  $\langle 243, 6 \rangle$  and  $\langle 243, 8 \rangle$  are mainline vertices having the mandatory total first transfer kernel  $\varkappa(1) = 0$  whereas the 5-groups  $\langle 3125, 12 \rangle$  and  $\langle 3125, 8 \rangle$  are terminal without total transfer.

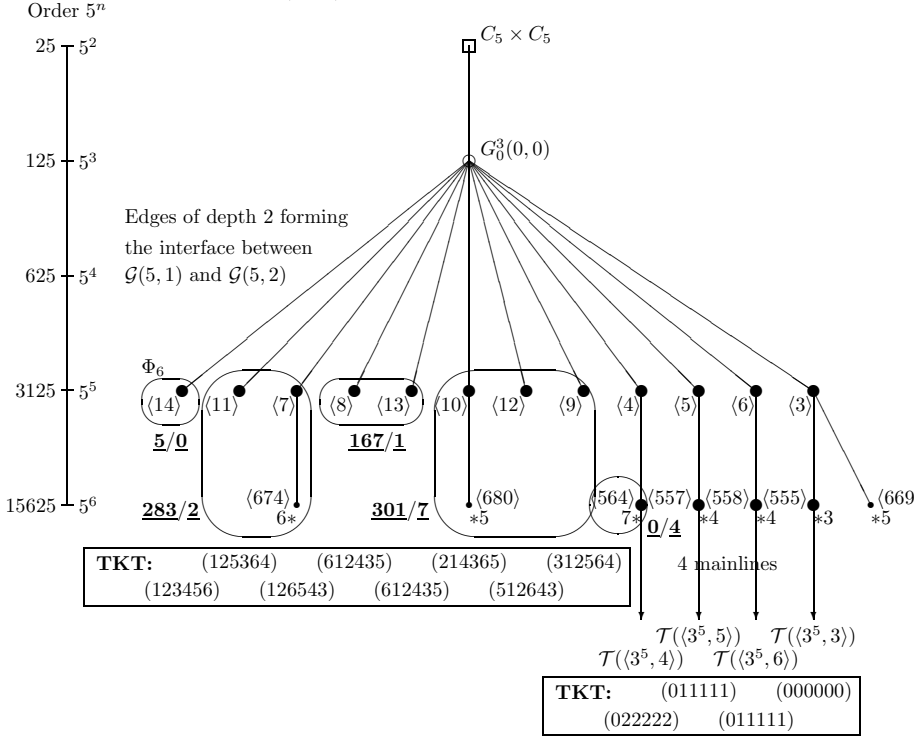
**3.5.1. Top vertices of type  $(5, 5)$  on  $\mathcal{G}(5, 2)$ .** Figure 3.8 shows the non-CF groups at the top of coclass graph  $\mathcal{G}(5, 2)$ . It was constructed by means of the SmallGroups library [14] of GAP [40] and MAGMA [56]. The groups are labelled by a number in angles, which is their identifier in that library. Additional confirmation was obtained by explicit descendant calculation with the aid of the ANUPQ package [39].

The vertices of the coclass graph  $\mathcal{G}(5, 2)$  in Figure 3.8 are classified by using different symbols:

- (1) a large contour square  $\square$  represents an abelian group,
- (2) a big contour circle  $\circ$  represents a metabelian group with abelian maximal subgroup,
- (3) big full discs  $\bullet$  represent metabelian groups with bicyclic centre of type  $(5, 5)$ ,
- (4) small full discs  $\bullet$  represent metabelian groups with cyclic centre of order 5.



FIGURE 3.8. Sporadic groups and roots of conjectural co-class trees on  $\mathcal{G}(5, 2)$



The actual distribution of the 959, resp. 377, second 5-class groups  $G_5^2(K)$  of complex, resp. real, quadratic number fields  $K = \mathbb{Q}(\sqrt{D})$  of type (5, 5) with discriminant  $-2\,270\,831 < D < 26\,695\,193$  is represented by underlined boldface counters (in the format complex/real) of the hits of vertices surrounded by the adjacent oval.

**3.5.2. Fixed point principalization problem.** We are pleased to present the solution of a problem posed in 1970 by Taussky [76, p. 438, Rem. 1]. It concerns the lack of realizations, in the form of second 5-class groups  $G_5^2(K)$  of number fields  $K$ , of the unique metabelian 5-group  $\langle 5^5, 14 \rangle$  with 6 fixed point transfer kernels, that is with coarse TKT  $\kappa = (AAAAAA)$ , but without total transfer kernels  $\varkappa(i) = 0$ . Actually, we now have 5 realizations of this very special TKT  $\varkappa = (123456)$  (identity permutation) for quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $D \in \{-89\,751, -235\,796, -1\,006\,931, -1\,996\,091, -2\,187\,064\}$ , in Table 3.13, and 4 further realizations for certain cyclic quartic fields [48]  $K = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{D})$ ,  $D \in \{581, 753, 2\,296, 4\,553\}$ .

**3.5.3. Statistical evaluation of second 5-class groups  $G_5^2(K)$ .** The possibilities for 5-groups of coclass 2 are more extensive than those for coclass 1.

For the 377 real quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $0 < D < 26\,695\,193$ , in Table 3.4, there occur 7 cases of coarse TKT  $\kappa = (\text{BBBBBB})$ , for  $D \in \{4\,954\,652, 7\,216\,401, 12\,562\,849, 16\,434\,245, 18\,434\,456, 19\,115\,293, 20\,473\,841\}$ , a single case of TKT  $\varkappa = (612435)$ , for  $D = 18\,070\,649$ , 2 cases of coarse TKT  $\kappa = (\text{AABBBB})$ , for  $D \in \{10\,486\,805, 18\,834\,493\}$ , and 4 cases of the first excited state of TKT  $\varkappa = (022222)$ , for  $D \in \{7\,306\,081, 11\,545\,953, 14\,963\,612, 22\,042\,632\}$ .

TABLE 3.12. 9 variants of  $G = G_5^2(K)$  for 959  $K = \mathbb{Q}(\sqrt{D})$ ,  $-2\,270\,831 \leq D < 0$

$D$	$\tau(K)$	$\tau(0)$	$\kappa(K)$	$G$	cc( $G$ )	#	%
-11 199	$(5, 5^2)^6$	$(5, 5, 5)$	$(B^6)$	$\langle 3125, 9 10 12 \rangle$	2	301	31.4
-12 451	$(5, 5, 5), (5, 5^2)^5$	$(5, 5, 5)$	$(A, B^5)$	$\langle 3125, 8 13 \rangle^*$	2	167	17.4
-30 263	$(5, 5, 5)^2, (5, 5^2)^4$	$(5, 5, 5)$	$(A^2, B^4)$	$\langle 3125, 7 11 \rangle$	2	283	29.5
-89 751	$(5, 5, 5)^6$	$(5, 5, 5)$	$(A^6)$	$\langle 3125, 14 \rangle^*$	2	5	
-62 632	$(5, 5, 5, 5^2), (5, 5, 5), (5, 5^2)^4$		$(?, A, B^4)$	$\langle 78125, \# \rangle$	2	124	12.9
-67 031	$(5, 5, 5, 5, 5), (5, 5^2)^5$		$(B^6)$	$\langle 78125, \# \rangle$	2	6	
-67 063	$(5, 5, 5, 5, 5), (5, 5, 5), (5, 5^2)^4$		$(B, A, B^4)$	$\langle 78125, \# \rangle$	2	37	3.9
-280 847	$(5, 5, 5, 5^2), (5, 5^2)^5$		$(?, B^5)$	$\langle 78125, \# \rangle$	2	32	3.3
-181 752	$(5, 5^2, 5^2, 5^2), (5, 5, 5), (5, 5^2)^4$		$(?, A, B^4)$	$\langle 1953125, \# \rangle$	2	4	

Among the 959 complex quadratic fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $-2\,270\,831 \leq D < 0$ , in Table 3.12, ground states (GS) appear exclusively with sporadic, and mostly terminal, top vertices of  $\mathcal{G}(5, 2)$ . The 5 cases of TKT  $\varkappa = (123456)$  have been presented separately in section § 3.5.2 as solutions of Taussky’s problem of 1970. Further, there are 167 cases (17.4%) of TKT  $\varkappa = (612435)$  (5-cycle with coarse TKT  $\kappa = (\text{BBBABB})$ ) starting with  $D = -12\,451$ , which was attempted but not analyzed completely in 1982 by Heider and Schmithals [45] and 283 cases (29.5%) of coarse TKT  $\kappa = (\text{AABBBB})$  starting with  $D = -30\,263$ . The remaining 301 cases (31.4%) of coarse TKT  $\kappa = (\text{BBBBBB})$ , starting with  $D = -11\,199$  are slightly dominating.

For excited states (ES) of coclass 2 as well as of coclass 1, the distinguished first 5-class group  $\text{Cl}_5(K_1)$  of the non-Galois absolute quintic subfield  $K_1$  of the unramified extension  $L_1|K$  is of 5-rank  $r_5(K_1) = 2$ , which shows impressively that the rank equation for  $p = 3$ ,  $r_3(K_i) = r_3(K) - 1$ , by Gras [42] and Gerth [41] generalizes to a double inequality for  $p \geq 5$ ,

$$r_p(K) - 1 \leq r_p(K_i) \leq \frac{p-1}{2} \cdot (r_p(K) - 1),$$

as predicted, and partially proved, by Bölling [17] and Lemmermeyer [55].

TABLE 3.13. 10 subvariants of  $G = G_5^2(K)$  for 31 fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $-89751 \leq D < 0$

$D$	$\tau(K)$	$\tau(0)$	$\varkappa(K)$	$G$	$\text{cc}(G)$	#
-11 199	$(5, 5^2)^6$	$(5, 5, 5)$	$(512643)$	$\langle 3125, 12 \rangle^*$	2	7
-17 944	$(5, 5^2)^6$	$(5, 5, 5)$	$(312564)$	$\langle 3125, 9 \rangle^*$	2	2
-42 871	$(5, 5^2)^6$	$(5, 5, 5)$	$(214365)$	$\langle 15625, 680 \rangle$	2	3
-12 451	$(5, 5, 5), (5, 5^2)^5$	$(5, 5, 5)$	$(612435)$	$\langle 3125, 8 \rangle^*$ or $\langle 3125, 13 \rangle^*$	2	5
-30 263	$(5, 5, 5)^2, (5, 5^2)^4$	$(5, 5, 5)$	$(126543)$	$\langle 15625, 674 \rangle$	2	4
-37 363	$(5, 5, 5)^2, (5, 5^2)^4$	$(5, 5, 5)$	$(125364)$	$\langle 3125, 11 \rangle^*$	2	2
-89 751	$(5, 5, 5)^6$	$(5, 5, 5)$	$(123456)$	$\langle 3125, 14 \rangle^*$	2	5
-62 632	$(5, 5, 5, 5^2), (5, 5, 5), (5, 5^2)^4$	$(5, 5, 5, 5, 5)$	$(322222)$	$\langle 78125, \# \rangle$	2	1
-67 031	$(5, 5, 5, 5, 5), (5, 5^2)^5$	$(5, 5, 5, 5, 5)$	$(211111)$	$\langle 78125, \# \rangle$	2	1
-67 063	$(5, 5, 5, 5, 5), (5, 5, 5), (5, 5^2)^4$	$(5, 5, 5, 5, 5)$	$(322222)$	$\langle 78125, \# \rangle$	2	2

The transfer target type (TTT)  $\tau(G)$  of second 5-class groups  $G_5^2(K)$  has been computed for all quadratic number fields  $K = \mathbb{Q}(\sqrt{D})$ , having discriminant  $-2270831 < D < 26695193$  and 5-class group  $\text{Cl}_5(K)$  of type  $(5, 5)$ , with the aid of MAGMA [56]. As a refinement, we calculated the transfer kernel type (TKT)  $\varkappa(G)$  for 31 fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $-89751 \leq D < 0$ , as given in Table 3.13. This also refines results of Bembom in [12, p. 129]. Observe that Bembom does not give TKTs in our sense and consequently was not able to discover the distinguished role of  $D = -89751$  with respect to the Taussky problem. Schur  $\sigma$ -groups are starred.

**3.5.4. Statistical evaluation of second 7-class groups  $G_7^2(K)$ .**

Among the 94 complex quadratic fields  $K$  of type  $(7, 7)$  with discriminants  $-10^6 \leq D < 0$ , we found 7 variants of the second 7-class group  $G = G_7^2(K)$ , characterized by different TTT  $\tau(K)$  and Taussky’s coarse TKT  $\varkappa(K)$ , which are related by [45, Thm. 7, p. 11].

TABLE 3.14. 7 variants of  $G = G_7^2(K)$  for 70 fields  $K = \mathbb{Q}(\sqrt{D})$ ,  $-751288 \leq D < 0$

$D$	$\tau(K)$	$\tau(0)$	$\varkappa(K)$	$G$	$\text{cc}(G)$	#	%
-63 499	$(7, 7^2)^8$	$(7, 7, 7)$	$(B^8)$	$\langle 16807, 10 14 15 16 \rangle$	2	40	43
-183 619	$(7, 7, 7)^2, (7, 7^2)^6$	$(7, 7, 7)$	$(A^2, B^6)$	$\langle 16807, 11 12 13 \rangle$	2	29	31
-227 860	$(7, 7, 7), (7, 7^2)^7$	$(7, 7, 7)$	$(A, B^7)$	$\langle 16807, 8 9 \rangle$	2	9	9.6
unknown	$(7, 7, 7)^8$	$(7, 7, 7)$	$(A^8)$	$\langle 16807, 7 \rangle$	2		
-159 592	$(7, 7, 7, 7, 7), (7, 7, 7), (7, 7^2)^6$		$(A^2, B^6)$	$\langle 823543, \# \rangle$	2	3	
-227 387	$(7, 7, 7, 7^2), (7, 7, 7), (7, 7^2)^6$		$(B, A, B^6)$	$\langle 823543, \# \rangle$	2	10	
-272 179	$(7, 7, 7, 7^2), (7, 7^2)^7$		$(B^8)$	$\langle 823543, \# \rangle$	2	2	
-673 611	$(7, 7, 7, 7, 7, 7^2), (7, 7, 7), (7, 7^2)^6$		$(?, A, B^6)$	$\langle 40353607, \# \rangle$	2	1	

In Table 3.14 we present the discriminants with smallest absolute values, corresponding to these variants.  $\tau(0)$  denotes the 7-class group of  $F_7^1(K)$ . Using the SmallGroups library [14], we identified 78 *ground states* (83%) having their  $G$  among the sporadic top vertices of  $\mathcal{G}(7, 2)$  in the stem of isoclinism family  $\Phi_6$ . Unfortunately, there didn't occur a solution of Taussky's 1970 fixed point capitulation problem for  $p = 7$ , in form of a realization of  $\langle 16807, 7 \rangle$ . However, there appeared 15 *first excited states* (16%) with  $G$  located on coclass trees of  $\mathcal{G}(7, 2)$ , where the non-Galois subfield  $K_1$  of the distinguished extension  $L_1$  has a 7-class group of type  $(7, 7)$ , and, particularly remarkable, a single *second excited state* for  $D = -673\,611$ , where the maximal 7-rank 3 in Bölling's inequality [17, 55] is attained in form of  $\text{Cl}_7(K_1) \simeq (7, 7, 7)$ .

#### 4. $p$ -Groups with double layered abelianization of type $(p^2, p)$ or $(p, p, p)$

For a number field  $K$  with  $p$ -class group  $\text{Cl}_p(K)$  of type  $(p^2, p)$ , resp.  $(p, p, p)$ , there exist *two layers* of unramified abelian extensions  $L|K$ , each containing  $p + 1$ , resp.  $p^2 + p + 1$ , members. Extensions in the *first layer* are of relative degree  $p$ , those in the *second layer* are of relative degree  $p^2$ . Consequently, the second layer tends to be out of the scope of actual computations. However, there are some exceptions of modest degree.

**4.1. Quadratic fields of type  $(9, 3)$ .** On the one hand, there is the case  $p = 3$  for quadratic fields  $K = \mathbb{Q}(\sqrt{D})$  with 3-class group  $\text{Cl}_3(K)$  of type  $(9, 3)$  or  $(3, 3, 3)$ , where extensions in the second layer are of absolute degree 18. From the viewpoint of 3-towers, there are no open problems for complex quadratic fields  $K$  of type  $(3, 3, 3)$ , since it is known that  $\ell_3(K) = \infty$  [50, 63], that is,  $G_3^\infty(K)$  is always an infinite pro-3 group.

Thus, we focussed on 3-class rank 2 and computed the first layer of the TTT and TKT of all 875 complex quadratic fields  $K$  of type  $(9, 3)$  with discriminant  $-10^6 < D < 0$  and of all 271 real quadratic fields  $K$  of type  $(9, 3)$  with discriminant  $0 < D < 10^7$ . In [61], we will show that this information is sufficient to identify the second 3-class group  $G_3^2(K)$  for 565 negative discriminants (65%) and for 188 positive discriminants (70%). For the remainder, the second layer of the TTT and TKT is required.

**4.2. Quadratic and biquadratic fields of type  $(2, 2, 2)$ .** On the other hand, we have the case  $p = 2$  for quadratic, resp. quartic, fields  $K$  with 2-class group  $\text{Cl}_2(K)$  of type  $(4, 2)$  or  $(2, 2, 2)$ , where extensions in the second layer are of absolute degree 8, resp. 16.

We were particularly interested in fields  $K$  of 2-class rank 3, where the 2-tower length  $\ell_2(K)$  is still an open problem. We found that the coclass

tree  $\mathcal{T}(\langle 16, 11 \rangle)$ , which is the unique tree of coclass graph  $\mathcal{G}(2, 2)$  containing groups with abelianization of type  $(2, 2, 2)$ , is populated by second 2-class groups  $G_2^2(K)$  of real quadratic fields  $K$  of type  $(2, 2, 2)$ . The tree  $\mathcal{T}(\langle 16, 11 \rangle)$  corresponds to the pro-2 group  $S_5$  in [35, 32] with periodic sequences  $K_x^i$ ,  $46 \leq i \leq 51$ , given by explicit parametrized presentations for  $x \geq 0$  in [35]. It also corresponds to the so-called family #59 with explicit pro-2 presentation given in [69].

Further, we obtained deeper results concerning second 2-class groups  $G = G_2^2(K)$  of complex quadratic fields of type  $(2, 2, 2)$ , which have been classified in terms of the smallest non-abelian lower central quotient  $G/\gamma_3(G)$ , usually coinciding with the root of the coclass tree  $\mathcal{T}$  such that  $G \in \mathcal{T}$ , by E. Benjamin, F. Lemmermeyer, and C. Snyder [54, 13]. Note that these authors use the Hall-Senior classification [43], whereas we give identifiers of the SmallGroups library [14]. The groups  $G$  are mainly, but not exclusively, located at the terminal top vertices  $\langle 32, 32 \rangle$  and  $\langle 32, 33 \rangle$  of coclass graph  $\mathcal{G}(2, 3)$  and on the coclass trees  $\mathcal{T}(\langle 32, 29 \rangle)$ ,  $\mathcal{T}(\langle 32, 30 \rangle)$ ,  $\mathcal{T}(\langle 32, 35 \rangle)$ , corresponding to the families #75, #76, #79 with explicit pro-2 presentations given in [69]. These subtrees of  $\mathcal{G}(2, 3)$  seem to be populated on every branch, with the only exception of the root.

We intend to include these results on complex quadratic fields in a forthcoming paper in cooperation with Azizi, Zekhnini, and Taous, where the principal aim is to investigate bicyclic biquadratic fields  $K = \mathbb{Q}(\sqrt{-1}, \sqrt{d})$ , called *special Dirichlet fields* by Hilbert [46], with 2-class groups of type  $(2, 2, 2)$ , based on work in [7, 8, 9]. The second 2-class groups  $G_2^2(K)$  for certain series of radicands  $d > 0$ , for example  $d \in \{170, 730, 2314\}$ , seem to be distributed on every branch of the coclass tree  $\mathcal{T}(\langle 64, 140 \rangle)$  of coclass graph  $\mathcal{G}(2, 3)$ , which corresponds to family #73 with explicit pro-2 presentation given in [69].

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