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2-Cohomology of semi-simple simply connected group-schemes over curves defined over p-adic fields

par Jean-Claude DOUAI

Résumé. Soit X une courbe propre, lisse, géométriquement connexe, définie sur un corps p-adique k. Lichtenbaum a prouvé l'existence d'une dualité parfaite :

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

entre le groupe de Brauer et le groupe de Picard de X et en a déduit l'existence d'une injection de $\mathrm{Br}(X)$ dans le produit des $\mathrm{Br}(k_P)$ où P décrit les point fermés de X et k_P désigne le corps résiduel du point P. Le but cet article est de montrer que si, $G = \widetilde{G}$ est un X_{et}-schéma en groupes semi-simples simplement connexes (groupes s.s.s.c), alors le résultat de Lichtenbaum implique la neutralité de chaque X_{et}-gerbe qui est localement liée par $\widetilde{G}$. En particulier, si $\mathfrak{X}$ est un modèle de X sur l'anneau $\mathcal{O}$ des entiers de k, i.e $X = \mathfrak{X} \times_{\mathcal{O}} k$, alors chaque $\mathfrak{X}_{et}$-gerbe localement liée par un $\mathfrak{X}$-groupe s.s.s.c est neutre (ceci étant une application du théorème de changement propre).

Plus généralement, reprenant un procédé du à Colliot-Thélène et Saito, nous pouvons montrer que si X est une k-variété propre, lisse, de dimension strictement plus grande que 1, alors chaque classe du quotient $H^2(X_{et}, \mathcal{L})/H^2(\mathfrak{X}_{et}, \mathcal{L})$ est neutre où $\mathfrak{X}$ est un $\mathcal{O}$-modèle de X et $\mathcal{L}$ un $\mathfrak{X}$-lien localement représentable par un schéma en groupes s.s.s.c sous la condition mineure que le cardinal de son centre soit premier à p. Nous donnerons ensuite des applications.

Abstract. Let X be a proper, smooth, geometrically connected curve over a p-adic field k. Lichtenbaum proved that there exists a perfect duality:

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

between the Brauer and the Picard group of X, from which he deduced the existence of an injection of $\mathrm{Br}(X)$ in $\prod_{P \in X} \mathrm{Br}(k_P)$ where $P \in X$ and k_P denotes the residual field of the point P. The aim of this paper is to prove that if $G = \widetilde{G}$ is an X_{et}- scheme of semi-simple simply connected groups (s.s.s.c groups), then we

can deduce from Lichtenbaum's results the neutrality of every X_{et}-gerb which is locally tied by $\widetilde{G}$. In particular, if $\mathfrak{X}$ is a model of X over the ring of integers $\mathcal{O}$ in k, i.e $X = \mathfrak{X} \times_{\mathcal{O}} k$, then every $\mathfrak{X}_{et}$-gerb which is locally tied by a s.s.s.c $\mathfrak{X}$-group is neutral (this being a variant of the proper base change theorem).

More generally, using a technique of Colliot-Thélène and Saito, we can prove that, if X is a proper smooth k-variety of dimension greater than 1, then every class of $H^2(X_{et}, \mathcal{L}) \diagup H^2(\mathfrak{X}_{et}, \mathcal{L})$ is neutral whenever $\mathcal{L}$ is a $\mathfrak{X}$-band that is locally represented by a s.s.s.c group under the condition that the cardinality of its center is coprime to p. We will then give some applications.

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1. Moduli field, definition field

Let k be a p-adic field (= finite extension of $\mathbb{Q}_p$), $\overline{k}/k$ a separable closure of k, X a projective, smooth, geometrically irreducible curve over k and $\overline{X} = X \bigotimes_k \overline{k}$.

Theorem 1.1. *Let $\widetilde{G}$ be a scheme of semi-simple, simply connected (or s.s.s.c) groups over X/k, $\overline{\widetilde{G}} = \widetilde{G} \bigotimes_k \overline{k}$. Then, there are the isomorphisms:*

$$H^1(X, \widetilde{G}) \simeq H^0(k, H^1(\overline{X}_{et}, \overline{\widetilde{G}})) \simeq H^0(k, H^1(\overline{X}_{Zar}, \overline{\widetilde{G}}))$$

where $H^1(X, G)$ means $H^1_{et}(X, G)$, i.e each X-torsor under $\widetilde{G}$ of moduli field k is defined over k and the descent from $\overline{k}$ is unique.

Remark. The second isomorphism comes from Nisnevich's isomorphism (**Corollaire 4.4** de [9]):

$$H^1(\overline{X}_{et}, \overline{\widetilde{G}}) \simeq H^1(\overline{X}_{Zar}, \overline{\widetilde{G}})$$

Proof. Leray's spectral sequence

$$H^p(k, H^q(\overline{X}, \overline{\widetilde{G}})) \Rightarrow H^{p+q}(X, \widetilde{G})$$

does not exist but we get the exact sequence (**Prop. 3.1.3**-Chap.V, p.323 of Giraud [4]):

$$1 \longrightarrow H^1(k, \widetilde{G}) \longrightarrow H^1(X, \widetilde{G}) \longrightarrow H^0(k, H^1(\bar{X}, \bar{\widetilde{G}}))\ .$$

To each class $[Z]$ of $H^0(k, H^1(\bar{X}, \bar{\widetilde{G}}))$, we associate the k-gerb $\mathcal{G}_Z$ whose objects over the open set

$$U = (\operatorname{Spec} K \longrightarrow \operatorname{Spec} k) \in Ob(\operatorname{Spec}(k)_{et})$$

are the $X \otimes_k K$-torsors under $\widetilde{G}$ which are isomorphic to $Z/_{X\otimes_k K}$. The k-gerb $\mathcal{G}_Z$ belongs to $Z^2(k,\mathcal{L})$, where the k-band $\mathcal{L}$ is locally (for the etale topology) represented by $\widetilde{G}$. □

Because k is a p-adic field and $\widetilde{G}$ semi-simple, $\mathcal{G}_Z$ is neutral (cf. [3],[3′],[1]) and by the **Prop.3.1.6 (i)** of Chap.V, p.325 of Giraud [4], $[Z]$ belongs to the image of $H^1(X,\widetilde{G})$.

Because $\widetilde{G}$ is s.s.s.c, $H^1(k,\widetilde{G}) = 0$ by Kneser's theorem 1 of [5]. From this, it results that $H^1(X,\widetilde{G}) \simeq H^0(k, H^1(\bar{X},\bar{\widetilde{G}}))$.

Remark. If G is only semi-simple (s.s), then we get the exact sequence:

$$1 \longrightarrow H^1(k,G) \longrightarrow H^1(X,G) \twoheadrightarrow H^0(k,H^1(\bar{X}_{Zar},\bar{G})) \longrightarrow *$$

i.e the descent is no longer unique.

2. The main Theorem

Let k be a p-adic field, X a projective, smooth, geometrically irreducible curve over k. Lichtenbaum [6] proved that there exists a perfect duality:

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

between the Brauer and the Picard group of X, from which he deduced the existence of an injection

$$\mathrm{Br}(X) \hookrightarrow \prod_{P\in X} \mathrm{Br}(k_P)$$

where k_P denotes the residual field of the point P.

Theorem 2.1. *Let k be a p-adic field, X a projective, smooth, geometrically irreducible curve over k, $\widetilde{G}$ being a s.s.s.c X-group, $\mathcal{L}$ a X-band that is locally (for the etale topology) representable by $\widetilde{G}$. Then each class of $H^2_{et}(X,\mathcal{L})$ is neutral.*

First, we shall prove **Theor.2.1**. In the §3, we shall see (**Rem. a)**) that **theor.3.1** gives again **theor.2.1** but only modulo the restriction $(|Z(\mathcal{L})|, p) = 1$, where $Z(\mathcal{L})$= center of $\mathcal{L}$.

Proof of theor. 2.1. As in [3], we can suppose that $\mathcal{L}$ has the form $\underline{lien}(\widetilde{G})$ and then establish the neutrality of each class of $H^2(X,\widetilde{G})$ with s.s.s.c and quasi-split $\widetilde{G}$. Let $(\widetilde{B},\widetilde{T})$ be a Killing pair of $\widetilde{G}$: by the **prop.3-1**3 exposé XXIV of [8], $\widetilde{T}$ is isomorphic to an induced torus $\prod_{X'/X} G_{m_{X'}}$, where X' is the X-scheme of Dynkin of $\widetilde{G}$. The injectivity of

$$\mathrm{Br}(X') \hookrightarrow \prod_{P'\in X'} \mathrm{Br}(k_P)$$

induces the injectivity of

$$H^2(X,\widetilde{T}) \hookrightarrow \prod_{P\in X} H^2(k_P,\widetilde{T}).$$

Consider the diagram

$$(D_{2.1})\qquad \begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ H^1(X,\widetilde{T}_{ad}) & \xrightarrow{\delta_1} & H^2(X,Z(\widetilde{G})) & \longrightarrow & H^2(X,\widetilde{T}) \\ \downarrow & & \downarrow & & \downarrow \\ \prod\limits_{P\in X} H^1(k_P,\widetilde{T}_{ad}) & \longrightarrow & \prod\limits_{P\in X} H^2(k_P,Z(\widetilde{G})) & \longrightarrow & \prod\limits_{P\in X} H^2(k_P,\widetilde{T}) \\ \| * & & & & \\ 0 & & & & \end{array}$$

The quotient of $H^2(X,Z(\widetilde{G}))$ by $Im.\delta_1$ is injected into $\prod\limits_{P\in X} H^2(k_P,Z(\widetilde{G}))$. The set $H^2(X,\widetilde{G})$ is a homogeneous principal space under $H^2(X,Z(\widetilde{G}))$. By analogy, $H^2(X,\widetilde{G})/Im.\delta_1$ is injected into $\prod\limits_{P\in X} H^2(k_P,\widetilde{G})$ and we obtain the following diagram:

$$(D_{2.2})\qquad \begin{array}{ccc} H^2(X,\widetilde{G})/Im.\delta_1 & \hookrightarrow & \prod\limits_{P\in X} H^2(k_P,\widetilde{G}) \\ \uparrow\!\circ & & \uparrow\!\circ \\ H^2(X,Z(\widetilde{G}))/Im.\delta_1 & \hookrightarrow & \prod\limits_{P\in X} H^2(k_P,Z(\widetilde{G})) \end{array}$$

where $\longrightarrow\!\circ\!\!\rightarrow$ is the relation of the non abelian 2-cohomology ([4], **Def. 3.1.4** Chap IV p.248).

We know ([3], [1]) that each class of $H^2(k_P,\widetilde{G})$ is neutral, therefore each class of $H^2(k_P,\widetilde{G})$ is in relation with the class 0 of $H^2(k_P,Z(\widetilde{G}))$ **(Cor. 3.3.7** Chap IV, p.258-259 of [4]).

From the previous diagram, we deduce that each class of $H^2(X,\widetilde{G})/Im.\delta_1$ is in relation with the class 0 of $H^2(X,Z(\widetilde{G}))/Im.\delta_1$, hence is neutral.

More, it is easy to see that each class of $H^2(X,\widetilde{G})$ which is in relation by $\longrightarrow\!\circ\!\!\rightarrow$ with a class of $Im.\delta_1$ is also neutral (Let $\alpha \in H^1(X,\widetilde{T}_{ad})$: the

* $\widetilde{T}_{ad}$ is also an induced torus

X-gerb $\delta_1(\alpha)\cdot Tors\ \widetilde{G}$ is X-equivalent with the gerb $Tors(\widetilde{G}^{\alpha'})$ where α' is the image of α in $H^1(X, \widetilde{G}_{ad})$, then is neutral). The result follows from these facts. □

Corollary 2.1 (to theorem 2.1)**.** *Let $\mathcal{L}$ be a X_{et}-band which is locally representable by s.s.s.c group. Denote*

$$\text{Ш}^2(k(X), \mathcal{L}) := \operatorname{Ker}\left\{ H^2(k(X), \mathcal{L}) \longrightarrow \prod_{P\in X} H^2(k(X)_P, \mathcal{L}) \right\}$$

Then, each class of $\text{Ш}^2(k(X), \mathcal{L})$ *is neutral.*

This result is a consequence of the exactness of the localization sequence

$$H^2_{et}(X, \mathcal{L}) \longrightarrow H^2(k(X), \mathcal{L}) \longrightarrow \prod_{P\in X} H^2(k(X)_P, \mathcal{L}),$$

each class of $H^2_{et}(X, \mathcal{L})$ being neutral ($H^2(k(X)_P, \mathcal{L})$ is pointed by $\mathrm{Tors}(\widetilde{G}_{\mathcal{L}})$, where $\widetilde{G}_{\mathcal{L}}$ represents the $k(X)_P$-band $\mathcal{L}$, cf. the analogy with the second line of the diagram D_1 of [3″] p.124).

3. Higher dimensional varieties

J.L Colliot-Thélène and S.Saito have generalized in [2] Lichtenbaum's duality for higher dimensional varieties : the evaluation map on the closed points together with the corestriction defines a pairing

$$\mathrm{Br}(X) \times \mathrm{CH}_0(X) \longrightarrow \mathrm{Br}(k) \tag{3.1}$$

(cf. the "Φ-Eigenshaft" property of Pop and Wiesend [7]).

More generally, if X is a smooth, projective, geometrically irreducible k-variety of dimension > 1, k always p-adic field, they have established ([2], **Cor. 2.4**) that

$$\mathrm{Br}(X)/\,\mathrm{Br}(\mathfrak{X}) \hookrightarrow \prod_{P\in X} \mathrm{Br}(k_P) \text{ (modulo } p\text{-primary torsion)}$$

where $\mathfrak{X}$ is a model of X over the ring $\mathcal{O}_k$ of integers of k.

Adapting their method we get:

Theorem 3.1. *Let X be a projective, smooth, geometrically irreducible scheme of dimension > 1 over a p-adic field. Then each class of $H^2_{et}(X, \mathcal{L})/H^2_{et}(\mathfrak{X}, \mathcal{L})$ is neutral, where $\mathcal{L}$ is a $\mathfrak{X}$-band which is locally representable by a scheme of s.s.s.c groups,* $(|Z(\mathcal{L})|, p) = 1$.

Proof of theor.3.1. We will continue on the lines of the proof of the **theor.** 2.1.

Consider the following diagram $(D_{3.1})$ that extends diagram $(D_{2.1})$ in the proof of the theor. 2.1.

$(D_{3.1})$

$$
\begin{array}{ccccccc}
H^1(\mathfrak{X}, \widetilde{T}_{ad}) & \longrightarrow & H^2(\mathfrak{X}, Z(\widetilde{G})) & \longrightarrow & H^2(\mathfrak{X}, \widetilde{T}) & = & \mathrm{Br}(\mathfrak{X}') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(X, \widetilde{T}_{ad}) & \xrightarrow{\delta_1} & H^2(X, Z(\widetilde{G})) & \longrightarrow & H^2(X, \widetilde{T}) & = & \mathrm{Br}(X') \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(X, \widetilde{T}_{ad}) / H^1(\mathfrak{X}, \widetilde{T}_{ad}) & \xrightarrow{\overline{\delta}_1} & H^2(X, Z(\widetilde{G})) / H^2(\mathfrak{X}, Z(\widetilde{G})) & \longrightarrow & H^2(X, \widetilde{T}) / H^2(\mathfrak{X}, \widetilde{T}) & & \\
 & & \downarrow & & \hookdownarrow \ \leftarrow \text{modulo } p\text{-primary torsion} & & \\
\prod_{P \in X} H^1(k_P, \widetilde{T}_{ad}) & \longrightarrow & \prod_{P \in X} H^2(k_P, Z(\widetilde{G})) & \longrightarrow & \prod_{P \in X} H^2(k_P, \widetilde{T}) & = & \prod_{P' \in X'} \mathrm{Br}(k_{P'}) \\
\| & & & & & & \\
0 & & & & & &
\end{array}
$$

where X'/k is defined by $\widetilde{T} = \prod_{X'/X} G_{m'_X}$, as in the proof of the theor. 2.1 and $\mathfrak{X}'$ is an $\mathcal{O}_k$-model of X'. □

From the diagram $(D_{3.1})$, we deduce the following diagram $(D_{3.2})$ corresponding with $(D_{2.2})$:

$(D_{3.2})$

$$
\begin{array}{ccc}
 & \text{modulo } p\text{-primary torsion} \downarrow & \\
H^2(X, \widetilde{G}) / H^2(\mathfrak{X}, \widetilde{G}) \big/ Im.\overline{\delta}_1 & \hookrightarrow & \prod_{P \in X} H^2(k_P, \widetilde{G}) \\
\uparrow \circ & & \uparrow \circ \\
H^2(X, Z(\widetilde{G})) / H^2(\mathfrak{X}, Z(\widetilde{G})) \big/ Im.\overline{\delta}_1 & \hookrightarrow & \prod_{P \in X} H^2(k_P, Z(\widetilde{G})) \\
 & \uparrow \text{modulo } p\text{-primary torsion} &
\end{array}
$$

We conlude as in **theor.2.1**, but modulo the p-torsion: each class of $H^2(X, \widetilde{G}) / H^2(\mathfrak{X}, \widetilde{G}) \big/ Im.\overline{\delta}_1$ is in relation mod. p-torsion with the class 0 of $H^2(X, Z(\widetilde{G})) / H^2(\mathfrak{X}, Z(\widetilde{G})) \big/ Im.\overline{\delta}_1$ and each class of

$H^2(X,\widetilde{G})\diagup H^2(\mathfrak{X},\widetilde{G})\diagup_{Im.\overline{\delta}_1}$ which is in relation by $\longrightarrow\!\!\circ\!\!\rightarrow$ with a class of $Im.\overline{\delta}_1$ is also neutral.

Remarks. a) If X is a projective, smooth, geometrically irreducible curve over a p-adic field k, $\mathrm{Br}(\mathfrak{X}) = \mathrm{Br}(\mathfrak{X}') = 0$: the cohomology of $\mathfrak{X}$ comes from the cohomology of the special fiber by " proper base change " theorem and each class of $H^2(\mathfrak{X},\mathcal{L})$ is neutral (cf. **Theor. 1.3** of [3″]). By Lichtenbaum's theorem, $\mathrm{Br}(X') \hookrightarrow \prod_{P'\in X'} \mathrm{Br}(k'_P)$ and the restriction " mod. p-torsion " is no more necessary.

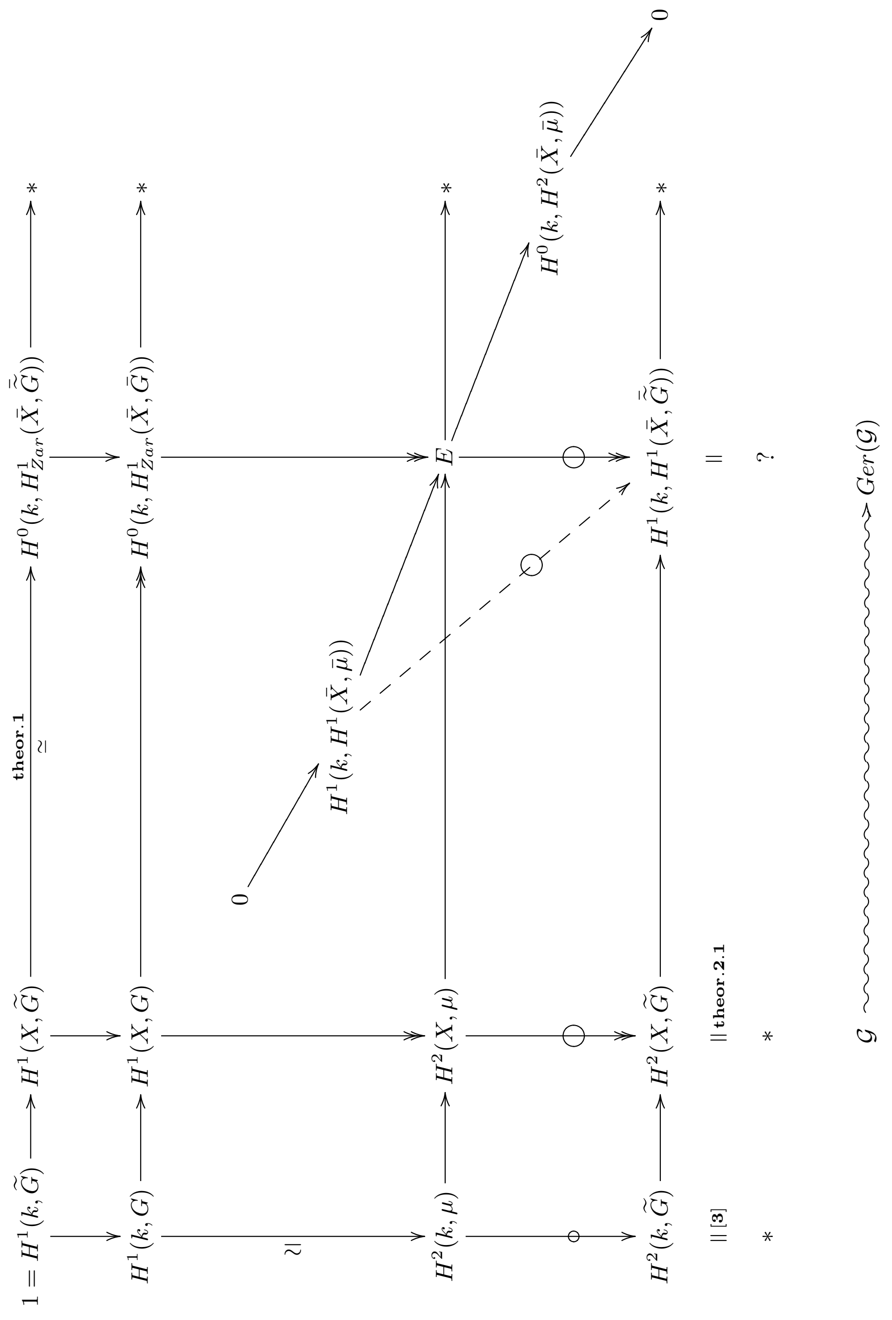
b) $\mathrm{Br}(\mathfrak{X})$ is the kernel of the pairing (3.1) and $H^2(\mathfrak{X},\widetilde{G})$ (resp. $H^2(\mathfrak{X},\mathcal{L})$) is the kernel of the application

$$H^2(X,\widetilde{G}) \times \mathrm{CH}_0(X) \longrightarrow H^2(k,\widetilde{G})$$

(resp. $H^2(X,\mathcal{L}) \times \mathrm{CH}_0(X) \longrightarrow H^2(k,\mathcal{L})$) in the non abelian sense, i.e $H^2(\mathfrak{X},\widetilde{G})$ (resp. $H^2(\mathfrak{X},\mathcal{L})$) is sent to the unit class of $H^2(k,\widetilde{G})$ pointed by $Tors\ \widetilde{G}$ (resp.of $H^2(k,\mathcal{L})$ pointed by $Tors\ \widetilde{G}_{\mathcal{L}}$ where $\widetilde{G}_{\mathcal{L}}$ represents $\mathcal{L}$).

4. Application

From **Theorem 1.1** and **Theorem 2.1**, we obtain the following diagram $(D_{4.1})$ where X is a proper, smooth, geometrically connected curve over a p-adic field k, and where G is a semi-simple group and $\widetilde{G}$ its universal covering, $\mu : \widetilde{G} \longrightarrow G$:

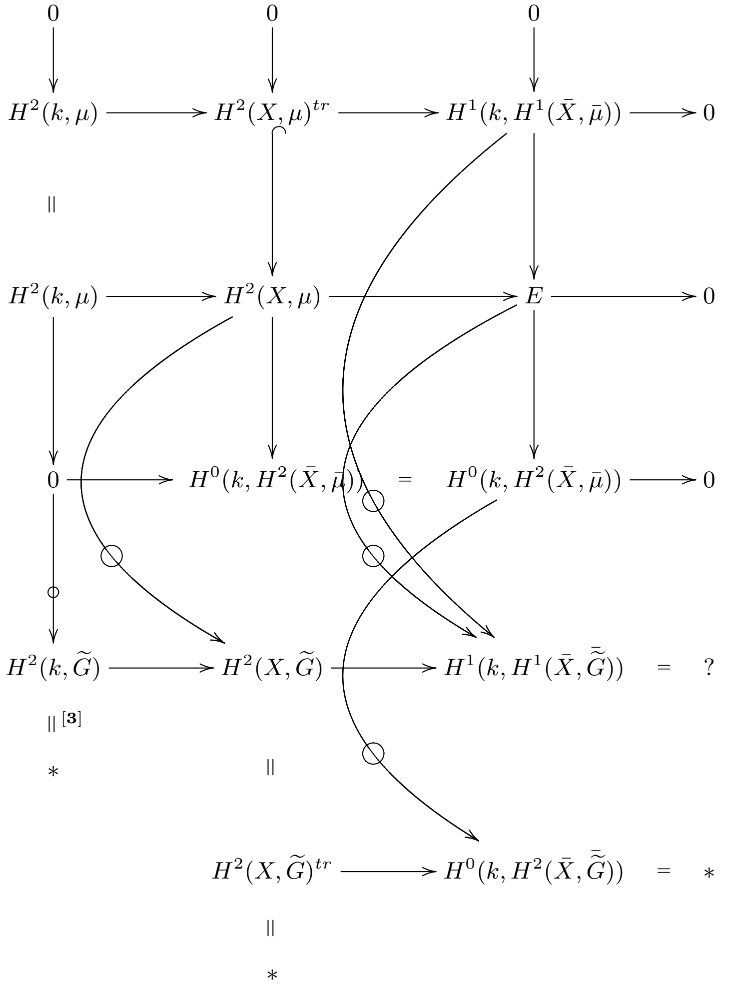


$(D_{4.1})$

Here $Ger(\mathcal{G})$ is the sheaf of maximal subgerbs of the direct image of $\mathcal{G}$ by $X \longrightarrow Spec(k)$ (cf. [4], p.131 and p.327, Ex 3.1.9.2).

E is calculated by the following diagram $(D_{4.2})$

$(D_{4.2})$



Note : $H^2(X,\cdot)^{tr} \ := \ \mathrm{Ker}\left\{H^2(X,\cdot) \longrightarrow H^0(k,H^2(\bar{X},\cdot))\right\}$ (cf.chap V, n° 3.1.9.3 de [4]).

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