JOURNAL
de Théorie des Nombres
de BORDEAUX
*anciennement Séminaire de Théorie des Nombres de Bordeaux*

Victor ABRASHKIN et Ruth JENNI

**The field-of-norms functor and the Hilbert symbol for higher local fields**

Tome 24, nº 1 (2012), p. 1-39.

<http://jtnb.cedram.org/item?id=JTNB_2012__24_1_1_0>

*Journal de Théorie des Nombres*
*de Bordeaux* **24** (2012), 1-39

# The field-of-norms functor and the Hilbert symbol for higher local fields

par Victor ABRASHKIN et Ruth JENNI

Résumé. Dans cet article nous appliquons le foncteur corps des normes pour déduire, dans le cas de caractéristique mixte, une formule explicite pour le symbole de Hilbert de la formule explicite pour le symbole de Witt en caractéristique $p > 2$ dans le contexte des corps locaux multidimensionnels. On montre que la formule explicite de Vostokov est un cas très particulier de notre construction.

Abstract. The field-of-norms functor is applied to deduce an explicit formula for the Hilbert symbol in the mixed characteristic case from the explicit formula for the Witt symbol in characteristic $p > 2$ in the context of higher local fields. Is is shown that a "very special case" of this construction gives Vostokov's explicit formula.

## Introduction

Throughout this paper, $M$ and $N$ are fixed natural numbers, $p$ is an odd prime number, $W(k)$ is the ring of Witt vectors with coefficients in a finite field $k$ of characteristic $p$, $W(k)_{\mathbb{Q}_p} = W(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $\sigma$ is the Frobenius automorphism of $W(k)$ induced by the $p$-th power map on $k$. In the main body of the paper we shall also use other notation from this Introduction without special reference.

Suppose $F$ is an $N$-dimensional local field of characteristic 0 with the (first) residue field $F^{(1)}$ (which is an $(N-1)$-dimensional local field) of characteristic $p$, $\bar{F}$ is a fixed algebraic closure of $F$ and $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$. Note that, by definition, the last residue field $F^{(N)}$ is a finite field of characteristic $p$ which we shall denote by $k$. Fix a system of local parameters $\pi_1, \dots, \pi_N$ in $F$. Let $v_F$ be the (first) valuation of $F$ such that $v_F(F^*) = \mathbb{Z}$. Then $v_F$ can be extended uniquely to $\bar{F}$ and we introduce for any $c \geqslant 0$, the ideals $\mathfrak{p}_F^c = \{a \in \bar{F} \mid v_F(a) \geqslant c\}$.

Manuscrit reçu le 15 novembre 2010.
*Mots clefs.* higher local fields, field-of-norms, Hilbert Symbol, Vostokov's pairing.
*Classification math.* 11S20, 11S31, 11S70.

Let $F_\bullet$ be a strictly deeply ramified (SDR) fields tower with parameters $(0,c)$, where $0 < c \leqslant v_F(p)$. This means that $F_\bullet = \{F_n \mid n \geqslant 0\}$ is an increasing tower of algebraic extensions of $F_0 = F$ such that for all $n \geqslant 0$,

— the last residue field of $F_n$ is $k$;

— there is a system of local parameters $\pi_1^{(n)}, \dots, \pi_N^{(n)}$ in $F_n$ such that $\pi_1^{(n+1)p} \equiv \pi_1^{(n)} \operatorname{mod} \mathfrak{p}_F^c, \dots, \pi_N^{(n+1)p} \equiv \pi_N^{(n)} \operatorname{mod} \mathfrak{p}_F^c$.

The construction of the field-of-norms functor $X$ from [17] attaches to $F_\bullet$ a field $X(F_\bullet) = \mathcal{F}$ of characteristic $p$. This field is the fraction field of the valuation ring $O_\mathcal{F} = \varprojlim_n O_{F_n}/\mathfrak{p}_F^c$, where $O_{F_n} = \{a \in F_n \mid v_F(a) \geqslant 0\}$ are the (first) valuation rings of $F_n$ for all $n \geqslant 0$. Note that $\mathcal{F}$ has a natural structure of an $N$-dimensional local field of characteristic $p$ with system of local parameters $\bar{t}_1 = \varprojlim_n \pi_1^{(n)}, \dots, \bar{t}_N = \varprojlim_n \pi_N^{(n)}$ and last residue field $\mathcal{F}^{(N)} = k$, i.e. $\mathcal{F}$ is the field of formal Laurent series $k((\bar{t}_N))\dots((\bar{t}_1))$. In addition, the field-of-norms functor $X$ provides us with a construction of a separable closure $\mathcal{F}_{sep}$ of $\mathcal{F}$ and identifies the Galois groups $\Gamma_\mathcal{F} = \operatorname{Gal}(\mathcal{F}_{sep}/\mathcal{F})$ and $\Gamma_{F_\infty} = \operatorname{Gal}(\bar{F}/F_\infty)$, where $F_\infty = \bigcup_{n\geqslant 0} F_n$.

We use the above system of local parameters $\bar{t}_1, \dots, \bar{t}_N$ to construct an absolutely unramified lift $L(\mathcal{F})$ of $\mathcal{F}$ of characteristic 0. Then $L(\mathcal{F})$ is an $(N+1)$-dimensional local field with system of local parameters $p, t_1, \dots, t_N$; its first residue field $L(\mathcal{F})^{(1)}$ coincides with $\mathcal{F}$ and for $1 \leqslant i \leqslant N$, we have $t_i \operatorname{mod} p = \bar{t}_i$.

For any higher local field $L$, let $K_N(L)$ be its $N$-th Milnor $K$-group. In this paper we mainly use the topological versions $K_N^t(L)$ of the Milnor $K$-groups, which have explicit systems of topological generators. Nevertheless, in the final statement we can return to Milnor $K$-groups due to the natural identification $K_N(L)/p^M = K_N^t(L)/p^M$, which we shall denote by $K_N(L)_M$.

The following maps play very important roles in the statement of the main result of this paper.

• $\mathcal{N}_{\mathcal{F}/F} : K_N^t(\mathcal{F}) \longrightarrow K_N^t(F)$.

In Subsection 4 we prove that for so-called special SDR towers $F_\bullet$, there is a natural identification $K_N^t(\mathcal{F}) = \varprojlim_n K_N^t(F_n)$, where the connecting morphisms are the norm maps $N_{F_{n+1}/F_n} : K_N^t(F_{n+1}) \longrightarrow K_N^t(F_n)$. Then $\mathcal{N}_{\mathcal{F}/F}$ is the corresponding projection from $K_N^t(\mathcal{F})$ to $K_N^t(F)$. For arbitrary SDR towers $F_\bullet$ we prove the analogous "modulo $p^M$" statement under the assumption that a primitive $p^M$-th root of unity $\zeta_M \in F_\infty$. In particular, this gives the map

$$\mathcal{N}_{\mathcal{F}/F} : K_N(\mathcal{F})_M \longrightarrow K_N(F)_M.$$

• $\mathrm{Col}: K_N^t(\mathcal{F}) \longrightarrow K_N^t(L(\mathcal{F}))$.

This map is obtained as a section of the natural map from $K_N^t(L(\mathcal{F}))$ to $K_N^t(L(\mathcal{F})^{(1)}) = K_N^t(\mathcal{F})$. Its construction, in which the concept of topological $K$-groups is essential, is a direct generalisation of Fontaine's 1-dimensional construction from [8].

• $\theta^1 : \mathrm{m}^0 \longrightarrow (1+\mathrm{m}^0)^\times$.

Here $\mathrm{m}^0$ consists of all series $\sum_{a>\bar{0}} w_a \underline{t}^a$, which are convergent in $L(\mathcal{F})$, where the indices $a = (a_1, \dots, a_N) \in \mathbb{Z}^N$ are provided with the lexicographical ordering, all $w_a \in W(k)$, and $\underline{t}^a := t_1^{a_1} \dots t_N^{a_N}$. The map $\theta^1$ is then a group homomorphism defined by the correspondence

$$\sum_{a>\bar{0}} w_a \underline{t}^a \mapsto \prod_{a>\bar{0}} E(w_a, \underline{t}^a).$$

Here for any $w \in W(k)$,

$$E(w, X) = \exp\Big(wX + \dots + \sigma^n(w)X^{p^n}/p^n + \dots\Big) \in W(k)[[X]]$$

is the Shafarevich generalisation of the Artin-Hasse exponential. Notice that the inverse of $\theta^1$ is the map given, for any $b \in 1+\mathrm{m}^0$, by the correspondence $b \mapsto (1/p)\log(b^p/\sigma(b))$, where $\sigma$ is the continuous map induced by the Frobenius on $W(k)$ and $t_i \mapsto t_i^p$, for all $1 \leqslant i \leqslant N$.

• $\gamma : (1+\mathrm{m}^0)^\times \longrightarrow \hat{F}_\infty^*$.

Here $\hat{F}_\infty$ is the completion (with respect to the valuation $v_F$) of $F_\infty = \bigcup_{n\geqslant 0} F_n$ and the map $\gamma$ is the continuous map uniquely determined by $t_i \mapsto \lim\limits_{n\to\infty} \pi_i^{(n)p^n}$, $1 \leqslant i \leqslant N$.

We now state the main result of this paper.

Let $F^{ab}$ be the maximal abelian extension of $F$, $\Gamma_F^{ab} = \mathrm{Gal}(F^{ab}/F)$ and $\widehat{K}_N^t(F) := \varprojlim_L K_N^t(F)/N_{L/F}K_N^t(L)$, where $L$ runs over the set of all finite extensions of $F$ in $F^{ab}$.

We denote by $\Theta_F : \Gamma_F^{ab} \longrightarrow \widehat{K}_N^t(F)$ the reciprocity map of local higher class field theory. For the field $\mathcal{F}$, we introduce similarly $\mathcal{F}^{ab}$, $\Gamma_{\mathcal{F}}^{ab}$, $\widehat{K}_N^t(\mathcal{F})$ and $\Theta_{\mathcal{F}}$. Then the compatibility of class field theories for the fields $F$ and $\mathcal{F}$ via the field-of-norms functor means that there is the following commutative diagram

$$\begin{tikzcd}
\Gamma_{\mathcal{F}}^{ab} \arrow[r, "\Theta_{\mathcal{F}}"] \arrow[d, "\iota_{\mathcal{F}/F}"] & \widehat{K}_N^t(\mathcal{F}) \arrow[d, "\widehat{\mathcal{N}}_{\mathcal{F}/F}"] & K_N^t(\mathcal{F}) \arrow[l] \arrow[d, "\mathcal{N}_{\mathcal{F}/F}"] \\
\Gamma(F)^{ab} \arrow[r, "\Theta_F"] & \widehat{K}_N^t(F) & K_N^t(F) \arrow[l]
\end{tikzcd} \tag{0.1}$$

Here $\iota_{\mathcal{F}/F}$ is induced by the identification $\Gamma_{\mathcal{F}} = \Gamma_{F_\infty}$ given by the field-of-norms functor, the horizontal maps on the right-hand side are the natural embeddings and the map $\widehat{\mathcal{N}}_{\mathcal{F}/F}$ is induced by $\mathcal{N}_{\mathcal{F}/F}$ on the corresponding completions. We prove the commutativity of the above diagram (0.1) only for $\mathcal{F} = X(F_\bullet)$, where $F_\bullet$ is a so-called special SDR tower, cf. Subsection 4.3. But under the additional assumption $\zeta_M \in F_\infty$ we prove the commutativity of the following "modulo $p^M$" version of (0.1) for any SDR tower $F_\bullet$ (we use the same notation for all involved maps taken modulo $p^M$)

$$\begin{tikzcd}
\Gamma^{ab}_{\mathcal{F}}/p^M \arrow[r, "\Theta_{\mathcal{F}}"] \arrow[d, "\iota_{\mathcal{F}/F}"] & \widehat{K}_N(\mathcal{F})_M \arrow[d, "\widehat{\mathcal{N}}_{\mathcal{F}/F}"] & K^t_N(\mathcal{F})_M \arrow[l] \arrow[d, "\mathcal{N}_{\mathcal{F}/F}"] \\
\Gamma^{ab}_F/p^M \arrow[r, "\Theta_F"] & \widehat{K}_N(F)_M & K_N(F)_M \arrow[l]
\end{tikzcd}$$

This property allows us to consider the $M$-th Hilbert pairing

$$(\ ,\ )^{F_\bullet}_M : \hat{F}^* \times \mathcal{N}_{\mathcal{F}/F}(K^t_N(\mathcal{F})) \longrightarrow \langle \zeta_M \rangle$$

under the condition that $\zeta_M \in F_\infty$. Namely, if $b \in \mathcal{N}_{\mathcal{F}/F}(K_N(\mathcal{F})_M)$ then there is $\tau \in \Gamma^{ab}_F/p^M$ such that $\tau|_{F_\infty} = \mathrm{id}$ and $\Theta_F(\tau) = b$. Then for any $a \in \hat{F}^*_\infty$, $(a, b)^{F_\bullet}_M := \tau(\xi)/\xi$, where $\xi \in \bar{F}$ is such that $\xi^{p^M} = a$.

Suppose $F_\bullet$ is an SDR tower with parameters $(0, c)$.

**Definition.** The tower $F_\bullet$ is called $\omega$-admissible, for $\omega \in \mathbb{Z}_{\geqslant 0}$, if $cp^\omega > 2v_F(p)/(p-1)$ and $F_\omega$ contains a primitive $p^{M+\omega}$-th root of unity $\zeta_{M+\omega}$.

For an $\omega$-admissible SDR tower $F_\bullet$, we define (not uniquely) an element $H_\omega = H_\omega(\zeta_{M+\omega}) \in \mathrm{m}^0$ as follows. Suppose $H' = 1 + \sum_{a > \bar{0}} w_a \underline{t}^a \in 1 + \mathrm{m}^0$ is such that $\gamma(H') \equiv \zeta_{M+\omega} \bmod \mathfrak{p}^c_F$. Then we set $H_\omega := H'^{p^{M+\omega}} - 1$. Note that the construction of $H_\omega$ does not require the knowledge of the whole tower $F_\bullet$, but only of the field $F_\omega$. In particular, if $\zeta_M \in F = F_0$ then the corresponding element $H_0 \in \mathrm{m}^0$ will be used later in the definition of Vostokov's pairing.

With the above notaion we have, for any $\omega$-admissible SDR tower, the following explicit formula for the $M$-th Hilbert symbol.

**Theorem 0.1.** *If $f \in \mathrm{m}^0$, $\beta \in K_N(\mathcal{F})$ and $\theta := \gamma \circ \theta_1$ then*

$$(\theta(f), \mathcal{N}_{\mathcal{F}/F}(\beta))^{F_\bullet}_M = \zeta^{p^\omega A}_{M+\omega} \tag{0.2}$$

*where* $A = (\mathrm{Tr} \circ \mathrm{Res})\left((f/H_\omega) d_{\log} \mathrm{Col}(\beta)\right)$.

Here (and everywhere below) Tr is the trace map for the field extension $W(k)_{\mathbb{Q}_p}/\mathbb{Q}_p$ and Res is $N$-dimensional residue.

The above Theorem 0.1 gives one of most general approaches to the explicit formulas for the Hilbert symbol. The proof uses the strategy from [1] and the construction of the field-of-norms functor for higher local fields from [17]. As a result, the explicit formula (0.2) is obtained from the explicit formula for the Witt symbol in characteristic $p$. Notice that symbol (0.2) depends not only on a fixed system of local parameters $\pi_1, \dots, \pi_N$ of $F$ but also involves special lifts of elements of $F$ to $L(\mathcal{F})$.

The result of the above Theorem 0.1 is related very closely to Vostokov's explicit formula for the $M$-th Hilbert symbol $F^* \times K_N^t(F) \longrightarrow \langle \zeta_M \rangle$. In this formula the elements of $F^*$ appear as the results of the substitution $t_i \mapsto \pi_i$, $i = 1, \dots, N$, into formal Laurent series with coefficients in $W(k)$ and indeterminants $t_1, \dots, t_N$. Vostokov's proof of this formula is based on a hard computation showing that the formula gives the same result for arbitrary choices of local parameters $\pi_1, \dots, \pi_N$.

In Section 3 we develop a slightly different approach to Vostokov's result. First of all, the Vostokov pairing has two different aspects. One is purely $K$-theoretic: it gives a (non-degenerate) pairing between $K_1(F)_M$ and $K_N(F)_M$ and factors through the canonical morphism

$$K_1(F)_M \times K_N(F)_M \longrightarrow K_{N+1}(F)_M. \tag{0.3}$$

(Note that Vostokov's formula gives also a pairing between $K_i(F)_M$ and $K_{N-i}(F)_M$ for $1 < i < N$.) We establish these properties following the strategy from [1] and using an idea of one calculation from [3]. Note that we can work throughout with our fixed system of local parameters $\pi_1, \dots, \pi_N$. Then the Galois-theoretic aspect of Vostokov's pairing, i.e. that it coincides with the Hilbert symbol, follows by an easy calculation from the following two elementary facts:

— the Hilbert symbol also factors through the map (0.3);

— $K_{N+1}(F)_M$ is generated by one element which can be written in terms of our fixed system of local parameters $\pi_1, \dots, \pi_N$.

At the end of Section 5 we show that symbol (0.2) from Theorem 0.1 coincides with Vostokov's pairing if we use a "very special " SDR tower $F^0_\bullet = \{F^0_n \mid n \geqslant 0\}$ such that $F^0_0 = F$ and for all $n \geqslant 0$, $F^0_n$ has a system of local parameters $\pi_1^{(n)}, \dots, \pi_N^{(n)}$ with $\pi_i^{(n+1)p} = \pi_i^{(n)}$ and $\pi_i^{(0)} = \pi_i$ for all $1 \leqslant i \leqslant N$.

Note that other interpretations of Vostokov's formula have been given by K.Kato [11] in terms of Fontaine-Messing theory and by S. Zerbes [21] in terms of $(\varphi, \Gamma)$-modules under an additional restriction on the basic field $F$. Note also the paper [9] where special cases of the constructions of the field-of-norms functor in the context of higher local fields were treated.

The structure of the paper is as follows. In Section 1 we discuss basic matters: the concept of higher local field, the $P$-topology, special systems of topological generators for the Milnor $K$-groups and the norm map in the context of $K$-groups. In Section 2 we give an invariant approach to the concept of residue, the Witt symbol and the Coleman map in the context of higher local fields. In Section 3 we recover the construction of Vostokov's pairing following mainly the strategy of the paper [1]. In Section 4 we use the field-of-norms functor $X$ to relate the behaviour of topological Milnor

$K$-groups in SDR towers. Finally, in Section 5 we prove the compatibility of the field-of-norms functor with class field theories for the fields $\mathcal{F} = X(F_\bullet)$ and $F = F_0$ and use the compatibility of the Kummer theory for $F$ and the Witt-Artin-Schreier theory for $\mathcal{F}$ from [2] to deduce the statement of Theorem 0.1.

## 1. Preliminaries

Most of the notation introduced in this Section will be used in the next sections without special references. In particular, this holds for the notation $F$, $\pi_1, \dots, \pi_N$, $\mathcal{F}$, $\bar{t}_1, \dots, \bar{t}_N$, $O(\mathcal{F})$ and $L(\mathcal{F})$.

**1.1. Higher local fields.** Let $L$ be an $N$-dimensional local field. This means that $L$ is a complete discrete valuation field and its (first) residue field $L^{(1)}$ is an $(N-1)$-dimensional local field. In our setting, 0-dimensional local fields are finite fields of charactersitic $p$. Let $L^{(N)}$ be the $N$-th residue field of $L$. By inductive definition this means that $L^{(N)} = (L^{(1)})^{(N-1)}$ and, therefore, it is a finite field of characteristic $p$. The system $u_1, \dots, u_N$ is a system of local parameters of $L$, if $u_1$ is a local parameter of $L$, $u_2, \dots, u_N$ belong to the valuation ring $O_L$ of $L$ and the images of $u_2, \dots, u_N$ in $L^{(1)}$ form a system of local parameters of $L^{(1)}$. The field $L$ is equipped with a special topology (we call it the $P$-topology) which relates all $N$ valuation topologies of $L$, $L^{(1)}$, $L^{(2)} := (L^{(1)})^{(1)}, \dots, L^{(N)} := (L^{(N-1)})^{(1)}$. The idea how to construct such topology appeared first in [15] and then was considerably developed and studied in [6, 22, 10]. We can sketch its definition as follows.

Fix a system of local parameters $u_1, \dots, u_N$ in $L$. Note that any element $x \in L$ can be written uniquely as a formal series

$$x = \sum_{a=(a_1,\dots,a_N)} [\alpha_a] u_1^{a_1} \cdots u_N^{a_N}, \tag{1.1}$$

where all coefficients $[\alpha_a]$ are the Teichmüller representatives of the elements $\alpha_a \in L^{(N)}$ in $L$. (Note that $\alpha_a = [\alpha_a]$ if $L$ has characteristic $p$.) Here $a \in \mathbb{Z}^N$ and there are (depending on the element $x$) integers $I_1, I_2(a_1), \dots, I_N(a_1, \dots, a_{N-1})$ such that $\alpha_a = 0$ if either $a_1 < I_1$ or $a_2 < I_2(a_1), \dots,$ or $a_N < I_N(a_1, \dots, a_{N-1})$.

**Remark.** The referee pointed out that this is equivalent to saying that the set $\{a \in \mathbb{Z}^N \mid \alpha_a \neq 0\}$ is well-ordered, i.e. any its subset has a minimal element (with respect to the lexicographical ordering); in the terminilogy of the papers [22] and [10] such set is also called admissible.

Then the $P$-topological structure on $L$ can be defined by induction on $N$ as follows. If $N = 0$ then it is discrete. If $N \geqslant 1$ then $\bar{u}_2 = u_2 \bmod u_1$, $\dots$, $\bar{u}_N = u_N \bmod u_1$ is a system of local parameters in $L^{(1)}$ and we can define a section $s : L^{(1)} \longrightarrow L$ by $\sum_a \alpha_a \bar{u}_2^{a_2} \dots \bar{u}_N^{a_N} \mapsto \sum_a u_2^{a_2} \dots u_N^{a_N}$. By definition, the basis of open neighbourhoods $\mathcal{C}_{L,\{u_1,\dots,u_N\}}$ in $L$ consists of the sets

$$L \cap \sum_{b\in\mathbb{Z}} u_1^b s(U_b) = \left\{ \sum_{b\gg-\infty} u_1^b s(U_b) \right\},$$

where all $U_b \in \mathcal{C}_{L^{(1)},\{\bar{u}_2,\dots,\bar{u}_N\}}$ and $U_b = L^{(1)}$ if $b \gg 0$. One can prove then that this does not depend on the initial choice of local parameters $u_1, \dots, u_N$. Then any compact subset in $L$ is a closed subset in the compact subset of the form $\sum_{b\in\mathbb{Z}} u_1^b s(C_b)$, where all $C_b \subset L^{(1)}$ are compact and $C_b = 0$ for $b \ll 0$. In particular, the set of all $\xi \in L$ given by (1.1) with fixed $I_1, I_2(a_1), \dots, I_N(a_1, \dots, a_{N-1}$, is compact. The following property explains that the concept of convergency in the $P$-topology just coincides with the concept of convergency of formal power series.

**Proposition 1.1.** *A sequence $\xi_n = \sum_a [\alpha_{an}] u_1^{a_1} \dots u_N^{a_N} \in L$ converges to $\xi = \sum_a [\alpha_a] u_1^{a_1} \dots u_N^{a_N} \in L$ if and only if*

*a) there is a compact $C \subset L$ such that all $\xi_n \in C$;*

*b) for any $a \in \mathbb{Z}^N$, the sequence $\alpha_{an}$ converges to $\alpha_a$ in $k$.*

**Remark.** The referee pointed out that the above Proposition identifies topologically $L$ with the inductive limit of $\prod_{a\in D} k$, where $D$ runs over all well-ordered subsets of $\mathbb{Z}^N$.

*Proof.* The proof can be easily reduced to the case $\xi = 0$. Then suppose that for any $b \in \mathbb{Z}$, the elements $\bar{\xi}_{bn} \in L^{(1)}$ are such that $\xi_n = \sum_b u_1^b s(\bar{\xi}_{bn})$.

Clearly, $\lim\limits_{n\to\infty} \xi_n = 0$ implies that for any $b \in \mathbb{Z}$, $\lim\limits_{n\to\infty} \bar{\xi}_{bn} = 0$ and that for $b \gg -\infty$ all $\xi_{bn} = 0$. Therefore, by induction on $N$ we obtain that all $\lim\limits_{n\to\infty} \alpha_{an} = 0$ and there is a compact $C \subset L$ containing all $\xi_n$.

Inversely, suppose that for all $a \in \mathbb{Z}^N$, $\lim\limits_{n\to\infty} \alpha_{an} = 0$ and all $\xi_n$ belong to a compact $C \subset L$. Then by induction on $N$, for any $b \in \mathbb{Z}$, $\lim\limits_{n\to\infty} \bar{\xi}_{bn} = 0$. Let $b_0 \in \mathbb{Z}$ be such that all $\bar{\xi}_{bn} = 0$ if $b < b_0$. Take any $U = \sum_b u_1^b s(U_b) \in \mathcal{C}_{L,\{u_1,\dots,u_N\}}$. Then there is $b_1 \in \mathbb{Z}$ such that $U_b = L^{(1)}$ for all $b > b_1$. For $b_0 < b \leqslant b_1$, let $m(b) \in \mathbb{Z}$ be such that $\bar{\xi}_{bn} \in U_b$ if $n \geqslant m(b)$. Then for $n \geqslant \max\{ m(b) \,|\, b_0 < b \leqslant b_1\}$, $\xi_n \in U$, i.e. $\lim\limits_{n\to\infty} \xi_n = 0$. $\square$

In terms of the power series (1.1), the $N$-dimensional valuation ring $\mathcal{O}_L$, resp. the maximal ideal $\mathrm{m}_L$, of $L$ consists of the elements $x$ such that all $\alpha_a = 0$ if $a < \bar{0} = (0, \dots, 0)$, resp. $a \leqslant \bar{0}$, with respect to the lexicographic ordering. Note that $L$, $\mathcal{O}_L$ and $\mathrm{m}_L$ are $P$-topological additive groups. Multiplication does not make $L^*$ into a topological group, but all operations in the field $L$ are sequentially $P$-continuous. The choice of local parameters

$u_1,\dots,u_N$ provides an isomorphism $L^*\simeq k^*\times\langle u_1\rangle\times\dots\langle u_N\rangle\times(1+\mathrm{m}_L)^\times$, where only the last factor has a non-trivial $P$-topological structure.

The concept of $P$-topology plays a very important role in this paper and we refer usually to the papers [22] and [10] for its detailed exposition. In particular, these papers contain the study of infinite products in $L$. The following fact clarifies the meaning of infinite products and will be used below without special references. Suppose $I_1,\dots,I_N(a_1,\dots,a_{N-1})$ are the above defined parameters. Consider the infinite product of the form $\prod\limits_{a>\bar{0}}(1+[\alpha_a]u_1^{a_1}\dots u_N^{a_N})$, where, as earlier, $a=(a_1,\dots,a_N)\in\mathbb{Z}^N$, $[\alpha_a]$ are the Teichmüller representatives of elements $\alpha_a\in k$ and $\alpha_a=0$ if either $a_1<I_1$, or $a_2<I_2(a_2)$, $\dots$, or $a_N<I_N(a_1,\dots,a_{N-1})$. Then any such infinite product converges in $L^*$ and any element from $1+\mathrm{m}_L$ can be presented uniquely as a such infinite product (with suitably chosen parameters $I_1,\dots,I_N(a_1,\dots,a_N)$). This follows from very general criterion 1.4.3 in [22].

The main object we shall deal with is an $N$-dimensional local field $F$ of characteristic 0 with first residue field $F^{(1)}$ of charactersitic $p$, $N$-th residue field $k$ (which is necessarily finite) and a fixed system of local parameters $\pi_1,\dots,\pi_N$. Fix an algebraic closure $\bar{F}$ of $F$, set $\Gamma_F=\mathrm{Gal}(\bar{F}/F)$ and denote by $\Gamma_F^{ab}$ the maximal abelian quotient of $\Gamma_F$.

We also consider $N$-dimensional local fields of characteristic $p$ with last residue field $k$. Any such field $\mathcal{F}$ is isomorphic to the field of formal Laurent power series $k((\bar{t}_N))\dots((\bar{t}_1))$, where $\bar{t}_1,\dots,\bar{t}_N$ is any system of local parameters of $\mathcal{F}$. We use this system of local parameters as a $p$-basis for $\mathcal{F}$ to construct a flat $\mathbb{Z}_p$-lift $O(\mathcal{F})$ of $\mathcal{F}$ to characteristic 0. By definition $O(\mathcal{F})=\varprojlim\limits_n O_n(\mathcal{F})$, where for all $n\in\mathbb{N}$,

$$O_n(\mathcal{F})=W_n((t_N))\dots((t_1))\subset W_n(\mathcal{F})$$

are $\mathbb{Z}/p^n$-flat lifts of $\mathcal{F}$ and for $1\leqslant i\leqslant N$, $t_i=[\bar{t}_i]$ are the Teichmüller representatives of $\bar{t}_i$.

The lift $O(\mathcal{F})$ is a complete discrete valuation ring of the $(N+1)$-dimensional local field $L(\mathcal{F})=\mathrm{Frac}\,O(\mathcal{F})$. Note that $L(\mathcal{F})^{(1)}=\mathcal{F}$ and $L(\mathcal{F})$ has a fixed system of local parameters $p,t_1,\dots,t_N$ such that for $1\leqslant i\leqslant N$, $t_i\bmod p=\bar{t}_i$. The elements of $L(\mathcal{F})$ can be written as formal power series $\sum_a\gamma_a t_1^{a_1}\cdots t_N^{a_N}$ with natural conditions on the coefficients $\gamma_a\in W(k)$, where $a=(a_1,\dots,a_N)\in\mathbb{Z}^N$.

**1.2. $P$-topological bases of $\mathcal{F}^*$ and $F^*/F^{*p^M}$.** The concept of $P$-topology allows us to describe explicitly the structure of the multiplicative groups $\mathcal{F}^*$ and of $F^*/F^{*p^M}$ under the additional assumption that $\zeta_M\in F$.

Consider the case of the field $\mathcal{F}=k((t_N))\dots((t_1))$. Choose an $\mathbb{F}_p$-basis $\theta_1,\dots,\theta_s$ of $k\simeq\mathbb{F}_{p^s}$. Then any element of $\mathcal{F}^*$ can be uniquely written as an infinite product as follows

$$\gamma\bar{t}_1^{a_1}\dots\bar{t}_N^{a_N}\prod_{j,b}(1+\theta_j\underline{t}^b)^{A_{jb}},$$

where $\gamma\in k^*$, $1\leqslant j\leqslant s$, $a_1,\dots,a_N\in\mathbb{Z}$, $b$ runs over the set of all multi-indices $(b_1,\dots,b_N)\in\mathbb{Z}^N\setminus p\mathbb{Z}^N$, $b>\bar{0}$, $\underline{\bar{t}}^b:=\bar{t}_1^{b_1}\dots\bar{t}_N^{b_N}$, and all $A_{jb}\in\mathbb{Z}_p$. The only essential condition on the above infinite product is that it must converge in $\mathcal{F}$ with respect to the $P$-topology. In particular, with the above notation the elements $\eta_{jb}:=1+\theta_j\underline{\bar{t}}^b$ form a set of free topological generators of the subgroup $(1+\mathrm{m}_{\mathcal{F}})^\times$ of $\mathcal{F}^*$.

Consider the case of the field $F$. In this case we have a similar description of the group $F^*/F^{*p^M}$ under the assumption that $F$ contains a primitive $p^M$-th root of unity $\zeta_M$.

Suppose $p=\pi_1^{e_1}\dots\pi_N^{e_N}\eta=\underline{\pi}^e\eta$, where $e=(e_1,\dots,e_N)\in\mathbb{Z}^N$ and $\eta\in\mathcal{O}_F^*$. Then Hensel's Lemma implies that any element of $F^*$ modulo $F^{*p^M}$ appears in the form

$$[\gamma]\pi_1^{a_1}\dots\pi_N^{a_N}\epsilon_0^{A_0}\prod_{j,b}\eta_{jb}^{A_{jb}},$$

where

— $a_1,\dots,a_N,A_0$ and all $A_{jb}$ are integers uniquely determined modulo $p^M$;

— $\eta_{jb}:=1+[\theta_j]\underline{\pi}^b$, where the multi-index $b=(b_1,\dots,b_N)$ runs over the set of all $b\in\mathbb{Z}^N\setminus p\mathbb{Z}^N$ such that $\bar{0}<b<e^*:=ep/(p-1)$;

— $\epsilon_0=1+[\theta_0]\underline{\pi}^{e^*}$, where $\theta_0\in k$ is such that $1+[\theta_0]\underline{\pi}^{e^*}\notin(1+\mathrm{m}_F)^p$.

**Remark.** 1) There is a more natural construction of the generator $\epsilon_0$ related to the concept of $p^M$-primary element. By definition, $\epsilon\in F^*$ is $p^M$-primary if the extension $F(\epsilon^{1/p^M})/F$ is purely unramified of degree $p^M$, i.e. the $N$-th residue fields satisfy $[F(\epsilon^{1/p^M})^{(N)}:F^{(N)}]=p^M$. Note that the images of $p^M$-primary elements in $F^*/F^{*p^M}$ form a cyclic group of order $p^M$. One of first explicit constructions of $p^M$-primary elements was given by Hasse, cf. [19], and can be explained as follows. Let $\xi\in\mathrm{m}_F$ be such that $E(1,\xi)=\zeta_M$. Let $\alpha_0\in W(k)$ be such that $\mathrm{Tr}(\alpha_0)=1$ and let $\beta\in W(\bar{k})$ be such that $\sigma(\beta)-\beta=\alpha_0$. Then $\epsilon_0=E(\beta,\xi)^{p^M}$ is a $p^M$-primary element of $F$. In Section 5 we shall use the $p^M$-primary element in the form $\epsilon_0=\theta(\alpha_0H_0)$, where $H_0=H_0(\zeta_M)\in\mathrm{m}^0$ was defined in the Introduction. A natural explanation of this construction of $p^M$-primary element appears there as a special case of the relation between the Witt-Artin-Schreier and Kummer theories.

2) The original construction of the Shafarevich basis [18] systematically uses the Shafarevich exponential $E(w,X)$ and establishes an explicit isomorphism $F^*/F^{*p^M}\simeq\langle\bar{\pi}_1\rangle^{\mathbb{Z}/p^M}\times\langle\bar{\epsilon}_0\rangle^{\mathbb{Z}/p^M}\times\prod_b W_M(k)_b$, where $0<b<$

$e^*$, $\bar{\pi}_1 = \pi_1 \bmod F^{*p^M}$, $\bar{\epsilon}_0 = \epsilon_0 \bmod F^{*p^M}$ and $gcd(b,p) = 1$, in the 1-dimensional case. This construction can be generalised to the $N$-dimensional case.

**1.3. Topological Milnor $K$-groups.** For a higher local field $L$ and a positive integer $n$, let $K_n(L)$ be the $n$-th Milnor $K$-group of $L$. Let $VK_n(L)$ be the subgroup of $K_n(L)$ generated by the symbols having at least one entry in $V_L := 1 + \mathrm{m}_L$. If $L$ is of dimension $N$ and $u_1, \dots, u_N$ is a system of local parameters of $L$, then, by [22],

$$K_N(L) \simeq VK_N(L) \oplus \mathbb{Z} \oplus \prod_{1 \leqslant i \leqslant N} A_{iN}(L), \tag{1.2}$$

where $\mathbb{Z}$ corresponds to the subgroup generated by $\{u_1, \dots, u_N\}$ and for all $1 \leqslant i \leqslant N$, the group $A_{iN}(L) \simeq L^{(N)*}$ consists of the symbols of the form $\{[\alpha], u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N\}$ with $\alpha \in L^{(N)*}$.

Following [6, 22] we introduce the $P$-topology on $K_N(L)$ as follows. The topology on $VK_N(L)$ is defined to be the finest topology such that the map of topological spaces $V_L \times (L^*)^{N-1} \to VK_N(L)$ is sequentially continuous. The other direct summands in (1.2) are equipped with the discrete topology. Then the topological Milnor $K$-groups $K_N^t(L)$ are defined to be $K_N(L)/\Lambda$, where $\Lambda$ is the intersection of all neighbourhoods of zero, with the induced topology. By [6], $\Lambda = \bigcap_{n \geq 1} nVK_N(L) = \bigcap_{m \geq 1} p^m VK_N(L)$, using $l$-divisibility of $VK_N(F)$, for any $l$ prime to $p$. In particular, for any $M \geqslant 1$, $K_N^t(L)_M = K_N(L)_M$ and the decomposition (1.2) induces the decomposition $K_N^t(L) \simeq \mathbb{Z} \oplus VK_N^t(L)$.

The advantage of the topological $K$-groups $K_N^t(L)$ is that they admit $P$-topological generators analogous to those of the multiplicative group $L^*$ from Subsection 1.2. Before stating these results notice that for any higher local field $K$ one can introduce a filtration of $K_N^t(K)$ by the subgroups $U_K^c K_N^t(K)$, where $c \geqslant 0$. These subgroups are generated by the symbols $\{\alpha_1, \dots, \alpha_N\} \in K_N^t(K)$ such that $v_K(\alpha_1 - 1) \geqslant c$. Here $v_K$ is the 1-dimensional valuation on $K$ such that $v_K(K^*) = \mathbb{Z}$. Then the classical identity

$$\{1-\alpha, 1-\beta\} = \{\alpha(1-\beta), 1 + \alpha\beta(1-\alpha)^{-1}\} \tag{1.3}$$

for 2-dimensional Milnor $K$-groups implies that

$$\{\alpha_1, \dots, \alpha_N\} \in U_K^{c_1 + \dots + c_N} K_N^t(K)$$

if $v_K(\alpha_i - 1) \geqslant c_i$ for $1 \leqslant i \leqslant N$.

- *Generators of $K_N^t(\mathcal{F})$.*

For $a = (a_1, \dots, a_N) \in \mathbb{Z}^N$, $a \notin p\mathbb{Z}^N$, $a > \bar{0}$, let $1 \leqslant i(a) \leqslant N$ be such that $a_1 \equiv \dots \equiv a_{i(a)-1} \equiv 0 \bmod p$ but $a_{i(a)} \not\equiv 0 \bmod p$. As earlier, choose an

$\mathbb{F}_p$-basis $\theta_1, \dots, \theta_s$ of $k$ and for all above multi-indices $a$ and $1 \leqslant j \leqslant s$, set

$$\varepsilon_{ja} = \{1 + \theta_j \underline{\bar{t}}^a, \bar{t}_1, \dots, \bar{t}_{i(a)-1}, \bar{t}_{i(a)+1}, \dots, \bar{t}_N\}. \tag{1.4}$$

This is a system of free topological generators of $VK_N^t(\mathcal{F})$ and $K_N^t(\mathcal{F}) = VK_N^t(\mathcal{F}) \oplus \langle \varepsilon_0 \rangle$, where $\varepsilon_0 = \{\bar{t}_1, \dots, \bar{t}_N\}$. This means that any element $\xi \in K_N^t(\mathcal{F})$ can be written in the form $\xi = A_0\varepsilon_0 + \sum_{j,b} A_{jb}\varepsilon_{jb}$, where $A_0$ and all $A_{jb}$ belong to $\mathbb{Z}_p$ and, for any $1 \leqslant i_0 \leqslant N$, the infinite product

$$\prod_{j,b,i_0(b)=i_0} (1 + \theta_j \underline{t}^b)^{A_{jb}}$$

converges in $\mathcal{F}$. This can be obtained from relation (1.3). Moreover, for a given $\xi \in K_N^t(\mathcal{F})$, the corresponding coefficients $A_0$ and $A_{jb}$ are uniquely determined by $\xi$, in other words the above system of symbols $\varepsilon_0$ and $\varepsilon_{jb}$ is a system of free topological generators for $K_N^t(\mathcal{F})$. This was established by Parshin [15] via an analogue of the Witt pairing, cf. Subsection 2.2 below. It can be also deduced from the Bloch-Kato theorem [4], which gives an explicit description of the grading of the filtration $U_{\mathcal{F}}^c(K_N^t(\mathcal{F}))$, $c \geqslant 0$.

• *Generators of $K_N^t(F)_M$, $\zeta_M \in F$.*

Introduce similarly the elements

$$\varepsilon_{ja} = \{1 + [\theta_j]\underline{\pi}^a, \pi_1, \dots, \pi_{i(a)-1}, \pi_{i(a)+1}, \dots, \pi_N\}, \tag{1.5}$$

where $1 \leqslant j \leqslant s$, $a \in \mathbb{Z}^N \setminus p\mathbb{Z}^N$ and $\bar{0} < a < e^* := ep/(p-1)$. Set $\varepsilon_0 = \{\pi_1, \dots, \pi_N\}$ and for $1 \leqslant i \leqslant N$,

$$\varepsilon_{ie^*} = \{\epsilon_0, \pi_1, \dots, \pi_{i-1}, \pi_{i+1}, \dots, \pi_N\}, \tag{1.6}$$

where $\epsilon_0$ was defined in Subsection 1.2.

Then for similar reasons to the case $L = \mathcal{F}$, the above elements $\varepsilon_0$, $\varepsilon_{ja}$ and $\varepsilon_{ie^*}$ give a set of $P$-topological generators of the $\mathbb{Z}/p^M$-module $K_N^t(F)_M$. The Bloch-Kato theorem [4] about the gradings of $U_F^c(K_N^t(F))$, where $0 \leqslant c \leqslant e' = v_F(p)p/(p-1)$, implies that the system of topological generators (1.6) is a topological $\mathbb{Z}/p$-basis of $K_N^t(F)_1$. The fact that we have a system of $\mathbb{Z}/p^M$-free topological generators can be deduced from the description of $p$-torsion in $K_N(F)$ from [6]. This fact can be also established directly from the non-degeneracy of Vostokov's pairing, cf. Section 3.

• *Generators of $K_{N-1}^t(F)_M$ and $K_{N+1}^t(F)_M$, $\zeta_M \in F$.*

A similar technique can be used to prove that $K_{N-1}^t(F)_M$ is topologically generated by the elements of the form:

— $\{1+[\theta_j]\underline{\pi}^a, \pi_{j_1}, \dots, \pi_{j_{N-2}}\}$, where $1 \leqslant j \leqslant s$, $a \in \mathbb{Z}^N \setminus p\mathbb{Z}^N$, $\bar{0} < a < e^*$, $1 \leqslant j_1 < \dots < j_{N-2} \leqslant N$ and $i(a) \notin \{j_1, \dots, j_{N-2}\}$.

— $\{\epsilon_0, \pi_{j_1}, \dots, j_{N_2}\}$, where $1 \leqslant j_1 < \dots < j_{N-2} \leqslant N$.

Similarly, in the case of $K^t_{N+1}(F)_M$ we have only one generator given by the symbol $\{\epsilon_0, \pi_1, \dots, \pi_N\}$.

**1.4. The Norm map.** For a finite extension of higher local fields $L/K$, the Norm-map of Milnor $K$-groups $N_{L/K} : K_n(L) \to K_n(K)$ was defined in [5] and [7]. It has the following properties:

(1) if $\alpha_1 \in L^*$ and $\alpha_2, \dots, \alpha_n \in K^*$ then

$$N_{L/K}\{\alpha_1, \alpha_2, \dots, \alpha_n\} = \{N_{L/K}(\alpha_1), \alpha_2, \dots, \alpha_n\};$$

(2) for a tower of finite field extensions $F \subset M \subset L$, it holds $N_{L/F} = N_{L/M} \circ N_{M/F}$;

(3) if $i_{L/K} : K_n(K) \longrightarrow K_n(L)$ is induced by the embedding $K \subset L$ then $i_{L/K} \circ N_{L/K} = [L : K] \,\mathrm{id}_{K_n(K)}$.

By [22], $N_{L/F}$ is sequentially $P$-continuous and therefore induces a continuous morphism of the corresponding topological $K$-groups which will be denoted by the same symbol.

Using the unique extension of $v_K$, define the subgroups $U^c_K(K^t_N(L)) \subset K^t_N(L)$ for all $c \geqslant 0$ and algebraic extensions $L$ of $K$, to be the groups generated by the symbols $\{\alpha_1, \dots, \alpha_N\}$ such that $v_K(\alpha_1 - 1) \geqslant c$. Then the general definition of the norm map $N_{L/K}$, e.g. cf. [7], implies that for all $c \geqslant 0$, $N_{L/K}$ maps $U^c_K(K^t_N(L))$ to $U^c_K(K^t_N(K))$ and preserves the decomposition $K^t_N(L) = \mathbb{Z} \oplus VK^t_N(L)$ from Subsection 1.3.

## 2. Pairings in the characteristic $p$ case

**2.1. Residues.** For any $n \geqslant 0$, denote by $\Omega^n_{L(\mathcal{F})}$ the $L(\mathcal{F})$-module of $P$-continuous differentials of degree $n$ for $L(\mathcal{F})$. For $n = N$, this module is free of rank 1 with the basis $dt_1 \wedge \dots \wedge dt_N$.

Suppose $\omega = f dt_1 \wedge \dots \wedge dt_N \in \Omega^N_{L(\mathcal{F})}$ with $f \in L(\mathcal{F})$. Then

$$f = \sum_{a=(a_0,\dots,a_N)} [\alpha_a] p^{a_0} t_1^{a_1} \dots t_N^{a_N}$$

and there is an $A_0(f) \in \mathbb{Z}$ such that $\alpha_a = 0$ if $a_0 < A_0(f)$, cf. Subsection 1.1. This makes sense for the following definition of the $L(\mathcal{F})$-residue $\mathrm{Res}_{L(\mathcal{F})}$ of $\omega$.

**Definition.** $\mathrm{Res}_{L(\mathcal{F})}(\omega) = \sum_{a=(a_0,-1,\dots,-1)} [\alpha_a] p^{a_0}$.

We have the following standard properties:

— if $\omega' \in \Omega^{N-1}_{L(\mathcal{F})}$ then $\mathrm{Res}_{L(\mathcal{F})}(d\omega') = 0$;

— if $\bar{t}'_1, \dots, \bar{t}'_N$ is another system of local parameters in $\mathcal{F}$ and $t'_1, \dots, t'_N$ are their lifts to $O(\mathcal{F})$ then

$$\mathrm{Res}_{L(\mathcal{F})}\left(\frac{dt'_1}{t'_1} \wedge \dots \wedge \frac{dt'_N}{t'_N}\right) = 1.$$

— if $\mathrm{Res}_{L(\mathcal{F})}\omega = c$ then there is an $\omega' \in \Omega^{N-1}_{L(\mathcal{F})}$ such that

$$\omega = d\omega' + c\frac{dt'_1}{t'_1} \wedge \dots \wedge \frac{dt'_N}{t'_N}$$

The above properties do not show that the residue $\mathrm{Res}_{L(\mathcal{F})}$ is apriori independent of the choice of local parameters of $\mathcal{F}$ because the construction of the lift $L(\mathcal{F})$ involves a choice of such system of local parameters. Therefore, we need to slightly modify the above approach to the concept of residue.

For any $i \in \mathbb{Z}$, denote by $O(\sigma^i\mathcal{F})$ the $\mathbb{Z}_p$-flat lifts of $\sigma^i\mathcal{F}$ via the system of local parameters $\bar{t}_1^{p^i}, \dots, \bar{t}_N^{p^i}$. Set $O_M(\sigma^i\mathcal{F}) := O(\sigma^i\mathcal{F})/p^M$. These flat $\mathbb{Z}/p^M$-lifts $O_M(\sigma^i\mathcal{F})$ of $\sigma^i(\mathcal{F})$ do depend on the system of local parameters $\bar{t}_1, \dots, \bar{t}_N$ but we have the following properties:

— $W_M(\sigma^{M-1}\mathcal{F}) \subset O_M(\mathcal{F}) \subset W_M(\mathcal{F}) \subset O_M(\sigma^{-M+1}\mathcal{F})$;

— $W_M(\mathcal{F}) = O_M(\mathcal{F}) + pO_{M-1}(\sigma^{-1}\mathcal{F}) + \dots + p^{M-1}O_1(\sigma^{-M+1}\mathcal{F})$.

Let $\widetilde{\Omega}(\mathcal{F}, M)$ be the $\mathbb{Z}_p$-submodule of $\Omega^N_{W_M(\mathcal{F})}$ consisting of differential forms $\omega = w\, d_{\log}a_1 \wedge \dots \wedge d_{\log}a_N$, where $w \in W_M(\sigma^{M-1}\mathcal{F})$ and all $a_i \in W_M(\mathcal{F})^*$. Then $w \in O_M(\mathcal{F})$, all $a_i \in O_M(\sigma^{1-M}\mathcal{F})^*$ and, therefore, $\omega \in \sum_i O_M(\mathcal{F})d_{\log}t_i$. As a result, we have a natural $W_M(k)$-linear embedding

$$\iota_{O_M(\mathcal{F})} : \widetilde{\Omega}(\mathcal{F}, M) \longrightarrow O_M(\sigma^{-M+1}\mathcal{F}) \otimes_{O_M(\mathcal{F})} \Omega^N_{O_M(\mathcal{F})}.$$

This means that for any $\omega \in \widetilde{\Omega}(\mathcal{F}, M)$, the image $\iota_{O_M(\mathcal{F})}(\omega)$ can be written uniquely as $f d_{\log}t_1 \wedge \dots \wedge d_{\log}t_N$, where $f = \sum_a \gamma_a t_1^{a_1} \dots t_N^{a_N}$, with the indices $a = (a_1, \dots, a_N) \in (p^{-M+1}\mathbb{Z})^N$ and the coefficients $\gamma_a \in W_M(k)$.

**Definition.** With above notation for any $\omega \in \widetilde{\Omega}(\mathcal{F}, M)$, define its $W_M(\mathcal{F})$-residue by the relation $\mathrm{Res}_{W_M(\mathcal{F})}(\omega) := \gamma_{(0,\dots,0)}$.

This definition is compatible with the earlier definition of the $L(\mathcal{F})$-residue $\mathrm{Res}_{L(\mathcal{F})}$ in the following sense. If $\omega \in \Omega^N_{O(\mathcal{F})} \subset \Omega^N_{L(\mathcal{F})}$ and $\omega \bmod p^M$ is in the image of $\widetilde{\Omega}(\mathcal{F}, M)$ in $O_M(\sigma^{-M+1}\mathcal{F}) \otimes_{O(\mathcal{F})} \Omega^N_{O(\mathcal{F})}$ then $\mathrm{Res}_{L(\mathcal{F})}(\omega) \in W(k)$ and $\mathrm{Res}_{L(\mathcal{F})}(\omega) \bmod p^M = \mathrm{Res}_{W_M(\mathcal{F})}(\omega \bmod p^M)$.

We now prove that the $W_M(\mathcal{F})$-residue $\mathrm{Res}_{W_M(\mathcal{F})}$ is independent of the choice of local parameters in $\mathcal{F}$. Suppose $\bar{t}'_1, \dots, \bar{t}'_N$ is another system of local parameters of $\mathcal{F}$. Consider, for all $i \in \mathbb{Z}$, the corresponding flat lifts $O'_M(\sigma^i\mathcal{F})$ and the $W_M(\mathcal{F})$-residue $\mathrm{Res}'_{W_M(\mathcal{F})}$ defined via an analogue $\iota_{O'_M(\mathcal{F})}$ of $\iota_{O_M(\mathcal{F})}$.

**Proposition 2.1.** *For any $\omega \in \widetilde{\Omega}(\mathcal{F}, M)$, $\mathrm{Res}_{W_M(\mathcal{F})}(\omega) = \mathrm{Res}'_{W_M(\mathcal{F})}(\omega)$.*

*Proof.* Note that any $\alpha \in W_M(\mathcal{F})^*$ can be written in the form $[\beta]\underline{t}^a\varepsilon\eta$, where $[\beta] \in W_M(k)$ is the Teichmüller representative of $\beta \in k$, $\underline{t}^a :=$

$t_1^{a_1}\dots t_N^{a_N}$ with $a=(a_1,\dots,a_N)\in\mathbb{Z}^N$, $\varepsilon\in(1+\mathrm{m}_{L(\mathcal{F})})\mathrm{mod}p^MO(\mathcal{F})$ and $\eta\in 1+pO_{M-1}(\sigma^{-1}\mathcal{F})+\dots+p^{M-1}O_1(\sigma^{-M+1}\mathcal{F})=1+pW_{M-1}(\sigma^{-1}\mathcal{F})$.

With this notation any element of $d_{\log}W_M(\mathcal{F})^*$ can be written as

$$\sum_{1\leqslant i\leqslant N}C_id_{\log}t_i+d_{\log}\varepsilon+d\eta',$$

where $C_1,\dots,C_N\in\mathbb{Z}$ and $\eta'=\log\eta\in pW_{M-1}(\sigma^{-1}\mathcal{F})$. Note that the $p$-adic logarithm establishes an isomorphism of the multiplicative group $1+pW_{M-1}(\sigma^{-1}\mathcal{F})$ with the additive group $pW_{M-1}(\sigma^{-1}\mathcal{F})$ and

$$d_{\log}\varepsilon=-\sum_a(\gamma_a\underline{t}^a)^nd_{\log}\underline{t}^a,$$

where $a=(a_1,\dots,a_n)\in\mathbb{Z}^N$, $a>\bar{0}$, $n\geqslant 0$, all $\gamma_a\in W_M(k)$ and the sum in the right-hand side converges in the $P$-topology. (Use that $\varepsilon$ can be written as an infinite product $\prod_{a>\bar{0}}(1-\gamma_a\underline{t}^a)$.)

Therefore, any element $\omega\in\widetilde{\Omega}(\mathcal{F},M)$ can be written as a sum of the following types of elements:

(i) $\gamma(\omega)d_{\log}t_1\wedge\dots\wedge d_{\log}t_N$ with $\gamma(\omega)\in W_M(k)$;

(ii) $md_{\log}t_1\wedge\dots\wedge d_{\log}t_N$ with $m\in\mathrm{m}_{L(\mathcal{F})}\mathrm{mod}p^MO(\mathcal{F})$;

(iii) $d_{\log}t_{i_1}\wedge\dots\wedge d_{\log}t_{i_s}\wedge d(\eta_1)\wedge\dots\wedge d(\eta_{N-s})$, where $0\le s<N$, $1\leqslant i_1<\dots<i_s\leqslant N$ and $\eta_1,\dots,\eta_{N-s}\in pW_{M-1}(\sigma^{-1}\mathcal{F})$.

This follows directly from the above description of the elements of $d_{\log}W_M(\mathcal{F})^*$ by taking into account that for $a\in\mathbb{Z}^N$ such that $a\geqslant\bar{0}$ and $\eta\in pW_{M-1}(\sigma^{-1}\mathcal{F})$, we have

$$(\underline{t}^a)^nd_{\log}(\underline{t}^a)\wedge d\eta=\sum_{1\leqslant i\leqslant N}a_id_{\log}t_i\wedge d(\underline{t}^{an}\eta),$$

and $d(w\eta)=wd(\eta)$ for any $\omega\in W_{M-1}(\sigma^{M-1}\mathcal{F})$.

Then it can be seen that $\mathrm{Res}_{W_M(\mathcal{F})}(\omega)=\gamma(\omega)$ by noting that the residues of elements of the form (ii)-(iii) are equal to 0.

Finally, it remains to verify that $\mathrm{Res}_{W_M(\mathcal{F})}(\omega)=1$ for differential forms $\omega=d_{\log}t'_1\wedge\dots\wedge d_{\log}t'_N$. This can be done also along the lines of the above calculations. $\square$

**2.2. The Witt symbol.** We introduce the $W_M(\mathbb{F}_p)$-linear pairing

$$[\ ,\ \}_M^{\mathcal{F}}:W_M(\mathcal{F})\times K_N^t(\mathcal{F})\longrightarrow W_M(k) \tag{2.1}$$

as follows. Suppose $w=(w_0,\dots,w_{M-1})\in W_M(\mathcal{F})$ and $\alpha\in K_N(\mathcal{F})$ is of the form $\alpha=\{\alpha_1,\dots,\alpha_N\}\in K_N(\mathcal{F})$. For $1\leqslant i\leqslant N$ and $\hat{\alpha}_i\in W_M(\mathcal{F})^*$ such that $\hat{\alpha}_i\,\mathrm{mod}\,p=\alpha_i$, set

$$[w,\alpha\}_M^{\mathcal{F}}=\mathrm{Res}_{W_M(\mathcal{F})}(\sigma^{M-1}(w)d_{\log}\hat{\alpha}_1\wedge\dots\wedge d_{\log}\hat{\alpha}_N)\in W_M(k). \tag{2.2}$$

It can be seen that $[w,\alpha\}_M^{\mathcal{F}}$ is well-defined and the pairing it induces factors through the natural projection $K_N(\mathcal{F})\longrightarrow K_N^t(\mathcal{F})$.

**Lemma 2.1.** *For any $w \in W_M(\mathcal{F})$ and $\alpha \in K_N^t(\mathcal{F})$, we have*

$$\sigma[w, \alpha\}_M^{\mathcal{F}} = [\sigma(w), \alpha\}_M^{\mathcal{F}}.$$

*Proof.* Note that for varying systems of local parameters $\bar{t}'_1, \dots, \bar{t}'_N$ of $\mathcal{F}$, the symbols $\{\bar{t}'_1, \dots, \bar{t}'_N\}$ generate the group $K_N^t(\mathcal{F})$. Therefore, it is sufficient to consider only the symbols $\alpha = \{\bar{t}'_1, \dots, \bar{t}'_N\}$. By Proposition 2.1, the symbol $[\ ,\ \}_M^{\mathcal{F}}$ is independent of the choice of local parameters. Therefore, we may assume that $\alpha = \{\bar{t}_1, \dots, \bar{t}_N\}$. Consider the expansion $\sigma^{M-1}w = \sum_a \gamma_a t_1^{a_1} \cdots t_N^{a_N}$, where all $\gamma_a \in W_M(k)$, with respect to the identification $W_M(\sigma^{M-1}\mathcal{F}) = O_M(\sigma^{M-1}\mathcal{F}) + \dots + p^{M-1}O_1(\mathcal{F})$. Clearly, $[w, \alpha)_M^{\mathcal{F}} = \gamma_{(0,\dots,0)}$ and $[\sigma(w), \alpha)_M^{\mathcal{F}} = \sigma(\gamma_{(0,\dots,0)})$. The lemma is proved. $\square$

Notice now that for any $w = (w_1, \dots, w_M) \in W_M(\mathcal{F})$, the element $\sigma^{M-1}w = [w_1]^{p^{M-1}} + p[w_2]^{p^{M-2}} + \dots + p^{M-1}[w_M]$ coincides, modulo $p^M$, with the $M$-th ghost component of $w$. Therefore, the classical Witt symbol, cf. [14],

$$[\ ,\ )_M^{\mathcal{F}} : W_M(\mathcal{F}) \times K_N^t(\mathcal{F}) \longrightarrow W_M(\mathbb{F}_p)$$

has the following invariant form:

— *If $w \in W_M(\mathcal{F})$ and $\alpha \in K_N^t(\mathcal{F})$ then* $[w, \alpha)_M^{\mathcal{F}} = \mathrm{Tr}\left([w, \alpha\}_M^{\mathcal{F}}\right)$.

Above Lemma 2.1 implies that the Witt symbol induces a $W_M(\mathbb{F}_p)$-linear pairing

$$W_M(\mathcal{F})/(\sigma - \mathrm{id})W_M(\mathcal{F}) \ \times\ K_N^t(\mathcal{F}) \longrightarrow W_M(\mathbb{F}_p) \tag{2.3}$$

and it can be verified that this pairing is non-degenerate using the explicit formula (2.2) for the above symbol $[\ ,\ \}_M^{\mathcal{F}}$.

**2.3. Coleman's lifts and Fontaine's pairing.** For 1-dimensional local fields, Fontaine [8] developed a version of the Witt symbol by defining a special multiplicative section $\mathrm{Col} : \mathcal{F}^* \longrightarrow O(\mathcal{F})^*$ of the natural projection $O(\mathcal{F}) \longrightarrow O(\mathcal{F})/p = \mathcal{F}$. His construction can be generalised in the context of topological $K$-groups as follows.

For any $x \in O_{L(\mathcal{F})}$, let $\bar{x} = (xp^{-v_p(x)}) \bmod p \in \mathcal{F}$. Consider the map $\Pi : K_N^t(L(\mathcal{F})) \to K_N^t(\mathcal{F})$ defined by the correspondences

$$\{x_1, \dots, x_N\} \mapsto \{\bar{x}_1, \dots, \bar{x}_N\}.$$

We use the free topological generators of $K_N^t(\mathcal{F})$ from Subsection 2.2 to define the $P$-continuous homomorphism $\mathrm{Col} : K_N^t(\mathcal{F}) \longrightarrow K_N^t(L(\mathcal{F}))$ by the following correspondences: $\{\bar{t}_1, \dots, \bar{t}_N\} \longmapsto \{t_1, \dots, t_N\}$ and $\varepsilon_{ja} \mapsto \{1 + [\theta_j]\underline{t}^a, t_1, \dots, t_{i(a)-1}, t_{i(a)+1}, \dots, t_N\}$. This definition makes sense because of the following property, cf. [22].

**Lemma 2.2.** *If for p-adic integers $A_{ja} \in \mathbb{Z}_p$, where $1 \leqslant j \leqslant s$ and $a \in \mathbb{Z}^N \setminus p\mathbb{Z}^N$, $a > \bar{0}$, the product $\prod_{ja}(1+\theta_j \underline{\bar{t}}^a)^{A_{ja}}$ converges in $\mathcal{F}$ then the product $\prod_{ja}(1+[\theta_j]\underline{t}^a)^{A_{ja}}$ converges in $L(\mathcal{F})$.*

The above defined morphism Col depends on the choice of local parameters $\bar{t}_1, \dots, \bar{t}_N$ of $\mathcal{F}$. As in Subsection 2.1 consider the lift $O(\sigma^{-1}\mathcal{F})$. Then $L(\sigma^{-1}\mathcal{F}) = \operatorname{Frac} O(\sigma^{-1}\mathcal{F})$ is a field extension of $L(\mathcal{F})$ of degree $p^M$. Let $\sigma^{-1}$ be the $\sigma^{-1}$-linear (with respect to the $W(k)_{\mathbb{Q}_p}$-module structure) field isomorphism $L(\mathcal{F}) \longrightarrow L(\sigma^{-1}\mathcal{F})$ given, for $1 \leqslant i \leqslant N$, by the correspondences $t_i \longrightarrow t_i^{1/p}$.

**Definition.** Call an element $x \in K_N^t(L(\mathcal{F}))$ Coleman if the norm map from $K_N^t(L(\sigma^{-1}\mathcal{F}))$ to $K_N^t(L(\mathcal{F}))$ maps $\sigma^{-1}(x)$ to $x$.

**Proposition 2.2.** *An element $\eta \in K_N^t(L(\mathcal{F}))$ is Coleman if and only if it belongs to* $\operatorname{Col}(K_N^t(\mathcal{F}))$.

*Proof.* Property (1) of Subsection 1.4 easily implies that all elements from $\operatorname{Col}(K_N^t(\mathcal{F}))$ are Coleman.

Suppose $x_0 \in K_N^t(L(\mathcal{F}))$ is Coleman. We prove that $x_0 \in \operatorname{Col} K_N^t(\mathcal{F})$. Shifting $x_0$ by the inverse to $\operatorname{Col}(\Pi(x_0))$ we may assume that $\Pi(x_0) = 0$.

Note that $L(\mathcal{F})$ has the system of local parameters $t_0 = p$, $t_1 = \operatorname{Col}(\bar{t}_1)$, $\dots, t_N = \operatorname{Col}(\bar{t}_N)$. Then the classical identity (1.5) implies that $K_N^t(L(\mathcal{F}))$ is topologically generated by the elements

$$\{t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_N\}$$

with $1 \leqslant i \leqslant N$, and the elements of the form

$$\{1 + [\theta_j]t_0^{a_0} \dots t_N^{a_N}, \alpha_2, \dots, \alpha_N\},$$

where $1 \leqslant j \leqslant s$, $\theta_1, \dots, \theta_s$ is an $\mathbb{F}_p$-basis of $k$, $a = (a_0, \dots, a_N) \in \mathbb{Z}^{N+1} \setminus p\mathbb{Z}^{N+1}$, $a > \bar{0}$ and for $2 \leqslant i \leqslant N$, $\alpha_i = t_{j_i}$ with $0 \leqslant j_2 < j_3 < \dots < j_N \leqslant N$.

These generators can be separated into the two following groups:

— the first group contains the generators belonging to $\operatorname{Ker} \Pi$ (in other words these generators do depend on $t_0$);

— the second group contains the generators from $\operatorname{Col}(K_N^t(\mathcal{F}))$.

Using that $K_N^t(\mathcal{F})$ is topologically free, we obtain for any $x \in K_N^t(\mathcal{F})$ the following properties:

— if $\Pi(x) = 0$ then $x$ is a product of generators from the first group;

— if $\Pi(x) = 0$ and $x = p^m x_1$ with $m \geqslant 0$ and $x_1 \in K_N^t(L(\mathcal{F}))$, then $\Pi(x_1) = 0$.

Returning to the Coleman element $x_0 \in K_N^t(L(\mathcal{F}))$, assume that there is an $m \geqslant 0$ such that $x_0 = p^m x_1$ with $x_1 \in K_N^t(L(\mathcal{F}))$ but $x \notin pK_N^t(L(\mathcal{F}))$. Then $\Pi(x_1) = 0$ and $x_1$ is a product of generators from the first group. But if $y$ is a generator from this group then property 1) of Subsection 1.4 implies that $N_{L(\sigma^{-1}\mathcal{F})/L(\mathcal{F})}(\sigma^{-1}y) \in pK_N^t(L(\mathcal{F}))$. This gives that $x_0 =$

$N_{L(\sigma^{-1}\mathcal{F})/L(\mathcal{F})}(\sigma^{-1}x_0) = p^m N_{L(\sigma^{-1}\mathcal{F})/L(\mathcal{F})}(\sigma^{-1}x_1) \in p^{m+1}K_N^t(L(\mathcal{F}))$, which is a contradiction. This means that $x_0$ is infinitely $p$-divisible and, therefore, is 0 in $K_N^t(L(\mathcal{F}))$. $\square$

We define an analogue of Fontaine's pairing [8]

$$[\ ,\ \rangle^{\mathcal{F}} : O(\mathcal{F}) \times K_N^t(\mathcal{F}) \longrightarrow \mathbb{Z}_p$$

by setting for $f \in O(\mathcal{F})$ and $\alpha \in K_N(\mathcal{F})$,

$$[f, \alpha\rangle^{\mathcal{F}} = \mathrm{Tr}\big(\mathrm{Res}_{L(\mathcal{F})} f d_{\log}\mathrm{Col}(\alpha)\big).$$

Here, for $\mathrm{Col}(\alpha) = \{\hat{\alpha}_1, \dots, \hat{\alpha}_N\}$, we set $d_{\log}\mathrm{Col}(\alpha) = d_{\log}\hat{\alpha}_1 \wedge \dots \wedge d_{\log}\hat{\alpha}_N$. This pairing is related to the Witt symbol by the following Proposition.

**Proposition 2.3.** *For all $f \in O(\mathcal{F})$ and $\alpha \in K_N^t(\mathcal{F})$, one has*

$$[f, \alpha\rangle^{\mathcal{F}} \bmod p^M = [f \bmod p^M, \alpha \bmod p^M)_M^{\mathcal{F}}.$$

*Proof.* We need to show that

$$\sigma^{M-1}\mathrm{Res}_{L(\mathcal{F})}(f d_{\log}\mathrm{Col}\alpha) = \mathrm{Res}_{L(\mathcal{F})}(\sigma^{M-1}(f) d_{\log}\mathrm{Col}(\alpha)). \tag{2.4}$$

By linearity and $P$-continuity this can be verified on the generating elements $f = \underline{t}^b = t_1^{b_1} \dots t_N^{b_N}$, $b = (b_1, \dots, b_N) \in \mathbb{Z}^N$, of $O(\mathcal{F})$ and the generators $\alpha = \{\bar{t}_1 1, \dots, \bar{t}_N\}$ and $\alpha = \varepsilon_{ja}$ of $K_N^t(\mathcal{F})$ from (2.4) of Subsection 1.3.

— *The case $f = \underline{t}^b$ and $\alpha = \{\bar{t}_1, \dots, \bar{t}_N\}$*. In this case the both sides of equality (2.4) are equal to the Kronecker symbol $\delta(b, \bar{0})$.

— *The case $f = \underline{t}^b$ and $\alpha = \varepsilon_{ja}$*. Here the left-hand side of (2.4) equals

$$(-1)^{i(a)-1}\sigma^{M-1}\mathrm{Res}_{L(\mathcal{F})}\left(\underline{t}^b \sum_{n\geqslant 0}(-1)^n[\theta_j^n]\underline{t}^{na}\frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_N}{t_N}\right)$$

and the corresponding right-hand side equals

$$(-1)^{i(a)-1}\mathrm{Res}_{L(\mathcal{F})}\left(\underline{t}^{p^{M-1}b} \sum_{n\geqslant )}(-1)^n[\theta_j^n]\underline{t}^{na}\frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_N}{t_N}\right).$$

Clearly, we may assume that $b \neq \bar{0}$. Then the left-hand side is non-zero if and only if there is an $n_0 \geqslant 1$ such that $b + n_0 a = \bar{0}$. This is equivalent to saying that the right-hand side is non-zero noting that $a \notin p\mathbb{Z}^N$. It can then be seen that both sides are equal to $(-1)^{i(a)+n_0-1}a_{i(a)}[\theta_j^{p^{M-1}n_0}]$. $\square$

## 3. Vostokov's pairing

As usual, $\pi_1, \dots, \pi_N$ is a fixed system of local parameters and $k$ is the $N$-th residue field of $F$. Let $L_0(\mathcal{F}) = W(k)((t_N))\dots((t_1)) \subset L(\mathcal{F})$ with the induced topological structure. Set

$$\mathrm{m}^0 = \left\{ \sum_{a > \bar{0}} w_a \underline{t}^a \mid w_a \in W(k) \right\}$$

and $O^0 = W(k) + \mathrm{m}^0$. Clearly, $\mathrm{m}^0 \subset O^0 \subset L_0(\mathcal{F})$ and $L_0(\mathcal{F}) = \bigcup_{a > \bar{0}} \underline{t}^{-a} O^0$. Let $\mathcal{R}$ be the multiplicative subgroup in $L_0(\mathcal{F})^*$ generated by the Teichmuller representatives of the elements of $k$, the indeterminants $t_1, \dots, t_N$ and the elements of $1 + \mathrm{m}^0$. Let $\kappa : \mathcal{R} \longrightarrow F$ be the epimorphic continuous morphism of $W(k)$-algebras such that $\kappa(t_i) = \pi_i$, where $1 \leqslant i \leqslant N$. We use the same notation $\kappa$ for the unique $P$-continuous epimorphism of $W(k)$-algebras $L_0(\mathcal{F}) \longrightarrow F$ such that $\kappa(t_i) = \pi_i$ for $1 \leqslant i \leqslant N$.

**3.1. The differential form $\Omega$.** For any $u_0, \dots, u_N \in \mathcal{R}$, denote by $\Omega = \Omega(u_0, \dots, u_N)$ the following differential form from $\Omega^N_{L_0(\mathcal{F})}$:

$$\sum_{0 \leqslant i \leqslant N} (-1)^i f_i \left( \frac{\sigma}{p} d_{\log} u_0 \right) \wedge \dots \wedge \left( \frac{\sigma}{p} d_{\log} u_{i-1} \right) \wedge d_{\log} u_{i+1} \wedge \dots \wedge d_{\log} u_N$$

where for $0 \leqslant i \leqslant N$, $f_i = (1/p) \log(u_i^p / \sigma u_i)$. Notice that all $f_i \in \mathrm{m}^0$ (use that $\sigma u_i / u_i^p \in 1 + \mathrm{m}^0$) and

$$df_i = d_{\log} u_i - (\sigma/p) d_{\log} u_i. \tag{3.1}$$

**Proposition 3.1.** $\Omega \bmod d\Omega^{N-1}_{O^0}$ *is skew symmetric in* $u_0, \dots, u_N$.

*Proof.* Prove that $\Omega \bmod d\Omega^{N-1}_{O^0}$ changes the sign under the transpositions $u_i \leftrightarrow u_{i+1}$, $0 \leqslant i < N$. Consider the identity (use (3.1))

$$f_i d_{\log} u_{i+1} - f_{i+1} (\sigma/p) d_{\log} u_i + f_{i+1} d_{\log} u_i - f_i (\sigma/p) d_{\log} u_{i+1} = d(f_i f_{i+1})$$

Then the form $\Omega(\dots, u_i, u_{i+1}, \dots) + \Omega(\dots, u_{i+1}, u_i, \dots)$ is congruent modulo $d\Omega^{N-1}_{O^0}$ to the form

$$(\sigma/p) d_{\log} u_0 \wedge \dots \wedge (\sigma/p) d_{\log} u_{i-1} \wedge d(f_i f_{i+1}) \wedge d_{\log} u_{i+1} \wedge \dots \wedge d_{\log} u_N$$

and, using again identity (3.1), we conclude that this form is exact. $\square$

Let $e = (e_1, \dots, e_N) \in \mathbb{Z}^N$ be such that $\pi_1^{e_1} \dots \pi_N^{e_N} / p \in \mathcal{O}_F^*$, where $\mathcal{O}_F$ is the $N$-dimensional valuation ring of $F$. We introduce the $W(k)$-algebra $\mathcal{L}^0 = O^0[[p/\underline{t}^{e(p-1)}, \underline{t}^{ep}/p]]$ and set $\mathcal{L} = \mathcal{L}^0 \otimes_{O^0} L_0(\mathcal{F})$. Clearly, we have $\mathcal{L} = \bigcup_{a > \bar{0}} \underline{t}^{-a} \mathcal{L}^0$.

The algebra $\mathcal{L}$ is a suitable completion of $L_0(\mathcal{F})$ and its elements can be treated as formal Laurent series in $t_1, \dots, t_N$ with coefficients in $W(k)_{\mathbb{Q}_p}$. Note first that any element of $\mathcal{L}^0$ can be written in the form

$$\sum_{n\geqslant 0} o_n \underline{t}^{epn}/p^n + \sum_{n\geqslant 1} o_{-n} p^n/\underline{t}^{e(p-1)n},$$

where all $o_n \in O^0$, $n \in \mathbb{Z}$. Therefore, $\mathcal{L}^0$ consists of formal Laurent series $\sum_{a\in\mathbb{Z}^N} w_a \underline{t}^a$ with coefficients $w_a \in W(k)_{\mathbb{Q}_p}$, such that for any $n \in \mathbb{Z}_{\geqslant 0}$:

— if $a \geqslant epn$ then $v_p(w_a) \geqslant -n$;

— if $-e(p-1)n > a \geqslant -e(p-1)(n+1)$ then $v_p(w_a) \geqslant n+1$.

We can use the above Laurent series to define the $\mathcal{L}$-residues $\operatorname{Res}_{\mathcal{L}} \omega$ for any $\omega \in \Omega^N_{\mathcal{L}}$. If any such form $\omega$ is the limit of $\omega_n \in \Omega^N_{L_0(\mathcal{F})}$, then $\operatorname{Res}_{\mathcal{L}} \omega$ is the limit of $\operatorname{Res} \omega_n$. Therefore, we can use for the $\mathcal{L}$-residue of $\omega$, the simpler notation $\operatorname{Res} \omega$.

**Lemma 3.1.** *Let $\pi_1^{e_1} \dots \pi_N^{e_N}/p = \eta \in \mathcal{O}_F^*$ and let $\hat{\eta} \in O^0$ be such that $\kappa(\hat{\eta}) = \eta$. Then the kernel of $\kappa : O^0(\mathcal{F}) \longrightarrow F$ is the principal ideal generated by $\underline{t}^e - p\hat{\eta}$.*

*Proof.* The proof follows easily from the fact that $\kappa$ induces a bijective map $O^0(\mathcal{F})/\underline{t}^e \to \mathcal{O}_F/p$. □

**Proposition 3.2.** *If $u_0 \in \mathcal{R}$ and $\kappa(u_0) = 1$ then there are $\omega^0, \omega^1 \in \Omega^{N-1}_{O^0}$ such that for $\Omega' = \log(u_0) d_{\log} u_1 \wedge \dots \wedge d_{\log} u_N - \frac{\sigma}{p} \log(u_0) \frac{\sigma}{p} d_{\log} u_1 \wedge \dots \wedge \frac{\sigma}{p} d_{\log} u_N$, it holds $\Omega = \Omega' + d(\log(u_0)\omega^0 + \omega^1)$.*

*Proof.* Clearly, $u_0 \in 1 + \mathrm{m}^0$. By above Lemma 3.1 the relation $\kappa(u_0) = 1$ implies that $\log(u_0) \in \mathcal{L}^0$. Then the statement of our Proposition is implied by the following identities: $f_0 = \log(u_0) - (\sigma/p)\log(u_0)$ and for $1 \leqslant i \leqslant N$,

$$f_i (\sigma/p) d_{\log} u_0 = d(f_i \log(u_0) - f_i f_0) - (\sigma/p)\log(u_0)(d_{\log} u_i - (\sigma/p) d_{\log} u_i).$$

□

**3.2. Element $H_0$.** As in the Introduction, choose a primitive $p^M$-th root of unity $\zeta_M \in F$ and introduce $H_0 \in \mathrm{m}^0$ such that $H_0 = H'^{p^M} - 1$, where $H' \in 1 + \mathrm{m}^0$ is such that $\kappa(H') \equiv \zeta_M \bmod pO_F$.

Clearly, we have $dH_0 \in p^M \Omega^1_{O^0}$.

**Lemma 3.2.** *a) There are $o_1 \in O^{0*}$ and $o_2, o_3 \in O^0$ such that*

*a) $H_0 = o_1 \underline{t}^{ep/(p-1)} + p o_2 \underline{t}^{e/(p-1)} + p^2 o_3$;*

*b) $H_0^{-1} \in \underline{t}^{-ep/(p-1)} O^0[[p\underline{t}^{-e}]] \subset \mathcal{L}$;*

*c) $H_0^{p-1}/p \in O^0[[\underline{t}^{ep}/p]] \subset \mathcal{L}^0$ and $O^0[[H_0^{p-1}/p]] = O^0[[\underline{t}^{ep}/p]]$;*

*d) $(\sigma/p)H_0 = H_0(1 + o_1 H_0 + o_2(H_0^{p-1}/p) + o_3(p^M/H_0))$, where the coefficients $o_1, o_2, o_3 \in O^0$ .*

*Proof.* In order to prove a) use that $H' \equiv 1 + o\underline{t}^{e/p^{M-1}(p-1)} \bmod (p, \underline{t}^e)$ with $o \in O^{0*}$. Then b) and c) are implied by a). For part d), use that $\sigma H' \equiv H'^p \bmod pO^0$ and therefore, $\sigma H_0 \equiv (1+H_0)^p - 1 \bmod p^{M+1}O^0$. □

**Lemma 3.3.** *If* $\omega = \log(u_0)\omega_1 + \omega_0$ *with* $\omega_0, \omega_1 \in \Omega^{N-1}_{O^0}$ *then we have* $\mathrm{Res}(H^{-1}d\omega) \in p^M W(k)$.

*Proof.* Note that $\mathrm{Res}(H^{-1}d\omega) = -\mathrm{Res}(d(1/H) \wedge \omega)$, because obviously $\omega/H^2 \in \underline{t}^{-2ep/(p-1)}\mathcal{L}^0\Omega^{N-1}_{O^0}$ and $dH \in p^M\Omega^1_{O^0}$. It follows that

$$d(1/H) \wedge \omega = (\omega/H^2) \wedge dH \in p^M \underline{t}^{-2ep/(p-1)}\mathcal{L}^0\Omega^N_{O^0}.$$

It remains to notice that $ep \geqslant 2ep/(p-1)$, which implies that

$$\mathrm{Res}(t^{-2ep/(p-1)}\mathcal{L}^0 dt_1 \wedge \cdots \wedge dt_N) \subset W(k).$$

□

**Corollary 3.1.** *With the notation from Proposition* 3.2*, we have*

$$\mathrm{Res}(\Omega/H) \equiv \mathrm{Res}(\Omega'/H) \bmod p^M.$$

**Proposition 3.3.** *If* $h \in O^0[[\underline{t}^{ep}/p]]$ *then*

$$\mathrm{Res}\left(\frac{h d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N}{H}\right) \equiv \mathrm{Res}\left(\frac{h d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N}{(\sigma/p)H}\right) \bmod p^M.$$

*Proof.* We follow the strategy from the proof of Lemma 3.1.3 in [1].

By Lemma 3.2d) it will be sufficient to prove the congruence

$$\mathrm{Res}(\mathcal{G} d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N) \equiv 0 \bmod p^M,$$

where

$$\mathcal{G} = \frac{h_1 H_0^{l_1}(H_0^{p-1}/p)^{l_2}(p^M/H)^{l_3}}{H_0}$$

with $h_1 \in O^0[[\underline{t}^{ep}/p]]$, $l_1, l_2, l_3 \in \mathbb{Z}_{\geqslant 0}$ and $l_1 + l_2 + l_3 \geqslant 1$.

If $l_3 = 0$ then $\mathcal{G} \in \mathcal{O}^0[[\underline{t}^{ep}/p]]$ and, therefore, the residue

$$\mathrm{Res}(\mathcal{G} d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N) = 0.$$

If $l_3 \geqslant 1$ then use that $H^{p-1}/p, p/H \in \mathcal{L}^0$ to obtain that

$$\mathcal{G} \in (p^M/H^2)\mathcal{L}^0 \subset p^M \underline{t}^{-2ep/-1)}\mathcal{L}^0.$$

Similarly to the proof of Lemma 3.3 this implies that the residue

$$\mathrm{Res}(\mathcal{G} d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N) \in p^M W(k).$$

□

**3.3. A construction of Vostokov's pairing $V$.** For any elements $u_0, \dots, u_N \in \mathcal{R}$, set

$$\widetilde{V}(u_0, \dots, u_N) = \mathrm{Tr}\left(\mathrm{Res}\frac{\Omega}{H}\right) \mathrm{mod} p^M, \tag{3.2}$$

where, as earlier, Tr is the trace map for the field extension $W(k)_{\mathbb{Q}_p}/\mathbb{Q}_p$. Then Proposition 3.1 and Lemma 3.3 imply that $\widetilde{V}$ is an $(N+1)$-linear skew-symmetric form on $\mathcal{R}$ with values in $\mathbb{Z}/p^M$.

**Proposition 3.4.** *If $\kappa(u_0) = 1$ then $\widetilde{V}(u_0, \dots, u_N) = 0$.*

*Proof.* By Propositions 3.3 and 3.7 it will be sufficient to prove that

$$\sigma \mathrm{Res}\left(\frac{\log(u_0)}{H_0} d_{\log} u_1 \wedge \dots \wedge d_{\log} u_N\right) \equiv$$
$$\mathrm{Res}\left(\frac{(\sigma/p)\log(u_0)}{(\sigma/p)H_0}(\sigma/p) d_{\log} u_1 \wedge \dots \wedge (\sigma/p) d_{\log} u_N\right) \mathrm{mod}\, p^M$$

Let $\mathcal{L}^{(-1)}$ be the subalgebra in $\mathcal{L}$ consisting of the formal Laurent series $l = \sum_{a \in \mathbb{Z}^N} w_a \underline{t}^a$ such that $\sigma(l) := \sum_{a \in \mathbb{Z}^N} \sigma(w_a) \underline{t}^{ap} \in \mathcal{L}$. Then one can verify that for any $r \in \mathcal{L}^{(-1)}$,

$$\sigma \mathrm{Res}(r d_{\log} u_1 \wedge \dots \wedge d_{\log} u_N) = \mathrm{Res}\left(\sigma(r)\frac{\sigma}{p} d_{\log} u_1 \wedge \dots \wedge \frac{\sigma}{p} d_{\log} u_N\right).$$

It remains to note that $\kappa(u_0) = 1$ implies (use Lemma 3.1) that $r = \log(u_0)/H_0 \in \mathcal{L}^{(-1)}$ and, therefore, $\sigma(r) = (\sigma/p)\log(u_0)/(\sigma/p)H_0$. □

**Corollary 3.2.** *The form $\widetilde{V}$ factors through the projection $\kappa : \mathcal{R} \longrightarrow F^*$ and defines an $(N+1)$-linear skew-symmetric form $\bar{V}$ on $F^*$ with values in $\mathbb{Z}/p^M$.*

We now verify the Steinberg relation for $\widetilde{V}$.

**Proposition 3.5.** *If $u_1 + u_0 = 1$ then $\widetilde{V}(u_0, u_1, \dots u_N) = 0$.*

*Proof.* As usually, it is sufficient to verify this property for $u_0 \in \mathrm{m}^0$. Then by Lemma 3.3 it will be sufficient to prove that

$$f_0 d_{\log} u_1 - f_1(\sigma/p) d_{\log} u_0 = dF \tag{3.3}$$

where $F \in O^0$.

For any $u \in \mathcal{R}$, set $l(u) := (1/p)\log(u^p/\sigma u)$. By computing in $L_0(\mathcal{F}) \otimes \mathbb{Q}_p$ we obtain the identity

$$l(u_0) d_{\log}(1 - u_0) = d(\mathrm{Li}_2(u_0) + \log(1 - u_0) l(u_0))$$

where $\mathrm{Li}_2(X) = \int \log(1 - X) X^{-1} dX = \sum_{n \geqslant 1} X^n/n^2$ is the dilogarithm function. This identity implies that (3.3) holds with

$$F = \mathrm{Li}_2(u_0) - (1/p^2)\mathrm{Li}_2(\sigma u_0) + \log(1 - u_0) l(u_0).$$

It remains to prove that $F \in O^0$.

Using the expansions for $\mathrm{Li}_2(X)$ and $\log(1-X)$ we can rewrite $F$ as a double sum $F = \sum_{m,s} F_{ms} u_0^{mp^s}$, where:

— the indices $s$ and $m$ run over the set of all non-negative integers with additional condition that $m$ is prime to $p$;

— for all (prime to $p$) indices $m$, we have

$$F_{m0} = 1/m^2 - (1/m)l(u_0)$$

and for all $s \geqslant 1$,

$$F_{ms} = \frac{1}{m^2p^{2s}}\left(1 - \frac{\sigma u_0^{mp^{s-1}}}{u_0^{mp^s}}\right) - \frac{1}{mp^s}l(u_0).$$

Clearly, $F_{m0} \in O^0$ and $F_{ms}$ appears as the result of the substitution of $X = -ml(u_0) \in \mathrm{m}^0$ into the $p$-integral power series of the function $(p^sX)^{-2}(1 + p^sX - \exp(p^sX))$. Therefore, all $F_{ms} \in O^0$ and $F \in O^0$. $\square$

**Corollary 3.3.** $\widetilde{V}$ *induces a bilinear continuous non-degenerate pairing* $V : K_1(F)_M \times K_N(F)_M \longrightarrow \mathbb{Z}/p^M$, *which factors through the canonical morphism of the left-hand side to* $K_{N+1}(F)_M$.

*Proof.* The only thing to verify is non-degeneracy. This can be done by routine calculations with the corresponding topological generators from Subsections 1.2 and 1.3. The most important fact is that

$$V(\epsilon_0, \{\pi_1, \dots, \pi_N\}) = 1,$$

where $\epsilon_0 = \theta(\alpha_0 H_0)$ is the $p^M$-primary element from Remark 1) of Subsection 1.2. $\square$

**Remark.** 1) The above construction of the pairing $V$ depends on the choice of a primitive $p^M$-th root of unity $\zeta_M$. However, Vostokov's pairing appears in the form

$$\mathcal{V} : K_1(F)_M \times K_N(F)_M \longrightarrow \langle \zeta_M \rangle. \tag{3.4}$$

where for any $\alpha \in K_1(F)_M$ and $\beta \in K_N(F)_M$, $\mathcal{V}(\alpha, \beta) = \zeta_M^{V(\alpha,\beta)}$, and this pairing is independent of the choice of $\zeta_M$.

2) The above Corollary immediately implies that Vostokov's pairing (3.4) coincides with the $M$-th Hilbert symbol

$$(\ ,\ )_M : K_1(F)_M \times K_N(F)_M \longrightarrow \langle \zeta_M \rangle$$

for the field $F$. Indeed, the norm property of the Hilbert symbol implies that it factors through the canonical morphism $K_1(F)_M \times K_N(F)_M$ to $K_{N+1}(F)/p^M$. Therefore, it is sufficient to verify that the Hilbert pairing is equal to $\zeta_M$ on the generator $\{\varepsilon_0, \pi_1, \dots, \pi_n\}$ of $K_{N+1}(F)_M$. But this is exactly the basic property of the $p^M$-primary element $\theta(\alpha_0 H_0)$, cf. Subsection 5.5.

## 4. The field-of-norms functor

**4.1. The field-of-norms functor.** $N$-dimensional local fields are special cases of the $(N-1)$-big fields used in [17] to construct a higher dimensional analogue of the field-of-norms functor. The main ideas of this construction are as follows.

Suppose $K$ is an $N$-dimensional local field and $v_K : K \longrightarrow \mathbb{Z}\cup\{\infty\}$ is the (first) valuation of $K$. If $\bar{K}$ is an algebraic closure of $K$, denote by the same symbol the unique extension of $v_K$ to $\bar{K}$. For any $c \geqslant 0$, let $\mathfrak{p}_K^c = \{x \in \bar{K} \mid v_K(x) \geqslant c\}$. If $L$ is a field extension of $K$ in $\bar{K}$, we use the simpler notation $\mathcal{O}_L/\mathfrak{p}^c$ instead of $\mathcal{O}_L/(\mathfrak{p}^c \cap \mathcal{O}_L)$. Clearly, if $[L : K] < \infty$ and $e(L/K)$ is the ramification index of $L/K$, then $\mathfrak{p}_K^c = \mathfrak{p}_L^{ce(L/K)}$.

An increasing fields tower $K_\bullet = (K_n)_{n\geqslant 0}$, where $K_0 = K$, is strictly deeply ramified (SDR) with parameters $(n_0, c)$, if for $n \geqslant n_0$, one has $[K_{n+1} : K_n] = p^N$, $c \leqslant v_K(p)$ and there is a surjective map $\Omega^1_{\mathcal{O}_{F_{n+1}}/\mathcal{O}_{F_n}} \longrightarrow (\mathcal{O}_{F_{n+1}}/\mathfrak{p}^c)^d$ or, equivalently, the $p$-th power map induces epimorphic maps

$$i_N(K_\bullet) : \mathcal{O}_{K_{n+1}}/\mathfrak{p}_K^c \longrightarrow \mathcal{O}_{K_n}/\mathfrak{p}_K^c. \tag{4.1}$$

This means that for $n \geqslant n_0$, $K_{n+1}^{(N)} = K_n^{(N)}$ and there are systems of local parameters $u_1^{(n)}, \dots, u_N^{(n)}$ in $K_n$ such that for all $1 \leqslant i \leqslant N$, $u_i^{(n+1)p} \equiv u_i^{(n)} \bmod \mathfrak{p}_K^c$.

The field-of-norms functor $X$ associates to any SDR tower $K_\bullet$ an $N$-dimensional field $\mathcal{K} = X(K_\bullet)$ of characteristic $p$ such that its $N$-dimensional valuation ring $\mathcal{O}_{\mathcal{K}}$ coincides with $\varprojlim_{i_n(K_\bullet)} \mathcal{O}_{K_n}/\mathfrak{p}_K^c$. Then for $n \geqslant n_0$, we have the following properties:

— the last residue fields of $\mathcal{K}$ and $K_n$ coincide;

— the natural projections from $\mathcal{O}_{\mathcal{K}}$ to $O_{K_n}/\mathfrak{p}_K^c$ induce isomorphisms of unitary rings

$$\mathcal{O}_{\mathcal{K}}/\mathfrak{p}_{\mathcal{K}}^{c_n} \simeq \mathcal{O}_{K_n}/\mathfrak{p}_K^c \tag{4.2}$$

where $c_n = p^{n-n_0}e(K_{n_0}/K)$;

— if $\bar{u}_1, \dots, \bar{u}_N$ is a system of local parameters in $\mathcal{K}$ then there are systems of local parameters $u_1^{(n)}, \dots, u_N^{(n)}$ in $K_n$ such that for $1 \leqslant i \leqslant N$, $\varprojlim_n u_i^{(n)} = \bar{u}_i$.

Suppose $L$ is a finite extension of $K$ in $\bar{K}$. Then the tower $L_\bullet = (LK_n)_{n\geqslant 0}$ is again SDR and $X(L_\bullet) = \mathcal{L}$ is a separable extension of $\mathcal{K}$ of degree $[LK_n : K_n]$, where $n \gg 0$. The extension $\mathcal{L}/\mathcal{K}$ is Galois if and only if, for $n \gg 0$, $LK_n/K_n$ is Galois. An analogue of the identification (4.2) can be used to identify $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ with $\mathrm{Gal}(LK_n/K_n)$.

Finally, $X(\bar{K}) := \varinjlim_{L} X(L_\bullet)$ is a separable closure $\mathcal{K}_{sep}$ of $\mathcal{K}$ and the functor $X$ identifies $\mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$ with $\mathrm{Gal}(\bar{K}/K_\infty)$, where $K_\infty = \varinjlim_{n} K_n$.

**4.2. Applications to *K*-groups.** Suppose there is an SDR tower $K_\bullet = (K_n)_{n\geqslant 0}$ with parameters $(n_0, c)$ and the ring epimorphisms $i_n = i_n(K_\bullet) : \mathcal{O}_{K_{n+1}}/\mathfrak{p}_K^c \longrightarrow \mathcal{O}_{K_n}/\mathfrak{p}_K^c$. Define for $n \geqslant n_0$, the morphisms

$$j_n = j_n(K_\bullet) : K_N^t(K_{n+1})/U_K^c K_N^t(K_{n+1}) \longrightarrow K_N^t(K_n)/U_K^c K_N^t(K_n).$$

as follows. Choose systems of local parameters $u_1^{(n)}, \dots, u_N^{(n)}$ of $K_n$ such that for $1 \leqslant i \leqslant N$, $u_i^{(n+1)p} \equiv u_i^{(n)} \bmod \mathfrak{p}_K^c$. Consider the lifts $\hat{\imath}_n$ of the morphisms $i_n$:

$$\hat{\imath}_n : \sum_a [\theta_a]\underline{\pi}^{(n+1)a} \mapsto \sum_a [\theta_a^p]\underline{\pi}^{(n)a},$$

where $a = (a_1, \dots, a_N) \in \mathbb{Z}^N$, $\theta_a \in K^{(N)}$, and $\underline{\pi}^{(n)a} = \pi_1^{(n)a_1} \dots \pi_N^{(n)a_N}$. Then $\hat{\imath}_n(K_{n+1}^*) \subset K_n^*$ and we may consider

$$\hat{\imath}_n^N : K_{n+1}^{*\otimes N} \longrightarrow K_N^t(K_n)$$

induced by $\alpha_1 \otimes \dots \otimes \alpha_N \mapsto \{\hat{\imath}_n(\alpha_1), \dots, \hat{\imath}_n(\alpha_N)\}$. Note that for any $\alpha, \beta \in K_{n+1}^*$, we have

— $\hat{\imath}_n(\alpha\beta) = \hat{\imath}_n(\alpha)\hat{\imath}_n(\beta)\gamma_1$ with $\gamma_1 \in \mathfrak{p}_K^c$;

— $\hat{\imath}_n(\alpha+\beta) = \hat{\imath}_n(\alpha)\gamma_2 + \hat{\imath}_n(\beta)$ with $v_K(\alpha) \leqslant v_K(\beta)$ and $\gamma_2 \in \mathfrak{p}_K^c$.

This implies that the images of the relations for the Milnor $K$-group $K_N(K_{n+1})$ lie in $U_K^c K_N^t(K_n)$. Therefore, $\hat{\imath}_n^N$ factors through the natural projection to $K_N^t(K_{n+1})$ and we can proceed modulo the subgroup $U_K^c K_N^t(K_{n+1})$ because it is mapped to $U_K^c K_N^t(K_n)$. It can be seen that the morphisms $j_n$, $n \geqslant n_0$, depend only on the tower $K_\bullet$ and its parameters $(n_0, c)$.

Note that

$$(K_N^t(K_n)/U_K^c K_N^t(K_n), j_n)_{n\geqslant n_0} \tag{4.3}$$

is a projective system. Consider the system of local parameters

$$\bar{u}_1 = \varprojlim_n u_1^{(n)}, \dots, \bar{u}_N = \varprojlim_n u_N^{(n)}$$

for $\mathcal{K} = X(F_\bullet)$. Similarly to the procedure of constructing the morphisms $j_n$, we use the identifications $\mathcal{O}_{\mathcal{K}}/\mathfrak{p}_{\mathcal{K}}^{c_n} \simeq \mathcal{O}_{K_n}/\mathfrak{p}_K^c$ to construct isomorphisms of topological $K$-groups

$$\iota_n : K_N^t(\mathcal{K})/U_{\mathcal{K}}^{c_n} K_N^t(\mathcal{K}) \longrightarrow K_N^t(K_n)/U_K^c K_N^t(K_n).$$

These morphisms are compatible with the morphisms $j_n$ introduced above and determine the isomorphism

$$\iota^{(c)} : K_N^t(\mathcal{K}) \longrightarrow \varprojlim_n K_N^t(K_n)/U_K^c K_N^t(K_n).$$

For any $0 < c' < c$, there is an induced projective system

$$(K_N^t(K_n)/U_K^{c'}K_N^t(K_n), j'_n)_{n\geqslant n_0}. \tag{4.4}$$

Its inverse limit coincides with $K_N^t(\mathcal{K})$ and the composition of $\iota^{(c)}$ with the natural projection from (4.3) to (4.4) coincides with $\iota^{(c')}$.

**4.3. Special SDR towers.** We need the following additional assumption about the SDR towers $K_\bullet$.

**Definition.** An SDR tower $K_\bullet$ will be called special if for any $n \geqslant n_0$, there is a fields tower of extensions of relative degree $p$

$$K_n = K_n^0 \subset K_n^1 \subset \cdots \subset K_n^{N-1} \subset K_n^N = K_{n+1}.$$

Our main applications are related to the following example of a special SDR-tower.

**Definition.** The tower $F_\bullet^0 = (F_n^0)_{n\geqslant 0}$ is very special if $F_0^0 = F$ and for all $n \geqslant 0$, the field $F_n^0$ has a system of local parameters $\pi_1^{(n)}, \dots \pi_N^{(n)}$ such that for $1 \leqslant i \leqslant N$, $\pi_i^{(n+1)p} = \pi_i^{(n)}$.

Clearly, a very special tower $F_\bullet^0$ is SDR with parameters $(0, v_F(p))$ and $X(F_\bullet^0) = \mathcal{F}$. In this case we have also an isomorphism $K_N(\mathcal{F}) \simeq \varprojlim_n K_N^t(F_n^0)$ induced in terms of generators from Subsection 1.3 by the morphisms $K_N^t(\mathcal{F}) \longrightarrow K_N^t(F_n^0)$ such that:

— $\{\bar{t}_1, \dots, \bar{t}_N\} \mapsto \{\pi_1^{(n)}, \dots, \pi_N^{(n)}\}$;

— $\{1 + [\theta_j]\underline{\bar{t}}^{\,a}, \bar{t}_1, \dots, \bar{t}_{i(a)-1}, \bar{t}_{i(a)+1}, \dots, \bar{t}_N\} \mapsto$

$$\{1 + [\theta_j^{p^{-n}}]\underline{\pi}^{(n)a}, \pi_1^{(n)}, \dots, \pi_{i(a)-1}^{(n)}, \pi_{i(a)+1}^{(n)}, \dots, \pi_N^{(n)}\}$$

This means that in the case of a very special tower $F_\bullet^0$, the group $K_N^t(\mathcal{F})$ coincides with the limit of the projective system $(K_N^t(F_n^0))_{n\geqslant 0}$, where the connecting morphisms are the norm maps $N_{F_{n+1}^0/F_n^0} : K_N^t(F_{n+1}^0) \longrightarrow K_N^t(F_n^0)$. We shall show that a similar property holds for any special SDR tower.

**Proposition 4.1.** *Suppose $K_\bullet$ is a special SDR tower with parameters $(n_0, c)$. Then for any $c_1 \geqslant c$ and $n \geqslant n_0$,*

$$N_{K_{n+1}/K_n} U_K^{c_1}(K_N^t(K_{n+1})) \subset U_K^{c_2}(K_N^t(K_n)),$$

*where $c_2 = c_1 + c/p - e(K_{n+1}/K_0)^{-1}$.*

*Proof.* Choose a field tower $K_n = L_0 \subset L_1 \subset \cdots \subset L_N = K_{n+1}$, where for $1 \leqslant i \leqslant N$, $[L_i : L_{i-1}] = p$. One can show the existence of:

— a system of local parameters $\eta_1, \dots, \eta_N$ of $L_0$;

— a system of local parameters $\eta'_1, \dots, \eta'_N$ of $L_N$;

— a permutation $\begin{pmatrix} 1 & 2 & \dots & N \\ j_1 & j_2 & \dots & j_N \end{pmatrix}$

such that for $1 \leqslant i \leqslant N$, $L_i = L_{i-1}(\eta'_{j_i})$ and $\eta_i'^p \equiv \eta_i \bmod \mathfrak{p}_K^c$.

For any field extension $E$ of $K$ in $\bar{K}$, set $U_K^{c_1}(E) := (1 + \mathfrak{p}_K^{c_1}) \cap E$.

**Lemma 4.1.** *For all $1 \leqslant i \leqslant N$, we have*

*a) $N_{L_i/L_{i-1}}(\eta'_{j_i}) \equiv \eta_{j_i} \bmod U_K^{c/p}(L_{i-1})$;*

*b) if $a \in L_i$ and $a - 1 \in \mathfrak{p}_K^{c_1}$ then $N_{L_i/L_{i-1}}(a) - 1 \in \mathfrak{p}_K^{c_1 + c/p - v_K(\eta'_{j_i})}$.*

*Proof.* The congruences $\eta_{j_i}'^p \equiv \eta_{j_i} \bmod \mathfrak{p}_K^c$ imply that all conjugates of $\eta'_{j_i}$ over $L_{i-1}$ are congruent modulo $\mathfrak{p}_K^{c/p}$. Therefore, $N_{L_i/L_{i-1}}(\eta'_{j_i}) \equiv \eta_i \bmod \mathfrak{p}_K^{c/p}$. This gives part a) of Lemma.

For property b), we can assume that $a = 1 + [\theta]\eta_{j_i}'^b \in U_K^{c_1}(L_i)$, where $\theta \in K^{(N)}$ and $b \in \mathbb{N}$. Then all conjugates of $a$ over $L_{i-1}$ are congruent modulo ${\eta'_{j_i}}^{b-1}\mathfrak{p}_K^{c/p} = {\eta'_{j_i}}^{-1}\mathfrak{p}_K^{c_1+c/p}$ and this implies that $N_{L_i/L_{i-1}}(a) \equiv a^p \equiv 1 \bmod ({\eta'_{j_i}}^{-1}\mathfrak{p}_K^{c_1+c/p})$. $\square$

We continue the proof of Proposition 4.1.

Consider the following system of topological generators of the group $U_K^{c_1}(K_N^t(K_{n+1}))$. For any $a = (a_1, \dots, a_N) \in \mathbb{Z}^N_{>\bar{0}}$, set $\varepsilon_{a\theta} = 1 + [\theta]\underline{\eta}^a$, where $\theta \in K^{(N)}$ and

$$\underline{\eta}^a = {\eta'}_{j_1}^{a_1} \dots {\eta'}_{j_{s(a)-1}}^{a_{s(a)-1}} \eta_{j_{s(a)+1}}^{p^{-1}a_{s(a)+1}} \dots \eta_{j_N}^{p^{-1}a_N}.$$

Here $0 \leqslant s(a) \leqslant N$ is such that $a_N \equiv \dots \equiv a_{s(a)+1} \equiv 0 \bmod p$ but $a_{s(a)} \not\equiv 0 \bmod p$. (Note that if $s(a) = 0$, i.e. $a \in p\mathbb{Z}^N$, then $\underline{\eta}^a \in K_n$.) With this notation, $U_K^{c_1}(K_{n+1})$ is topologically generated by all $\varepsilon_{a\theta}$ such that $v_K(\underline{\eta}^a) \geqslant c_1$. Therefore, $U_K^{c_1}(K_N^t(K_{n+1}))$ is topologically generated by the symbols

$$\alpha_{a\theta i} = \{\varepsilon_{a\theta}, \eta'_{j_1}, \dots, \eta'_{j_{i-1}}, \dots, \eta'_{j_{i+1}}, \dots, \eta'_{j_N}\}$$

with $a \in \mathbb{Z}^N$, $a > \bar{0}$, $1 \leqslant i \leqslant N$ and $\varepsilon_{a\theta} \in U_K^{c_1}(K_{n+1})$. If $a \notin p\mathbb{Z}^N$ then it will be enough to keep in this list only the generators with $i = s(a)$ and denote them by $\alpha_{a\theta} := \alpha_{a\theta s(a)}$.

By Lemma 4.1a) and the properties of the norm map from Subsection 1.4, we obtain that $N_{L_N/L_{s(a)}}(\alpha_{a\theta})$ is congruent to the symbol

$$\{\varepsilon_{a\theta}, \eta'_{j_1}, \dots, \eta'_{j_{s(a)-1}}, \eta_{j_{s(a)+1}}, \dots, \eta_{j_N}\}$$

modulo $U^{c'}(K_N^t(L_{s(a)}))$ if $a \notin p\mathbb{Z}^N$ with

$$c' = c_1 + c/p - \max\{v_K(\eta'_{j_{s(a)+1}}), \dots, v_K(\eta'_{j_N})\} \leqslant c_1 + c/p - e(K_{n+1}/K)^{-1}$$

Applying Lemma 4.1b) and using that $\varepsilon_{a\theta} \in L_{s(a)}$ but $\eta'_{j_1}, \dots, \eta'_{j_{s(a)-1}} \in L_{s(a)-1}$, we obtain that the norm $N_{L_N/L_{s(a)-1}}(\alpha_{a\theta}) \in U_K^{c_2}(K_N^t(L_{s(a)-1}))$ for $c_2 = c_1 + c/p - e(K_{n+1}/K)^{-1}$. This implies that $N_{L_N/L_0}(\alpha_{a\theta}) \in U_K^{c_2}(K_N^t(L_0))$.

Similarly, we can consider the case $a \in p\mathbb{Z}^N$, where the index $i$ plays the role of $s(a)$ and we use that $\varepsilon_{a\theta} \in L_0 \subset L_{i-1}$. □

**Corollary 4.1.** *For a special SDR tower $K_\bullet$, the limit of the projective system $(U_K^c K_N^t(K_n), N_{K_{n+1}/K_n})_{n\geqslant 0}$ is trivial.*

**Proposition 4.2.** *Suppose $K_\bullet$ is a special SDR tower with parameters $(n_0, c)$. Then there is $0 < c^0 < c$ such that for all $n \gg 0$, the morphisms $j_n$ in the projective system (4.3) with $c$ replaced by $c^0$, are induced by the norm maps $N_{K_{n+1}/K_n} : K_N^t(K_{n+1}) \longrightarrow K_N^t(K_n)$.*

*Proof.* Let $n'_0 \geqslant n_0$ be such that $e(K_{n'_0+1}/K) < c/p$ and $c^0 := c/p - e(K_{n'_0+1}/K)$. Take $n \geqslant n'_0$ and use the notation from Proposition 4.1. The group $K_N^t(K_{n+1})$ is topologically generated by the symbol $\{\eta'_{j_1}, \dots, \eta'_{j_N}\}$ and all $\alpha_{a\theta} = \{\varepsilon'_{a\theta}, \eta_{j_1}, \dots, \eta'_{j_N}\}$, where $a \in \mathbb{Z}^N \setminus p\mathbb{Z}^N$, $a > \bar{0}$. Applying arguments from the proof of Proposition 4.1, we obtain that

$$N_{K_{n+1}/K_n}(\{\eta'_{j_1}, \dots, \eta'_{j_N}\}) \equiv \{\eta'^p_{j_1}, \dots, \eta'^p_{j_N}\} \bmod U_K^{c^0}(K_N^t(K_n))$$

$$N_{K_{n+1}/K_n}(\alpha_{a\theta}) \equiv \{\varepsilon^p_{a\theta}, \eta'^p_{j_1}, \dots, \eta'^p_{j_{s(a)-1}}, \eta'^p_{j_{s(a)+1}}, \dots, \eta'^p_{j_N}\} U_K^{c^0}(K_N^t(K_n)).$$

□

**Proposition 4.3.** *Suppose $K_\bullet$ is a special SDR tower. Then for all $n \geqslant 0$, there are homomorphisms $\mathcal{N}_{\mathcal{K}/K_n} : K_N^t(\mathcal{K}) \longrightarrow K_N^t(K_n)$ such that*

*a) the system of morphisms $\{\mathcal{N}_{\mathcal{K}/K_n}\}_{n\geqslant 0}$ maps $K_N^t(\mathcal{K})$ to the projective system $(K_N^t(K_n), N_{K_{n+1}/K_n})_{n\geqslant 0}$ and defines a group isomorphism*

$$K_N^t(\mathcal{K}) \simeq \varprojlim_n K_N^t(K_n);$$

*b) for sufficiently small $c > 0$, the projective limit of the compositions of $\mathcal{N}_{\mathcal{K}/K_n}$ with projections $K_N^t(K_n) \to K_N^t(K_n)/U_K^c(K_N^t(K_n))$ coincides with the isomorphism $\iota^{(c)}$ from Subsection 4.2.*

*Proof.* Suppose $(n'_0, c^0)$ are the parameters for $K_\bullet$ from Proposition 4.2. For any $a \in K_N^t(\mathcal{K})$ and $m \geqslant n'_0$, choose $a_m \in K_N^t(K_m)$ such that $\iota_m(a \bmod U_{\mathcal{K}}^{c^0}(K_N^t(\mathcal{K})) = a_m \bmod U_K^{c^0}(K_N^t(K_m))$, where $\iota_m$ are isomorphisms from Subsection 4.2.

This allows us to set $\mathcal{N}_{\mathcal{K}/K_n}(a) = \lim_{m\to\infty} N_{K_m/K_n}(a_m)$ for any $n \geqslant n'_0$.

Then Corollary 4.1 can be used to prove that the maps $\mathcal{N}_{\mathcal{K}/K_n}$ are well-defined and satisfy the requirements of our Proposition. □

Finally, notice the following properties:

**Corollary 4.2.** *Suppose $K_\bullet$ is a special SDR tower, $\mathcal{K} = X(K_\bullet)$ and $\bar{u}_1, \dots, \bar{u}_N$ is a system of local parameters of $\mathcal{K}$. Then*

*a)* $\bigcap\limits_{n\geqslant 0} N_{K_n/K_0} K_N^t(K_n) = \mathcal{N}_{\mathcal{K}/K_0} K_N^t(\mathcal{K})$;

*b) for all $n \gg 0$, $\mathcal{N}_{\mathcal{K}/K_n}(\{\bar{u}_1, \dots, \bar{u}_N\}) = \{u_1^{(n)}, \dots, u_N^{(n)}\}$, where the elements $u_1^{(n)}, \dots, u_N^{(n)}$ form a system of local parameters of $K_n$.*

**4.4. The case of an arbitrary SDR tower $F_\bullet$.** Suppose $L/K$ is a finite Galois extension of $N$-dimensional local fields in a fixed algebraic closure $\bar{K}$ of $K$ and $G = \mathrm{Gal}(\bar{K}/K)$. Then there is a unique maximal purely unramified extension $K_0$ of $K$ in $L$. This means that $[K_0 : K] = [k_0 : k]$, where $k_0$ and $k$ are the $N$-th residue fields of $L$ and $K$, respectively. Let $G_0 = \mathrm{Gal}(L/K_0)$ and let $G_1 = \{\tau \in G_0 \mid \tau(a)/a \in 1 + \mathrm{m}_L, \forall a \in L\}$, where $\mathrm{m}_L$ is the maximal ideal of the $N$-dimensional valuation ring $\mathcal{O}_L$ of $L$. Then $G_1$ is a $p$-group, it is normal in $G$ and the order of $G_0/G_1$ is prime to $p$. The field extension $K_1 := L^{G_1}$ of $K$ is the maximal tamely ramified extension of $K$ in $L$. Note that in our setting, any tamely ramified field extension is always assumed to be Galois. Keeping the above notation we obtain the following property.

**Proposition 4.4.** *Let $\widetilde{K}_1$ be a tamely ramified extension of $K$ in $\bar{K}$ with the $N$-th residue field $k_0$ and $d = [\widetilde{K}_1 : K_0]$. If $[\alpha]$ is the Teichmüller representative of a generator $\alpha$ of $k_0^*$ and $u_1, \dots, u_N$ is a system of local parameters in $K$ then $\widetilde{K}_1 \subset K(\sqrt[d]{[\alpha]}, \sqrt[d]{u_1}, \dots, \sqrt[d]{u_N})$.*

**Corollary 4.3.** *Suppose $\tilde{d}$ is a natural number and $\tilde{d} = p^m d$, where $d$ is prime to $p$. Let $\tilde{k}/k$ be a field extension of degree $\tilde{d}$. Choose a generator $\alpha$ of $\tilde{k}^*$, a system of local parameters $u_1, \dots, u_N$ of $K$ and set $K(\tilde{d}) := K(\sqrt[d]{[\alpha]}, \sqrt[d]{u_1}, \dots, \sqrt[d]{u_N})$. Then $K(\tilde{d})$ contains any tamely ramified extension of $K$ of degree dividing $\tilde{d}$.*

**Proposition 4.5.** *Suppose $K_\bullet$ is an SDR tower with parameters $(n_0, c)$. Then there is a tamely ramified extension $K'_{n_0}$ of $K_{n_0}$ such that the tower $K'_\bullet = \{K_n K'_{n_0} \mid n \geqslant 0\}$ is a special SDR tower.*

*Proof.* We need the following Lemma.

**Lemma 4.2.** *Suppose $\widetilde{N} \in \mathbb{N}$ and $L/K$ is a totally ramified separable extension of $K$ of degree $p^{\widetilde{N}}$, i.e. $L^{(N)} = k$. Let $\tilde{d} = (p^{\widetilde{N}})! = p^m d$ where $m, d \in \mathbb{N}$ and $d$ is prime to $p$. Then there is a fields tower $L'_0 \subset L'_1 \subset \cdots \subset L'_{\widetilde{N}}$ such that*

*a) $L'_0$ is a tamely ramified extension of $K$ of degree dividing $\tilde{d}$;*

*b) for $1 \leqslant i \leqslant N$, $[L'_i : L'_{i-1}] = p$;*

*c) $L'_{\widetilde{N}} = L'_0 L$.*

*Proof of Lemma.* Since $L/K$ is separable there is $\theta \in L$ such that $L = K(\theta)$. Therefore, there is a Galois extension $\widetilde{K}$ of $K$ of degree dividing $\tilde{d} := (p^{\widetilde{N}})!$ such that $L \subset \widetilde{K}$. Let $L'_0$ be the maximal tamely ramified extension of $K$ in $\widetilde{K}$. Then $L'_0 \subset L'_{\widetilde{N}} := L'_0 L \subset \widetilde{K}$, $[L'_{\widetilde{N}} : L'_0] = p^{\widetilde{N}}$ and $\widetilde{G} := \mathrm{Gal}(\widetilde{K}/L'_0)$ is a finite $p$-group. Then elementary group theoretic arguments (e.g. any finite $p$-group has a central subgroup of order $p$) show the existence of a decreasing sequence of subgroups

$$\widetilde{H}_0 := \widetilde{G} \supset \widetilde{H}_1 \supset \cdots \supset \widetilde{H}_{\widetilde{N}-1}. \supset H_{\widetilde{N}} := \mathrm{Gal}(\widetilde{K}/L'_{\widetilde{N}})$$

such that for $1 \leqslant i \leqslant \widetilde{N}$, $(H_i : H_{i-1}) = p$ and we can take $L'_i = \widetilde{K}^{H_i}$. $\square$

For every $n \geqslant n_0$, let $K'_n := L'_0$ where $L'_0$ is the field from the above Lemma chosen for $K = K_n$ and $L = K_{n+1}$. Since $[K'_n : K_n]$ divides $\tilde{d} = (p^N)!$, the field $K'_n$ is contained in the tamely ramified extension $K_n(\tilde{d})$ of $K_n$ from Corollary 4.3. It remains to notice that for every $n \geqslant n_0$, $K_n(\tilde{d}) = K_{n_0}(\tilde{d})K_n$. The Proposition is proved. $\square$

Remind that for any $N$-dimensional local field $K$ we use the notation $K_N(K)_M := K_N(K)/p^M$. If $L$ is a finite extension of $K$ and $c \geqslant 0$ denote by $U^c_K K_N(L)_M$ the image of $U^c_K K^t_N(L)$ in $K_N(L)_M$.

**Proposition 4.6.** *Suppose $F_\bullet$ is an SDR tower such that a primitive $p^M$-th root of unity $\zeta_M \in F_\infty = \bigcup_{n \geqslant 0} F_n$. Then for any $c > 0$, $\varprojlim_n U^c_F K_N(F_n)_M = 0$.*

*Proof.* We can assume that $F_\bullet$ has $(0, c)$ and $\zeta_M \in F := F_0$. Let $F'$ be a tamely ramified extension of $F$ such that the SDR tower $F'_\bullet = \{F'F_n \mid n \geqslant 0\}$ is special. Let $C_n$ be the kernel of the natural map $U^c_F K_N(F_n)_M$ to $U^c_F K_N(F'_n)_M$ induced by the embedding $F_n \subset F'_n$. By Corollary 4.1, $\varprojlim_n U^c_F K_N(F'_n)_M = 0$. It remains to prove that $\varprojlim_n C_n = 0$.

Suppose $\widetilde{F}$ is the maximal absolutely unramified extension of $F$ in $F'$ of $p$-power degree. Then property (2) of norm maps from Subsection 1.4 implies that the natural map $K_N(F_n\widetilde{F})_M \longrightarrow K_N(F_nF')_M$ is injective. Therefore, for $n \geqslant 0$, $C_n$ is the kernel of the natural map from $K_N(F_n)_M$ to $K_N(F_n\widetilde{F})_M$.

Consider the Shafarevich bases of these $K_N$-groups from Subsection 1.3 using the Hasse $p^M$-primary elements $\epsilon_0$. By extending an $\mathbb{F}_p$-basis of $F_n^{(N)}$ to an $\mathbb{F}_p$-basis of $(F_n\widetilde{F})^{(N)}$, we can assume that the corresponding part of the Shafarevich basis for $K_N(F_n\widetilde{F})_M$ extends the corresponding part of the Shafarevich basis for $K_N(F_n)_M$. This implies that $C_n$ is contained in the subgroup of $K_N(F_n)_M$ which is generated by the elements of the Shafarevich basis which depend on $\epsilon_0$. Going again to the SDR tower $F'_\bullet$

we obtain that if $y$ is a such generator in $K_n(F_{n+1})_M$ then $N_{F_{n+1}/F_n}(y)\in pK_N(F_n)_M$. Therefore, $\varprojlim_n C_n=0$, as required. □

**Corollary 4.4.** *Suppose $F_\bullet$ is an SDR tower such that a primitive $p^M$-th root of unity $\zeta_M$ belongs to $F_\infty$ and $\mathcal{F}=X(F_\bullet)$. Then*

*a) for $n\geqslant 0$, there is a system of homomorphisms*

$$\mathcal{N}_{\mathcal{F}/F_n}:K_N(\mathcal{F})_M\longrightarrow K_N(F_n)_M$$

*mapping $K_N(\mathcal{F})_M$ to the projective system $(K_N(F_n)_M, N_{F_{n+1}/F_n})_{n\geqslant 0}$ and defining a group isomorphism $K_N(\mathcal{F})_M\simeq\varprojlim_n K_N(F_n)_M$;*

*b) for any $n\geqslant 0$, there is an $m\geqslant n$ such that*

$$\mathcal{N}_{\mathcal{F}/F_n}(K_N(\mathcal{F})_M)=N_{F_m/F_n}(K_N(F_m)_M).$$

*c) for all $n\gg 0$, $\mathcal{N}_{\mathcal{F}/F_n}(\{\bar t_1,\dots,\bar t_N\}_M)=\{\pi_1^{(n)},\dots,\pi_N^{(n)}\}_M$, where $\bar t_1,\dots,\bar t_N$ and $\pi_1^{(n)},\dots,\pi_N^{(n)}$ are the systems of local parameters in $\mathcal{F}$ and, resp., $F_n$, and $\{\ ,\dots,\ \}_M$ denotes the symbol $\{\ ,\dots,\ \}$ taken modulo $p^M$.*

## 5. Applications to the Hilbert symbol

As earlier let $F$ be an $N$-dimensional local field of characteristic 0 with the first residue field of characteristic $p$ and a fixed system of local parameters $\pi_1,\ \dots,\ \pi_N$; we use the notation $\mathcal{F}$ for an $N$-dimensional local field of characteristic $p$ with a system of local parameters $\bar t_1,\dots,\bar t_N$. The last residue field of both $F$ and $\mathcal{F}$ will be denoted by $k$.

**5.1. Parshin's reciprocity map.** Consider the non-degenerate perfect Witt-Artin-Schreier pairing

$$W_M(\mathcal{F})/\wp W_M(\mathcal{F})\times\Gamma_{\mathcal{F}}(p)/p^M\to\mathbb{Z}/p^M, \tag{5.1}$$

where for any $w\in W_M(\mathcal{F})$, $\wp(w)=\sigma(w)-w$, and $\Gamma_{\mathcal{F}}(p)$ is the Galois group of the maximal abelian $p$-extension $\mathcal{F}(p)$ of $\mathcal{F}$. Then the Witt symbol, cf. Subsection 2.2, implies the existence of injective group homomorphisms

$$\widetilde\Theta^P_{\mathcal{F}}:K^t_N(\mathcal{F})_M\longrightarrow\Gamma_{\mathcal{F}}(p)/p^M=\mathrm{Hom}(W_M(\mathcal{F})/\wp(W_M(\mathcal{F})),\mathbb{Z}/p^M).$$

By taking the projective limit over $M\in\mathbb{N}$, we obtain an injective homomorphism $\widetilde\Theta^P_{\mathcal{F}}:K^t_N(\mathcal{F})\longrightarrow\Gamma_{\mathcal{F}}(p)$. Here, as in the Introduction, we use the same notation for homomorphisms and their reductions modulo $p^M$. Then the explicit formula for the Witt symbol from Subsection 2.2 implies that $\widetilde\Theta^P_{\mathcal{F}}$ is $P$-continuous. In [14, 15, 16] Parshin used $\widetilde\Theta^P_{\mathcal{F}}$ to develop class field theory of higher local fields of characteristic $p$ by proving that for all finite abelian extensions $\mathcal{E}$ of $\mathcal{F}$ in $\mathcal{F}(p)$, $\widetilde\Theta^P_{\mathcal{F}}$ induces the group isomorphisms

$$\widetilde\Theta^P_{\mathcal{E}/\mathcal{F}}:K^t_N(\mathcal{F})/N_{\mathcal{E}/\mathcal{F}}K^t_N(\mathcal{E})\longrightarrow\mathrm{Gal}(\mathcal{E}/\mathcal{F}).$$

The explicit formula for the Witt symbol implies that the intersection of all $N_{\mathcal{E}/\mathcal{F}}K_N^t(\mathcal{E})$, where $\mathcal{E}$ runs over the family of all finite abelian extensions of $\mathcal{F}$, is trivial. Therefore, $\widetilde{\Theta}_{\mathcal{F}}^P$ is the composition of the canonical embedding

$$j_{\mathcal{F}} : K_N^t(\mathcal{F}) \longrightarrow \widehat{K}_N^t(\mathcal{F}) := \varprojlim_{\mathcal{E}} K_N^t(\mathcal{F})/N_{\mathcal{E}/\mathcal{F}}K_N^t(\mathcal{E})$$

and the isomorphism $\widehat{\Theta}_{\mathcal{F}}^P := \varprojlim_{\mathcal{E}} \widetilde{\Theta}_{\mathcal{E}/\mathcal{F}}^P : \widehat{K}_N^t(\mathcal{F}) \longrightarrow \Gamma_{\mathcal{F}}(p)$.

This gives the morphisms

$$\Gamma_{\mathcal{F}}^{ab}/p^M \xrightarrow{\Theta_{\mathcal{F}}^P} \widehat{K}_N^t(\mathcal{F})_M \xleftarrow{j_{\mathcal{F}}} K_N^t(\mathcal{F})_M, \tag{5.2}$$

where $\Theta_{\mathcal{F}}^P = \left(\widetilde{\Theta}_{\mathcal{E}/\mathcal{F}}^P\right)^{-1}$ is a group isomorphism and $j_{\mathcal{F}}$ is embedding with a dense image.

Let $\mathcal{F}^{ur}$ be the maximal absolutely unramified extension of $\mathcal{F}$ in $\mathcal{F}(p)$. Denote by $\varphi_{\mathcal{F}} \in \mathrm{Gal}(\mathcal{F}^{ur}/\mathcal{F})$ the Frobenius automorphism of this extension. Let $\mathcal{F}^+$ be the subfield in $\mathcal{F}(p)$ invariant under the action of $\widetilde{\Theta}_{\mathcal{F}}^P(\{\bar{t}_1, \ldots, \bar{t}_N\})$. Using the explicit formula for the pairing (2.2) from Subsection 2.2 one can easily obtain the following properties:

a) $\mathcal{F}(p) = \mathcal{F}^+\mathcal{F}^{ur}$ and $\mathcal{F}^+ \cap \mathcal{F}^{ur} = \mathcal{F}$;

b) $\{\bar{t}_1, \ldots, \bar{t}_N\}$ generates the group $\bigcap_{\mathcal{E}} N_{\mathcal{E}/\mathcal{F}}K_N^t(\mathcal{E})$, where $\mathcal{E}$ runs over the family of all $\mathcal{F} \subset \mathcal{E} \subset \mathcal{F}^+$ such that $[\mathcal{E} : \mathcal{F}] < \infty$;

c) $\widetilde{\Theta}_{\mathcal{F}}^P(VK_N(\mathcal{F}))$ is a closed subgroup in $\Gamma_{\mathcal{F}}(p)^{ab}$ and its invariant subfield is $\mathcal{F}^{ur}$;

d) $\widetilde{\Theta}_{\mathcal{F}}^P(\{\bar{t}_1, \ldots, \bar{t}_N\})|_{\mathcal{F}^{ur}} = \varphi_{\mathcal{F}}$;

e) $VK_N^t(\mathcal{F}) = \bigcap_{\mathcal{E}} N_{\mathcal{E}/\mathcal{F}}K_N^t(\mathcal{E})$, where $\mathcal{E}$ runs over the family of all finite extensions of $\mathcal{F}$ in $\mathcal{F}^{ur}$.

**5.2. Fesenko's reciprocity map.** In [6], Fesenko defined the reciprocity map for higher local fields of arbitrary characteristic. This construction can be specified in our situation as follows.

Let $K$ be either $\mathcal{F}$ or $F$. Let $L$ be a finite extension of $K$ in its maximal abelian $p$-extension $K(p)$. Denote by $L^{ur}$ and $K^{ur}$ the maximal absolutely unramified extensions of $L$ and, resp. $K$, in $K(p)$. Set $L_0 = K^{ur} \cap L$.

For any $\tau \in \mathrm{Gal}(L/K)$, consider its lift $\tilde{\tau} \in \mathrm{Gal}(L^{ur}/K)$ such that $\tilde{\tau}|_{K^{ur}} = \varphi_K^i$, where $i \in \mathbb{N}$ and $\varphi_K$ is the Frobenius automorphism of $K^{ur}/K$. Such lift exists because $K^{ur}$ and $L$ are linearly disjoint over $L_0$. Let $\Sigma$ be the subfield of $\tilde{\tau}$-invariants in $L^{ur}$. Then $\Sigma$ is a finite extension of $K$ such that $\Sigma K^{ur} = L^{ur}$. Denote by $\pi_1^{\Sigma}, \ldots, \pi_N^{\Sigma}$ a system of local parameters in $\Sigma$. Then Fesenko's reciprocity map is defined to be

$$\Theta_{L/K}^{\Phi} : \mathrm{Gal}(L/K) \longrightarrow K_N^t(K)/N_{L/K}(K_N^t(L))$$

such that for any $\tau \in \mathrm{Gal}(L/K)$, $\Theta^{\Phi}_{L/K}(\tau) = \mathrm{cl}_{L/K}(N_{\Sigma/K}(\{\pi_1^{\Sigma}, \dots, \pi_N^{\Sigma}\})$, where for any $\alpha \in K_N^t(K)$, $\mathrm{cl}_{L/K}(\alpha)$ is the image of $\alpha$ under the natural projection of $K_N^t(K)$ to $K_N^t(K)/N_{L/K}(K_N^t(L))$. The maps $\Theta^{\Phi}_{L/K}$ are well-defined group homomorphisms. The proof generalises Neukirch's 1-dimensional approach from [13].

**Remark.** Fesenko establishes his construction of class field theory for higher local fields by proving that all $\Theta^{\Phi}_{L/K}$ are isomorphisms. We do not assume this until the introduction of the $M$-th Hilbert symbol in Subsection 5.5.

Taking projective limit we obtain Fesenko's reciprocity map in the form

$$\Theta^{\Phi}_K : \Gamma_K^{ab} \longrightarrow \varprojlim_L K_N^t(K)/N_{L/K}K_N^t(L) := \widehat{K}_N^t(K).$$

The following properties follow directly from the above definitions.

a) Suppose $L \subset K^{ur}$ and $\tau = \varphi_K|_L$. Then $\Theta^{\Phi}_{L/\mathcal{K}}(\tau) = \mathrm{cl}_{L/\mathcal{K}}(\{u_1, \dots, u_N\})$, where $u_1, \dots, u_N$ is a system of local parameters in $K$.

b) Suppose $L/K$ is a finite abelian extension, $K_1$ is a finite field extension of $K$ and $L_1 = LK_1$. Then there is a natural group homomorphism $\kappa : \mathrm{Gal}(L_1/K_1) \longrightarrow \mathrm{Gal}(L/K)$ and the diagram

$$\begin{array}{ccc}
\mathrm{Gal}(L_1/K_1) & \xrightarrow{\Theta^{\Phi}_{L_1/K_1}} & K_N^t(K_1)/N_{L_1/K_1}K_N^t(L_1) \\
\downarrow \kappa & & \downarrow N_{K_1/K} \\
\mathrm{Gal}(L/K) & \xrightarrow{\Theta^{\Phi}_{L/K}} & K_N^t(K)/N_{L/K}K_N^t(L)
\end{array}$$

is commutative.

The following proposition shows that Parshin's and Fesenko's reciprocity maps coincide in the case of fields of characteristic $p$.

**Proposition 5.1.** *Suppose $\mathcal{E}$ is a finite abelian extension of $\mathcal{F}$ in $\mathcal{F}(p)$. Then the diagram*

$$\begin{array}{ccc}
\Gamma_{\mathcal{F}}(p) & \xleftarrow{\widetilde{\Theta}^{P}_{\mathcal{F}}} & K_N^t(\mathcal{F}) \\
\downarrow & & \downarrow \\
\Gamma_{\mathcal{E}/\mathcal{F}} & \xrightarrow{\Theta^{\Phi}_{\mathcal{E}/\mathcal{F}}} & K_N^t(\mathcal{F})/N_{\mathcal{E}/\mathcal{F}}K_N^t(\mathcal{E})
\end{array}$$

*is commutative, where the vertical maps are the natural projections.*

*Proof.* The group $K_N^t(\mathcal{F})$ is generated by the symbols $\{\bar{t}'_1, \dots, \bar{t}'_N\}$, where $\bar{t}'_1, \dots, \bar{t}'_N$ run over all systems of local parameters in $\mathcal{F}$. Therefore, it will be sufficient to consider the images of $\alpha = \{\bar{t}_1, \dots, \bar{t}_N\}$. By Subsection 3.1

we have $\widetilde{\Theta}^P_{\mathcal{F}}(\alpha)|_{\mathcal{F}^{ur}} = \varphi_{\mathcal{F}}$ and $\widetilde{\Theta}^P_{\mathcal{F}}(\alpha)|_{\mathcal{F}^+} = \mathrm{id}$. According to properties a) and b) of Subsection 5.2, this implies that $\Theta^{\Phi}_{\mathcal{E}/\mathcal{F}}(\widetilde{\Theta}^P(\alpha))$ belongs to

— $\mathrm{cl}_{\mathcal{E}/\mathcal{F}}\left(\bigcap_{\mathcal{E}_1} N_{\mathcal{E}_1/\mathcal{F}}(K^t_N(E_1))\right)$ , where $\mathcal{E}_1$ runs over the set of all finite extensions of $\mathcal{F}$ in $\mathcal{F}^+$, which is a group generated by $\mathrm{cl}_{\mathcal{E}/\mathcal{F}}(\alpha)$;

— $\mathrm{cl}_{\mathcal{E}/\mathcal{F}}(\alpha \bmod VK_N(\mathcal{F}))$.

Therefore, $\Theta^{\Phi}_{\mathcal{E}/\mathcal{F}}(\widetilde{\Theta}^P(\alpha)) = \mathrm{cl}_{\mathcal{E}/\mathcal{F}}(\alpha)$. □

**5.3. Compatibility of class-field theories.** Suppose $E$ is a finite field extension of $F$ and $F_\bullet = (F_n)_{n\geqslant 0}$ with $F_0 = F$, is a special SDR tower. Then $E_\bullet = (E_n)_{n\geqslant 0}$, where $E_n = EF_n$, is also a special SDR tower and $\mathcal{E} = X(E_\bullet)$ is a finite separable extension of $\mathcal{F} = X(F_\bullet)$. Notice that for any $n \geqslant 0$, there is a commutative diagram

$$\begin{tikzcd} K^t_N(\mathcal{E}) \arrow[r, "N_{\mathcal{E}/\mathcal{F}}"] \arrow[d, "\mathcal{N}_{\mathcal{E}/E_n}"] & K^t_N(\mathcal{F}) \arrow[d, "\mathcal{N}_{\mathcal{F}/F_n}"] \\ K^t_N(E_n) \arrow[r, "N_{E_n/F_n}"] & K^t_N(F_n) \end{tikzcd}$$

We shall use the notation $\mathcal{N}_{\mathcal{E}/F,n}$ for the morphism

$$K^t_N(\mathcal{F})/N_{\mathcal{E}/\mathcal{F}}(K^t_N(\mathcal{E}) \longrightarrow K^t_N(F_n)/N_{E_n/F_n}K^t_N(E_n)$$

induced by $\mathcal{N}_{\mathcal{F}/F_n}$.

Suppose that $E$ is abelian over $F$. Then $\mathcal{E}$ is also abelian over $\mathcal{F}$, and for any $n \geqslant 0$, we have natural homomophisms

$$\iota_{\mathcal{E}/F,n} : \mathrm{Gal}(\mathcal{E}/\mathcal{F}) \longrightarrow \mathrm{Gal}(E_n/F_n)$$

which are isomorphisms for $n \gg 0$.

Let $\iota_{\mathcal{E}/\mathcal{F},0} := \iota_{\mathcal{E}/\mathcal{F}}$ and $\mathcal{N}_{\mathcal{E}/\mathcal{F},0} := \mathcal{N}_{\mathcal{E}/\mathcal{F}}$.

**Proposition 5.2.** *The diagram*

$$\begin{tikzcd} \mathrm{Gal}(\mathcal{E}/\mathcal{F}) \arrow[r, "\Theta^{\Phi}_{\mathcal{E}/\mathcal{F}}"] \arrow[d, "\iota_{\mathcal{E}/F}"] & K^t_N(\mathcal{F})/N_{\mathcal{E}/\mathcal{F}}K^t_N(\mathcal{E}) \arrow[d, "\mathcal{N}_{\mathcal{E}/\mathcal{F}}"] \\ \mathrm{Gal}(E/F) \arrow[r, "\Theta^{\Phi}_{E/F}"] & K^t_N(F)/N_{E/F}K^t_N(E) \end{tikzcd}$$

*is commutative.*

*Proof.* Let $\tau \in \mathrm{Gal}(\mathcal{E}/\mathcal{F})$. Construct $\tilde{\tau} \in \mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{F})$ and $\mathcal{S} = \mathcal{E}^{ur}|_{\tilde{\tau}=\mathrm{id}}$ to define Fesenko's element

$$\Theta^{\Phi}_{\mathcal{E}/\mathcal{F}}(\tau) = N_{\mathcal{S}/\mathcal{F}}(\{\bar{u}_1, \ldots, \bar{u}_N\}) \in K^t_N(\mathcal{K})/N_{\mathcal{L}/\mathcal{K}}K^t_N(\mathcal{L}).$$

where $\bar{u}_1, \ldots, \bar{u}_N$ is a system of local parameters of $\mathcal{S}$.

For any $n \geqslant 0$, consider an analogue

$$\iota_{\mathcal{E}^{ur}/\mathcal{F},n} : \mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{F}) \longrightarrow \mathrm{Gal}(E_n^{ur}/F_n)$$

of $\iota_{\mathcal{E}/\mathcal{K},n}$. Let $\tau_n = \iota_{\mathcal{E}/\mathcal{F},n}(\tau)$, $\tilde{\tau}_n = \iota_{\mathcal{E}^{ur}/\mathcal{F},n}(\tilde{\tau})$ and set $\Sigma_n = L_n^{ur}|_{\tilde{\tau}_n=\mathrm{id}}$. Then from the construction of the field-of-norms functor $X$ it follows that $\Sigma_\bullet = (\Sigma_n)_{n\geqslant 0}$ is an SDR tower and $X(\Sigma_\bullet) = \mathcal{S}$. Therefore, for $n \gg 0$,

$$\mathcal{N}_{\mathcal{E}/\mathcal{F},n}(\Theta^{\Phi}_{\mathcal{E}/\mathcal{F}}(\tau)) = \mathcal{N}_{\mathcal{E}/\mathcal{F},n}(N_{\mathcal{S}/\mathcal{F}}(\{\bar{u}_1, \dots, \bar{u}_N\}) \bmod N_{\mathcal{E}/\mathcal{F}} K_N^t(\mathcal{E}))$$

$$= N_{\Sigma_n/F_n}(\mathcal{N}_{\mathcal{S}/\Sigma_n}(\{\bar{u}_1, \dots, \bar{u}_N\}) \bmod N_{E_n/F_n} K_N^t(E_n)) = \Theta^{\Phi}_{E_n/F_n}(\tau_n).$$

It remains to apply the property b) from Subsection 5.2. □

Finally, we can use the results of Subsection 4.4 to establish the compatibility of class field theories for the fields $F$ and $\mathcal{F} = X(F_\bullet)$ if $F_\bullet$ is an arbitrary SDR tower such that $\zeta_M \in F_\infty$. This property can be stated in the following form.

**Corollary 5.1.** *With the above notation and assumptions one has the following commutative diagram*

$$\begin{tikzcd}
\Gamma^{ab}_{\mathcal{F}}/p^M \arrow[r, "\Theta^{\Phi}_{\mathcal{F}}"] \arrow[d, "\iota_{\mathcal{F}/F}"] & \widehat{K}_N(\mathcal{F})_M \arrow[d, "\widehat{\mathcal{N}}_{\mathcal{F}/F}"'] & K_N(\mathcal{F})_M \arrow[l] \arrow[d, "\mathcal{N}_{\mathcal{F}/F}"] \\
\Gamma^{ab}_{F}/p^M \arrow[r, "\Theta^{\Phi}_{F}"] & \widehat{K}_N(F)_M & K_N(F)_M \arrow[l]
\end{tikzcd}$$

*where the right horizontal maps are natural embeddings and $\iota_{\mathcal{F}/F} : \Gamma_{\mathcal{F}} \longrightarrow \Gamma_F$ is given by the field-of-norms functor.*

**5.4. Relating Witt-Artin-Schreier and Kummer theories.** Consider an $N$-dimensional analogue $R(N)$ of Fontaine's ring. By definition, $R(N) = \varprojlim_n (O_{\bar{F}}/p)_n$, where the connecting morphisms are induced by the $p$-th power map on $O_{\bar{F}}$. If $r = (r_n \bmod p)_{n\geqslant 0} \in R(N)$ with all $r_n \in \mathcal{O}_{\bar{F}}$ and $m \in \mathbb{Z}$, set $r^{(m)} := \lim_{n\to\infty} r_n^{p^{n+m}} \in \mathcal{O}_{\hat{\bar{F}}_\infty}$ and consider Fontaine's map $\gamma : W(R(N)) \longrightarrow \mathcal{O}_{\hat{\bar{F}}_\infty}$ given by the correspondence

$$(w_0, \dots, w_n, \dots) \mapsto \sum_{n\geqslant 0} p^n w_n^{(0)}.$$

Let $F_\bullet$ be an SDR tower with parameters $(0, c)$. Then we have natural embeddings

$$\mathcal{O}_{\mathcal{F}} = \varprojlim_n \mathcal{O}_{F_n}/\mathfrak{p}^c_F \subset \varprojlim_n \mathcal{O}_{\bar{F}}/\mathfrak{p}^c_F = \varprojlim_n \mathcal{O}_{\bar{F}}/p = R(N),$$

where $\mathcal{O}_{\mathcal{F}}, \mathcal{O}_{F_n}$ and $\mathcal{O}_{\bar{F}}$ are the corresponding $N$-dimensional valuation rings. This implies that $\mathcal{F} \subset R_0(N) := \mathrm{Frac}\, R(N)$. Note that $R_0(N)$ is algebraicly closed and equals the completion (with respect to the first valuation) of the algebraic closure of $\mathcal{F}$ in $R_0(N)$. We have also a natural embedding $\mathcal{O}_{L(\mathcal{F})} \subset W(R_0(N))$. In particular, $\Gamma_{\mathcal{F}}$ acts on $R_0(N)$. In terms of the fixed system of local parameters in $L(\mathcal{F})$, $\mathcal{F}$ and $F_n$, where $n \geqslant 0$, we have for all $1 \leqslant i \leqslant N$, that $\bar{t}_i = \varprojlim_n \pi_i^{(n)}$, $t_i = [\bar{t}_i] \in W(R(N))$ and $\gamma(t_i) = \lim_{n\to\infty} \pi_i^{(n)p^n}$.

Suppose $\omega \in \mathbb{Z}_{\geqslant 0}$ and $F_\bullet$ is $\omega$-admissible, cf. Introduction for the deinition of an $\omega$-admissible tower. Then we can fix a primitive $p^{M+\omega}$-th primitive root of unity $\zeta_{M+\omega} \in F_\omega$ and introduce an element $H_\omega \in \mathcal{O}_{L(\mathcal{F})}$ as follows.

Let $H' \in \mathcal{F}$ be such that

$$H' \operatorname{mod} \mathfrak{p}_{\mathcal{F}}^{p^\omega c} = \zeta_{M+\omega} \operatorname{mod} \mathfrak{p}_F^c \tag{5.3}$$

under the identification $\mathcal{O}_{\mathcal{F}}/\mathfrak{p}_{\mathcal{F}}^{p^\omega c} = \mathcal{O}_{F_\omega}/\mathfrak{p}_F^c$ from the definition of $\mathcal{F} = X(F_\bullet)$. Take any $H \in \mathcal{O}_{L(\mathcal{F})}$ such that $H \operatorname{mod} p = H'$ and set $H_\omega = H^{p^{M+\omega}} - 1$.

Suppose $f \in \mathrm{m}^0 = \{\sum_{a>\bar{0}} w_a \underline{t}^a \mid w_a \in W(k)\}$. For any $\tau \in \Gamma_{\mathcal{F}}$, let $a_\tau(f) = \tau(T) - T$, where $T \in W(R_0(N))$ is such that $\sigma(T) - T = f/H_\omega$. Clearly, $a_\tau(f) \in \mathbb{Z}_p$ for all $\tau \in \Gamma_{\mathcal{F}}$.

Suppose $g \in \hat{F}_\infty^*$. For any $\tau \in \Gamma_{\mathcal{F}}$, define $b_\tau(g) \in \mathbb{Z}/p^M$ such that $(\tau U)U^{-1} = \zeta_{M+\omega}^{p^\omega b_\tau(g)}$, where $U \in \hat{\bar{F}}$ is such that $U^{p^M} = g$. We use the identification $\Gamma_{\mathcal{F}} = \mathrm{Gal}(\bar{F}/F_\infty)$ given by the field-of-norms functor $X$.

Recall that

$$\sum_{a>\bar{0}} w_a \underline{t}^a \mapsto \gamma\left(\prod_{a>\bar{0}} E(w_a, \underline{t}^a)\right)$$

defines a homomorphism $\theta : \mathrm{m}^0 \longrightarrow \hat{F}_\infty^*$, where $E(w, X)$ is the Shafarevich generalisation of the Artin-Hasse exponential, cf. Introduction.

**Proposition 5.3.** *For any $f \in \mathrm{m}^0$ and $\tau \in \Gamma_{\mathcal{F}}$, we have the equality $a_\tau(f) \operatorname{mod} p^M = b_\tau(\theta(f))$.*

*Proof.* Let $\varepsilon \in R(N)$ be such that $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$. We assume that $\varepsilon^{(M+\omega)} = \zeta_{M+\omega}$. Then relation (5.3) implies that

$$H \operatorname{mod} p = \varepsilon^{p^{-(M+\omega)}} \operatorname{mod} \mathfrak{p}_{\mathcal{F}}^{p^\omega c}.$$

The ideal $\mathfrak{p}_{\mathcal{F}}^{p^\omega c}$ is generated by the element $(\varepsilon - 1)^{c(p-1)/pv_F(p)}$ and, therefore,

$$H^{p^{\omega+1}} \operatorname{mod} p = \varepsilon^{p^{-(M-1)}} \operatorname{mod} (\varepsilon - 1)^{cp^\omega (p-1)/v_F(p)}.$$

Using that $F_\bullet$ is $\omega$-admissible, we obtain the existence of $w_1 \in W(R(N))$ and $w_1^0 \in W(R_0(N))$ such that

$$H^{p^{\omega+1}} = [\varepsilon]^{p^{-(M-1)}} + ([\varepsilon]-1)^2 w_1 + p w_1^0.$$

Therefore, there are $w_2 \in W(R(N))$ and $w_2^0 \in W(R_0(N))$ such that

$$H_\omega = [\varepsilon] - 1 + ([\varepsilon]-1)^2 w_2 + p^M w_2^0$$

and for some $w \in W(R(N))$ we have

$$\frac{1}{H_\omega} \equiv \left(\frac{1}{[\varepsilon]-1} + w\right) \operatorname{mod} p^M W(R_0(N)).$$

For any $\tau \in \Gamma_{\mathcal{F}}$, let $a'_\tau(f) = \tau T' - T' \in \mathbb{Z}_p$, where $T' \in W(R_0(N))$ is such that $\sigma(T') - T' = f/([\varepsilon]-1)$. Clearly, $\lim\limits_{s\to\infty} \sigma^s(fw) = 0$ and this implies that $a'_\tau(f) \equiv a_\tau(f) \operatorname{mod} p^M$.

Now one can proceed along the lines of the Main Lemma from [1] (or cf also [2]) to establish that $a'_\tau(f) \operatorname{mod} p^M = b_\tau(\theta(f))$. □

**Corollary 5.2.** *If* $\alpha_0 \in W(k)$ *is such that* $\operatorname{Tr}(\alpha_0) = 1$ *then*
a) $\theta(\alpha_0 H_0)$ *is a* $p^M$*-primary element of* $F$*;*
b) *if* $\phi$ *denotes the Frobenius automorphism of the extension* $F_{ur}/F$ *then* $\phi(\theta(\alpha_0 H_0)) = \zeta_M(\theta(\alpha_0 H_0))$.

*Proof.* Indeed, $\theta(\alpha_0 H_0) \in 1 + \mathrm{m}_F$ and therefore we can study $\Gamma_F$-properties of the extension $F(\sqrt[p^M]{\theta(\alpha_0 H_0)})$ by studying $\Gamma_{\mathcal{F}}$-properties of the extension $\mathcal{F}(T)$, where $T \in W_M(R_0(N))$ is such that $\sigma(T) - T = \alpha_0$. But this extension is absolutely unramified of degree $p^M$ by Witt's explicit formula from Section 2). This proves part a). In order to prove b), it is sufficient to note that $\phi(T) - T = \sigma^s(T) - T = \operatorname{Tr}(\alpha_0) = 1$, where $[k : \mathbb{F}_p] = s$ and then to apply Proposition 5.3. □

**5.5. Proof of Theorem 0.1.** Suppose $F_\bullet$ is an SDR $\omega$-admissible tower with parameters $(0, c)$, $F_0 = F$, $\mathcal{F} = X(F_\bullet)$ and $\beta \in K_N(\mathcal{F})$. Then there is an $\tau \in \Gamma_F^{ab}$ such that (in the notation of Subsection 4)

$$\Theta_F^\Phi(\tau \operatorname{mod} p^M) = \mathcal{N}_{\mathcal{F}/F}(\beta \operatorname{mod} p^M).$$

By Corollary 5.1, there is an $\tilde\tau \in \Gamma_{\mathcal{F}}^{ab}$ such that $\tau = \iota_{\mathcal{F}/F}(\tilde\tau)$ and $\Theta_F^\Phi(\tilde\tau) = \beta$, using the notation from the Introduction.

For any $f \in \mathrm{m}^0$, let $(\Theta(f), \mathcal{N}_{\mathcal{F}/F}(\beta))_M^{F_\bullet} = \zeta_{M+\omega}^{p^\omega A}$, where $A \in \mathbb{Z}/p^M$.

We construct the corresponding $H_\omega \in \mathrm{m}^0$, cf. Introduction and use Proposition 5.3 to deduce that:

— if $U \in W(R_0(N))$ is such that $\sigma(U) - U = f/H_\omega$ then

$$\tilde\tau U - U) \operatorname{mod} p^M = A.$$

Finally, by Proposition 2.3,

$$A = \mathrm{Tr}\left(\mathrm{Res}_{L(\mathcal{F})}\frac{f}{H_\omega}d_{\log}(\mathrm{Col}\beta)\right).$$

Theorem 0.1 is proved.

**5.6. Relation to Vostokov's pairing.** Suppose that $\zeta_M \in F$ and $F^0_\bullet = \{F^0_n \mid n \geqslant 0\}$ is a very special tower given in notation of Subsection 4.3 such that $F^0_0 = F$. Recall that each $F^0_n$ has a system of local parameters $\pi_1^{(n)}, \dots, \pi_N^{(n)}$ such that for $1 \leqslant i \leqslant N$, $\pi_i^{(0)} = \pi_i$ and $\pi_i^{(n+1)p} = \pi_i^{(n)}$. Then $F^0_\bullet$ is a 0-admissible SDR tower. As earlier, $\mathcal{F} = X(F^0_\bullet)$ with system of local parameters $\bar{t}_i = \varprojlim_n \pi_i^{(n)}$, where $1 \leqslant i \leqslant N$, and $L(\mathcal{F})$ is the corresponding absolutely unramified lift of $\mathcal{F}$ to characteristic 0 with local parameters $p, t_1, \dots, t_N$ such that $t_i \bmod p = \bar{t}_i$, $1 \leqslant i \leqslant N$. We have the following result.

**Theorem 5.1.** *For above SDR tower $F^0_\bullet$, the explicit formula for the $M$-th Hilbert symbol from Theorem 0.1 coincides with Vostokov's pairing. In other words, for very special towers the field-of-norms functor transforms Witt's pairing to Vostokov's pairing.*

*Proof.* Note that the very special tower $F^0_\bullet$ has the following advantages:

— the map $\gamma : 1+\mathrm{m}^0 \longrightarrow F$ is given by the correspondences $t_i \mapsto \pi_i$, $1 \leqslant i \leqslant N$, and hence coincides with the evaluation map $\kappa$ from the beginning of section 3;

— the Coleman map $\mathrm{Col} : K_N^t(\mathcal{F}) \longrightarrow K_N^t(L(\mathcal{F}))$ has a very simple explicit description in terms of the standard topological generators of the corresponding $K$-groups, cf. the beginning of Subsection 2.3.

It will be sufficient to verify the coincidence of the both explicit formulae on the standard topological generators of $F^*/p^M$ and $K_N(F)/p^M$ from Subsections 1.2 and 1.3. It can be seen that on these generators (due to the above mentioned properties of very special towers) the formula from Theorem 0.1 coincides with the "$i = 0$"-term of Vostokov's formula. In the notation of Section 3 we, therefore, need to verify that for $1 \leqslant i \leqslant N$, the $i$-parts

$$V_i := \mathrm{Tr}(\mathrm{Res} H_0^{-1} f_i(\sigma/p)d_{\log}u_0 \wedge \dots \wedge (\sigma/p)d_{\log}u_{i-1} \wedge d_{\log}u_{i+1} \wedge \dots \wedge d_{\log}u_N)$$

of Vostokov's formula give a zero contribution on standard generators.

The variable $u_0$ can take the following values:

$a_1$) $t_j$, where $1 \leqslant j \leqslant N$;

$a_2$) $1 + [\theta]\underline{t}^a$, where $\theta \in k$ and $a = (a_1, \dots, a_N) \in \mathbb{Z}^N \setminus p\mathbb{Z}^N$, $a > \bar{0}$;

$a_3$) $\alpha_0 H_0$ in the notation from Subsection 5.5.

The symbol $\{u_1, \dots, u_N\}$ can take the following values (the generators containing $\epsilon_0$ do not come from $K_N(\mathcal{F})$):

$b_1)$ $\{t_1,\dots,t_N\}$;

$b_2)$ $\{1+[\theta']\underline{t}^b, t_1,\dots,t_{i(b)-1},t_{i(b)+1},\dots,t_N\}$, where $\theta'\in k$, $b=(b_1,\dots,b_N)$ belongs to $\mathbb{Z}^N\setminus p\mathbb{Z}^N$, $b>\bar{0}$, $b_N\equiv\dots b_{i(b)+1}\equiv 0 \bmod p$ and $b_{i(b)}\not\equiv 0\bmod p$.

In the case $b_1)$ $V_i=0$ for any $u_0$, because $f_i=0$.

In the case $a_3)$ $V_i=0$ for any $\{u_1,\dots,u_N\}$, because $(\sigma/p)d_{\log}H_0\in p^M\Omega_{O^0}$.

In the case $a_1b_2)$ we can assume that $j=i(b)$ and $i=1$. Then the differential form from the expression of $V_1$ is a linear combination of the differential forms $H_0^{-1}\underline{t}^{ub}d_{\log}t_1\wedge\dots\wedge d_{\log}t_N$ for $u\in\mathbb{N}$, $u\not\equiv 0\bmod p$. All these differential forms have zero residue because $H_0\in\sigma(\mathrm{m}^0)\mathrm{mod}p^M$.

Similarly, in the case $a_2b_2)$ we can assume that $i=1$ and that the corresponding differential form is a linear combination of the forms

$$H_0^{-1}\underline{t}^{spa+ub}d_{\log}t_1\wedge\dots\wedge d_{\log}t_N$$

for $s,u\in\mathbb{Z}_{\geqslant 0}$ and $u\not\equiv 0\mathrm{mod}\,p$. All these forms have zero residue for the same reason.

The Theorem is proved. $\square$

## References


[1] V. Abrashkin *The field of norms functor and the Brueckner-Vostokov formula*. Math. Annalen **308** (1997), 5–19.

[2] V. Abrashkin *An analogue of the field-of-norms functor and of the Grothendieck Conjecture*. J. Algebraic Geom. **16** (2007), no. 4, 671–730.

[3] T. V. Belyaeva, S. V. Vostokov *The Hilbert symbol in a complete multidimensional field.I.* J. Math. Sci (N.Y.) **120** (2004), no. 4, 1483–1500.

[4] S. Bloch, K. Kato *p-adic étale cohomology.* Publ. Math. IHES **63** (1986), 107–152.

[5] H. Bass, J. Tate, *The Milnor ring of a global field.* Lect. Notes Math. **342**, Springer-Verlag, Berlin, 1973, 474–486.

[6] I. B. Fesenko, *Sequential topologies and quotients of the Milnor $K$-groups of higher local fields.* Algebra i Analiz **13** (2001), no.3, 198–221; English translation in: St. Petersburg Math. J. **13** (2002), issue 3, 485–501.

[7] I. Fesenko, S. Vostokov, *Local Fields and their Extensions.* Translations of Mathematical Monographs, vol. **121**, Amer. Math. Soc., Providence, Rhode Island, 2002.

[8] J.-M.Fontaine, *Representations p-adiques des corps locaux (1-ere partie).* In: The Grothendieck Festschrift, A Collection of Articles in Honor of the 60th Birthday of Alexander Grothendieck, vol. II, 1990, 249–309.

[9] T. Fukaya, *The theory of Coleman power series for $K_2$.* J. Algebraic Geom. **12** (2003), No. 1, 1–80.

[10] A. I. Madunts, I. B. Zhukov, *Multidimensional complete fields: topology and other basic constructions.* Trudy S.Peterb. Mat. Obshch. (1995); English translation in: Amer. Math. Soc. Transl., (Ser.2) **165**, 1–34.

[11] K. Kato, *The explicit reciprocity law and the cohomology of Fontaine-Messing.* Bull. Soc. Math. France **109** (1991), no.4, 397–441.

[12] F. Laubie, *Extensions de Lie et groupes d'automorphismes de corps locaux.* Comp. Math. **67** (1988), 165–189.

[13] J. Neukirch, *Class Field theory.* Springer-Verlag, Berlin and New York, 1986.

[14] A. N. Parshin, *Class fields and algebraic $K$-theory. (Russian).* Uspekhi Mat. Nauk **30** (1975), 253–254.

[15] A. N. Parshin, *Local class field theory. (Russian).* Algebraic Geometry and its applications, Trudy Mat. Inst. Steklov **165** (1985), 143–170; English translation in: Proc. Steklov Inst. Math., 1985, issue 3, 157–185.

[16] A. N. Parshin, *Galois cohomology and Brauer group of local fields.* Trudy Mat. Inst. Steklov (1990); English translation in: Proc. Steklov Inst. Math., 1991, issue 4, 191–201.

[17] T. Scholl, *Higher fields of norms and* $(\varphi, \Gamma)$*-modules.* Doc. Math., Extra vol. (2006), 685–709 (electronic).

[18] I. R. Shafarevich *A general reciprocity law*(Russian), Mat. Sb. **26(68)** (1950), 113–146

[19] S. Vostokov, *An explicit form of the reciprocity law.* Izv. Akad. Nauk SSSR, Ser. Mat. (1978); English translation in: Math. USSR Izv. **13** (1979), 557–588.

[20] S. Vostokov, *Explicit construction of the theory of class fields of a multidimensional local field.* Izv. Akad. Nauk SSSR Ser. Mat., **49** (1985), 283–308.

[21] S.Zerbes, *The higher Hilbert pairing via* $(\phi, G)$*-modules.* ArXiv:0705.4269

[22] I. Zhukov, *Higher dimensional local fields.* (Münster, 1999), Geom. Topol. Monogr. (2000), no.3, 5–18.

Victor Abrashkin
Department of Mathematical Sciences
Durham University
Science Laboratories
South Rd, Durham DH1 3LE
United Kingdom
*E-mail*: `victor.abrashkin@durham.ac.uk`

Ruth Jenni
*E-mail*: `r.c.jenni@gmail.com`