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Invariants and coinvariants of semilocal units modulo elliptic units

par Stéphane VIGUIÉ

Résumé. Soient p un nombre premier, et k un corps quadratique imaginaire dans lequel p se décompose en deux idéaux maximaux $\mathfrak{p}$ et $\bar{\mathfrak{p}}$. Soit k_∞ l'unique $\mathbb{Z}_p$-extension de k non ramifiée en dehors de $\mathfrak{p}$, et soit K_∞ une extension finie de k_∞, abélienne sur k. Soit $\mathcal{U}_\infty/\mathcal{C}_\infty$ la limite projective du module des unités semi-locales principales modulo le module des unités elliptiques. Nous prouvons que les différents modules des invariants et des co-invariants de $\mathcal{U}_\infty/\mathcal{C}_\infty$ sont finis. Notre approche utilise les distributions et la fonction L p-adique, définie dans [5].

Abstract. Let p be a prime number, and let k be an imaginary quadratic number field in which p decomposes into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Let k_∞ be the unique $\mathbb{Z}_p$-extension of k which is unramified outside of $\mathfrak{p}$, and let K_∞ be a finite extension of k_∞, abelian over k. Let $\mathcal{U}_\infty/\mathcal{C}_\infty$ be the projective limit of principal semi-local units modulo elliptic units. We prove that the various modules of invariants and coinvariants of $\mathcal{U}_\infty/\mathcal{C}_\infty$ are finite. Our approach uses distributions and the p-adic L-function, as defined in [5].

1. Introduction

Let p be a prime number, and let k be an imaginary quadratic number field in which p decomposes into two distinct primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Let k_∞ be the unique $\mathbb{Z}_p$-extension of k which is unramified outside of $\mathfrak{p}$, and let K_∞ be a finite extension of k_∞, abelian over k. Let G_∞ be the Galois group of K_∞/k. We choose a decomposition of G_∞ as a direct product of a finite group G (the torsion subgroup of G_∞) and a topological group Γ isomorphic to $\mathbb{Z}_p$, $G_\infty = G \times \Gamma$. For all $n \in \mathbb{N}$, let K_n be the field fixed by $\Gamma_n := \Gamma^{p^n}$, and let $G_n := \mathrm{Gal}\,(K_n/k)$. Remark that there may be different choices for Γ, but when p^n is larger than the order of the p-part of G, the group Γ_n does not depend on the choice of Γ.

Let F/k be a finite abelian extension of k. We denote by $\mathcal{O}_F$ the ring of integers of F. Then we write $\mathcal{O}_F^\times$ for the group of global units of F, and C_F for the group of elliptic units of F (see section 3). We set $\mathcal{C}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_F$.

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For all prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ above $\mathfrak{p}$, we write $F_{\mathfrak{q}}$, $\mathcal{O}_{F_{\mathfrak{q}}}$, and $\mathcal{O}_{F_{\mathfrak{q}}}^{\times}$ respectively for the completion of F at $\mathfrak{q}$, the ring of integers of $F_{\mathfrak{q}}$, and the group of units of $\mathcal{O}_{F_{\mathfrak{q}}}$. Then we write $\mathcal{U}_F$ for the pro-p-completion of $\prod_{\mathfrak{q}|\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{q}}}^{\times}$. The injection $\mathcal{O}_F^{\times} \hookrightarrow \prod_{\mathfrak{q}|\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{q}}}^{\times}$ induces a canonical map $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times} \to \mathcal{U}_F$. The Leopoldt conjecture, which is known to be true for abelian extensions of k, states that this map is injective. For all $n \in \mathbb{N}$, we write $\mathcal{C}_n$ and $\mathcal{U}_n$ for $\mathcal{C}_{K_n}$ and $\mathcal{U}_{K_n}$ respectively. We define $\mathcal{C}_\infty := \varprojlim \mathcal{C}_n$ and $\mathcal{U}_\infty := \varprojlim \mathcal{U}_n$ by taking projective limit under the norm maps. The injections $\mathcal{C}_n \hookrightarrow \mathcal{U}_n$ are norm compatible and taking the limit we obtain an injection $\mathcal{C}_\infty \hookrightarrow \mathcal{U}_\infty$.

For any profinite group $\mathcal{G}$, and any commutative ring R, we define the Iwasawa algebra

$$R\,[[\mathcal{G}]] := \varprojlim\, R\,[\mathcal{H}]\,,$$

where the projective limit is over all finite quotients $\mathcal{H}$ of $\mathcal{G}$. Then $\mathcal{C}_\infty$ and $\mathcal{U}_\infty$ are naturally $\mathbb{Z}_p\,[[G_\infty]]$-modules. It is well known that they are finitely generated over $\mathbb{Z}_p[[\Gamma]]$. Moreover one can show that $\mathcal{U}_\infty/\mathcal{C}_\infty$ is torsion over $\mathbb{Z}_p[[\Gamma]]$ (see [17, Proposition 3.1]). Let us fix a topological generator γ of Γ, and set $T := \gamma - 1$. We denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. For any complete subfield L of $\mathbb{C}_p$, finitely ramified over $\mathbb{Q}_p$, we denote by $\mathcal{O}_L$ the complete discrete valuation ring of integers of L. Then the ring $\mathcal{O}_L[[\Gamma]]$ is isomorphic to $\mathcal{O}_L[[T]]$. It is well known that $\mathcal{O}_L[[T]]$ is a noetherian, regular, local domain. We also recall that $\mathcal{O}_L[[T]]$ is a unique factorization domain. If $\mathfrak{u}_L$ is a uniformizer of $\mathcal{O}_L$, then the maximal ideal $\mathfrak{M}$ of $\mathcal{O}_L\,[[T]]$ is generated by $\mathfrak{u}_L$ and T, and $\mathcal{O}_L[[T]]$ is a complete topological ring with respect to its $\mathfrak{M}$-adic topology. A morphism $f : M \to N$ between two finitely generated $\mathcal{O}_L[[T]]$-module is called a pseudo-isomorphism if its kernel and its cokernel are finitely generated and torsion over $\mathcal{O}_L$. If a finitely generated $\mathcal{O}_L[[T]]$-module M is given, then one may find elements P_1, ..., P_r in $\mathcal{O}_L[T]$, irreducible in $\mathcal{O}_L[[T]]$, and nonnegative integers n_0, ..., n_r, such that there is a pseudo-isomorphism

$$M \longrightarrow \mathcal{O}_L[[T]]^{n_0} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_L[[T]]/\left(P_i^{n_i}\right).$$

Moreover, the integer n_0 and the ideals $(P_1^{n_1})$, ..., $(P_r^{n_r})$, are uniquely determined by M. If $n_0 = 0$, then the ideal generated by $P_1^{n_1} \cdots P_r^{n_r}$ is called the characteristic ideal of M, and is denoted by $\mathrm{char}_{\mathcal{O}_L[[T]]}(M)$.

Let χ be an irreducible $\mathbb{C}_p$-character of G. Let $L(\chi) \subset \mathbb{C}_p$ be the abelian extension of L generated by the values of χ. The group G acts naturally on $L(\chi)$ if we set, for all $g \in G$ and all $x \in L(\chi)$, $g.x := \chi(g)x$. For any $\mathcal{O}_L[G]$-module Y, we define the χ-quotient Y_χ by $Y_\chi := \mathcal{O}_{L(\chi)} \otimes_{\mathcal{O}_L[G]} Y$. If Y is an $\mathcal{O}_L\,[[G_\infty]]$-module, then Y_χ is an $\mathcal{O}_{L(\chi)}[[T]]$-module in a natural way. Moreover if L contains a $[K_0 : k]$-th primitive root of unity, then there is $(a, b) \in \mathbb{N}^2$ such that

$$\mathfrak{u}_L^a \mathrm{char}_{\mathcal{O}_L[[T]]}(M) = \mathfrak{u}_L^b \prod_{\chi} \mathrm{char}_{\mathcal{O}_L[[T]]}(M_\chi), \tag{1.1}$$

where the product is over all irreducible $\mathbb{C}_p$-character on G.

For any profinite group $\mathcal{G}$, any normal subgroup $\mathcal{H}$ of $\mathcal{G}$ and any $\mathcal{O}_L[[\mathcal{G}]]$-module M, we denote by $M^{\mathcal{H}}$ the module of $\mathcal{H}$-invariants of M, that is to say the maximal submodule of M which is invariant under the action of $\mathcal{H}$. We denote by $M_{\mathcal{H}}$ the module of $\mathcal{H}$-coinvariants of M, which is the quotient of M by the closed submodule topologically generated by the elements $(h-1)m$ with $h \in \mathcal{H}$ and $m \in M$.

In this article, we prove that for all $n \in \mathbb{N}$, the module of Γ_n-invariants and the module of Γ_n-coinvariants of $\mathcal{U}_\infty/\mathcal{C}_\infty$ are finite (see Theorem 6.1). It generalizes a part of a result of Coates-Wiles [4, Theorem 1], where this result is shown at the χ^i-parts, for $i \not\equiv 0$ modulo $p-1$, and for χ the character giving the action of G on the $\mathfrak{p}$-torsion points of a certain elliptic curve. But the result of [4] is stated for non-exceptional primes p (in particular $p \notin \{2, 3\}$), and under the assumption that $\mathcal{O}_k$ is principal. Here we prove the general case.

Moreover we would like to mention an application of Theorem 6.1 to the main conjecture of Iwasawa theory. For all $n \in \mathbb{N}$, we set $\mathcal{E}_n := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{K_n}^\times$ and we denote by A_n the p-part of the class-group $\mathrm{Cl}(\mathcal{O}_{K_n})$ of $\mathcal{O}_{K_n}$. We define $\mathcal{E}_\infty := \varprojlim \mathcal{E}_n$ and $A_\infty := \varprojlim A_n$, projective limits under the norm maps. A formulation of the (one variable) main conjecture says that $\mathrm{char}_{\mathbb{Z}_p(\chi)[[T]]}(\mathcal{E}_\infty/\mathcal{C}_\infty)_\chi = \mathrm{char}_{\mathbb{Z}_p(\chi)[[T]]}(A_{\infty,\chi})$, where $\mathbb{Z}_p(\chi)$ is the ring of integers of $\mathbb{Q}_p(\chi)$. It has been proved in many cases by the use of Euler systems. We refer the reader to the pioneering work of Rubin in [15, Theorem 4.1] and [16, Theorem 2], adapted to the cyclotomic case by Greither in [7, Theorem 3.2]. The method is now classical, applied by many authors, see [2, Theorem 3.1], [11] and [17]. However the result of Gillard [6] which implies the nullity of the μ-invariant of A_∞ is stated for $p \notin \{2, 3\}$, and for $p \in \{2, 3\}$ we just obtain a divisibility relation

$$\mathrm{char}_{\mathbb{Z}_p(\chi)[[T]]}(A_{\infty,\chi}) \quad \text{divides} \quad p^a \mathrm{char}_{\mathbb{Z}_p(\chi)[[T]]}(\mathcal{E}_\infty/\mathcal{C}_\infty)_\chi, \tag{1.2}$$

for some $a \in \mathbb{N}$ (see [11] and [17]). Following the ideas of Belliard in [1], in [18] we deduce from Theorem 6.1 that for $p \in \{2, 3\}$ the $\mathbb{Z}_p[[\Gamma]]$-modules $\mathcal{E}_\infty/\mathcal{C}_\infty$ and A_∞ have the same Iwasawa's μ and λ invariants. This result, together with (1.2), implies that there is $(a, b) \in \mathbb{N}^2$ such that the following raw form of the main conjecture holds,

$$\mathfrak{u}_\chi^a \mathrm{char}_{\mathbb{Z}_p(\chi)[[T]]}(A_{\infty,\chi}) = \mathfrak{u}_\chi^b \mathrm{char}_{\mathbb{Z}_p(\chi)[[T]]}(\mathcal{E}_\infty/\mathcal{C}_\infty)_\chi,$$

where $\mathfrak{u}_\chi$ is a uniformizer of $\mathbb{Z}_p(\chi)$.

2. Distributions.

In this section, let A be a commutative ring and let $\mathcal{G}$ be a profinite group. We denote by $\mathfrak{X}(\mathcal{G})$ the set of compact-open subsets of $\mathcal{G}$. Remark that for any $X \in \mathfrak{X}(\mathcal{G})$, one can find a finite subset F of X, and an open normal subgroup $\mathcal{H}$ of $\mathcal{G}$, such that $X = \underset{x\in F}{\cup} x\mathcal{H}$.

Definition 1. An A-distribution on $\mathcal{G}$ is an application $\mu : \mathfrak{X}(\mathcal{G}) \to A$, such that for all $(X_1, X_2) \in \mathfrak{X}(\mathcal{G})^2$, if $X_1 \cap X_2 = \varnothing$, then

$$\mu\left(X_1 \cup X_2\right) = \mu(X_1) + \mu(X_2).$$

We denote by $\mathcal{M}\left(\mathcal{G}, A\right)$ the A-module of A-distributions on $\mathcal{G}$. Moreover for $X \in \mathfrak{X}(\mathcal{G})$ and $\mu \in \mathcal{M}\left(\mathcal{G}, A\right)$, we denote by $\mu_{|X}$ the A-distribution on $\mathcal{G}$ defined by

$$\mu_{|X} : \mathfrak{X}(X) \longrightarrow A, \quad Y \longmapsto \mu(Y \cap X).$$

Let $\pi : \mathcal{G} \to \mathcal{G}'$ be a continuous map between two profinite groups. To any distribution $\mu \in \mathcal{M}\left(\mathcal{G}, A\right)$ we attach the unique A-distribution $\pi_*\mu$ on $\mathcal{G}'$, such that for all $X \in \mathfrak{X}\left(\mathcal{G}'\right)$,

$$\pi_*\mu(X) = \mu\left(\pi^{-1}(X)\right).$$

For any $\sigma \in \mathcal{G}$, let us also denote by $\sigma_*\mu$ the unique A-distribution on $\mathcal{G}$, such that for all $X \in \mathfrak{X}(\mathcal{G})$,

$$\sigma_*\mu(X) = \mu\left(\sigma^{-1}X\right).$$

Assume moreover that π is an open (continuous) group morphism, such that $\mathrm{Ker}(\pi)$ is finite. To any distribution $\mu' \in \mathcal{M}\left(\mathcal{G}', A\right)$ we attach the unique A-distribution $\pi^\sharp\mu'$ on $\mathcal{G}$, such that for all $g \in \mathcal{G}$, and all open subgroup $\mathcal{H}$ of $\mathcal{G}$,

$$\pi^\sharp\mu'\left(g\mathcal{H}\right) = \#\left(\mathcal{H} \cap \mathrm{Ker}(\pi)\right)\, \mu'\left(\pi\left(g\mathcal{H}\right)\right). \tag{2.1}$$

Then we have

$$\pi^\sharp\pi_*\mu = \sum_{\sigma\in\mathrm{Ker}(\pi)} \sigma_*\mu \quad \text{and} \quad \pi_*\pi^\sharp\mu' = \#\left(\mathrm{Ker}(\pi)\right)\, \mu'_{|\mathrm{Im}(\pi)}. \tag{2.2}$$

For $(\alpha_1, \alpha_2) \in \mathcal{M}\left(\mathcal{G}, A\right)^2$, there is a unique A-distribution β on $\mathcal{G} \times \mathcal{G}$ such that for all $(X_1, X_2) \in \mathfrak{X}(\mathcal{G})^2$, $\beta\left(X_1 \times X_2\right) = \alpha_1(X_1)\alpha_2(X_2)$. Then the convolution product $\alpha_1\alpha_2$ of α_1 and α_2 is defined by $\alpha_1\alpha_2 := m_*\beta$, where $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(\sigma_1, \sigma_2) \mapsto \sigma_1\sigma_2$. Once equipped with the convolution product, $\mathcal{M}\left(\mathcal{G}, A\right)$ is an A-algebra. For any A-distribution μ on $\mathcal{G}$, let us denote by $\underline{\mu}$ the unique element in $A\left[\left[\mathcal{G}\right]\right]$ such that for all open normal subgroup $\mathcal{H}$ of $\mathcal{G}$, the image $\underline{\mu}_{\mathcal{H}}$ of $\underline{\mu}$ in $A\left[\mathcal{G}/\mathcal{H}\right]$ is given by

$$\underline{\mu}_{\mathcal{H}} = \sum_{g\in\mathcal{G}/\mathcal{H}} \mu\left(\tilde{g}\mathcal{H}\right) g,$$

where for any $g \in \mathcal{G}/\mathcal{H}$, $\tilde{g} \in \mathcal{G}$ is an arbitrary preimage of g. Then we have a canonical isomorphism

$$\mathcal{M}\left(\mathcal{G}, A\right) \xrightarrow{\sim} A\left[\left[\mathcal{G}\right]\right], \quad \mu \longmapsto \underline{\mu},$$

and for any $\mu \in \mathcal{M}\left(\mathcal{G}, A\right)$ and any $\sigma \in \mathcal{G}$, we have $\underline{\sigma_*\mu} = \sigma\underline{\mu}$. Also we mention that if $\tilde{\pi} : A[[\mathcal{G}]] \to A[[\mathcal{G}']]$ is the canonical morphism defined by π, then we have the following commutative squares,

$$\begin{array}{ccc} \mathcal{M}\left(\mathcal{G}, A\right) & \xrightarrow{\sim} & A\left[\left[\mathcal{G}\right]\right] \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ \mathcal{M}\left(\mathcal{G}', A\right) & \xrightarrow{\sim} & A\left[\left[\mathcal{G}'\right]\right] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{M}\left(\mathcal{G}, A\right) & \xrightarrow{\sim} & A\left[\left[\mathcal{G}\right]\right] \\ {\scriptstyle \pi^\sharp}\uparrow & & \uparrow \\ \mathcal{M}\left(\mathcal{G}', A\right) & \xrightarrow{\sim} & A\left[\left[\mathcal{G}'\right]\right] \end{array} \qquad \begin{array}{c} \Sigma h \\ \uparrow \\ g \end{array}$$

where for all $g \in \mathcal{G}'$, Σh is the sum over all $h \in \mathcal{G}$ such that $\pi(h) = g$.

Proposition 2.1. *Let $\pi : \mathcal{G}_1 \to \mathcal{G}_2$ be an open continuous morphism of profinite groups, such that* $\mathrm{Ker}(\pi)$ *is finite. The morphism $\pi^\sharp : \mathcal{M}\left(\mathcal{G}_2, A\right) \to \mathcal{M}\left(\mathcal{G}_1, A\right)$ is injective if and only if π is surjective. Moreover if $\#\left(\mathrm{Ker}(\pi)\right)$ is not a zero divisor in A, then the image of $\pi^\sharp$ is $\mathcal{M}\left(\mathcal{G}_1, A\right)^{\mathrm{Ker}(\pi)}$.*

Proof. Let $\mu_2 \in \mathcal{M}\left(\mathcal{G}_2, A\right)$. From (2.1) it is straightforward to check that $\pi^\sharp\mu_2 = 0$ if and only if $\mu_2(X) = 0$ for all $X \in \mathfrak{X}(\mathrm{Im}(\pi))$, and then we deduce that $\pi^\sharp$ is injective if and only if π is surjective. For any $\sigma \in \mathrm{Ker}(\pi)$, any $g \in \mathcal{G}_1$, and any open subgroup $\mathcal{H}$ of $\mathcal{G}_1$, we have

$$\begin{aligned} \sigma_*\pi^\sharp\mu_2\left(g\mathcal{H}\right) &= \#\left(\mathcal{H} \cap \mathrm{Ker}(\pi)\right)\mu_2\left(\pi\left(\sigma^{-1}g\mathcal{H}\right)\right) \\ &= \#\left(\mathcal{H} \cap \mathrm{Ker}(\pi)\right)\mu_2\left(\pi\left(g\mathcal{H}\right)\right) \\ &= \pi^\sharp\mu_2\left(g\mathcal{H}\right), \end{aligned}$$

hence $\sigma_*\pi^\sharp\mu_2 = \pi^\sharp\mu_2$, and $\mathrm{Im}\left(\pi^\sharp\right) \subseteq \mathcal{M}\left(\mathcal{G}_1, A\right)^{\mathrm{Ker}(\pi)}$.

Now let $\mu_1 \in \mathcal{M}\left(\mathcal{G}_1, A\right)^{\mathrm{Ker}(\pi)}$. Let $\mathcal{H}$ be an open subgroup of $\mathrm{Im}(\pi)$, and $g \in \mathcal{G}_1$. Let $\mathcal{W}$ be an open normal subgroup of $\pi^{-1}(\mathcal{H})$ such that $\mathcal{W}\cap\mathrm{Ker}(\pi)$ is trivial. Let R be a complete representative system of $\pi^{-1}(\mathcal{H})$ modulo $\mathcal{W}\mathrm{Ker}(\pi)$. Then $(\sigma r)_{(\sigma,r)\in\mathrm{Ker}(\pi)\times R}$ is a complete representative system of

$\pi^{-1}(\mathcal{H})$ modulo $\mathcal{W}$, and we have

$$\begin{aligned}\mu_1\left(\pi^{-1}(\mathcal{H})\,g\right) &= \sum_{(\sigma,r)\in\mathrm{Ker}(\pi)\times R}\mu_1(\sigma r\mathcal{W}g)\\ &= \sum_{(\sigma,r)\in\mathrm{Ker}(\pi)\times R}\left(\sigma^{-1}\right)_*\mu_1(r\mathcal{W}g)\\ &= \sum_{(\sigma,r)\in\mathrm{Ker}(\pi)\times R}\mu_1(r\mathcal{W}g)\\ &= \#\left(\mathrm{Ker}(\pi)\right)\sum_{r\in R}\mu_1(r\mathcal{W}g).\end{aligned}$$

Hence $\pi_*\mu_1$ takes values in $\#\left(\mathrm{Ker}(\pi)\right)A$, and we deduce the equality $\mu_1 = \pi^\sharp\left(\#\left(\mathrm{Ker}(\pi)\right)^{-1}\pi_*\mu_1\right)$ from (2.2). □

Now assume $A := \mathcal{O}_L$ for some complete subfield L of $\mathbb{C}_p$, finitely ramified over $\mathbb{Q}_p$. An A-distribution on $\mathcal{G}$ is called a measure. Let $\mu\in\mathcal{M}(\mathcal{G},A)$ be such a measure, and let V be a complete separated topological A-module, such that the open submodules of V form a neighborhood basis for 0. Let $\mathcal{C}(\mathcal{G},V)$ be the A-module of continuous maps from $\mathcal{G}$ to V, equipped with the uniform convergence topology. For any $X\in\mathfrak{X}(\mathcal{G})$, we denote by $1_X:\mathcal{G}\to A$ the map such that $1_X(x)=1$ for $x\in X$ and $1_X(x)=0$ for $x\in\mathcal{G}\setminus X$. Then there is a unique continuous A-linear map

$$\mathcal{C}(\mathcal{G},V)\longrightarrow V,\qquad f\longmapsto\int f(t).\mathrm{d}\mu(t),$$

such that for all $X\in\mathfrak{X}(\mathcal{G})$ and all $v\in V$, $\int 1_X(t)v.\mathrm{d}\mu(t)=\mu(X)v$ (see [9, Chapter 4, §1]). For $X\in\mathfrak{X}(\mathcal{G})$ and $f\in\mathcal{C}(\mathcal{G},V)$, we write $\int_X f.\mathrm{d}\mu$ for $\int 1_X f.\mathrm{d}\mu$. Then for $\sigma\in\mathcal{G}$, we have

$$\int_X f(t).\mathrm{d}\sigma_*\mu(t)=\int_{\sigma^{-1}X} f(\sigma t).\mathrm{d}\mu(t), \tag{2.3}$$

the equality being obvious if f is locally constant, and then extended to all $f\in\mathcal{C}(\mathcal{G},V)$ by continuity. Then for $\mu\in\mathcal{M}(\Gamma,A)$, we have

$$\underline{\mu}=\int(1+T)^{\kappa(\sigma)}.\mathrm{d}\mu(\sigma)\quad\text{in}\quad A[[T]], \tag{2.4}$$

where $\kappa:\Gamma\to\mathbb{Z}_p$ is the unique isomorphism of profinite groups such that $\kappa(\gamma)=1$. Moreover if we write $\mathfrak{m}_{\mathbb{C}_p}$ for the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, then for any $x\in\mathfrak{m}_{\mathbb{C}_p}$ we have

$$\underline{\mu}(x)=\int(1+x)^{\kappa(\sigma)}.\mathrm{d}\mu(\sigma)\quad\text{in}\quad\mathbb{C}_p, \tag{2.5}$$

see [9, Chapter 4, §1, Theorem 1.2, p. 98].

3. Elliptic units.

For L and L' two $\mathbb{Z}$-lattices of $\mathbb{C}$ such that $L \subseteq L'$ and $[L' : L]$ is prime to 6, we denote by $z \mapsto \psi\left(z; L, L'\right)$ the elliptic function defined in [14]. For $\mathfrak{m}$ a nonzero proper ideal of $\mathcal{O}_k$, and $\mathfrak{a}$ a nonzero ideal of $\mathcal{O}_k$ prime to $6\mathfrak{m}$, G. Robert proved that $\psi\left(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}\right) \in k(\mathfrak{m})$, where $k(\mathfrak{m})$ is the ray class field of k, modulo $\mathfrak{m}$. If $\varphi_{\mathfrak{m}}(1) \in k(\mathfrak{m})^{\times}$ is the Robert-Ramachandra invariant, as defined in [12, p. 15], or in [5, p. 55], we have by [13, Corollaire 1.3, (iii)]

$$\psi\left(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}\right)^{12m} = \varphi_{\mathfrak{m}}(1)^{N(\mathfrak{a}) - (\mathfrak{a}, k(\mathfrak{m})/k)}, \tag{3.1}$$

where m is the positive generator of $\mathfrak{m} \cap \mathbb{Z}$, $N(\mathfrak{a}) := \#\left(\mathcal{O}_k/\mathfrak{a}\right)$ and $(\mathfrak{a}, k(\mathfrak{m})/k)$ is the Artin automorphism of $k(\mathfrak{m})/k$ defined by $\mathfrak{a}$. Let $S(\mathfrak{m})$ be the set of maximal ideals of $\mathcal{O}_k$ which divide $\mathfrak{m}$. Then $\psi\left(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}\right)$ and $\varphi_{\mathfrak{m}}(1)$ are units if and only if $|S(\mathfrak{m})| \geq 2$. More precisely, if we denote by $w_{\mathfrak{m}}$ the number of roots of unity of k which are congruent to 1 modulo $\mathfrak{m}$, and if we write w_k for the number of roots of unity in k, then by [13, (iv'), p. 21], we have

$$\varphi_{\mathfrak{m}}(1)\mathcal{O}_{k(\mathfrak{m})} = \begin{cases} (1) & \text{if} \quad 2 \leq |S(\mathfrak{m})| \\ (\mathfrak{q})_{k(\mathfrak{m})}^{12mw_{\mathfrak{m}}/w_k} & \text{if} \quad S(\mathfrak{m}) = \{\mathfrak{q}\}, \end{cases} \tag{3.2}$$

where $(\mathfrak{q})_{k(\mathfrak{m})}$ is the product of the prime ideals of $\mathcal{O}_{k(\mathfrak{m})}$ which lie above $\mathfrak{q}$. Moreover, if $\mathfrak{a}$ is prime to $6\mathfrak{m}\mathfrak{q}$, then by [13, Corollaire 1.3, (ii-1)] we have

$$N_{k(\mathfrak{m}\mathfrak{q})/k(\mathfrak{m})}\left(\psi\left(1; \mathfrak{m}\mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{m}\mathfrak{q}\right)\right)^{w_{\mathfrak{m}} w_{\mathfrak{m}\mathfrak{q}}^{-1}} = \begin{cases} \psi\left(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}\right)^{1 - (\mathfrak{q}, k(\mathfrak{m})/k)^{-1}} & \text{if} \quad \mathfrak{q} \nmid \mathfrak{m}, \\ \psi\left(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}\right) & \text{if} \quad \mathfrak{q} \mid \mathfrak{m}. \end{cases} \tag{3.3}$$

Definition 2. Let $F \subset \mathbb{C}$ be a finite abelian extension of k, and write $\mu(F)$ for the group of roots of unity in F. Let $\mathfrak{m}$ be a nonzero proper ideal of $\mathcal{O}_k$. We define the $\mathbb{Z}\left[\mathrm{Gal}(F/k)\right]$-submodule $\Psi(F, \mathfrak{m})$ of $F^{\times}$, generated by the $w_{\mathfrak{m}}$-roots of all $N_{k(\mathfrak{m})/k(\mathfrak{m})\cap F}\left(\psi\left(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}\right)\right)$, where $\mathfrak{a}$ is any nonzero ideal of $\mathcal{O}_k$ prime to $6\mathfrak{m}$. Also, we set $\Psi'(F, \mathfrak{m}) := \mathcal{O}_F^{\times} \cap \Psi(F, \mathfrak{m})$.

Then, we let C_F be the group generated by $\mu(F)$ and by all $\Psi'(F, \mathfrak{m})$, for any nonzero proper ideal $\mathfrak{m}$ of $\mathcal{O}_k$.

Remark 1. Let $\mathfrak{m}$ and $\mathfrak{g}$ be two nonzero proper ideals of $\mathcal{O}_k$, such that the conductor of F/k divides $\mathfrak{m}$. Let us denote by $\mathfrak{g} \wedge \mathfrak{m}$ the gcd of $\mathfrak{g}$ and $\mathfrak{m}$. If $\mathfrak{g} \wedge \mathfrak{m} = 1$, then $\Psi'\left(F, \mathfrak{g}\right) \subseteq C_F \cap \mathcal{O}_{k(1)}^{\times}$. Else by (3.3) we have $\Psi'\left(F, \mathfrak{g}\right) \subseteq \Psi'\left(F, \mathfrak{g} \wedge \mathfrak{m}\right)$.

We define $\mathcal{C}_n := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_{K_n}$, and $\mathcal{C}_\infty := \varprojlim (\mathcal{C}_n)$, projective limit under the norm maps. For any nonzero ideal $\mathfrak{g}$ of $\mathcal{O}_k$, we define

$$\Psi (K_n, \mathfrak{g}\mathfrak{p}^\infty) := \bigcup_{i=1}^{\infty} \Psi \left(K_n, \mathfrak{g}\mathfrak{p}^i\right) \quad \text{and} \quad \Psi' (K_n, \mathfrak{g}\mathfrak{p}^\infty) := \bigcup_{i=1}^{\infty} \Psi' \left(K_n, \mathfrak{g}\mathfrak{p}^i\right).$$

Then the projective limits under the norm maps are denoted by

$$\overline{\Psi} (K_\infty, \mathfrak{g}\mathfrak{p}^\infty) := \varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi (K_n, \mathfrak{g}\mathfrak{p}^\infty)),$$
$$\overline{\Psi}' (K_\infty, \mathfrak{g}\mathfrak{p}^\infty) := \varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi' (K_n, \mathfrak{g}\mathfrak{p}^\infty)).$$

Let us write $\mathcal{I}$ for the set of nonzero ideals of $\mathcal{O}_k$ which are prime to $\mathfrak{p}$. For $\mathfrak{g} \in \mathcal{I}$, we set $K_{\mathfrak{g},\infty} := k(\mathfrak{g}\mathfrak{p}^\infty) = \bigcup_{n\in\mathbb{N}} k(\mathfrak{g}\mathfrak{p}^n)$, and $G_{\mathfrak{g},\infty} := \mathrm{Gal}(K_{\mathfrak{g},\infty}/k)$. Then we write $G_{\mathfrak{g}}$ for the torsion subgroup of $G_{\mathfrak{g},\infty}$. We denote by $\mathcal{I}'$ the subset of $\mathcal{I}$ containing all the $\mathfrak{g} \in \mathcal{I}$ such that $w_{\mathfrak{g}} = 1$. In the sequel, we fix once and for all $\mathfrak{f} \in \mathcal{I}'$ such that $K_\infty \subseteq K_{\mathfrak{f},\infty}$. We choose arbitrarily a subgroup of $G_{\mathfrak{f},\infty}$, isomorphic to $\mathbb{Z}_p$, such that its image in G_∞ is Γ. Then for any $\mathfrak{g} \in \mathcal{I}$ such that $\mathfrak{g}|\mathfrak{f}$, we have the decomposition $G_{\mathfrak{g},\infty} = G_{\mathfrak{g}} \times \Gamma$.

Remark 2. From Remark 1, $\mathcal{C}_\infty$ is generated by all the $\overline{\Psi}' (K_\infty, \mathfrak{g}\mathfrak{p}^\infty)$, where $\mathfrak{g} \in \mathcal{I}$ is such that $\mathfrak{g}|\mathfrak{f}$.

From (3.3), for $\mathfrak{g} \in \mathcal{I}$ such that $\mathfrak{g}|\mathfrak{f}$, and for any nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_k$ which is prime to $6\mathfrak{g}\mathfrak{p}$, there is a unique

$$\psi \langle \mathfrak{g}, \mathfrak{a} \rangle \in \overline{\Psi} (K_{\mathfrak{g},\infty}, \mathfrak{g}\mathfrak{p}^\infty)$$

such that for large enough $n \in \mathbb{N}$, the canonical image of $\psi \langle \mathfrak{g}, \mathfrak{a} \rangle$ in $\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi (k(\mathfrak{g}\mathfrak{p}^n), \mathfrak{g}\mathfrak{p}^\infty)$ is $1 \otimes \psi \left(1; \mathfrak{g}\mathfrak{p}^n, \mathfrak{a}^{-1}\mathfrak{g}\mathfrak{p}^n\right)$.

4. From semilocal units to measures.

Let $\mathbb{Q}_p^{\mathrm{nr}} \subseteq \mathbb{C}_p$ be the maximal unramified algebraic extension of $\mathbb{Q}_p$, and let L be the completion of $\mathbb{Q}_p^{\mathrm{nr}}$. We denote by $\mathcal{O}_{\mathfrak{f}}$ the ring $\mathcal{O}_L[\zeta]$, where ζ is any primitive $[K_{\mathfrak{f},0} : k]$-th root of unity in $\mathbb{C}_p$. For all $(\mathfrak{g}_1, \mathfrak{g}_2) \in \mathcal{I}^2$ such that $\mathfrak{g}_1|\mathfrak{g}_2$, we denote by $\pi_{\mathfrak{g}_2,\mathfrak{g}_1} : G_{\mathfrak{g}_2,\infty} \to G_{\mathfrak{g}_1,\infty}$ the restriction map. We write $N_{\mathfrak{g}_2,\mathfrak{g}_1} : \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2,\infty} \to \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1,\infty}$ for the norm map and we write $\upsilon_{\mathfrak{g}_2,\mathfrak{g}_1} : \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1,\infty} \to \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2,\infty}$ for the canonical injection.

For all $\mathfrak{g} \in \mathcal{I}'$, de Shalit defined in [5, I.3.4, II.4.6, and II.4.7] an injective morphism of $\mathbb{Z}_p[[G_\infty]]$-modules $i^0_{\mathfrak{g}} : \mathcal{U}_{\mathfrak{g},\infty} \to \mathcal{M}(G_{\mathfrak{g},\infty}, \mathcal{O}_L)$, which we extend by linearity to a morphism $i_{\mathfrak{g}}$ from $\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g},\infty}$ to $\mathcal{M}(G_{\mathfrak{g},\infty}, \mathcal{O}_{\mathfrak{f}})$.

Lemma 4.1. *There is a unique way to extend the family* $(i_{\mathfrak{g}})_{\mathfrak{g}\in\mathcal{I}'}$ *to* $\mathcal{I}$ *such that for all* $(\mathfrak{g}_1,\mathfrak{g}_2)\in\mathcal{I}^2$*, if* $\mathfrak{g}_1|\mathfrak{g}_2$ *then the following squares are commutative,*

(4.1)

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_{\mathfrak{g}_2,\infty} & \xrightarrow{i_{\mathfrak{g}_2}} & \mathcal{M}\left(G_{\mathfrak{g}_2,\infty},\mathcal{O}_{\mathfrak{f}}\right) \\ \downarrow N_{\mathfrak{g}_2,\mathfrak{g}_1} & & \downarrow (\pi_{\mathfrak{g}_2,\mathfrak{g}_1})_* \\ \mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_{\mathfrak{g}_1,\infty} & \xrightarrow{i_{\mathfrak{g}_1}} & \mathcal{M}\left(G_{\mathfrak{g}_1,\infty},\mathcal{O}_{\mathfrak{f}}\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_{\mathfrak{g}_1,\infty} & \xrightarrow{i_{\mathfrak{g}_1}} & \mathcal{M}\left(G_{\mathfrak{g}_1,\infty},\mathcal{O}_{\mathfrak{f}}\right) \\ \downarrow \upsilon_{\mathfrak{g}_2,\mathfrak{g}_1} & & \downarrow (\pi_{\mathfrak{g}_2,\mathfrak{g}_1})^\sharp \\ \mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_{\mathfrak{g}_2,\infty} & \xrightarrow{i_{\mathfrak{g}_2}} & \mathcal{M}\left(G_{\mathfrak{g}_2,\infty},\mathcal{O}_{\mathfrak{f}}\right) \end{array}$$

Proof. This was proved by de Shalit in the case $p\neq 2$ (see [5, III.1.2 and III.1.3]). Let $\mathfrak{g}_1\in\mathcal{I}\setminus\mathcal{I}'$, and let $\mathfrak{g}_2\in\mathcal{I}'$ be such that $\mathfrak{g}_1|\mathfrak{g}_2$. When $p\neq 2$ de Shalit uses the surjectivity of $N_{\mathfrak{g}_2,\mathfrak{g}_1}$ in order to construct $i_{\mathfrak{g}_1}$. If $p=2$, $N_{\mathfrak{g}_2,\mathfrak{g}_1}$ may not be surjective. However we have $\operatorname{Im}\left(i_{\mathfrak{g}_2}\circ\upsilon_{\mathfrak{g}_2,\mathfrak{g}_1}\right)\subseteq \mathcal{M}\left(G_{\mathfrak{g}_2,\infty},\mathcal{O}_{\mathfrak{f}}\right)^{\operatorname{Ker}\left(\pi_{\mathfrak{g}_2,\mathfrak{g}_1}\right)}$. But by Proposition 2.1, $(\pi_{\mathfrak{g}_2,\mathfrak{g}_1})^\sharp$ is injective and $\operatorname{Im}(\pi_{\mathfrak{g}_2,\mathfrak{g}_1})^\sharp=\mathcal{M}\left(G_{\mathfrak{g}_2,\infty},\mathcal{O}_{\mathfrak{f}}\right)^{\operatorname{Ker}\left(\pi_{\mathfrak{g}_2,\mathfrak{g}_1}\right)}$. Hence there is a unique map $i_{\mathfrak{g}_1}$ such that the right hand square of (4.1) is commutative. The rest of the proof is identical to [5]. □

Lemma 4.2. *For all* $\mathfrak{g}\in\mathcal{I}$*,* $i_{\mathfrak{g}}$ *is an injective pseudo-isomorphism of* $\mathcal{O}_{\mathfrak{f}}[[T]]$*-modules.*

Proof. Let $\mathfrak{g}_1\in\mathcal{I}$, and let $\mathfrak{g}_2\in\mathcal{I}'$ be such that $\mathfrak{g}_1|\mathfrak{g}_2$. Then $(\pi_{\mathfrak{g}_2,\mathfrak{g}_1})^\sharp$, $\upsilon_{\mathfrak{g}_2,\mathfrak{g}_1}$, and $i_{\mathfrak{g}_2}$ are injective, and by (4.1) we deduce the injectivity of $i_{\mathfrak{g}_1}$.

By class field theory, one can show that for any prime $\mathfrak{q}$ of $K_{\mathfrak{g}_2,\infty}$ above $\mathfrak{p}$, the number of p-power roots of unity in $\left(K_{\mathfrak{g}_2,n}\right)_{\mathfrak{q}}$ is bounded independantly of n (see [17, Lemma 2.1]). Then it follows from [5, I.3.7, Theorem] that $i_{\mathfrak{g}_2}$ is a pseudo-isomorphism. Since $\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_{\mathfrak{g}_2,\infty}\right)^{\operatorname{Ker}\left(\pi_{\mathfrak{g}_2,\mathfrak{g}_1}\right)}=\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_{\mathfrak{g}_1,\infty}$ and since $(\pi_{\mathfrak{g}_2,\mathfrak{g}_1})^\sharp$ is injective, it follows from (4.1) that $\mathcal{M}\left(G_{\mathfrak{g}_1,\infty},\mathcal{O}_{\mathfrak{f}}\right)/\operatorname{Im}\left(i_{\mathfrak{g}_1}\right)$ is a submodule of $\mathcal{M}\left(G_{\mathfrak{g}_2,\infty},\mathcal{O}_{\mathfrak{f}}\right)/\operatorname{Im}\left(i_{\mathfrak{g}_2}\right)$, which is pseudo-nul since $i_{\mathfrak{g}_2}$ is a pseudo-isomorphism. □

An element of the total fraction ring of $\mathcal{M}\left(G_{\mathfrak{g},\infty},\mathcal{O}_L\right)$ is called an $\mathcal{O}_L$-pseudo-measure. For $\mathfrak{g}\in\mathcal{I}$, let $\mu(\mathfrak{g})$ be the $\mathcal{O}_L$-pseudo-measure on $G_{\mathfrak{g},\infty}$ defined in [5, II.4.12, Theorem]. It is a measure if $\mathfrak{g}\neq(1)$, and $\alpha\mu(1)$ is a measure for all $\alpha\in\mathcal{J}_{(1)}$, where we write $\mathcal{J}_{(1)}$ for the augmentation ideal of $\mathcal{O}_{\mathfrak{f}}\left[\left[G_{(1),\infty}\right]\right]$. By definition of $\mu(\mathfrak{g})$, we have

$$i_{\mathfrak{g}}\left(\psi\left\langle\mathfrak{g},\mathfrak{a}\right\rangle\right)=\left(\left(\mathfrak{a},K_{\mathfrak{g},\infty}/k\right)_*-N(\mathfrak{a})\right)\mu(\mathfrak{g}). \tag{4.2}$$

Moreover, for $(\mathfrak{g}_1,\mathfrak{g}_2)\in\mathcal{I}^2$ such that $\mathfrak{g}_1|\mathfrak{g}_2$, we have

$$\left(\pi_{\mathfrak{g}_2,\mathfrak{g}_1}\right)_*\mu\left(\mathfrak{g}_2\right)=\prod_{\substack{\mathfrak{l}\text{ prime of }\mathcal{O}_k\\ \mathfrak{l}|\mathfrak{g}_2\text{ and }\mathfrak{l}\nmid\mathfrak{g}_1}}\left(1-\left(\mathfrak{l},K_{\mathfrak{g}_1,\infty}/k\right)_*^{-1}\right)\mu\left(\mathfrak{g}_1\right). \tag{4.3}$$

Lemma 4.3. *For $\mathfrak{g} \in \mathcal{I}$, we denote by $\mu_{p^\infty}(K_{\mathfrak{g},\infty})$ the group of p-power roots of unity in $K_{\mathfrak{g},\infty}$. Then we have*

$$i_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\overline{\Psi}'\left(K_{\mathfrak{g},\infty},\mathfrak{g}\mathfrak{p}^\infty\right)\right)=\mathcal{J}_{\mathfrak{g}}\mu(\mathfrak{g}),$$

where $\mathcal{J}_{\mathfrak{g}}$ is the annihilator of the $\mathcal{O}_{\mathfrak{f}}[[G_{\mathfrak{g},\infty}]]$-module $\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mu_{p^\infty}(K_{\mathfrak{g},\infty})$ if $\mathfrak{g}\neq(1)$, and where $\mathcal{J}_{(1)}$ is the augmentation ideal of $\mathcal{O}_{\mathfrak{f}}\left[\left[G_{(1),\infty}\right]\right]$.

Proof. We refer the reader to [5, III.1.4]. □

5. Generation of the characteristic ideal.

For any $\mathfrak{g}\in\mathcal{I}$ such that $\mathfrak{g}|\mathfrak{f}$, and any irreducible ($\mathbb{C}$ or $\mathbb{C}_p$) character χ of $G_{\mathfrak{g}}$, let $\mathfrak{f}_\chi\in\mathcal{I}$ be such that the conductor of χ is $\mathfrak{f}_\chi\mathfrak{p}^n$ for some $n\in\mathbb{N}$. Then χ defines a character on $G_{\mathfrak{f}_\chi}$, which we denote by χ_0. We have

$$\mathcal{O}_{\mathfrak{f}}\left[\left[G_{\mathfrak{g},\infty}\right]\right]_\chi\simeq\mathcal{O}_{\mathfrak{f}}\left[\left[\Gamma\right]\right]\quad\text{and}\quad\mathcal{M}\left(G_{\mathfrak{g},\infty},\mathcal{O}_{\mathfrak{f}}\right)_\chi\simeq\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right),$$

where the isomorphisms are induced by the following maps,

$$\tilde{\chi}:\mathcal{O}_{\mathfrak{f}}\left[\left[G_{\mathfrak{g},\infty}\right]\right]\to\mathcal{O}_{\mathfrak{f}}\left[\left[\Gamma\right]\right]\quad\text{and}\quad\chi':\mathcal{M}\left(G_{\mathfrak{g},\infty},\mathcal{O}_{\mathfrak{f}}\right)\to\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right),$$

such that for any $(g,\sigma)\in G_{\mathfrak{g}}\times\Gamma$, $\tilde{\chi}(\sigma g)=\chi(g)\sigma$, and such that for any $\mu\in\mathcal{M}\left(G_{\mathfrak{g},\infty},\mathcal{O}_{\mathfrak{f}}\right)$, $\underline{\chi'(\mu)}=\tilde{\chi}\left(\underline{\mu}\right)$. Moreover, remark that we have

$$\chi'(\mu)=\chi_0'\left(\left(\pi_{\mathfrak{g},\mathfrak{f}_\chi}\right)_*\mu\right)\quad\text{for all}\quad\mu\in\mathcal{M}\left(G_{\mathfrak{g},\infty},\mathcal{O}_{\mathfrak{f}}\right),\tag{5.1}$$

and

$$\chi'\circ\left(\pi_{\mathfrak{g},\mathfrak{h}}\right)^\sharp=0\quad\text{for all }\mathfrak{h}\in\mathcal{I}\text{ such that }\mathfrak{h}\neq\mathfrak{f}_\chi\text{ and }\mathfrak{h}|\mathfrak{f}_\chi.\tag{5.2}$$

For any finite group $\mathcal{G}$, any irreducible $\mathbb{C}_p$-character χ of $\mathcal{G}$, and any morphism $f:M\to N$ of $\mathcal{O}_{\mathfrak{f}}[\mathcal{G}]$-modules, we denote by $f_\chi:M_\chi\to N_\chi$ the morphism defined by f. For any $x\in M$, we write x_χ for the canonical image of x in M_χ.

Lemma 5.1. *Let $\mathfrak{g}\in\mathcal{I}$ be such that $\mathfrak{g}|\mathfrak{f}$. Let $\chi\neq1$ be an irreducible $\mathbb{C}_p$-character of $G_{\mathfrak{g}}$. Then*

$$\left(i_{\mathfrak{g}}\right)_\chi\left(\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{C}_{\mathfrak{g},\infty}\right)_\chi\right)\subseteq\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}\left(\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\overline{\Psi}'\left(K_{\mathfrak{f}_\chi,\infty},\mathfrak{f}_\chi\mathfrak{p}^\infty\right)\right)_{\chi_0}\right),$$

and the quotient is a pseudo-null $\mathcal{O}_{\mathfrak{f}}[[T]]$-module.

Proof. Let $\mathfrak{h}\in\mathcal{I}$ be such that $\mathfrak{h}|\mathfrak{g}$, and let $x\in\overline{\Psi}'\left(K_{\mathfrak{g},\infty},\mathfrak{h}\mathfrak{p}^\infty\right)$. From Remark 1, there is $y\in\overline{\Psi}'\left(K_{\mathfrak{h}\wedge\mathfrak{f}_\chi,\infty},\left(\mathfrak{h}\wedge\mathfrak{f}_\chi\right)\mathfrak{p}^\infty\right)$ such that $N_{\mathfrak{g},\mathfrak{f}_\chi}(x)=$

$\upsilon_{\mathfrak{f}_\chi,\mathfrak{h}\wedge\mathfrak{f}_\chi}(y)$. From (5.1), and then from (4.1), one has

$$\begin{aligned}(i_{\mathfrak{g}})_\chi(x_\chi) = \chi'\circ i_{\mathfrak{g}}(x) &= \chi'_0\circ\left(\pi_{\mathfrak{g},\mathfrak{f}_\chi}\right)_*\circ i_{\mathfrak{g}}(x)\\ &= \chi'_0\circ i_{\mathfrak{f}_\chi}\circ N_{\mathfrak{g},\mathfrak{f}_\chi}(x)\\ &= \chi'_0\circ i_{\mathfrak{f}_\chi}\circ \upsilon_{\mathfrak{f}_\chi,\mathfrak{h}\wedge\mathfrak{f}_\chi}(y)\\ &= \chi'_0\circ\left(\pi_{\mathfrak{f}_\chi,\mathfrak{h}\wedge\mathfrak{f}_\chi}\right)^\sharp\circ i_{\mathfrak{h}\wedge\mathfrak{f}_\chi}(y).\end{aligned}\tag{5.3}$$

From (5.2) and (5.3), we deduce $(i_{\mathfrak{g}})_\chi(x_\chi)=0$ if $\mathfrak{f}_\chi\nmid\mathfrak{h}$, and $(i_{\mathfrak{g}})_\chi(x_\chi)=\chi'_0\circ i_{\mathfrak{f}_\chi}(y)=\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}(y_{\chi_0})$ if $\mathfrak{f}_\chi|\mathfrak{h}$. By Remark 2, this states the inclusion $\mathcal{B}\subseteq\mathcal{A}$, where we set

$$\mathcal{A}:=\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}\left(\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\overline{\Psi}'\left(K_{\mathfrak{f}_\chi,\infty},\mathfrak{f}_\chi\mathfrak{p}^\infty\right)\right)_{\chi_0}\right)$$

and

$$\mathcal{B}:=(i_{\mathfrak{g}})_\chi\left(\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{C}_{\mathfrak{g},\infty}\right)_\chi\right).$$

Let $m:=[k(\mathfrak{g}\mathfrak{p}^\infty):k(\mathfrak{f}_\chi\mathfrak{p}^\infty)]$, and let $x\in\overline{\Psi}'\left(K_{\mathfrak{f}_\chi,\infty},\mathfrak{f}_\chi\mathfrak{p}^\infty\right)$. Then $mx=N_{\mathfrak{g},\mathfrak{f}_\chi}\circ\upsilon_{\mathfrak{g},\mathfrak{f}_\chi}(x)$, and from (4.1) and (5.1), we obtain

$$\begin{aligned}m\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}(x_{\chi_0}) &= \chi'_0\circ i_{\mathfrak{f}_\chi}\circ N_{\mathfrak{g},\mathfrak{f}_\chi}\circ\upsilon_{\mathfrak{g},\mathfrak{f}_\chi}(x)\\ &= \chi'_0\circ\left(\pi_{\mathfrak{g},\mathfrak{f}_\chi}\right)_*\circ i_{\mathfrak{g}}\circ\upsilon_{\mathfrak{g},\mathfrak{f}_\chi}(x)\\ &= \chi'\circ i_{\mathfrak{g}}\circ\upsilon_{\mathfrak{g},\mathfrak{f}_\chi}(x)\\ &= (i_{\mathfrak{g}})_\chi\left(\upsilon_{\mathfrak{g},\mathfrak{f}_\chi}(x)_\chi\right),\end{aligned}$$

and we deduce that m annihilates $\mathcal{A}/\mathcal{B}$. Let $\alpha:=\prod_{\substack{\mathfrak{l}\text{ prime of }\mathcal{O}_k\\ \mathfrak{l}|\mathfrak{g}\text{ and }\mathfrak{l}\nmid\mathfrak{f}_\chi}}\left(1-\tilde{\chi}_0\left(\sigma_{\mathfrak{l}}^{-1}\right)\right)$, where $\sigma_{\mathfrak{l}}$ is the Fröbenius of $\mathfrak{l}$ in $K_{\mathfrak{f}_\chi,\infty}/k$. Let $x\in\overline{\Psi}'\left(K_{\mathfrak{f}_\chi,\infty},\mathfrak{f}_\chi\mathfrak{p}^\infty\right)$. From (3.3), there is $y\in\overline{\Psi}'(K_{\mathfrak{g},\infty},\mathfrak{g}\mathfrak{p}^\infty)$ such that $\alpha x=N_{\mathfrak{g},\mathfrak{f}_\chi}(y)$. Then by (4.1) and (5.1), we have

$$\begin{aligned}\alpha\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}(x_{\chi_0}) &= \chi'_0\circ i_{\mathfrak{f}_\chi}\circ N_{\mathfrak{g},\mathfrak{f}_\chi}(y)\\ &= \chi'_0\circ\left(\pi_{\mathfrak{g},\mathfrak{f}_\chi}\right)_*\circ i_{\mathfrak{g}}(y)\\ &= \chi'\circ i_{\mathfrak{g}}(y)\\ &= (i_{\mathfrak{g}})_\chi(y_\chi).\end{aligned}$$

Hence α annihilates $\mathcal{A}/\mathcal{B}$. As a particular case, if there is no maximal ideal $\mathfrak{l}$ of $\mathcal{O}_k$ such that $\mathfrak{l}|\mathfrak{g}$ and $\mathfrak{l}\nmid\mathfrak{f}_\chi$, then $\alpha=1$, $\mathcal{A}=\mathcal{B}$, and Lemma 5.1 is proved in this case. Now assume that there is a maximal ideal $\mathfrak{l}$ of $\mathcal{O}_k$ such that $\mathfrak{l}|\mathfrak{g}$ and $\mathfrak{l} \nmid \mathfrak{f}_\chi$. By class field theory, the decomposition group of $\mathfrak{l}$ in $K_{\mathfrak{f}_\chi,\infty}/k$ has a finite index in $\mathrm{Gal}\left(K_{\mathfrak{f}_\chi,\infty}/k\right)$. Hence $\sigma_\mathfrak{l} \notin G_{\mathfrak{f}_\chi}$, and there are a topological generator $\tilde{\gamma}$ of Γ, $n \in \mathbb{N}$, and $g \in G_{\mathfrak{f}_\chi}$ such that $\sigma_\mathfrak{l}^{-1} = g\tilde{\gamma}^{p^n}$. Then

$$1 - \tilde{\chi}_0\left(\sigma_\mathfrak{l}^{-1}\right) = 1 - \chi_0(g)\tilde{\gamma}^{p^n} = 1 - \chi_0(g)\sum_{i=0}^{p^n}\left(\begin{array}{c} p^n \\ i \end{array}\right)\tilde{T}^j, \tag{5.4}$$

where $\tilde{T} := \tilde{\gamma} - 1$. Since m and $\chi_0(g)$ are coprime, and since $-\chi_0(g)$ is the coefficient of $\tilde{T}^{p^n}$ in the decomposition (5.4), we deduce that m and $1 - \tilde{\chi}_0\left(\sigma_\mathfrak{l}^{-1}\right)$ are coprime. Then m and α are coprime, and annihilate $\mathcal{A}/\mathcal{B}$, so that Lemma 5.1 follows. □

Lemma 5.2. *Let $\mathfrak{g} \in \mathcal{I}$ be such that $\mathfrak{g}|\mathfrak{f}$. Let $\chi \neq 1$ be an irreducible $\mathbb{C}_p$-character of $G_\mathfrak{g}$.*

(i) *If $p \neq 2$ or if $w_\mathfrak{g} = w_{\mathfrak{f}_\chi}$, then* $\mathrm{Im}\,(i_\mathfrak{g})_\chi = \mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}$.

(ii) *If $p = 2$, then* $\mathrm{Im}\,(i_\mathfrak{g})_\chi \subseteq \mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}$, *and the quotient is annihilated by* 2.

Proof. For $x \in \mathcal{U}_{\mathfrak{g},\infty}$, by (5.1) and (4.1), we have

$$\begin{aligned}(i_\mathfrak{g})_\chi(x_\chi) = \chi_0' \circ \left(\pi_{\mathfrak{g},\mathfrak{f}_\chi}\right)_* \circ i_\mathfrak{g}(x) &= \chi_0' \circ i_{\mathfrak{f}_\chi} \circ N_{\mathfrak{g},\mathfrak{f}_\chi}(x) \\ &= \left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}\left(N_{\mathfrak{g},\mathfrak{f}_\chi}(x)_{\chi_0}\right).\end{aligned} \tag{5.5}$$

We deduce $\mathrm{Im}\,(i_\mathfrak{g})_\chi \subseteq \mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}$. For n large enough, the ramification index of the primes above $\mathfrak{p}$ in $K_{\mathfrak{g},n}/K_{\mathfrak{f}_\chi,n}$ is $w_{\mathfrak{f}_\chi}w_\mathfrak{g}^{-1}$. If $p \neq 2$, then $w_{\mathfrak{f}_\chi}w_\mathfrak{g}^{-1}$ is prime to p. Hence in case (i), $K_{\mathfrak{g},n}/K_{\mathfrak{f}_\chi,n}$ is tamely ramified. Then $N_{\mathfrak{g},\mathfrak{f}_\chi}$ is a surjection from $\mathcal{U}_{\mathfrak{g},\infty}$ onto $\mathcal{U}_{\mathfrak{f}_\chi,\infty}$, and we deduce $\mathrm{Im}\,(i_\mathfrak{g})_\chi \supseteq \mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}$ from (5.5). If $p = 2$, $\mathcal{U}_{\mathfrak{f}_\chi,\infty}/N_{\mathfrak{g},\mathfrak{f}_\chi}(\mathcal{U}_{\mathfrak{g},\infty})$ is annihilated by $w_{\mathfrak{f}_\chi}w_\mathfrak{g}^{-1}$ which is 1 or 2, and we deduce (ii) from (5.5). □

For $p \neq 2$, Theorems 5.1 and 5.2 below were already proved by de Shalit in [5, III.1.10].

Theorem 5.1. *Let $\mathfrak{g} \in \mathcal{I}$ be such that $\mathfrak{g}|\mathfrak{f}$. Let $\mathfrak{u}$ be a uniformizer of $\mathcal{O}_\mathfrak{f}$. Let $\chi \neq 1$ be an irreducible $\mathbb{C}_p$-character of $G_\mathfrak{g}$.*

(i) *If $p \neq 2$ or if $w_\mathfrak{g} = w_{\mathfrak{f}_\chi}$, then* $\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty})\right)_\chi$ *is generated by* $\tilde{\chi}_0\left(\underline{\mu(\mathfrak{f}_\chi)}\right)$.

(ii) *If $p = 2$, then the ideal* $\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty})\right)_\chi$ *is generated by* $\mathfrak{u}^{-m_\chi}\tilde{\chi}_0\left(\underline{\mu(\mathfrak{f}_\chi)}\right)$, *for some $m_\chi \in \mathbb{N}$.*

(In case $\mathfrak{f}_\chi = (1)$, we have expanded $\tilde{\chi}_0$ to the total fraction ring of $\mathcal{O}_\mathfrak{f}\left[\left[G_{(1),\infty}\right]\right]$ and to the fraction field of $\mathcal{O}_\mathfrak{f}[[\Gamma]]$. We still have $\tilde{\chi}_0\left(\underline{\mu(1)}\right) \in \mathcal{O}_\mathfrak{f}[[\Gamma]]$.)

Proof. Let us set $\tilde{\mathcal{C}}_\mathfrak{g} := \left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{C}_{\mathfrak{g},\infty}\right)_\chi$. We have the tautological exact sequence below,

$$(5.6)\quad 0 \to \mathrm{Im}\left(i_\mathfrak{g}\right)_\chi / \left(i_\mathfrak{g}\right)_\chi\left(\tilde{\mathcal{C}}_\mathfrak{g}\right) \to \mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0} / \left(i_\mathfrak{g}\right)_\chi\left(\tilde{\mathcal{C}}_\mathfrak{g}\right) \to \mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0} / \mathrm{Im}\left(i_\mathfrak{g}\right)_\chi \to 0.$$

From Lemma 5.2, we deduce the existence of $m_\chi \in \mathbb{N}$ such that

$$(5.7)\qquad \mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0} / \mathrm{Im}\left(i_\mathfrak{g}\right)_\chi\right) = \left(\mathfrak{u}^{m_\chi}\right),$$

with $m_\chi = 0$ in case (i). Since $\mathrm{Im}\left(i_\mathfrak{g}\right)_\chi / \left(i_\mathfrak{g}\right)_\chi\left(\tilde{\mathcal{C}}_\mathfrak{g}\right) \simeq \left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_\chi$, from (5.6) and (5.7), we deduce that

$$(5.8)\quad \mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_\chi = \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0} / \left(i_\mathfrak{g}\right)_\chi\left(\tilde{\mathcal{C}}_\mathfrak{g}\right)\right).$$

We set $\tilde{\Psi} := \left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\overline{\Psi}'\left(K_{\mathfrak{f}_\chi,\infty}, \mathfrak{f}_\chi\mathfrak{p}^\infty\right)\right)_{\chi_0}$. From (5.8) and Lemma 5.1, we deduce

$$(5.9)\quad \mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_\chi = \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0} / \left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}\left(\tilde{\Psi}\right)\right).$$

Since $\mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right)_{\chi_0} / \left(i_{\mathfrak{f}_\chi}\right)_{\chi_0}\left(\tilde{\Psi}\right) \simeq \left(\mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right) / \left(i_{\mathfrak{f}_\chi}\right)\left(\tilde{\Psi}\right)\right)_{\chi_0}$ and since $i_{\mathfrak{f}_\chi}$ is a pseudo-isomorphism, we deduce from (5.9) and Lemma 4.3 that

$$\begin{aligned}(5.10)\qquad \mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}&\left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_\chi \\ &= \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathrm{Im}\left(i_{\mathfrak{f}_\chi}\right) / \left(i_{\mathfrak{f}_\chi}\right)\left(\tilde{\Psi}\right)\right)_{\chi_0} \\ &= \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathcal{M}\left(G_{\mathfrak{f}_\chi,\infty}, \mathcal{O}_\mathfrak{f}\right) / \mathcal{J}_{\mathfrak{f}_\chi}\mu\left(\mathfrak{f}_\chi\right)\right)_{\chi_0} \\ &= \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_\mathfrak{f}[[T]]}\left(\mathcal{M}\left(\Gamma, \mathcal{O}_\mathfrak{f}\right) / \chi_0'\left(\mathcal{J}_{\mathfrak{f}_\chi}\mu\left(\mathfrak{f}_\chi\right)\right)\right).\end{aligned}$$

First we assume that $\mathfrak{f}_\chi \neq (1)$. Then $\chi_0'\left(\mu\left(\mathfrak{f}_\chi\right)\right)\mathcal{M}\left(\Gamma, \mathcal{O}_\mathfrak{f}\right) / \chi_0'\left(\mathcal{J}_{\mathfrak{f}_\chi}\mu\left(\mathfrak{f}_\chi\right)\right)$ is isomorphic to $\left(\mathcal{O}_\mathfrak{f}\widehat{\otimes}_{\mathbb{Z}_p}\mu_{p^\infty}\left(K_{\mathfrak{f}_\chi,\infty}\right)\right)_{\chi_0}$, hence pseudo-nul since $\mu_{p^\infty}\left(K_{\mathfrak{f}_\chi,\infty}\right)$

is finite. Then from (5.10) we deduce

$$\begin{aligned}\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_{\chi}&\\ &= \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)/\chi_0'\left(\mu\left(\mathfrak{f}_\chi\right)\right)\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)\right)\\ &= \mathfrak{u}^{-m_\chi}\tilde{\chi}_0\left(\underline{\mu\left(\mathfrak{f}_\chi\right)}\right)\mathcal{O}_{\mathfrak{f}}[[T]],\end{aligned}$$

and Theorem 5.1 follows in this case. Now assume $\mathfrak{f}_\chi = (1)$. Then we expand χ_0' to the total faction ring of $\mathcal{M}\left(G_{(1),\infty},\mathcal{O}_{\mathfrak{f}}\right)$ and to the fraction field of $\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)$. There is $\sigma\in G_{\mathfrak{g}}$ such that $\chi(\sigma)\neq 1$. Then

$$\chi_0'\left(\mu\left(1\right)\right)\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)/\chi_0'\left(\mathcal{J}_{(1)}\mu\left(1\right)\right)$$

is pseudo-nul, annihilated by $1-\chi(\sigma)$ and T. Since we have

$$\chi_0'\left(\mathcal{J}_{(1)}\mu\left(1\right)\right)\subseteq\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right),$$

we deduce the inclusion $\chi_0'\left(\mu\left(1\right)\right)\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)\subseteq\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)$ and from (5.10) we obtain

$$\begin{aligned}\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_{\chi}&\\ &= \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)/\chi_0'\left(\mu\left(1\right)\right)\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)\right).\end{aligned}$$

(i) and (ii) follow immediately in this case. □

Theorem 5.2. *Let $\mathfrak{g}\in\mathcal{I}$ be such that $\mathfrak{g}|\mathfrak{f}$. Let χ be the trivial character on $G_{\mathfrak{g}}$.*

(i) *If $p\neq 2$ or if $w_{\mathfrak{g}} = |\mu(k)|$, then* $\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_{\chi}$ *is generated by $\tilde{\chi}_0\left(T\underline{\mu(1)}\right)$.*

(ii) *If $p=2$, then the ideal* $\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_{\chi}$ *is generated by $\mathfrak{u}^{-m_\chi}\tilde{\chi}_0\left(T\underline{\mu(1)}\right)$, for some $m_\chi\in\mathbb{N}$.*

Proof. As in the proof of Theorem 5.1, we have

$$\begin{aligned}(5.11)\quad \mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\left(\mathcal{U}_{\mathfrak{g},\infty}/\mathcal{C}_{\mathfrak{g},\infty}\right)\right)_{\chi}&\\ &= \mathfrak{u}^{-m_\chi}\mathrm{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)/\chi_0'\left(\mathcal{J}_{(1)}\mu\left(1\right)\right)\right),\end{aligned}$$

where $m_\chi\in\mathbb{N}$ is zero in case (i). But $\chi_0'\left(\mathcal{J}_{(1)}\mu\left(1\right)\right) = \chi_0'\left(T\mu\left(1\right)\right)\mathcal{M}\left(\Gamma,\mathcal{O}_{\mathfrak{f}}\right)$, and the theorem follows. □

6. Finiteness of invariants and coinvariants.

For any $\mathfrak{h}\in\mathcal{I}$, we write $\mathrm{L}_{p,\mathfrak{h}}$ for the p-adic L-function of k with modulus $\mathfrak{h}$, as defined in [5, II.4.16]. It is the map defined on the set of all continuous group morphisms ξ from $\mathrm{Gal}\left(K_{\mathfrak{h},\infty}/k\right)$ to $\mathbb{C}_p^\times$ (with $\xi \neq 1$ if $\mathfrak{h} = (1)$), such that

$$\mathrm{L}_{p,\mathfrak{h}}\left(\xi\right) = \int \xi(\sigma)^{-1}.\mathrm{d}\mu(\mathfrak{h})(\sigma). \tag{6.1}$$

Let $n \in \mathbb{N}$, and let χ be an irreducible $\mathbb{C}_p$-character on $\mathrm{Gal}\left(k\left(\mathfrak{h}\mathfrak{p}^n\right)/k\right)$ (with $\chi \neq 1$ if $\mathfrak{h} = (1)$). We write F_χ for the subfield of $k\left(\mathfrak{h}\mathfrak{p}^n\right)$ fixed by $\mathrm{Ker}(\chi)$, and we write χ_{pr} for the character on $\mathrm{Gal}\left(F_\chi/k\right)$ defined by χ. By inflation we can consider χ as a group morphism $\mathrm{Gal}\left(K_{\mathfrak{h},\infty}/k\right) \to \mathbb{C}_p^\times$, so that the notation $\mathrm{L}_{p,\mathfrak{h}}\left(\chi\right)$ makes sense. As in [5, II.5.2], if $n > 0$ we set

$$\mathrm{L}_{p,\mathfrak{h}\mathfrak{p}^n}\left(\chi\right) := \begin{cases} \left(1 - \chi_{\mathrm{pr}}\left(\mathfrak{p}, F_\chi/k\right)\right)\mathrm{L}_{p,\mathfrak{h}}\left(\chi\right) & \text{if} \quad \mathfrak{p} \text{ is unramified in } F_\chi, \\ \mathrm{L}_{p,\mathfrak{h}}\left(\chi\right) & \text{if} \quad \mathfrak{p} \text{ is ramified in } F_\chi. \end{cases} \tag{6.2}$$

Lemma 6.1. *Let $\mathfrak{g} \notin \{(0),(1)\}$ be an ideal of $\mathcal{O}_k$, and let χ be an irreducible $\mathbb{C}_p$-character on $\mathrm{Gal}\left(k(\mathfrak{g})/k\right)$. If $\chi \neq 1$ and if none of the prime ideals dividing $\mathfrak{g}$ are totally split in F_χ/k, then $\mathrm{L}_{p,\mathfrak{g}}\left(\chi\right) \neq 0$. If $\chi = 1$, if $\mathfrak{g}$ is a power of a prime ideal, and if $\mathfrak{p} \nmid \mathfrak{g}$, then $\mathrm{L}_{p,\mathfrak{g}}\left(\chi\right) \neq 0$.*

Proof. We set $H := \mathrm{Gal}\left(k(\mathfrak{g})/k\right)$. For all maximal ideal $\mathfrak{r}$ of $\mathcal{O}_{k(\mathfrak{g})}$, let us denote by $v_\mathfrak{r}$ the normalized valuation at $\mathfrak{r}$. Let $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ be such that $v_\mathfrak{r}\left(\varphi_\mathfrak{g}(1)\right) = 0$ for all maximal ideal $\mathfrak{r}$ of $\mathcal{O}_{k(\mathfrak{g})}$ not lying above $\mathfrak{q}$. Let $U \subset k(\mathfrak{g})^\times$ be the subgroup of all the numbers $x \in k(\mathfrak{g})^\times$ verifying the two following conditions,

- $v_\mathfrak{r}(x) = 0$ for all maximal ideal $\mathfrak{r}$ of $\mathcal{O}_{k(\mathfrak{g})}$ not lying above $\mathfrak{q}$,
- $v_\mathfrak{r}(x) = v_\mathfrak{s}(x)$ for all maximal ideals $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{O}_{k(\mathfrak{g})}$ above $\mathfrak{q}$.

Using Dirichlet's theorem and the product formula, we see that $\mathbb{Q} \otimes_\mathbb{Z} U \simeq \mathbb{Q}\left[H\right]$. Hence we can fix $u \in U$ such that $\mathbb{Q} \otimes_\mathbb{Z} U$ is freely generated by $1 \otimes u$ over $\mathbb{Q}\left[H\right]$. Let us fix an embedding $\iota_p : k^{\mathrm{alg}} \hookrightarrow \mathbb{C}_p$. We define the morphism of $k^{\mathrm{alg}}\left[H\right]$-modules below,

$$\ell_p : k^{\mathrm{alg}} \otimes_\mathbb{Z} U \to \mathbb{C}_p\left[H\right], \quad a \otimes x \mapsto \iota_p\left(a\right) \sum_{\sigma \in H} \log_p\left(\iota_p\left(x^\sigma\right)\right)\sigma^{-1},$$

where $\log_p$ is the p-adic logarithm, as defined in [8, §4]. Let us show that ℓ_p is injective on $k^{\mathrm{alg}} \otimes_\mathbb{Z} U$. We assume that it is not injective, and a contradiction will arise. There is an irreducible $\mathbb{C}_p$-character ξ of H such that $e_\xi \ell_p\left(1 \otimes u\right) = 0$, and then the family $\left(\log_p\left(\iota_p\left(u^\sigma\right)\right)\right)_{\sigma \in H}$ is not linearly independant over $\iota_p\left(k^{\mathrm{alg}}\right)$. By a theorem of Brumer [3, Theorem 1], we deduce that there are integers $\lambda_\sigma \in \mathbb{Z}$, $\sigma \in H$, with $\lambda_{\sigma_0} \neq 0$ for some $\sigma_0 \in H$, such that

$$\log_p\left(\iota_p\left(\prod_{\sigma \in H} u^{\lambda_\sigma \sigma}\right)\right) = \sum_{\sigma \in H} \lambda_\sigma \log_p\left(\iota_p\left(u^\sigma\right)\right) = 0.$$

It is well known that $\operatorname{Ker}\left(\log_p\right)$ is generated by the roots of powers of p, hence $\prod_{\sigma\in H} u^{\lambda_\sigma \sigma}$ is a root of unity. Then we must have $\lambda_\sigma = 0$ for all $\sigma \in H$, which contradicts $\lambda_{\sigma_0} \neq 0$. Thus we have verified the injectivity of ℓ_p. Now assume $\mathrm{L}_{p,\mathfrak{g}}(\chi) = 0$. From the p-adic version of the Kronecker limit formula [5, II.5.2, Theorem], we deduce that $e_{\chi^{-1}}\ell_p\left(1 \otimes \varphi_{\mathfrak{g}}(1)\right) = 0$ in $\mathbb{C}_p[H]$. Then

$$e_{\chi^{-1}}\left(1 \otimes \varphi_{\mathfrak{g}}(1)\right) = 0 \quad \text{in} \quad k^{\mathrm{alg}} \otimes_{\mathbb{Z}} U, \tag{6.3}$$

where χ is identified to a group morphism $H \to k^{\mathrm{alg}}$ via ι_p. If $\chi \neq 1$, then from [12, Théorème 10] we deduce the existence of a maximal ideal $\mathfrak{r}$ of $\mathcal{O}_k$, unramified in F_χ/k, such that $\mathfrak{r}|\mathfrak{g}$, and such that $\chi_{\mathrm{pr}}\left(\mathfrak{r}, F_\chi/k\right) = 1$ (hence totally split in F_χ/k). If $\chi = 1$, from (6.3) we deduce $N_{k(\mathfrak{g})/k}\left(\varphi_{\mathfrak{g}}(1)\right) \in \mu(k)$. Then $\mathfrak{g}$ must be divisible by at least two distinct prime ideals in virtue of (3.2). □

Theorem 6.1. *For all $n \in \mathbb{N}$, the module of Γ_n-invariants and the module of Γ_n-coinvariants of $\mathcal{U}_\infty/\mathcal{C}_\infty$ are finite.*

Proof. By [10, p. 254, Exercise 3], it is sufficient to verify that

$$\operatorname{char}_{\mathbb{Z}_p[[T]]}\left(\mathcal{U}_\infty/\mathcal{C}_\infty\right) \text{ is prime to } \left((1+T)^{p^n} - 1\right) \text{ in } \mathbb{Z}_p[[T]], \text{ for all } n \in \mathbb{N}. \tag{6.4}$$

For n large enough, $K_{\mathfrak{f},n}/K_n$ is tamely ramified if $p \neq 2$, and if $p = 2$ the ramification index is 1 or 2. Hence we deduce that the cokernel of the norm maps $\mathcal{U}_{\mathfrak{f},\infty} \to \mathcal{U}_\infty$ and $\mathcal{U}_{\mathfrak{f},\infty}/\mathcal{C}_{\mathfrak{f},\infty} \to \mathcal{U}_\infty/\mathcal{C}_\infty$ are annihilated by 2. Then we have

$$\operatorname{char}_{\mathbb{Z}_p[[T]]}\left(\mathcal{U}_\infty/\mathcal{C}_\infty\right) \quad \text{divides} \quad 2^a \operatorname{char}_{\mathbb{Z}_p[[T]]}\left(\mathcal{U}_{\mathfrak{f},\infty}/\mathcal{C}_{\mathfrak{f},\infty}\right), \tag{6.5}$$

for some $a \in \mathbb{N}$. By (6.5), we are reduced to prove (6.4) in the case $K_\infty = K_{\mathfrak{f},\infty}$. Then by (1.1), in order to verify (6.4) we only have to show that the ideal $\operatorname{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}\left(\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{U}_\infty/\mathcal{O}_{\mathfrak{f}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{C}_\infty\right)_\chi$ is prime to $\left((1+T)^{p^n} - 1\right)$ in $\mathcal{O}_{\mathfrak{f}}[[T]]$, for all $n \in \mathbb{N}$, and all irreducible $\mathbb{C}_p$-character χ on $G_{\mathfrak{f}}$. Let χ be such a character, and let $\zeta \in \mu_{p^\infty}(\mathbb{C}_p)$. We choose a maximal ideal ℓ of $\mathcal{O}_k$, prime to $\mathfrak{f}\mathfrak{p}$, such that $\chi_{\mathrm{pr}}\left(\ell, F_\chi/k\right) \neq 1$ if $\chi \neq 1$, and such that ℓ is not totally split in k_1 (the subfield of k_∞ fixed by Γ^p) if $\chi = 1$. By Theorem 5.1 and Theorem 5.2, it suffices to prove $\tilde{\chi}_0\left(\left(1 - \sigma_\ell^{-1}\right)\underline{\mu\left(\mathfrak{f}_\chi\right)}\right)|_{T=\zeta-1} \neq 0$, where $\sigma_\ell := \left(\ell, K_{\mathfrak{f},\infty}/k\right)$. By (4.3) and by (2.5), we have

$$\begin{aligned}
\tilde{\chi}_0\left(\left(1 - \sigma_\ell^{-1}\right)\underline{\mu\left(\mathfrak{f}_\chi\right)}\right)|_{T=\zeta-1} &= \tilde{\chi}_0\left(\tilde{\pi}_{\mathfrak{f}_\chi\ell,\mathfrak{f}_\chi}\left(\underline{\mu\left(\mathfrak{f}_\chi\ell\right)}\right)\right)|_{T=\zeta-1} \\
&= \int_\Gamma \zeta^{\kappa(\sigma)}.\mathrm{d}\chi'_{\mathfrak{f}_\chi\ell}\left(\mu\left(\mathfrak{f}_\chi\ell\right)\right)(\sigma),
\end{aligned} \tag{6.6}$$

where $\chi_{\mathfrak{f}_\chi\ell}$ is the character on $G_{\mathfrak{f}_\chi\ell}$ defined by χ_0, and where $\kappa : \Gamma \to \mathbb{Z}_p$ is the unique morphism of topological groups such that $\kappa(\gamma) = 1$. From (6.6) and (2.3) we deduce

$$\tilde{\chi}_0\left(\left(1-\sigma_\ell^{-1}\right)\underline{\mu\left(\mathfrak{f}_\chi\right)}\right)|_{T=\zeta-1} = \sum_{g\in G_{\mathfrak{f}_\chi\ell}} \chi_{\mathfrak{f}_\chi\ell}(g)\int_\Gamma \zeta^{\kappa(\sigma)}.\mathrm{d}\left(g^{-1}\right)_* \mu\left(\mathfrak{f}_\chi\ell\right)(\sigma).$$

$$= \sum_{g\in G_{\mathfrak{f}_\chi\ell}} \chi_{\mathfrak{f}_\chi\ell}(g)\int_{g\Gamma} \zeta^{\kappa\left(g^{-1}\sigma\right)}.\mathrm{d}\mu\left(\mathfrak{f}_\chi\ell\right)(\sigma)$$

$$(6.7)\qquad = \int_{G_{\mathfrak{f}_\chi\ell,\infty}} \zeta^{\kappa\left(g_\sigma^{-1}\sigma\right)}\chi_{\mathfrak{f}_\chi\ell}\left(g_\sigma\right).\mathrm{d}\mu\left(\mathfrak{f}_\chi\ell\right)(\sigma),$$

where for any $\sigma \in G_{\mathfrak{f}_\chi\ell,\infty}$, g_σ is the image of σ through the projection $G_{\mathfrak{f}_\chi\ell,\infty} \to G_{\mathfrak{f}_\chi\ell}$. We define $\xi : G_{\mathfrak{f}_\chi\ell,\infty} \to \mathbb{C}_p^\times$, $\sigma \mapsto \zeta^{\kappa\left(g_\sigma^{-1}\sigma\right)}\chi_{\mathfrak{f}_\chi\ell}\left(g_\sigma\right)$. Then ξ is a group morphism, and if $n \in \mathbb{N}$ is such that $\zeta^{p^n} = 1$, then ξ defines an irreducible $\mathbb{C}_p$-character on $G_{\mathfrak{f}_\chi\ell,n} := \mathrm{Gal}\left(K_{\mathfrak{f}_\chi\ell,n}/k\right)$. Let $\mathfrak{g}$ be the conductor of F_ξ. Since the restriction of ξ to $G_{\mathfrak{f}_\chi\ell} \hookrightarrow G_{\mathfrak{f}_\chi\ell,n}$ is $\chi_{\mathfrak{f}_\chi\ell}$, we deduce that there is $m \in \mathbb{N}$ such that $\mathfrak{g} = \mathfrak{f}_\chi\mathfrak{p}^m$, and from (6.2) we deduce that

$$(6.8)\qquad \mathrm{L}_{p,\mathfrak{g}\ell}\left(\xi^{-1}\right) = \mathrm{L}_{p,\mathfrak{f}_\chi\ell}\left(\xi^{-1}\right).$$

Then from (6.7) and (6.1) we deduce

$$(6.9)\qquad \left(1-\tilde{\chi}_0\left(\sigma_\ell^{-1}\right)\right)\tilde{\chi}_0\left(\underline{\mu\left(\mathfrak{f}_\chi\right)}\right)|_{T=\zeta-1}$$

$$= \int_{G_{\mathfrak{f}_\chi\ell,\infty}} \xi(\sigma).\mathrm{d}\mu\left(\mathfrak{f}_\chi\ell\right)(\sigma) = \mathrm{L}_{p,\mathfrak{f}_\chi\ell}\left(\xi^{-1}\right).$$

If $\chi \neq 1$, then $\chi_{\mathrm{pr}}\left(\ell, F_\chi/k\right) \neq 1$ implies that ℓ is not totally split in F_ξ/k. If $\chi = 1$ and $\zeta \neq 1$, then $k_1 \subseteq F_\xi$ and ℓ is not totally split in F_ξ/k. If $\chi = 1$ and $\zeta = 1$, then $\xi = 1$ and $\mathfrak{g} = (1)$. From (6.9), (6.8), and Lemma 6.1, we deduce

$$\left(1-\tilde{\chi}_0\left(\sigma_\ell^{-1}\right)\right)\tilde{\chi}_0\left(\underline{\mu\left(\mathfrak{f}_\chi\right)}\right)|_{T=\zeta-1} = \mathrm{L}_{p,\mathfrak{g}\ell}\left(\xi^{-1}\right) \neq 0.$$

□

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