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The algebraic groups leading to the Roth inequalities

par Masami FUJIMORI

Résumé. On détermine les groupes algébriques qui ont une étroite relation avec les inégalités de Roth.

Abstract. We determine the algebraic groups which have a close relation to the Roth inequalities.

Introduction

Let α be a real algebraic number; r, s indeterminates; $|\cdot|$ the usual absolute value on the field of real numbers; and ε an arbitrary positive constant. Suppose that α is not a rational number. Finiteness of the number of rational integral solutions to the famous Roth inequality

$$\left|\alpha - \frac{s}{r}\right| < \frac{1}{|r|^{2+\varepsilon}}$$

is deduced by a simple argument from finiteness of the number of rational integral solutions to the linear inequalities

$$|\alpha r - s| < Q^{-1-\delta}, \quad |r| < Q^{1-\delta} \quad (Q > 1),$$

where Q is a (variable) real parameter and δ is an arbitrarily fixed positive number (cf. e.g. [12, VI §3]). We call this latter system (of linear inequalities) a *classical* Roth *system* in this paper. We denote respectively by $\mathbb{Q}$ and by $\bar{\mathbb{Q}}$ the field of rational numbers and its algebraic closure considered in the field $\mathbb{C}$ of complex numbers: $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Given the classical Roth system, we attach to the vector space $\check{V} = \mathbb{Q}r \oplus \mathbb{Q}s$ a filtration $F_\alpha^\cdot \check{V}$ over $\bar{\mathbb{Q}}$ defined as

$$F_\alpha^i \check{V} = \begin{cases} \bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \check{V} & (i \leq -1) \\ \bar{\mathbb{Q}}(\alpha r - s) & (-1 < i \leq 1) \\ 0 & (i > 1). \end{cases}$$

Let $x_1, \ldots, x_n$ be indeterminates; $l_1, \ldots, l_n$ linearly independent linear forms in $x_1, \ldots, x_n$ with real algebraic coefficients; and $c(1), \ldots, c(n)$ real

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constant numbers with $\sum_{q=1}^{n} c(q) = 0$. The system $\mathcal{S} = (l_1, \ldots, l_n; c(1), \ldots, c(n))$ is called a *general* ROTH *system* if the simultaneous linear inequalities

$$|l_q| < Q^{-c(q)-\delta} \quad (Q > 1;\ q = 1, \ldots, n)$$

have only a finite number of rational integral solutions for each arbitrarily fixed positive number δ. We generally attach to the vector space $V = \mathbb{Q}x_1 \oplus \cdots \oplus \mathbb{Q}x_n$ a filtration $F^{\cdot}_{\mathcal{S}}V$ over $\bar{\mathbb{Q}}$ defined as

$$F^i_{\mathcal{S}}V = \sum_{c(q) \geq i} \bar{\mathbb{Q}}\, l_q \quad (i \in \mathbb{R}).$$

The filtration thus obtained is descending, exhaustive, separated, and left-continuous in the sense that we have

$$F^i_{\mathcal{S}}V \supset F^j_{\mathcal{S}}V \quad (i \leq j), \qquad \bigcup_{i \in \mathbb{R}} F^i_{\mathcal{S}}V = \bar{\mathbb{Q}} \otimes_{\mathbb{Q}} V,$$

$$\bigcap_{i \in \mathbb{R}} F^i_{\mathcal{S}}V = 0, \qquad \text{and} \qquad F^i_{\mathcal{S}}V = \bigcap_{j < i} F^j_{\mathcal{S}}V.$$

For a finite dimensional non-zero vector space V over $\mathbb{Q}$ equipped with a filtration $F^{\cdot}V$ $(i \in \mathbb{R})$ over $\bar{\mathbb{Q}}$ as above, the real number

$$\mu(V) = \mu\left(V, F^{\cdot}V\right) = \frac{1}{\dim_{\mathbb{Q}} V} \sum_{w \in \mathbb{R}} w \dim_{\bar{\mathbb{Q}}} \mathrm{gr}^w \left(F^{\cdot}V\right),$$

where $\mathrm{gr}^w\left(F^{\cdot}V\right) = F^wV/F^{w+}V$, $F^{w+}V = \cup_{j>w}F^jV$, is called the slope of the filtered vector space $V = (V, F^{\cdot}V)$. A filtered vector space V or its filtration is said to be semi-stable if for any non-zero subspace W over $\mathbb{Q}$ of V with the (induced) sub-filtration over $\bar{\mathbb{Q}}$, the inequality $\mu(W) \leq \mu(V)$ is valid. We denote by $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ the category of finite dimensional vector spaces over $\mathbb{Q}$ with semi-stable filtration over $\bar{\mathbb{Q}}$ of slope zero. The morphisms in $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ are the linear maps over $\mathbb{Q}$ which respect filtrations when linearly extended over $\bar{\mathbb{Q}}$.

Theorem 0.1 (Schmidt, cf. e.g. [12, VI Theorem 2B])**.** *The filtration $F^{\cdot}_{\mathcal{S}}V$ derived from a general* ROTH *system $\mathcal{S}$ is semi-stable of slope zero. Conversely, every object of $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ whose filtration is defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$ is derived from a general* ROTH *system.*

For objects $V = (V, F^{\cdot}V)$ and $W = (W, F^{\cdot}W)$ in $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$, their tensor product $V \otimes W$ is the vector space $V \otimes_{\mathbb{Q}} W$ equipped with the filtration

$$F^i(V \otimes_{\mathbb{Q}} W) = \sum_{j+q=i} F^jV \otimes_{\bar{\mathbb{Q}}} F^qW \quad (i \in \mathbb{R}).$$

The claim that the tensor product $V \otimes W$ is again semi-stable is the heart of the next:

Theorem 0.2 (Faltings [6], Totaro [13])**.** *Let* $\omega_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *be the forgetful tensor functor of* $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *to the tensor category* $\mathrm{Vec}_{\mathbb{Q}}$ *of finite dimensional vector spaces over* $\mathbb{Q}$*. The tensor category* $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *is equivalent to the tensor category* $\mathrm{Rep}_{\mathbb{Q}} \mathrm{Aut}\, \omega_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *of finite dimensional representations over* $\mathbb{Q}$ *of the affine group scheme* $\mathrm{Aut}\, \omega_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *of natural equivalences of the functor* $\omega_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$*.*

We prove in the present paper the following:

Theorem 0.3. *If the real algebraic number* α *is* not *quadratic over* $\mathbb{Q}$*, then there exists a fully faithful tensor functor* ι *of the category* $\mathrm{Rep}_{\mathbb{Q}} \mathrm{SL}_2$ *of finite dimensional representations over* $\mathbb{Q}$ *of the special linear group* SL_2 *of degree* 2 *into the tensor category* $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *such that the functor* ι *commutes with the forgetful tensor functors to* $\mathrm{Vec}_{\mathbb{Q}}$ *and such that the image of* ι *contains the filtered vector space* $\left(\breve{V}, F_\alpha^{\cdot} \breve{V}\right)$ *derived from a classical* Roth *system.*

Similarly, if the real algebraic number α *is quadratic over* $\mathbb{Q}$*, then there exists a fully faithful tensor functor* ι *of the category* $\mathrm{Rep}_{\mathbb{Q}} T_\alpha$ *of finite dimensional representations over* $\mathbb{Q}$ *of a one-dimensional anisotropic torus* T_α *over* $\mathbb{Q}$ *into* $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ *such that the group* $T_\alpha(\mathbb{Q})$ *of* $\mathbb{Q}$*-valued points of the torus* T_α *is isomorphic to the kernel of the norm map of the quadratic number field* $\mathbb{Q}(\alpha)$ *over* $\mathbb{Q}$*, such that the functor* ι *is compatible with the forgetful tensor functors to* $\mathrm{Vec}_{\mathbb{Q}}$*, and such that its image contains the filtered vector space* $\left(\breve{V}, F_\alpha^{\cdot} \breve{V}\right)$*.*

Denote by $\breve{G}$ the anisotropic torus T_α or the special linear group SL_2 according as α is quadratic over $\mathbb{Q}$ or not. Theorem 0.3 implies that the action of $\mathrm{Aut}\, \omega_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ on the filtered vector space $\left(\breve{V}, F_\alpha^{\cdot} \breve{V}\right)$ factors through an action of the linear algebraic group $\breve{G}$. In this way, the classical Roth system corresponds to a representation of a one-dimensional anisotropic torus T_α or the special linear group SL_2 of degree 2 according as the coefficient α is quadratic over $\mathbb{Q}$ or not.

The definition of the category $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$ is motivated by the strong (or parametric) subspace theorem of Schmidt (e.g. [12, VI §3]) in Diophantine Approximation ([8], [5]). It originates from the observations of Faltings and Wüstholz ([7], [6]) that the condition of a linear system being a general Roth system agrees with the semi-stability in Geometric Invariant Theory.

For an arbitrary field K and any Galois extension field L of K, we can define a category $\mathcal{C}_0^{\mathrm{ss}}(K, L)$ in a parallel way to the case of $\mathcal{C}_0^{\mathrm{ss}}(\mathbb{Q}, \bar{\mathbb{Q}})$. We shall also determine in that case the algebraic group $\breve{G}$ as above (less explicitly when the base field K has positive characteristic). Note that the concept of a vector space with filtration in this paper coincides with the one in the theory of period domains over finite and local fields of RAPOPORT [10].

Now we state our plan of the present paper. In Section 1, we recall several facts on categories of vector spaces with filtrations and tensor functors of the category of finite dimensional representations of an algebraic group to a category of vector spaces with filtrations. In Section 2, we define a linear algebraic group whose representation leads to a classical ROTH system. Through Sections 3 and 4, we prove Theorem 0.3 in a slightly general setting.

When we take into account solutions to linear inequalities in, for example, the ring of integers in a number field and consider more linear inequalities, the category of vector spaces with multiple filtrations indexed by a set emerges. For multiple filtrations, we can naturally define the notion of slope and semi-stability from the viewpoint of linear inequalities. Let $\mathfrak{M}$ be an index set and K^{sep} a separable algebraic closure of an arbitrary field K. We denote by $\mathcal{C}_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ the category of finite dimensional vector spaces over K with multiple filtrations over K^{sep} indexed by $\mathfrak{M}$ which is semi-stable of slope zero. Let $\omega_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ be the forgetful tensor functor of $\mathcal{C}_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ to the tensor category Vec_K of finite dimensional vector spaces over K. As in the single filtration case, a tensor category $\mathcal{C}_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ is equivalent to the tensor category $\mathrm{Rep}_K \operatorname{Aut} \omega_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ of finite dimensional representations over K of the affine group scheme $\operatorname{Aut} \omega_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ of natural equivalences of the functor $\omega_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ by the result of FALTINGS [6] or of TOTARO [13].

In the appendix of the present paper, we prove in particular the following:

Theorem 0.4. *Any connected reductive group G over K occurs (up to isomorphism) as a quotient of the affine group scheme* $\operatorname{Aut} \omega_0^{\mathrm{ss}}(K, K^{\mathrm{sep}}, \mathfrak{M})$ *for an index set $\mathfrak{M}$ with sufficiently large cardinality depending on G.*

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1. Filtration and representation

A functorial way is formulated to associate the finite dimensional representations of an algebraic group over a field with finite dimensional vector spaces which are filtered over an extension field. We look at some of its properties.

Let K be an arbitrary field and L a (finite or infinite) Galois extension field of K. To confirm the terminology we use in this paper, we first recall the definitions of some basic notions.

Definition 1.1 (filtration over an extension field [10] [8])**.** For a finite dimensional vector space V over K, a family V^i $(i \in \mathbb{R})$ of subspaces over L of $L \otimes_K V$ is called a *filtration over* L of V if

$$V^i \supset V^j \ (i \leq j), \quad \bigcup_{i \in \mathbb{R}} V^i = L \otimes_K V, \quad \bigcap_{i \in \mathbb{R}} V^i = 0,$$

and

$$V^i = \bigcap_{j<i} V^j$$

are satisfied. The condition says that the filtration is defined over L and that it is descending, exhaustive, separated, and left-continuous.

Remark 1.2. Rapoport [10] is calling such a filtration an $\mathbb{R}$-filtration over L. In our previous paper [8], we have called it an L-filtration for simplicity.

Definition 1.3 (filtered homomorphism)**.** For vector spaces V and W over K with filtrations over L, a linear map $f\colon V \to W$ over K is called *filtered* if it satisfies $f\bigl(V^i\bigr) \subset W^i$ for all $i \in \mathbb{R}$ when the map f is linearly extended over L.

Definition 1.4 (sub-filtration and quotient filtration)**.** Let V be a vector space over K with a filtration over L as above. For a subspace W over K of V, the *sub-filtration* over L on W is given by

$$W^i = (L \otimes_K W) \cap V^i \quad (i \in \mathbb{R}).$$

The filtration over L on V/W defined as

$$(V/W)^i = (V^i + L \otimes_K W)/L \otimes_K W \quad (i \in \mathbb{R})$$

is called the *quotient filtration*.

Definition 1.5 (tensor product)**.** For vector spaces V and W with filtrations, the *tensor product* $(V \otimes W)^{\cdot}$ of their filtrations is a filtration over L of the vector space $V \otimes_K W$ over K defined as

$$(V \otimes W)^i = \sum_{j+q=i} V^j \otimes_L W^q \quad (i \in \mathbb{R}).$$

Remark 1.6. The tensor product of filtrations is associative and commutative in the obvious sense. It is compatible with the associativity law and the commutativity law of the underlying tensor product of vector spaces.

Let G be an algebraic group over K, $\kappa\colon T \times_K L \to G \times_K L$ a group homomorphism over L of a torus T over K which splits over L, Y the cocharacter group over L of T, and $e \in \mathbb{R} \otimes_{\mathbb{Z}} Y$. We associate a finite dimensional representation over K of G with a filtration over L as follows.

Let V be a finite dimensional representation space over K of G (sometimes called a G-representation over K in the following) and X the character group over L of T. We have a direct sum decomposition

$$L \otimes_K V = \bigoplus_{\chi \in X} V_\chi,$$

where V_χ is the vector space over L on which $T \times_K L$ acts by multiplication of a character χ via the group homomorphism κ. We define a filtration over L of V as

$$V^i_{\kappa,\, e} = \bigoplus_{\langle \chi, e \rangle \geq i} V_\chi.$$

Here the pairing $\langle \cdot, \cdot \rangle$ is the canonical $\mathbb{Z}$-valued one between the elements of X and those of Y, extended linearly to an $\mathbb{R}$-valued functional for the elements of $\mathbb{R} \otimes_{\mathbb{Z}} Y$.

For another representation space W over K of G, we have a similar decomposition

$$L \otimes_K W = \bigoplus_{\chi \in X} W_\chi.$$

Since a G-equivariant linear map of V to W sends each V_χ to W_χ, the linear map sends $V^i_{\kappa,\, e}$ to $W^i_{\kappa,\, e}$, hence it is filtered.

The representation on the tensor product $V \otimes_K W$ is defined diagonally, hence we have an equality

$$(V \otimes_K W)_\chi = \bigoplus_{\phi + \psi = \chi} V_\phi \otimes_L W_\psi.$$

Thus

$$\begin{aligned} (V \otimes_K W)^i_{\kappa,\, e} &= \bigoplus_{\langle \chi, e \rangle \geq i} (V \otimes_K W)_\chi \\ &= \bigoplus_{\langle \phi + \psi, e \rangle \geq i} V_\phi \otimes_L W_\psi \\ &= \sum_{j+q=i} V^j_{\kappa,\, e} \otimes_L W^q_{\kappa,\, e}. \end{aligned}$$

This means the following.

Lemma 1.7 (compatibility)**.** *The filtration on the G-representation $V \otimes_K W$ is the same as the filtration on the tensor product of vector spaces with filtrations.*

Remark 1.8. A functorial filtration on the finite dimensional representations of an algebraic group usually appears in this way ([11, IV 2.2.5 Proposition 3)], [3, Theorem 4.2.13]).

Let $\mathfrak{M}$ be a non-empty finite or countable infinite set of indices.

Definition 1.9 (category of vector spaces with multiple filtrations)**.** We denote by $\mathcal{C}(K, L, \mathfrak{M})$ the tensor category composed of the following objects and morphisms.

An object is a finite dimensional vector space V over K equipped with a family of filtrations $V_v^{\cdot}$ $(v \in \mathfrak{M})$ over L of V such that for except a finite number of indices v, the filtrations are trivial:

$$V_v^i = \begin{cases} L \otimes_K V & (i \leq 0) \\ 0 & (i > 0) \end{cases}$$

For vector spaces with multiple filtrations V and W as above, a morphism of V to W is a linear map over K of V to W which is filtered with respect to every index $v \in \mathfrak{M}$.

The tensor product $V \otimes W$ of V and W is a vector space $V \otimes_K W$ over K equipped with the tensor product of filtrations $V_v^{\cdot}$ and $W_v^{\cdot}$ for each $v \in \mathfrak{M}$.

Remark 1.10. The tensor product of objects is apparently associative and commutative. The associativity law and the commutativity law are compatible with those of the underlying vector spaces. A one-dimensional vector space over K equipped with the trivial filtrations for all $v \in \mathfrak{M}$ gives a unit object.

It is easy to check that the tensor category $\mathcal{C}(K, L, \mathfrak{M})$ with the above prescribed structure is a K-linear additive tensor category [4, Definition 1.1]. Moreover, we can prove without difficulty that $\mathcal{C}(K, L, \mathfrak{M})$ is rigid. It has kernels and cokernels, but is not abelian (e.g. [8, Example 1.4]).

Remark 1.11. The above filtration is of Faltings-Rapoport type (cf. [6], [9], and [1, §14.1]). The category $\mathcal{C}(K, L, \mathfrak{M})$ is the inductive limit of the categories n-$Fil_{L/K}$ in [1, §14.1] with respect to the number n of filtrations. The category n-$Fil_{L/K}$ is in turn a mixed variant of the categories Fil_K^L and Fil_K^n in [3, Definition 1.1.3 & Variant 1.1.5].

For an algebraic group G over K and for every $v \in \mathfrak{M}$, we select a group homomorphism $\kappa(v) \colon T_v \times_K L \to G \times_K L$ of a torus T_v over K which splits over L and an element $e(v)$ of the coefficient extension $\mathbb{R} \otimes_{\mathbb{Z}} Y(v)$ of the cocharacter group $Y(v)$ over L of T_v in such a way that for except a finite number of the indices v, we have $e(v) = 0$. With these data (G, κ, e) of an algebraic group G, a family κ of group homomorphisms of tori to G, and a family e of 'cocharacters', we can match each G-representation V over K with an object of $\mathcal{C}(K, L, \mathfrak{M})$ as follows.

The underlying vector space is V itself. For each $v \in \mathfrak{M}$, we associate V with the filtration $V_v^{\cdot}$ defined by $\kappa(v)$ and $e(v)$ as before:

$$V_v^i = V_{\kappa(v),\, e(v)}^i \quad (i \in \mathbb{R})$$

A G-equivariant linear map between representation spaces obviously gives a morphism in $\mathcal{C}(K, L, \mathfrak{M})$. Thus we obtain a functor $\iota_{G,\kappa,e}$ of the category $\mathrm{Rep}_K(G)$ of finite dimensional representation spaces over K of G to the category $\mathcal{C}(K, L, \mathfrak{M})$:

$$\mathrm{Rep}_K(G) \overset{\iota_{G,\kappa,e}}{\longrightarrow} \mathcal{C}(K, L, \mathfrak{M})$$

By Lemma 1.7, we see that the functor is a K-linear tensor functor [4, Definition 1.8].

Now we recall a certain full tensor subcategory of $\mathcal{C}(K, L, \mathfrak{M})$. Before that, to make sure of definitions, we remember the slope and the semi-stability of an object of $\mathcal{C}(K, L, \mathfrak{M})$.

Definition 1.12 (slope)**.** For a non-zero object V of $\mathcal{C}(K, L, \mathfrak{M})$, the *slope* $\mu(V)$ is a real number given by

$$\mu(V) = \sum_{v \in \mathfrak{M}} \frac{1}{\dim_K V} \sum_{w \in \mathbb{R}} w \dim_L \mathrm{gr}^w \left(V_v^{\cdot}\right),$$

where $\mathrm{gr}^{\cdot}$ is the graduation derived from filtration. For later use, we write μ_v for its partial sum with respect to the real index w:

$$\mu_v(V) = \frac{1}{\dim_K V} \sum_{w \in \mathbb{R}} w \dim_L \mathrm{gr}^w \left(V_v^{\cdot}\right)$$

Definition 1.13 (semi-stability)**.** A non-zero object V is *semi-stable* if it satisfies the condition that for any monomorphism $W \to V$ in $\mathcal{C}(K, L, \mathfrak{M})$ of a non-zero object W, we have $\mu(W) \leq \mu(V)$.

Remark 1.14. For V to be semi-stable, it is enough to meet the condition for all the strict subobjects (kernels) of V. In fact, if the underlying linear map of a morphism $U \to W$ is isomorphic, then we have $\mu(U) \leq \mu(W)$ [8, Lemma 1.8].

Remark 1.15. The slope in Definition 1.12 is a slope function [1, 3.1.1 Definition] and the slope filtration (HARDER-NARASIMHAN filtration) exists [1, 4.2.3 Theorem] (for a direct proof, see e.g. [3] or [5]).

Definition 1.16. The category $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is defined to be the full subcategory of $\mathcal{C}(K, L, \mathfrak{M})$ consisting of semi-stable objects of slope zero and of zero objects.

Theorem 1.17 (Faltings [6], Totaro [13])**.** *The category $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is a rigid abelian K-linear tensor subcategory. The forgetful tensor functor $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ of $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ to the tensor category of finite dimensional vector spaces over K is a fiber functor which makes the category $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ a neutral Tannakian category over K.*

Remark 1.18. In our previous paper [8], we have given a third proof that $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is Tannakian when the base field K is a so-called number field (and when some non-essential restriction is placed on the filtrations). As noted earlier in the paper [13, Remark p. 88] of TOTARO, we can prove the fact over any base field using the method in the same paper of TOTARO. A detailed proof over an arbitrary base field is indicated in [3, Variant 1.2.13]. For another approach, see [6].

Prior to proceeding ahead, we record two auxiliary facts. One says that the slope in Definition 1.12 is 'determinantal' [1, 8.2.1 Definition]:

Lemma 1.19. *Let V be a non-zero object of $\mathcal{C}(K, L, \mathfrak{M})$ and $n = \dim_K V$. We denote by $\det V$ the n-th exterior product $\bigwedge^n V$ with the quotient filtrations of the n-times tensor product $\bigotimes^n V$ of V. We have for each $v \in \mathfrak{M}$*

$$\mu_v(V) = \frac{1}{\dim_K V}\, \mu_v(\det V)\,.$$

Proof. See e.g. [3, Lemma 1.1.8 (iii)]. □

The other says that the functor $\iota_{G,\kappa,e}$ sends a cokernel to a cokernel:

Lemma 1.20. *Let V be a finite dimensional representation space over K of G and let W be a G-stable subspace over K of V. The quotient filtration over L on V/W as a strict quotient object (cokernel) of $\iota_{G,\kappa,e}(V)$ coincides with the filtration of $\iota_{G,\kappa,e}(V/W)$, the vector space V/W being regarded as a quotient G-representation over K.*

Proof. A proof is indirectly contained in [3, Definition 4.2.6 (iv)]. It is readily checked as follows.

Fix an index $v \in \mathfrak{M}$. Let $\tilde{u}_1, \ldots, \tilde{u}_d$ be a basis over L of $L \otimes_K W$ such that the torus T_v acts on $L\tilde{u}_j$ by a character $\tilde{\chi}(j)$ via $\kappa(v)\colon T_v \times_K L \to G \times_K L$ for each $j = 1, \ldots, d$. Let $u_1, \ldots, u_{n-d}$ be elements of $L \otimes_K V$ such that T_v acts on Lu_i by a character $\chi(i)$ via $\kappa(v)$ for each $i = 1, \ldots, n-d$ and such that together with $\tilde{u}_1, \ldots, \tilde{u}_d$ they form a basis over L of $L \otimes_K V$.

By definition, we have

$$V_v^q = \left(\bigoplus_{\langle \chi(i), e(v) \rangle \geq q} Lu_i \right) \oplus \left(\bigoplus_{\langle \tilde{\chi}(j), e(v) \rangle \geq q} L\tilde{u}_j \right),$$

hence

$$V_v^q + L \otimes_K W = \left(\bigoplus_{\langle \chi(i), e(v) \rangle \geq q} Lu_i \right) \oplus (L \otimes_K W)\,.$$

Denote by $\bar{u}_i$ $(i = 1, \ldots, n-d)$ the image of u_i through the quotient map

$$L \otimes_K V \to L \otimes_K (V/W).$$

The family $\bar{u}_1, \ldots, \bar{u}_{n-d}$ is a basis over L of $L \otimes_K (V/W)$ such that T_v acts on $L\bar{u}_i$ by the character $\chi(i)$. Hence

$$(V/W)_v^q = \bigoplus_{\langle \chi(i), e(v) \rangle \geq q} L\bar{u}_i.$$

This is the same as $(V_v^q + L \otimes_K W) / L \otimes_K W$, which is the q-th subspace of $L \otimes_K (V/W)$ as a strict quotient of $V_v^{\cdot}$. ☐

Remark 1.21. As is easily seen, the functor $\iota_{G,\kappa,e}$ also sends a kernel to a kernel.

2. Definition of a group

We define a linear algebraic group one of whose representation leads to a classical ROTH system (of linear inequalities). The relation of the group to the classical ROTH system is described in Section 3.

Let K be an arbitrary field and u, t, s_{ij} $(i, j = 1, 2)$ indeterminates. We consider s_{ij} the coefficient in the i-th row and the j-th column of a matrix of degree two. We denote respectively by $\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{m}}$, and SL_2 an additive group $\operatorname{Spec}(K[u])$, a multiplicative group $\operatorname{Spec}\left(K\left[t, t^{-1}\right]\right)$, and a special linear group $\operatorname{Spec}\left(K[s_{ij}] / (1 - \det(s_{ij}))\right)$ of degree two.

Let L be a (finite or infinite) GALOIS extension field of K, $\alpha \in L \setminus K$, and $\sigma \in \operatorname{Gal}(L/K)$ such that $\sigma(\alpha) \neq \alpha$. Let e, λ_+, λ_- be embeddings defined over L respectively of $\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{a}}$ into SL_2 given using the usual identifications $\mathbb{G}_{\mathrm{m}}(R) \simeq R^\times$ and $\mathbb{G}_{\mathrm{a}}(R) \simeq R$ for a K-algebra R by

$$\begin{aligned}
e(c) &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \qquad (c \in R^\times), \\
\lambda_+(c) &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \qquad (c \in R), \\
\lambda_-(c) &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \qquad (c \in R),
\end{aligned}$$

where $\beta = \sigma^{-1}(\alpha)$. We denote respectively by $T_{\alpha,\sigma}; U_+; U_-$ the subgroups over L of SL_2 which are the images of $e; \lambda_+; \lambda_-$. Denoting by l_{ij} $(i, j = 1, 2)$ the linear forms in s_{ij} given as

$$(l_{ij}) = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} (s_{ij}) \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix},$$

we have the subgroup $T_{\alpha,\sigma}$ as an affine subvariety over L of SL_2 which corresponds to the ideal in $L[s_{ij}]$ generated by

$$1 - l_{11} l_{22}, \; l_{12}, \; l_{21}.$$

The group U_+ corresponds to the ideal generated by

$$1 - l_{1\,1},\ 1 - l_{2\,2},\ l_{2\,1}$$

and U_- corresponds to the ideal generated by

$$1 - l_{1\,1},\ 1 - l_{2\,2},\ l_{1\,2}.$$

The image of the GALOIS conjugate $\sigma(e)$ of the embedding e of $\mathbb{G}_{\mathrm{m}}$ into SL_2 is the base change of $T_{\alpha,\sigma}$ by σ. We denote it by $\sigma(T_{\alpha,\sigma})$:

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m}} \times_K L & \xrightarrow{\ e\ } & T_{\alpha,\sigma} \subset \mathrm{SL}_2 \times_K L \\ \downarrow \sigma & & \downarrow \sigma \\ \mathbb{G}_{\mathrm{m}} \times_K L & \xrightarrow{\ \sigma(e)\ } & \sigma(T_{\alpha,\sigma}) \subset \mathrm{SL}_2 \times_K L \end{array}$$

Since the homomorphism $\sigma(e)$ is given by

$$\sigma(e)(c) = \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix}^{-1} \quad (c \in R),$$

the subgroup $\sigma(T_{\alpha,\sigma})$ of SL_2 corresponds to the ideal in $L\,[s_{i\,j}]$ generated by

$$1 - \sigma\,(l_{1\,1})\,\sigma\,(l_{2\,2})\,,\ \sigma\,(l_{1\,2})\,,\ \sigma\,(l_{2\,1})\,.$$

We use symbols $\sigma(U_-)$, $\sigma^{-1}(T_{\alpha,\sigma})$, $\tau(T_{\alpha,\sigma})$ ($\tau \in \mathrm{Gal}(L/K)$) and so forth in the same way.

Lemma 2.1. $\sigma\,(U_-) = U_+$

Proof. We have by the definition of β

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} &= \frac{1}{\beta - \alpha} \begin{pmatrix} -1 & -\beta \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \\ &= \frac{1}{\beta - \alpha} \begin{pmatrix} -\sigma(\alpha) + \beta & -\sigma(\beta) + \beta \\ \sigma(\alpha) - \alpha & \sigma(\beta) - \alpha \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{-\sigma(\alpha) + \sigma^{-1}(\alpha)}{\sigma^{-1}(\alpha) - \alpha} & 1 \\ \dfrac{\sigma(\alpha) - \alpha}{\sigma^{-1}(\alpha) - \alpha} & 0 \end{pmatrix}. \end{aligned}$$

Putting

$$\gamma = \frac{\sigma^{-1}(\alpha) - \sigma(\alpha)}{\sigma^{-1}(\alpha) - \alpha}, \qquad \delta = \frac{\sigma(\alpha) - \alpha}{\sigma^{-1}(\alpha) - \alpha},$$

we can rewrite the above expression as

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} &= \begin{pmatrix} \gamma & 1 \\ \delta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

We see that

$$\begin{aligned}
&\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \\
&\cdot \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \\
= &\begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \left(\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \right)^{-1} \\
= &\begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix} \begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & \delta^{-1} \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix} \\
= &\begin{pmatrix} 1 & \delta^{-1}u \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix}^{-1} \\
= &\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \delta^{-1}u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1},
\end{aligned}$$

which shows the claim. □

Let γ and δ be the elements of L defined in the proof of Lemma 2.1 and let A be an L-valued point of U_+ given by

$$A = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1}.$$

Lemma 2.2. *The relations*

(2.1) $$\sigma(e) = \mathrm{Int}\left(A^{-1}\right) \circ e \circ (\textit{inverse in } \mathbb{G}_{\mathrm{m}}),$$

(2.2) $$e = \mathrm{Int}\,(A) \circ \sigma(e) \circ (\textit{inverse in } \mathbb{G}_{\mathrm{m}}),$$

and

$$\sigma\left(T_{\alpha,\sigma} U_-\right) = T_{\alpha,\sigma} U_+$$

hold, where the symbol Int *is the (left) conjugation action in* SL_2*. As a diagram, we have the following:*

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m}} \times_K L & \xrightarrow{\ e\ } & T_{\alpha,\sigma} U_+ \\ {\scriptstyle .^{-1}}\downarrow & & \uparrow{\scriptstyle \mathrm{Int}(A)} \\ \mathbb{G}_{\mathrm{m}} \times_K L & \xrightarrow{\ \sigma(e)\ } & \sigma\left(T_{\alpha,\sigma} U_-\right) \end{array}$$

Proof. With the same notation as in the proof of Lemma 2.1, we see that

$$\begin{aligned} &\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \\ &\cdot \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \\ =& \begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & \delta^{-1} \end{pmatrix} \\ =& \begin{pmatrix} 1 & \gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \\ =& \begin{pmatrix} 1 & \gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This implies the relation (2.1), equivalently the relation (2.2), and in particular that

$$\sigma\left(T_{\alpha,\sigma}\right) \subset T_{\alpha,\sigma} U_+$$

and

$$T_{\alpha,\sigma} \subset \sigma\left(T_{\alpha,\sigma}\right) U_+ = \sigma\left(T_{\alpha,\sigma}\right) \sigma\left(U_-\right) = \sigma\left(T_{\alpha,\sigma} U_-\right).$$

□

Let X be the character group of the torus $T_{\alpha,\sigma}$. For any character ϕ of $T_{\alpha,\sigma}$, we write $\sigma(\phi)$ for the homomorphism determined by the following diagram:

$$\begin{array}{ccccc} \mathbb{G}_{\mathrm{m}} \times_K L & \xrightarrow{\ e\ } & T_{\alpha,\sigma} & \xrightarrow{\ \phi\ } & \mathbb{G}_{\mathrm{m}} \times_K L \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathbb{G}_{\mathrm{m}} \times_K L & \xrightarrow{\ \sigma(e)\ } & \sigma(T_{\alpha,\sigma}) & \xrightarrow{\ \sigma(\phi)\ } & \mathbb{G}_{\mathrm{m}} \times_K L \end{array}$$

The map $\sigma(\phi)$ is a character of a base change $\sigma(T_{\alpha,\sigma})$ of the torus $T_{\alpha,\sigma}$. Denoting in general by $\langle \cdot, \cdot \rangle$ the canonical $\mathbb{Z}$-valued pairing between a character and a cocharacter of a torus, we see that

$$\langle \phi, e \rangle = \langle \sigma(\phi), \sigma(e) \rangle \quad \text{for all } \phi \in X.$$

We write $\sigma(X)$ for the character group of the torus $\sigma(T_{\alpha,\sigma})$.

Let χ be a generator of X determined by

$$\chi \circ e = (\text{identity of } \mathbb{G}_{\mathrm{m}})\,.$$

We have

$$1 = \langle \chi, e \rangle = \langle \sigma(\chi), \sigma(e) \rangle.$$

The character $\sigma(\chi)$ is a generator of $\sigma(X)$ such that

$$\sigma(\chi) \circ \sigma(e) = (\text{identity of } \mathbb{G}_{\mathrm{m}})\,.$$

Since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t^{-2}u & 1 \end{pmatrix},$$

the group U_- is the unipotent subgroup over L of SL_2 which corresponds to the character $(-2)\chi$ (additive notation) of the maximal torus $T_{\alpha,\sigma}$.

Lemma 2.3. *When $\sigma^2(\alpha) \neq \alpha$, the tori $T_{\alpha,\sigma}$ and $\sigma(T_{\alpha,\sigma})$ generate a* BOREL *subgroup $T_{\alpha,\sigma}U_+ = \sigma\left(T_{\alpha,\sigma}U_-\right)$ of $\mathrm{SL}_2 \times_K L$.*

Proof. For an algebraic closure $\bar{L}$ of L and $c \in \bar{L}^\times = \bar{L} \setminus \{0\}$, let B and C be $\bar{L}$-valued points respectively of $T_{\alpha,\sigma}$ and $\sigma(T_{\alpha,\sigma})$ determined by the matrices

$$B = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1}$$

and

$$C = \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \sigma(\alpha) & \sigma(\beta) \\ -1 & -1 \end{pmatrix}^{-1}.$$

From the relation (2.1), we see that

$$C = A^{-1}B^{-1}A.$$

We have on the other hand

$$\begin{aligned} &BA^{-1}B^{-1} \\ &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & \gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & c^2\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \end{aligned}$$

and

$$B^{-1}A^{-1}B = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & c^{-2}\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1}.$$

We obtain

$$\begin{aligned} BCB^{-1}C^{-1} &= \left(BA^{-1}B^{-1}\right) A \left(B^{-1}A^{-1}B\right) A \\ &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & c^2\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \\ &\qquad \cdot \begin{pmatrix} 1 & c^{-2}\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \left(c - c^{-1}\right)^2 \gamma\delta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1}. \end{aligned}$$

Thus the commutator group of $T_{\alpha,\sigma}$ with $\sigma\left(T_{\alpha,\sigma}\right)$ contains the unipotent group $U_+ = \sigma\left(U_-\right)$. □

Corollary 2.4. *The special linear group* $\mathrm{SL}_2 \times_K L$ *is generated by the three tori* $\sigma^{-1}(T_{\alpha,\sigma})$, $T_{\alpha,\sigma}$, *and* $\sigma(T_{\alpha,\sigma})$ *if* $\sigma^2(\alpha) \neq \alpha$.

Proof. Applying σ^{-1} respectively to $T_{\alpha,\sigma}$, $\sigma(T_{\alpha,\sigma})$, and $T_{\alpha,\sigma}U_+ = \sigma\left(T_{\alpha,\sigma}U_-\right)$, we see that when $\sigma^2(\alpha) \neq \alpha$, the tori $\sigma^{-1}(T_{\alpha,\sigma})$ and $T_{\alpha,\sigma}$ generate a Borel subgroup $\sigma^{-1}\left(T_{\alpha,\sigma}U_+\right) = T_{\alpha,\sigma}U_-$ of $\mathrm{SL}_2 \times_K L$. Since $\mathrm{SL}_2 \times_K L$ is generated by U_+ and U_-, the conclusion holds. □

When $\sigma^2(\alpha) = \alpha$, that is, when $\gamma = 0$, the L-valued point A of SL_2 equals the neutral element given by the identity matrix, hence we get $T_{\alpha,\sigma} = \sigma(T_{\alpha,\sigma})$. In this case, the torus $T_{\alpha,\sigma}$ is defined over the fixed field of $\sigma \in \mathrm{Gal}(L/K)$. We calculate its defining equation.

To ease notation, we put

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Lemma 2.5. *The torus* $T_{\alpha,\sigma}$ *is naturally identified with a one-dimensional torus*

$$\mathrm{Spec}\left(L[r,s]/\left(1 - (\alpha r - s)\left(\sigma^{-1}(\alpha) r - s\right)\right)\right),$$

the function $\alpha r - s$ *being considered a generator of its character group.*

When $\sigma^2(\alpha) = \alpha$, *letting* N *be the* Galois *closure of the field generated by* α *over* K *and* k *the fixed field of* σ *in* N, *the torus* $T_{\alpha,\sigma}$ *is defined and anisotropic over* k.

Proof. The latter part is immediate from the former one. The torus $T_{\alpha,\sigma}$ is anisotropic, for otherwise the character $\alpha r - s$ became rational over K, hence $\alpha \in K$. We give the proof of the former part.

By the definition of linear forms l_{ij}, we have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}.$$

On the congruence $l_{12} \equiv l_{21} \equiv 0$, we see

$$\begin{aligned} \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} &\equiv \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} l_{11} & 0 \\ 0 & l_{22} \end{pmatrix} \\ &\equiv \begin{pmatrix} l_{11} & l_{22} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} &\equiv \begin{pmatrix} -1 & -1 \end{pmatrix} \begin{pmatrix} l_{11} & 0 \\ 0 & l_{22} \end{pmatrix} \\ &\equiv \begin{pmatrix} l_{11} & l_{22} \end{pmatrix} (-1). \end{aligned}$$

We get

$$\begin{aligned} \begin{pmatrix} p & q \end{pmatrix} &\equiv \begin{pmatrix} l_{11} & l_{22} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \\ &\equiv \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \\ &\equiv \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} -\alpha^2 & -\beta^2 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \\ &\equiv \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} -\alpha^2 & -\beta^2 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} -1 & -\beta \\ 1 & \alpha \end{pmatrix} \frac{1}{\beta - \alpha} \\ &\equiv \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} -(\alpha + \beta) & -\alpha\beta \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} l_{11} & l_{22} \end{pmatrix} &\equiv \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} (-1) \\ &\equiv \begin{pmatrix} \alpha r - s & \beta r - s \end{pmatrix} (-1). \end{aligned}$$

Hence we obtain in particular

$$l_{11} l_{22} \equiv (\alpha r - s)(\beta r - s) = (\alpha r - s) \left(\sigma^{-1}(\alpha) r - s \right).$$

Since the ideal of $T_{\alpha,\sigma}$ is generated by $l_{12},\ l_{21},\ 1 - l_{11} l_{22}$, we see the result. □

Remark 2.6. On the torus $T_{\alpha,\sigma}$, the functions $p = s_{11}$ and $q = s_{12}$ are respectively identified with $s - \left(\alpha + \sigma^{-1}(\alpha)\right) r$ and $-\alpha\, \sigma^{-1}(\alpha) r$, which are seen (or have been seen) in the proof of Lemma 2.5.

Definition 2.7 (group leading to the ROTH inequality)**.** We denote by $\breve{G}$ the group closure of the union of GALOIS conjugates $\tau(T_{\alpha,\sigma})$ $(\tau \in \mathrm{Gal}(L/K))$ in $\mathrm{SL}_2 \times_K L$.

Remark 2.8. The group $\breve{G}$ is the ZARISKI closure of the group generated by $\tau(T_{\alpha,\sigma})$ ($\tau \in \mathrm{Gal}(L/K)$) and defined over the base field K [2, 2.1].

As we know, when $\sigma^2(\alpha) \neq \alpha$, we have $\breve{G} = \mathrm{SL}_2$. When α is quadratic over K, we see from Lemma 2.5 that $\breve{G}$ is a one-dimensional anisotropic torus over K. In Section 4, we shall explicitly determine the group $\breve{G}$ in another case.

3. Representation of the group

We associate with filtrations as in Section 1 the representation spaces of the group defined in Section 2. We prove that a filtered homomorphism between them is nothing but an equivariant homomorphism between the representation spaces. We also prove that the associated filtrations are semi-stable of slope zero. Thus the tensor category of finite dimensional representations of the group defined in Section 2 is identified with a full subcategory of the tensor category of vector spaces with semi-stable filtrations of slope zero. The filtered vector space derived from a classical ROTH system is regarded as a representation space of the group in such a way.

In this section, we assume that the index set $\mathfrak{M}$ consists of a single element, say ∞: $\mathfrak{M} = \{\infty\}$. In addition, we omit the subscript $v = \infty$ and the index set $\mathfrak{M}$ from the symbols for simplicity.

To the triple of the group $\breve{G}$, the inclusion map $\kappa = \mathrm{incl}\colon T_{\alpha,\sigma} \hookrightarrow \breve{G} \times_K L$, and the cocharacter $e\colon \mathbb{G}_{\mathrm{m}} \times_K L \to T_{\alpha,\sigma}$ given in Section 2, apply the construction of the tensor functor $\iota_{\breve{G},\kappa,e}\colon \mathrm{Rep}_K(\breve{G}) \to \mathcal{C}(K,L) = \mathcal{C}(K,L,\mathfrak{M})$ in Section 1. Recall that for a finite dimensional representation space V over K of $\breve{G}$, we have defined a filtration over L of $\iota_{\breve{G},\kappa,e}(V)$ as

$$V^i = V^i_\infty = V^i_{\kappa,e} = \bigoplus_{\langle \phi, e\rangle \geq i} V_\phi \quad (i \in \mathbb{R}),$$

where V_ϕ is the subspace over L of $L \otimes_K V$ on which $T_{\alpha,\sigma}$ acts by multiplication of a character ϕ via the map $\kappa = \mathrm{incl}$.

Example 3.1 (representation corresponding to a classical ROTH system). Put $r = s_{21}$ and $s = s_{22}$. The indeterminate s_{21} or s_{22} is a function on SL_2 defined as the matrix coefficient either in the second row and the first column or in the second row and the second column as in Section 2. Let $\breve{V}$ be the vector space generated by r and s over K in the ring of functions over K on SL_2. By right translation of SL_2, the vector space $\breve{V}$ becomes a representation space of $\breve{G}$, which is a closed subgroup of SL_2. Since the action of the torus $T_{\alpha,\sigma}$ is given by for $c \in R^\times \simeq T_{\alpha,\sigma}(R)$

$$\begin{pmatrix} r & s \end{pmatrix} \longmapsto \begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1},$$

where R is a K-algebra and $\beta = \sigma^{-1}(\alpha)$, we see that

$$\check{V}_\chi = L(\alpha r - s)$$

and

$$\check{V}_{-\chi} = L(\beta r - s) = L\left(\sigma^{-1}(\alpha) r - s\right).$$

The character χ is the same χ as the one in Section 2 such that $\langle \chi, e \rangle = 1$. Hence the filtration of $\check{V}$ is defined as

$$F_\alpha^i \check{V} = \begin{cases} L \otimes_K \check{V} & \text{for } i \leq -1 \\ L(\alpha r - s) & \text{for } -1 < i \leq 1 \\ 0 & \text{for } i > 1. \end{cases}$$

This is the filtration derived from a classical ROTH system. Note that the filtration $F_\alpha^{\cdot} \check{V}$ does not depend on the choice of the element $\sigma \in \mathrm{Gal}(L/K)$.

For an L-valued point g of $T_{\alpha,\sigma}$, recall that its GALOIS conjugate $\sigma(g)$ by $\sigma \in \mathrm{Gal}(L/K)$ is an L-valued point of the base change $\sigma(T_{\alpha,\sigma})$:

$$\begin{array}{ccc} \operatorname{Spec} L & \xrightarrow{g} & T_{\alpha,\sigma} \\ \downarrow \sigma & & \downarrow \sigma \\ \operatorname{Spec} L & \xrightarrow{\sigma(g)} & \sigma(T_{\alpha,\sigma}) \end{array}$$

If x is an element of V_ϕ, then by definition

$$gx = \phi(g) x.$$

Since the representation in $\mathrm{GL}(V)$ of $\check{G}$ is defined over K and the action of $\mathrm{Gal}(L/K)$ on $L \otimes_K V$ is induced from the action on the coefficient field L, we have

$$\begin{aligned} \sigma(g)\sigma(x) = \sigma(gx) &= \sigma\left(\phi(g)x\right) \\ &= \sigma\left(\phi(g)\right)\sigma(x) \\ &= \sigma(\phi)\left(\sigma(g)\right)\sigma(x). \end{aligned}$$

This says that the GALOIS conjugate $\sigma\left(V_\phi\right)$ equals the L-vector space $V_{\sigma(\phi)}$ on which $\sigma(T_{\alpha,\sigma})$ acts by multiplication of a character $\sigma(\phi)$:

$$\sigma\left(V_\phi\right) = V_{\sigma(\phi)} \tag{3.1}$$

We introduce a useful quantity.

Definition 3.2. For a non-zero element x of $L \otimes_K V$, let

$$m(x) = \max\left\{ i \mid V^i \ni x \right\} \quad \text{and} \quad m(0) = +\infty.$$

From the definition, we see immediately that

$$m(x+y) \geq \min\{m(x), m(y)\}$$

and that if $m(x) \neq m(y)$, then

$$m(x+y) = \min\{m(x), m(y)\}.$$

Proposition 3.3. *We have*

$$m(\sigma(x)) = -\langle \phi, e\rangle \quad \textit{for any} \quad x \in V_\phi \setminus \{0\},$$

where σ is the element of $\mathrm{Gal}(L/K)$ *used to define the torus $T_{\alpha,\sigma}$ in Section 2.*

Proof. First, we assume that $\sigma^2(\alpha) \neq \alpha$. We know that $\breve{G} = \mathrm{SL}_2$. As we have seen in Section 2, a unipotent subgroup U_- corresponds to the character $(-2)\chi$ of the maximal torus $T_{\alpha,\sigma}$, where χ is the character such that $\langle \chi, e\rangle = 1$. As is well-known, the group U_- sends an element x of V_ϕ to an affine space $x + \bigoplus_{k\in\mathbb{Z},k>0} V_{\phi+(-2)k\chi}$. Applying $\sigma \in \mathrm{Gal}(L/K)$, we have

$$\sigma(U_-)(L)\,\sigma(x) \;\subset\; \sigma(x) + \bigoplus_{k\in\mathbb{Z},k>0} V_{\sigma(\phi)+(-2)k\,\sigma(\chi)}.$$

In particular, for the L-valued point A of $\sigma(U_-)$ in Section 2, we get an expression

$$A^{-1}\sigma(x) = \sigma(x) + \sum_{k\in\mathbb{Z},k>0} \sigma(x_k), \quad x_k \in V_{\phi+(-2)k\chi}.$$

We identify $c \in L^\times = L\setminus\{0\}$ with an L-valued point of $\mathbb{G}_\mathrm{m}$ given in terms of pullback of functions by

$$K\left[t, t^{-1}\right] \ni t \mapsto c.$$

Let $w = \langle \sigma(\phi), \sigma(e)\rangle$, which is equal to $\langle \phi, e\rangle$. We have

$$\begin{aligned}
&\left(\sigma(e)\left(c^{-1}\right)A^{-1}\right)\sigma(x) = \sigma(e)\left(c^{-1}\right)\left(A^{-1}\sigma(x)\right)\\
&= \left(c^{-1}\right)^{\langle\sigma(\phi),\sigma(e)\rangle}\sigma(x) + \sum_{k\in\mathbb{Z},k>0}\left(c^{-1}\right)^{\langle\sigma(\phi)+(-2)k\,\sigma(\chi),\sigma(e)\rangle}\sigma(x_k)\\
&= c^{-w}\sigma(x) + \sum_{k\in\mathbb{Z},k>0} c^{-w+2k}\sigma(x_k).
\end{aligned}$$

From the relation (2.2) between e and $\sigma(e)$ in Section 2, we obtain

$$\begin{aligned}
e(c)\,\sigma(x) &= \left(A\,\sigma(e)\left(c^{-1}\right)A^{-1}\right)\sigma(x)\\
&= A\left(\sigma(e)\left(c^{-1}\right)A^{-1}\sigma(x)\right)\\
&= c^{-w}A\,\sigma(x) + \sum_{k\in\mathbb{Z},k>0} c^{-w+2k}A\,\sigma(x_k).
\end{aligned}$$

Since $A\,\sigma(x) \neq 0$ if $x \neq 0$, this means that

$$\sigma(x) \in V^{-w} \quad \text{and} \quad \sigma(x) \notin V^{i}\ (i > -w),$$

hence

$$m\left(\sigma(x)\right) = -w = -\langle \phi, e\rangle.$$

Next, assume that $\sigma^2(\alpha) = \alpha$. In this case, the L-valued point A equals the identity, hence we know by the relation (2.2)

$$e = \sigma(e) \circ (\text{inverse in } \mathbb{G}_{\mathrm{m}})\,.$$

Using the same notation as above, we have

$$\begin{aligned} e(c)\,\sigma(x) &= \sigma(e)\Big(c^{-1}\Big)\,\sigma(x) \\ &= \Big(c^{-1}\Big)^{\langle \sigma(\phi), \sigma(e)\rangle}\,\sigma(x) = c^{-w}\sigma(x). \end{aligned}$$

Thus we get the same conclusion as in the first case. □

Lemma 3.4. *Any one-dimensional representation defined over K of $\breve{G}$ is trivial.*

Proof. Let V be a one-dimensional representation space over K of $\breve{G}$. The torus $T_{\alpha,\sigma}$ acts on $L \otimes_K V$ via a character ϕ, in other words, $L \otimes_K V = V_\phi$. We have

$$m(x) = \langle \phi, e\rangle \quad \text{for all} \ \ x \in L \otimes_K V \setminus \{0\}.$$

Since $\sigma\left(L \otimes_K V\right) = L \otimes_K V$, we get from Proposition 3.3 that

$$m(x) = -\langle \phi, e\rangle \quad \text{for all} \ \ x \in L \otimes_K V \setminus \{0\}.$$

This forces $\phi = 0$.

For an L-valued point g of $T_{\alpha,\sigma}$, $\tau \in \mathrm{Gal}(L/K)$, and $x \in L \otimes_K V$, we see that

$$\begin{aligned} \tau(g)x &= \tau(g)\,\tau\Big(\tau^{-1}(x)\Big) \\ &= \tau\left(g\,\tau^{-1}(x)\right) \\ &= \tau\left(\tau^{-1}(x)\right) = x. \end{aligned}$$

This means that the base change $\tau(T_{\alpha,\sigma})$ of the torus $T_{\alpha,\sigma}$ is contained in the stabilizer of any element of $L \otimes_K V$. Hence $\breve{G}$ itself is the stabilizer, that is, the group $\breve{G}$ acts trivially on V, for $\tau(T_{\alpha,\sigma})$ $(\tau \in \mathrm{Gal}(L/K))$ generate a dense subgroup of $\breve{G}$ and stabilizers are closed in general. □

Lemma 3.5. *For any finite dimensional non-zero representation space V over K of $\breve{G}$, we have $\mu(\iota_{\breve{G},\kappa,e}(V)) = \mu_\infty(\iota_{\breve{G},\kappa,e}(V)) = 0$.*

Proof. Once we know $\breve{G}$ is reductive, Lemma 3.4 says that $\breve{G}$ has anisotropic center (cf. the discussion at the beginning of [3, Chapter V]), so we can resort to [3, Remark 5.1.3]. In general, we are able to prove our present lemma as follows. The essence of the following proof is the same as the essence of the proof of [3, Remark 5.1.3].

Let n be the dimension over K of V. From Lemma 1.19 and Lemma 1.20, we see

$$\begin{aligned}\mu(\iota_{\breve{G},\kappa,e}(V)) &= \frac{1}{n}\,\mu(\det \iota_{\breve{G},\kappa,e}(V)) \\ &= \frac{1}{n}\,\mu\left(\operatorname{Coker}\left(\iota_{\breve{G},\kappa,e}(W) \to \iota_{\breve{G},\kappa,e}\left(\bigotimes^{n} V\right)\right)\right) \\ &= \frac{1}{n}\,\mu\left(\iota_{\breve{G},\kappa,e}\left(\bigwedge^{n} V\right)\right).\end{aligned}$$

Here W in the second line of the displayed expression is an appropriate $\breve{G}$-stable subspace of $\bigotimes^n V$. Due to Lemma 3.4, the action of $\breve{G}$ is trivial on the one-dimensional representation space $\bigwedge^n V$ over K. By the definition of filtration on representation spaces, we have

$$\left(\bigwedge^{n} V\right)^{i} = \begin{cases} L \otimes_K \left(\bigwedge^n V\right) & (i \leq 0) \\ 0 & (i > 0), \end{cases}$$

hence the result. □

Lemma 3.6. *To an arbitrary one-dimensional vector subspace W over K of a $\breve{G}$-representation V over K, attach the sub-filtration over L of $\iota_{\breve{G},\kappa,e}(V)$. We have $\mu(W) \leq 0$.*

Proof. In the coefficient extension $L \otimes_K V$, a non-zero vector $w \in W$ is written

$$w = w_1 + \cdots + w_r, \quad w_i \in V_{\psi(i)} \setminus \{0\},$$

where $V_{\psi(i)}$ is as in Section 2 the subspace over L of $L \otimes_K V$ on which the torus $T_{\alpha,\sigma}$ acts via a character $\psi(i)$. We may assume that the characters $\psi(i)$ are pairwise distinct. By the definition of the sub-filtration and the quantity $m(\cdot)$ in Definition 3.2, we have

$$\mu(W) = m(w) = \min_{1 \leq i \leq r} \langle \psi(i), e \rangle .$$

Applying $\sigma \in \mathrm{Gal}(L/K)$, we also have

$$w = \sigma(w_1) + \cdots + \sigma(w_r).$$

From Proposition 3.3, we see that

$$m(\sigma(w_i)) = -\langle \psi(i), e \rangle .$$

We get

$$\begin{aligned} m(w) &= m\left(\sigma\left(w_1\right)+\cdots+\sigma\left(w_r\right)\right) \\ &= \min_{1\le i\le r} m\left(\sigma\left(w_i\right)\right) \\ &= -\max_{1\le i\le r}\left\langle \psi(i), e\right\rangle . \end{aligned}$$

This is possible only when $0 \ge m(w) = \mu(W)$. □

Proposition 3.7. *For any finite dimensional representation space V over K of $\breve{G}$, the filtered vector space $\iota_{\breve{G},\kappa,e}(V)$ is semi-stable of slope zero, hence the functor $\iota_{\breve{G},\kappa,e}$ factors through $\mathcal{C}_0^{\mathrm{ss}}(K,L) = \mathcal{C}_0^{\mathrm{ss}}(K,L,\mathfrak{M})$.*

Proof. We are going to prove that for any non-zero vector subspace W over K of V with the sub-filtration of $\iota_{\breve{G},\kappa,e}(V)$, we have $\mu(W) \le 0$.

Let $d = \dim_K W$. When we consider $\bigotimes^d W$ the d-times tensor product in $\mathcal{C}(K,L)$, the natural map

$$\bigotimes^d W \to \bigotimes^d \iota_{\breve{G},\kappa,e}(V) \simeq \iota_{\breve{G},\kappa,e}\left(\bigotimes^d V\right)$$

is of course a morphism in $\mathcal{C}(K,L)$. By the definition of quotient filtration and by Lemma 1.20, the induced map

$$f\colon \det W \to \bigwedge^d \iota_{\breve{G},\kappa,e}(V) \simeq \iota_{\breve{G},\kappa,e}\left(\bigwedge^d V\right)$$

is also a morphism in $\mathcal{C}(K,L)$. Since the underlying linear map of the morphism

$$\det W \to \operatorname{Im} f$$

is an isomorphism, we have $\mu(\det W) \le \mu(\operatorname{Im} f)$. (See e.g. [8, Lemma 1.8]. In our present case, we can prove that $\det W \simeq \operatorname{Im} f$.) Thanks to Lemma 1.19 and Lemma 3.6, we obtain

$$\mu(W) = \frac{1}{d}\,\mu(\det W) \le \frac{1}{d}\,\mu(\operatorname{Im} f) \le 0.$$

□

Let V and W be the underlying vector spaces over K of objects in $\mathcal{C}(K,L)$. Remember that a linear map f over K of V to W is filtered if and only if

$$f\left(V^i\right) \subset W^i$$

for all $i \in \mathbb{R}$. In terms of the quantity $m(\cdot)$, the map f is filtered if and only if

$$m(x) \le m\left(f(x)\right)$$

for all $x \in L \otimes_K V$.

Theorem 3.8. *The functor* $\iota_{\breve{G},\kappa,e}\colon \mathrm{Rep}_K(\breve{G}) \to \mathcal{C}_0^{\mathrm{ss}}(K,L)$ *is fully faithful.*

Proof. Let $f\colon V \to W$ be a filtered linear map over K between finite dimensional representation spaces. Take any $x \in V_\phi \setminus \{0\}$. We know from Proposition 3.3 that

$$m\left(\sigma(x)\right) = -\langle \phi, e\rangle.$$

We have an expression

$$f(x) = \sum_{\psi \in X} y_\psi, \quad y_\psi \in W_\psi,$$

where X is the character group of the torus $T_{\alpha,\sigma}$. On the assumption that f is filtered, we have

$$y_\psi = 0 \quad \text{if } \langle \psi, e\rangle < \langle \phi, e\rangle.$$

Suppose that $f(x) \neq 0$ and let

$$M = \max\left\{\langle \psi, e\rangle \;\middle|\; y_\psi \neq 0\right\}.$$

We see that

$$M \geq \langle \phi, e\rangle.$$

Since we have by Proposition 3.3

$$m\left(\sigma\left(y_\psi\right)\right) = -\langle \psi, e\rangle$$

for $y_\psi \neq 0$, we get

$$\begin{aligned} m\left(\sigma\left(f(x)\right)\right) &= m\left(\sum_{\psi\in X} \sigma\left(y_\psi\right)\right) = \min\left\{m\left(\sigma\left(y_\psi\right)\right) \;\middle|\; y_\psi \neq 0\right\} \\ &= \min\left\{-\langle \psi, e\rangle \;\middle|\; y_\psi \neq 0\right\} = -\max\left\{\langle \psi, e\rangle \;\middle|\; y_\psi \neq 0\right\} \\ &= -M. \end{aligned}$$

Thus we obtain

$$\begin{aligned} -\langle \phi, e\rangle &= m\left(\sigma(x)\right) \\ &\leq m\left(f\left(\sigma(x)\right)\right) = m\left(\sigma\left(f(x)\right)\right) = -M, \end{aligned}$$

hence

$$M \leq \langle \phi, e\rangle,$$

for f is filtered and defined over K. Consequently, we see that $M = \langle \phi, e\rangle$. Thus

$$y_\psi = 0 \quad \text{if } \langle \psi, e\rangle > \langle \phi, e\rangle.$$

This implies that

$$f(x) = y_\phi \in W_\phi \quad \text{for all } x \in V_\phi,$$

that is, that the map f commutes with the action of $T_{\alpha,\sigma}$. Since f is defined over K, the map commutes with all Galois conjugates $\tau(T_{\alpha,\sigma})$ ($\tau \in \mathrm{Gal}(L/K)$), and so with $\breve{G}$. □

4. The group in characteristic zero

We shall determine an explicit form of the group defined in Section 2 when the characteristic of the base field is zero, using the one-to-one correspondence between LIE algebras and LIE groups.

We calculate the LIE algebra $\mathfrak{t}_{\alpha,\sigma}$ of the torus $T_{\alpha,\sigma}$ defined in Section 2 as a subalgebra of the LIE algebra $\mathfrak{sl}_2$ of the special linear group SL_2 of degree two. We identify the LIE algebra $\mathfrak{gl}_2$ of the general linear group GL_2 of degree two with the LIE algebra of all matrices of degree two and regard $\mathfrak{sl}_2$ as the subalgebra of trace zero. The element of $\mathfrak{gl}_2$ defined as the partial differentiation by s_{ij} at the unity as one of the coordinate functions $(s_{ij})_{i,j}$ on GL_2 is identified with the matrix with the coefficient in the i-th row and the j-th column one and with the others zero.

Lemma 4.1. *The* LIE *algebra* $\mathfrak{t}_{\alpha,\sigma}$ *of* $T_{\alpha,\sigma}$ *is the one-dimensional subspace of* $\mathfrak{sl}_2$ *generated by the element*

$$\begin{pmatrix} -\left(\alpha+\sigma^{-1}(\alpha)\right) & -2\alpha\,\sigma^{-1}(\alpha) \\ 2 & \alpha+\sigma^{-1}(\alpha) \end{pmatrix}.$$

Proof. Recall that the pullbacks of functions s_{ij} on SL_2 (or GL_2) by the morphism $e\colon \mathbb{G}_\mathrm{m}\times_K L \to T_{\alpha,\sigma} \hookrightarrow \mathrm{SL}_2\times_K L$ ($\hookrightarrow \mathrm{GL}_2\times_K L$) are written in matrix notation

$$(e^* s_{ij}) = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1},$$

where $\beta=\sigma^{-1}(\alpha)$. Differentiating by t, we have

$$\left(\frac{d}{dt}\,e^* s_{ij}\right) = \begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -t^{-2} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1}.$$

By calculation, we get

$$\begin{aligned} &\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \alpha & -\beta \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -\beta \\ 1 & \alpha \end{pmatrix}\frac{1}{\beta-\alpha} \\ &= \begin{pmatrix} -\alpha-\beta & -2\alpha\beta \\ 2 & \beta+\alpha \end{pmatrix}\frac{1}{\beta-\alpha} \\ &= \begin{pmatrix} -\left(\alpha+\sigma^{-1}(\alpha)\right) & -2\alpha\,\sigma^{-1}(\alpha) \\ 2 & \alpha+\sigma^{-1}(\alpha) \end{pmatrix}\frac{1}{\sigma^{-1}(\alpha)-\alpha}. \end{aligned}$$

□

We see similarly the following:

Corollary 4.2. *For any $\tau \in \mathrm{Gal}(L/K)$, the* Lie *algebra of the torus $\tau(T_{\alpha,\sigma})$ is the one-dimensional subspace of $\mathfrak{sl}_2$ generated by the element*

$$\begin{pmatrix} -\tau\big(\alpha+\sigma^{-1}(\alpha)\big) & -2\tau\big(\alpha\,\sigma^{-1}(\alpha)\big) \\ 2 & \tau\big(\alpha+\sigma^{-1}(\alpha)\big) \end{pmatrix},$$

which is just the Galois *conjugate $\tau(\mathfrak{t}_{\alpha,\sigma})$ in $\mathfrak{sl}_2$ of the subalgebra $\mathfrak{t}_{\alpha,\sigma}$.*

To determine the Lie algebra $\breve{\mathfrak{g}}$ of the group $\breve{G}$ defined in Section 2 as the group closure of the union of $\tau(T_{\alpha,\sigma})$ $(\tau \in \mathrm{Gal}(L/K))$, put for an element τ of $\mathrm{Gal}(L/K)$

$$a = \alpha + \sigma^{-1}(\alpha), \quad b = \alpha\,\sigma^{-1}(\alpha), \quad c = \tau(a), \quad d = \tau(b)$$

and set

$$\begin{pmatrix} -a & -2b \\ 2 & a \end{pmatrix} = Z \in \mathfrak{t}_{\alpha,\sigma}, \quad \begin{pmatrix} -c & -2d \\ 2 & c \end{pmatrix} = W \in \tau(\mathfrak{t}_{\alpha,\sigma})\,.$$

We have as a (simple) product of matrices

$$\begin{pmatrix} -a & -2b \\ 2 & a \end{pmatrix}\begin{pmatrix} -c & -2d \\ 2 & c \end{pmatrix} = \begin{pmatrix} ac-4b & 2ad-2bc \\ 2a-2c & ac-4d \end{pmatrix},$$

so we get as a bracket product

$$\left[\begin{pmatrix} -a & -2b \\ 2 & a \end{pmatrix},\begin{pmatrix} -c & -2d \\ 2 & c \end{pmatrix}\right] = \begin{pmatrix} 4(d-b) & 4(ad-bc) \\ 4(a-c) & 4(b-d) \end{pmatrix}.$$

Hence we obtain

$$\begin{pmatrix} -(d-b) & bc-ad \\ c-a & d-b \end{pmatrix} = -\frac{1}{4}\,[Z,W] \in \breve{\mathfrak{g}}.$$

We see that the dimension of the Lie algebra $\breve{\mathfrak{g}}$ of the group $\breve{G}$ is at least the rank of the matrix

$$D = \begin{pmatrix} 2 & a & -2b \\ 2 & c & -2d \\ c-a & d-b & bc-ad \end{pmatrix}.$$

We calculate its determinant.

Adding a half of a times the second row to the third row and subtracting a half of c times the first row from the third row, we have

$$\det D = \det\begin{pmatrix} 2 & a & -2b \\ 2 & c & -2d \\ 0 & d-b & 2(bc-ad) \end{pmatrix}.$$

Subtracting the first row from the second row, we get

$$\det D = \det\begin{pmatrix} 2 & a & -2b \\ 0 & c-a & 2(b-d) \\ 0 & d-b & 2(bc-ad) \end{pmatrix}.$$

Thus we obtain

$$\begin{aligned}\frac{1}{4}\det D &= (c-a)(bc-ad)+(d-b)^2\\ &= (\tau(a)-a)\left(b\tau(a)-a\tau(b)\right)+(\tau(b)-b)^2\,.\end{aligned}$$

Lemma 4.3. *On the assumption that $\omega^2(\alpha)=\alpha$ for every $\omega\in\operatorname{Gal}(L/K)$, the* GALOIS *closure N of the field generated by α over K is a finite abelian extension of K. Non-trivial elements of $\operatorname{Gal}(N/K)$ are all of order two.*

Proof. It is sufficient to prove that for any τ and $\omega\in\operatorname{Gal}(L/K)$, we have $\omega^2\tau(\alpha)=\tau(\alpha)$, for it is easily seen (or well-known) that any group whose elements are all of order two is abelian. On our assumption, we see that

$$\omega^2\tau(\alpha)=\tau\left(\tau^{-1}\omega^2\tau\right)(\alpha)=\tau\left(\tau^{-1}\omega\tau\right)^2(\alpha)=\tau(\alpha).$$

□

Lemma 4.4. *On the assumption that $\omega^2(\alpha)=\alpha$ for any $\omega\in\operatorname{Gal}(L/K)$ and that α is not quadratic over K, we have $\breve{G}=\mathrm{SL}_2$.*

Proof. If $\tau(a)=a$ and $\tau(b)=b$ for all $\tau\in\operatorname{Gal}(L/K)$, then $a=\alpha+\sigma^{-1}(\alpha)$ and $b=\alpha\sigma^{-1}(\alpha)$ would belong to the base field K, hence α must have been quadratic over K. Fix an element τ of $\operatorname{Gal}(L/K)$ such that $\tau(a)\neq a$ or $\tau(b)\neq b$.

When $\tau(a)=a$, we have $\tau(b)\neq b$. We see that

$$\det D=4\left(\tau(b)-b\right)^2\neq 0,$$

so $3\leq\dim\breve{\mathfrak{g}}\leq\dim\mathfrak{sl}_2=3$. This implies that $\breve{G}=\mathrm{SL}_2$.

When $\tau(a)\neq a$, let N be the GALOIS closure of the field generated by α over K. We denote by k the fixed field of σ in N. We also denote by $\tilde{K}$ the fixed field of σ and τ in N. By Lemma 4.3, the extension $k/\tilde{K}$ is quadratic. Since a belongs to k, we have in particular

$$k=\tilde{K}\oplus\tilde{K}a.$$

Write $b=\xi+\eta a$ $(\xi,\eta\in\tilde{K})$. We see

$$\begin{aligned}\tau(b)-b &= (\xi+\eta\tau(a))-(\xi+\eta a)\\ &= \eta\left(\tau(a)-a\right)\end{aligned}$$

and

$$\begin{aligned}b\tau(a)-a\tau(b) &= \xi\tau(a)+\eta a\tau(a)-a\xi-a\eta\tau(a)\\ &= \xi\left(\tau(a)-a\right),\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{4}\,\det D &= (\tau(a)-a)\,(b\tau(a)-a\tau(b)) + (\tau(b)-b)^2 \\ &= \xi\,(\tau(a)-a)^2 + \eta^2\,(\tau(a)-a)^2 \\ &= \left(\xi+\eta^2\right)(\tau(a)-a)^2\,.\end{aligned}$$

So, when $\tau(a)\neq a$, if $\det D=0$, then we would have $\xi+\eta^2=0$. If $\xi+\eta^2=0$, then we would get

$$a^2-4b = a^2-4\xi-4\eta a = a^2+4\eta^2-4\eta a = (a-2\eta)^2.$$

We would obtain

$$\alpha = \frac{a+\sqrt{(a-2\eta)^2}}{2} = a-\eta \quad \text{or} \quad \eta,$$

in either case α had to be in k. This is a contradiction. □

Theorem 4.5. *If the base field K is of characteristic zero, then the filtered vector space $\left(\breve{V}, F^{\cdot}_{\alpha}\breve{V}\right)$ in Example 3.1 (derived from a classical* Roth *system) is in the image of a fully faithful tensor functor of the category of finite dimensional representation spaces over K of a one-dimensional anisotropic torus over K or* SL_2 *according as the coefficient α is quadratic over K or not, the functor being compatible with the forgetful tensor functors to* Vec_K.

Proof. When the number α is quadratic over K, we have already seen in Section 2 that the conclusion holds. When α is not quadratic over K and if there is an element $\sigma\in\mathrm{Gal}(L/K)$ such that $\sigma^2(\alpha)\neq\alpha$, then we have a chance to choose such an element σ defining the group $\breve{G}$ in Section 2. With such a choice, we know that the result has been obtained in Section 2. When α is not quadratic over K and if $\omega^2(\alpha)=\alpha$ for all $\omega\in\mathrm{Gal}(L/K)$, then we have confirmed in Lemma 4.4 that the group $\breve{G}$ is identical to SL_2, hence we are done. □

Appendix A. Notes on the representation of algebraic groups that are generated by tori

In this appendix, we investigate the way how the category of finite dimensional representations of an algebraic group can be realized as a full tensor subcategory of a category of vector spaces with semi-stable multiple filtrations of slope zero, using tori and morphisms of tori defined over the base field.

A.1. Anisotropic torus. When the group considered is an anisotropic torus which splits over the extension field, we prove that the functor defined in Section 1 factors through the full subcategory of semi-stable objects of slope zero. Moreover, in certain circumstances, the functor is fully faithful.

Let $K, L, \mathfrak{M}$ be as in Section 1 throughout Section A. Let $G = T$ be an anisotropic torus over K which splits over L. The maps $\kappa(v)$ $(v \in \mathfrak{M})$ are all defined to be the identity of T: $\kappa(v) = \mathrm{id}_T \times_K L$. We denote respectively by X and by Y the character group and the cocharacter group over L of T. The canonical $\mathbb{Z}$-valued pairing between elements of X and those of Y is denoted by $\langle \cdot, \cdot \rangle$. We extend it bi-linearly to an $\mathbb{R}$-valued pairing between elements of $\mathbb{R} \otimes_{\mathbb{Z}} X$ and $\mathbb{R} \otimes_{\mathbb{Z}} Y$. Note that we have a natural diagonal action of $\mathrm{Gal}(L/K)$ on $X \times_{\mathbb{Z}} Y$ and that the value of the pairing is invariant under the action.

Choose a function e of $\mathfrak{M}$ to $\mathbb{R} \otimes_{\mathbb{Z}} Y$ such that for except a finite number of $v \in \mathfrak{M}$, we have $e(v) = 0$. Let V be a finite dimensional representation space over K of T. We have a direct sum decomposition

$$L \otimes_K V = \bigoplus_{\chi \in X} V_\chi,$$

where V_χ is the vector space over L on which $T \times_K L$ acts by multiplication of a character χ. When we associate V with a family of filtrations $V_v^{\cdot} = V_{\mathrm{id},e(v)}^{\cdot}$ as in Section 1 defined by the cocharacter $e(v)$ for each $v \in \mathfrak{M}$, we have

$$V_v^i = \bigoplus_{\langle \chi, e(v) \rangle \geq i} V_\chi \quad (i \in \mathbb{R},\ v \in \mathfrak{M}).$$

Lemma A.1. *We have $\mu_v(\iota_{T,\mathrm{id},e}(V)) = 0$ for any non-zero representation space V over K of T and for each $v \in \mathfrak{M}$, in particular, $\mu(\iota_{T,\mathrm{id},e}(V)) = 0$.*

Proof. Lemma A.1 is a special case of [3, Remark 5.1.3]. A direct proof goes as follows.

Let n be the dimension over K of V. Since T is anisotropic over K, its action is trivial on the one-dimensional representation space $\bigwedge^n V$ over K. Hence the same proof as that of Lemma 3.5 applies. □

Remark A.2. Compare with the sophisticated definition of slope in the book [3, Definition 5.1.1], where the anisotropic part of a maximal torus is not taken into account from the beginning.

Lemma A.3. *To an arbitrary one-dimensional vector subspace W over K of a T-representation V over K, attach the sub-filtration over L of $\iota_{T,\mathrm{id},e}(V)$. We have $\mu_v(W) \leq 0$ for each $v \in \mathfrak{M}$, in particular, $\mu(W) \leq 0$.*

Proof. Let u be a non-zero element of W. We have an expression

$$u = \sum_{\chi \in X} u_\chi, \quad u_\chi \in V_\chi.$$

Making $\sigma \in \mathrm{Gal}(L/K)$ act on both sides, we get

$$u = \sum_{\chi \in X} \sigma\left(u_\chi\right).$$

We know $\sigma\left(u_\chi\right) \in V_{\sigma(\chi)}$ (cf. (3.1) in Section 3). By uniqueness of the decomposition into the elements of subspaces V_χ, we obtain

$$\sigma\left(u_\chi\right) = u_{\sigma(\chi)} \quad \left(\chi \in X,\ \sigma \in \mathrm{Gal}(L/K)\right),$$

which means that the GALOIS group $\mathrm{Gal}(L/K)$ acts on the set of vectors $\{u_\chi\}_{\chi \in X}$ by permutation. Hence the representation space over L of linear combinations of $\{u_\chi\}_{\chi \in X}$ comes from scalar extension of a subspace U over K of V (HILBERT's theorem 90) which contains W. The vector space U is a non-zero T-representation over K.

For each $v \in \mathfrak{M}$, let

$$m_v = \min\left\{\langle \chi, e(v)\rangle \middle| u_\chi \neq 0\right\}.$$

As W is of dimension one, we see by the definition of sub-filtration

$$W_v^i = \begin{cases} L \otimes_K W & (i \leq m_v) \\ 0 & (i > m_v). \end{cases}$$

Hence we have $\mu_v(W) = m_v$. Since

$$\mathrm{gr}^w\left(U_v^\cdot\right) \simeq \bigoplus_{\langle \chi, e(v)\rangle = w} L u_\chi,$$

we get

$$\begin{aligned} \mu_v\left(\iota_{T,\mathrm{id},e}(U)\right) &= \frac{1}{\dim_K U} \sum_{w \in \mathbb{R}} w \dim_L \mathrm{gr}^w\left(U_v^\cdot\right) \\ &= \frac{1}{\dim_K U} \sum_{u_\chi \neq 0} \langle \chi, e(v)\rangle \\ &\geq \frac{1}{\dim_K U} \sum_{u_\chi \neq 0} m_v \\ &= m_v. \end{aligned}$$

Lemma A.1 tells us that

$$0 = \mu_v(\iota_{T,\mathrm{id},e}(U)) \geq m_v = \mu_v(W).$$

□

Proposition A.4. *For any finite dimensional representation space V over K of T, the vector space $\iota_{T,\mathrm{id},e}(V)$ with multiple filtrations is semi-stable of slope zero, hence the functor $\iota_{T,\mathrm{id},e}$ factors through $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$.*

Proof. Completely in the same way as in the proof of Proposition 3.7, we can show that $\mu_v(W) \leq 0$ for each $v \in \mathfrak{M}$ and for any non-zero vector subspace W over K of V with the sub-filtration of $\iota_{T,\mathrm{id},e}(V)$. □

Now we assume that the image of the map $e\colon \mathfrak{M} \to \mathbb{R} \otimes_{\mathbb{Z}} Y$ and its GALOIS conjugates span the vector space $\mathbb{R}\otimes_{\mathbb{Z}}Y$ over $\mathbb{R}$. Namely, we assume that we have an equation

$$\sum_{v\in\mathfrak{M}} \sum_{\sigma\in\mathrm{Gal}(L/K)} \mathbb{R}\,\sigma(e(v)) = \mathbb{R} \otimes_{\mathbb{Z}} Y. \tag{A.1}$$

Theorem A.5. *On the assumption* (A.1)*, the functor* $\iota_{T,\mathrm{id},e}\colon \mathrm{Rep}_K(T) \to \mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ *is fully faithful.*

Proof. Let V and W be T-representations over K and let $f\colon V \to W$ be the underlying linear map over K of a morphism in $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$. In other words, the map f is filtered with respect to every index $v \in \mathfrak{M}$. We have to show that f is T-equivariant.

Let V_χ be as before. For an element x of V_χ, decompose the image $f(x)$ as

$$f(x) = \sum_{\phi\in X} y_\phi, \quad y_\phi \in W_\phi.$$

From our definition of filtration on a representation space, we have $x \in V_v^{\langle\chi,e(v)\rangle}$ for all $v \in \mathfrak{M}$. Since f is filtered, we see

$$f(x) \in W_v^{\langle\chi,e(v)\rangle} \quad \text{for any} \quad v \in \mathfrak{M},$$

hence

$$y_\phi = 0 \quad \text{if} \quad \langle\phi, e(v)\rangle < \langle\chi, e(v)\rangle \quad \text{for some} \quad v \in \mathfrak{M}.$$

On the other hand, we have for all $\sigma \in \mathrm{Gal}(L/K)$

$$f(\sigma(x)) = \sigma(f(x)) = \sum_\phi \sigma(y_\phi).$$

As we have seen in the equation (3.1) in Section 3, we know

$$\sigma(x) \in V_{\sigma(\chi)} \quad \text{and} \quad \sigma(y_\phi) \in W_{\sigma(\phi)}.$$

By the same reasoning as above, if $\langle\sigma(\phi), e(v)\rangle < \langle\sigma(\chi), e(v)\rangle$ for some σ and v, then $y_\phi = \sigma^{-1}(\sigma(y_\phi)) = \sigma^{-1}(0) = 0$. As the canonical pairing $\langle\cdot,\cdot\rangle\colon \mathbb{R} \otimes_{\mathbb{Z}} X \times \mathbb{R} \otimes_{\mathbb{Z}} Y \to \mathbb{R}$ is $\mathrm{Gal}(L/K)$-invariant, this means that

$$y_\phi = 0 \quad \text{if} \quad \left\langle \phi, \sigma^{-1}(e(v)) \right\rangle < \left\langle \chi, \sigma^{-1}(e(v)) \right\rangle$$

for some $\sigma \in \mathrm{Gal}(L/K)$ and some $v \in \mathfrak{M}$.

We need the following simple lemma.

Lemma A.6. *Let ψ be a character over L on T and assume that $\langle\psi,\tau(e(v))\rangle \neq 0$ for some τ and v. There exists $\sigma \in \mathrm{Gal}(L/K)$ such that*

$$\langle\sigma(\psi),\tau(e(v))\rangle = \left\langle\psi,\sigma^{-1}\tau(e(v))\right\rangle < 0.$$

Proof. When $\langle\psi,\tau(e(v))\rangle < 0$, the identity element of $\mathrm{Gal}(L/K)$ meets the requirement.

Suppose that $\langle\psi,\tau(e(v))\rangle > 0$. To draw the conclusion, we may assume that the extension degree of the field L over K is finite. If for all $\sigma \in \mathrm{Gal}(L/K)$

$$\langle\sigma(\psi),\tau(e(v))\rangle \geq 0,$$

then

$$\left\langle\sum_{\sigma\in\mathrm{Gal}(L/K)}\sigma(\psi),\tau(e(v))\right\rangle = \sum_{\sigma\in\mathrm{Gal}(L/K)}\langle\sigma(\psi),\tau(e(v))\rangle > 0.$$

Since the group multiplication in $\mathbb{G}_{\mathrm{m}}$ is defined over K (in fact, over $\mathbb{Z}$), the character $\sum_{\sigma\in\mathrm{Gal}(L/K)}\sigma(\psi)$ is invariant by the action of $\mathrm{Gal}(L/K)$, hence it is defined over K. When T is anisotropic, the last inequality is impossible. $\square$

If some element τ of $\mathrm{Gal}(L/K)$ and some index $v \in \mathfrak{M}$ satisfy

$$\langle\phi,\tau(e(v))\rangle \neq \langle\chi,\tau(e(v))\rangle\,,$$

then apparently $\langle\phi-\chi,\tau(e(v))\rangle = \langle\phi,\tau(e(v))\rangle - \langle\chi,\tau(e(v))\rangle \neq 0$. Thanks to Lemma A.6, there exists $\sigma \in \mathrm{Gal}(L/K)$ fulfilling

$$\langle\phi-\chi,\sigma(e(v))\rangle < 0, \quad \text{i.e.} \quad \langle\phi,\sigma(e(v))\rangle < \langle\chi,\sigma(e(v))\rangle\,.$$

From what we have seen, we get $y_\phi = 0$ for such a character $\phi \in X$.

Consequently, we observe that

$$y_\phi = 0 \quad \text{unless} \quad \langle\phi,\sigma(e(v))\rangle = \langle\chi,\sigma(e(v))\rangle$$

for all $\sigma \in \mathrm{Gal}(L/k)$ and all $v \in \mathfrak{M}$. As the Galois conjugates of $e(v)$ ($v \in \mathfrak{M}$) span $\mathbb{R}\otimes_{\mathbb{Z}} Y$ on the assumption (A.1), only when $\phi = \chi$, it occurs that $y_\phi \neq 0$, which means $f(x)$ ($= y_\chi$) is an element of W_χ. It is obvious that any linear map which sends V_χ to W_χ commutes with the action of T. Hence the map f is T-equivariant. $\square$

A.2. Split torus. When the group is a split torus, the functor defined in Section 1 does not necessarily factor through the full subcategory of semi-stable objects of slope zero. But with an appropriate choice of the index set and of a family of 'cocharacters', we can realize the category of finite dimensional representations of a split torus as a full subcategory of the category of vector spaces with semi-stable multiple filtrations of slope zero.

Let $G = T$ be a split torus over K and Y the cocharacter group of T. The maps $\kappa(v)$ ($v \in \mathfrak{M}$) are all defined to be the identity of T. Choose a function e of $\mathfrak{M}$ to $\mathbb{R} \otimes_{\mathbb{Z}} Y$ such that for except a finite number of $v \in \mathfrak{M}$, we have $e(v) = 0$ and such that

$$\sum_{v \in \mathfrak{M}} e(v) = 0.$$

Other symbols are the same as in Section 1 and in Subsection A.1.

Remark A.7. As mentioned earlier in Remark 1.8, any functorial filtration stems from a 'cocharacter'. Since the category of finite dimensional representations of a split torus is generated by one-dimensional objects, we observe that the above vanishing condition of the sum of values of e must be placed in order to realize functorially the representation spaces as vector spaces with filtrations of slope zero (cf. Lemma A.9 below).

Remark A.8. On the same line of thought, the category of representations of a *finite* split multiplicative group cannot be realized as a full subcategory of $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$. It is because any non-trivial power (self tensor product) of a one-dimensional object other than a unit object in $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ never becomes a unit object. This means that the group of connected components of $\operatorname{Aut} \omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ does not have such a finite group as a quotient group. We would like to return to the issue of connectedness of $\operatorname{Aut} \omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ in a future.

Lemma A.9. *For any finite dimensional non-zero representation space V over K of T, we have $\mu\left(\iota_{T,\mathrm{id},e}(V)\right) = 0$.*

Proof. Let n be the dimension over K of V. As in the proof of Lemma 3.5, we see

$$\mu_v(\iota_{T,\mathrm{id},e}(V)) = \frac{1}{n}\, \mu_v\left(\iota_{T,\mathrm{id},e}\left(\bigwedge^n V\right)\right).$$

Call χ the character of the one-dimensional representation space $\bigwedge^n V$ over K. By our definition of filtration on representation spaces, we have for each $v \in \mathfrak{M}$

$$\mu_v\left(\iota_{T,\mathrm{id},e}(V)\right) = \frac{1}{n}\, \langle \chi, e(v) \rangle.$$

Summing them up all over $v \in \mathfrak{M}$, we get

$$\begin{aligned} \mu\left(\iota_{T,\mathrm{id},e}(V)\right) &= \sum_{v \in \mathfrak{M}} \mu_v\left(\iota_{T,\mathrm{id},e}(V)\right) \\ &= \frac{1}{n} \left\langle \chi, \sum_{v \in \mathfrak{M}} e(v) \right\rangle \\ &= \frac{1}{n}\, \langle \chi, 0 \rangle = 0. \end{aligned}$$

□

Lemma A.10. *To an arbitrary one-dimensional vector subspace W over K of a T-representation V over K, attach the sub-filtration over L of $\iota_{T,\mathrm{id},e}(V)$. We have $\mu(W) \leq 0$.*

Proof. In the same way (more easily) as in the proof of Lemma A.3, we see that there exists a sub-representation U over K of V such that

$$\mu_v(\iota_{T,\mathrm{id},e}(U)) \geq \mu_v(W) \quad \text{for any} \quad v \in \mathfrak{M}.$$

Adding the respective sides of the inequalities all over the indices v, we obtain

$$\mu(\iota_{T,\mathrm{id},e}(U)) \geq \mu(W).$$

Lemma A.9 says that the left hand side is equal to zero. □

Proposition A.11. *For any finite dimensional representation space V over K of T, the vector space $\iota_{T,\mathrm{id},e}(V)$ with multiple filtrations is semi-stable of slope zero, hence the functor $\iota_{T,\mathrm{id},e}$ factors through $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$.*

Proof. It is enough to prove that for any non-zero vector subspace W over K of V with the sub-filtrations of $\iota_{T,\mathrm{id},e}(V)$, we have $\mu(W) \leq 0$.

Let $d = \dim_K W$. Completely in the same way as in the proof of Proposition 3.7, we find a non-trivial canonical homomorphism $f\colon \det W \to \iota_{T,\mathrm{id},e}\left(\bigwedge^d V\right)$ in $\mathcal{C}(K, L, \mathfrak{M})$ so that for all $v \in \mathfrak{M}$

$$\mu_v(\det W) \leq \mu_v(\operatorname{Im} f).$$

Summation over $v \in \mathfrak{M}$ of the respective sides of the inequalities gives

$$\mu(W) = \frac{1}{d}\,\mu(\det W) \leq \frac{1}{d}\,\mu(\operatorname{Im} f).$$

By Lemma A.10, the right hand side is not positive. □

We further assume (that the cardinality of the index set $\mathfrak{M}$ is greater than $\dim T$ and) that the equation

$$\sum_{v\in\mathfrak{M}} \mathbb{R}\, e(v) = \mathbb{R} \otimes_{\mathbb{Z}} Y \tag{A.2}$$

holds.

Theorem A.12. *On the assumption* (A.2)*, the functor $\iota_{T,\mathrm{id},e}\colon \mathrm{Rep}_K(T) \to \mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is fully faithful.*

Proof. Let V and W be T-representations over K and let $f\colon V \to W$ be the underlying linear map over K of a morphism in $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$. We have to show that f is T-equivariant.

Let V_χ be the subspace over L of $L \otimes_K V$ on which $T \times_K L$ acts by multiplication of a character χ. For an element x of V_χ, decompose the image $f(x)$ as

$$f(x) = \sum_{\phi\in X} y_\phi, \quad y_\phi \in W_\phi,$$

where the symbol W_ϕ has a similar meaning to V_ϕ. As in the proof of Theorem A.5, we see

$$y_\phi = 0 \quad \text{if} \quad \langle \phi, e(v)\rangle < \langle \chi, e(v)\rangle \quad \text{for some} \quad v \in \mathfrak{M}.$$

If a character ϕ satisfies

$$\langle \phi, e(v)\rangle \geq \langle \chi, e(v)\rangle \quad \text{for all} \quad v \in \mathfrak{M},$$

then we have

$$\begin{aligned} 0 \leq \langle \phi, e(v)\rangle - \langle \chi, e(v)\rangle &\leq \sum_{v\in\mathfrak{M}} \left(\langle \phi, e(v)\rangle - \langle \chi, e(v)\rangle\right) \\ &= \sum_{v\in\mathfrak{M}} \langle \phi - \chi, e(v)\rangle \\ &= \left\langle \phi - \chi, \sum_{v\in\mathfrak{M}} e(v) \right\rangle. \end{aligned}$$

The assumption $\sum_{v\in\mathfrak{M}} e(v) = 0$ forces

$$\langle \phi, e(v)\rangle = \langle \chi, e(v)\rangle \quad \text{for all} \quad v \in \mathfrak{M}.$$

Since $e(v)$ $(v \in \mathfrak{M})$ span $\mathbb{R} \otimes_{\mathbb{Z}} Y$ on the assumption (A.2), this occurs only when $\phi = \chi$, which means $f(x)$ $(= y_\chi)$ is an element of W_χ. Thus the map f is T-equivariant. □

A.3. Algebraic groups generated by tori. Let G be an algebraic group over K such that the images of a finite number of morphisms defined over K of tori to G generate a dense subgroup of G and such that the tori split over L. As an example, any connected reductive group over K satisfies these conditions for a suitable GALOIS extension L which depends of course on the reductive group.

We fix tori $T^{(1)}, \ldots, T^{(l)}$ over K which split over L and morphisms $\kappa^{(i)}\colon T^{(i)} \to G$ $(i = 1, \ldots, l)$ over K such that the images of $\kappa^{(i)}$ $(i = 1, \ldots, l)$ generate a dense subgroup of G. We may suppose that the tori $T^{(1)}, \ldots, T^{(h)}$ are anisotropic over K and the tori $T^{(h+1)}, \ldots, T^{(l)}$ are split over K, for any torus is generated by an anisotropic subtorus and a split subtorus (see e.g. [2, 8.15 Proposition]). We denote by $Y^{(i)}$ $(i = 1, \ldots, l)$ their respective cocharacter groups over L. Select a family $\kappa(\cdot)$ of morphisms and a family $e(\cdot)$ of 'cocharacters' as follows: For each $v \in \mathfrak{M}$, the morphism $\kappa(v)$ is $\kappa^{(i)} \times_K L\colon T^{(i)} \times_K L \to G \times_K L$ for some i and the 'cocharacter' $e(v)$ is an element of $\mathbb{R} \otimes_{\mathbb{Z}} Y^{(i)}$ for the same i such that as before $e(v) = 0$ for except a finite number of $v \in \mathfrak{M}$ and such that

$$\sum_{v\in\mathfrak{M}} e(v) \in \bigoplus_{i=1}^{h} \mathbb{R} \otimes_{\mathbb{Z}} Y^{(i)}$$

(the 'split part' of the sum vanishes, cf. Subsection A.2).

Proposition A.13. *For any finite dimensional representation space V over K of G, the vector space $\iota_{G,\kappa,e}(V)$ with multiple filtrations is semi-stable of slope zero, hence the functor $\iota_{G,\kappa,e}$ factors through $\mathcal{C}_0^{\mathrm{ss}}(K,L,\mathfrak{M})$.*

Proof. Let

$$\mathfrak{M}(i) = \{v \in \mathfrak{M} \mid \kappa(v) = \kappa^{(i)} \times_K L\} \quad (i = 1, \dots, l)$$

and

$$\mu^{(i)}(\cdot) = \sum_{v \in \mathfrak{M}(i)} \mu_v(\cdot) \quad (i = 1, \dots, l).$$

We have

$$\mathfrak{M} = \bigsqcup_{i=1}^{l} \mathfrak{M}(i) \ \ \text{(disjoint union)}, \quad \mu(\cdot) = \sum_{i=1}^{l} \mu^{(i)}(\cdot).$$

Since the morphisms $\kappa(v)$ ($v \in \mathfrak{M}$) are defined over K, restricting the group and the index set to $T^{(i)}$ and $\mathfrak{M}(i)$ for any fixed i and applying the results of Subsection A.1 and Subsection A.2, we see that the vector space V with $\mathfrak{M}(i)$-indexed filtration(s) is semi-stable of slope zero with respect to the slope function $\mu^{(i)}$ for each $i = 1, \dots, l$. As μ is the sum of $\mu^{(i)}$ ($i = 1, \dots, l$), the vector space V with $\mathfrak{M}$-indexed filtration(s) is semi-stable of slope zero with respect to the slope function μ. □

We assume in addition (that the cardinality of the index set $\mathfrak{M}$ is sufficiently large so) that we possess the equation

$$\sum_{v \in \mathfrak{M}} \sum_{\sigma \in \mathrm{Gal}(L/K)} \mathbb{R}\, \sigma(e(v)) = \bigoplus_{i=1}^{l} \mathbb{R} \otimes_{\mathbb{Z}} Y^{(i)}. \tag{A.3}$$

Theorem A.14. *The condition* (A.3) *imposed, the functor $\iota_{G,\kappa,e} \colon \mathrm{Rep}_K(G) \to \mathcal{C}_0^{\mathrm{ss}}(K,L,\mathfrak{M})$ is fully faithful.*

Proof. Let V and W be G-representations over K and let $f \colon V \to W$ be the underlying linear map over K of a morphism in $\mathcal{C}_0^{\mathrm{ss}}(K,L,\mathfrak{M})$. We have to show that f is G-equivariant.

By the same method as in the proof of Proposition A.13, we see that f is $T^{(i)}$-equivariant for every $i = 1, \dots, l$.

We regard f as a K-valued point of an affine variety $\mathrm{Hom}_K(V,W)$ over K, on which algebraic groups $\mathrm{Aut}_K(V)$ and $\mathrm{Aut}_K(W)$ act over K respectively from the right and from the left. In particular, the group G acts over K on $\mathrm{Hom}_K(V,W)$ by conjugation. The stability group G_f in G of f is a closed subvariety [2, 1.7 Proposition (b)]. As we know already, it includes the images of tori $T^{(1)}, \dots, T^{(l)}$. Since they generate a dense subgroup of G, we obtain $G_f = G$. □

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