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## Flows of Mellin transforms with periodic integrator

par TITUS HILBERDINK

RÉSUMÉ. Nous étudions les transformées de Mellin  $\hat{N}(s) = \int_{1-}^{\infty} x^{-s} dN(x)$  pour lesquelles  $N(x) - x$  est périodique de période 1 dans le but d'examiner les "flots" de telles fonctions vers la fonction  $\zeta(s)$  de Riemann et la possibilité de prouver l'hypothèse de Riemann avec cette approche. Nous montrons que, à part le cas trivial  $N(x) = x$ , la borne supérieure des parties réelles des zéros de n'importe quelle telle fonction est au moins  $\frac{1}{2}$ .

Nous examinons un flot particulier de telles fonctions  $\{\hat{N}_{\lambda}\}_{\lambda \geq 1}$  qui converge localement uniformément vers  $\zeta(s)$  quand  $\lambda \rightarrow 1$ , et montrons qu'elles présentent un aspect similaire à  $\zeta(s)$ . Par exemple,  $\hat{N}_{\lambda}(s)$  a à peu près  $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$  zéros dans la bande critique jusqu'à la hauteur  $T$ , et une infinité de zéros négatifs, environ aux points  $\lambda - 1 - 2n$  ( $n \in \mathbb{N}$ ).

ABSTRACT. We study Mellin transforms  $\hat{N}(s) = \int_{1-}^{\infty} x^{-s} dN(x)$  for which  $N(x) - x$  is periodic with period 1 in order to investigate 'flows' of such functions to Riemann's  $\zeta(s)$  and the possibility of proving the Riemann Hypothesis with such an approach. We show that, excepting the trivial case where  $N(x) = x$ , the supremum of the real parts of the zeros of any such function is at least  $\frac{1}{2}$ .

We investigate a particular flow of such functions  $\{\hat{N}_{\lambda}\}_{\lambda \geq 1}$  which converges locally uniformly to  $\zeta(s)$  as  $\lambda \rightarrow 1$ , and show that they exhibit features similar to  $\zeta(s)$ . For example,  $\hat{N}_{\lambda}(s)$  has roughly  $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$  zeros in the critical strip up to height  $T$  and an infinite number of negative zeros, roughly at the points  $\lambda - 1 - 2n$  ( $n \in \mathbb{N}$ ).

### Introduction

One idea of approaching the Riemann Hypothesis (RH) is to construct a sequence or a flow of holomorphic functions converging to  $\zeta(s)$ , uniformly on compact subsets of  $\mathbb{C} \setminus \{1\}$  in such a way that all the functions in the

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sequence have no zeros in<sup>1</sup>  $H_{\frac{1}{2}}$ . Then by Hurwitz's Theorem on the zeros of the limit function, RH would follow. Less stringently, we would only require that there are no zeros in half-planes converging to  $H_{\frac{1}{2}}$ . To make it worthwhile, it should be easier to locate the zeros of the sequence than of  $\zeta(s)$  itself.

The problem with such an approach is of course how to choose your sequence or flow (if indeed this is possible). We shall restrict ourselves to Mellin transforms; i.e.

$$\hat{N}_\lambda(s) = \int_0^\infty x^{-s} dN_\lambda(x),$$

where  $\lambda$  ranges over some interval, say  $\lambda \in [0, 1]$  with  $N_\lambda(x) \rightarrow [x]$  as  $\lambda \rightarrow 1$ . Thus  $\hat{N}_\lambda(s) \rightarrow \zeta(s)$ .

For instance, one can imagine starting from very 'smooth' generalised primes and integers and 'flowing' to the actual primes and integers as time progresses. For example, we could start from  $N_0(x) = x$  ( $x \geq 1$ ) and zero otherwise and 'flow' to the function  $N_1(x) = [x]$ . Then  $\hat{N}_0(s) = \frac{s}{s-1}$  'flows' to  $\hat{N}_1(s) = \zeta(s)$ .

There are many 'natural' properties that a typical integrator  $N(x)$  (or its Mellin transform) in such a flow could be assumed to have, by analogy with  $[x]$  and its Mellin transform  $\zeta(s)$ . One property we shall assume at the outset is that  $N(x) = 0$  for  $x < 1$  and  $N(1) = 1$ . Thus  $N$  has a jump at 1 and so  $\hat{N}(s) = 1 + \int_1^\infty x^{-s} dN(x)$ , ensuring that  $\hat{N}(s)$  is bounded away from zero in half-planes far enough to the right (given that the integral converges absolutely here). In this paper we shall further assume that for  $x \geq 1$ ,  $N(x) - x$  is periodic with period 1. (This is true for the cases  $N(x) = x$  and  $N(x) = [x]$  mentioned above). A further property that could be considered is that  $N(x)$  forms part of a generalised prime system; i.e.  $N(x) = \exp_* \Pi(x)$  for some increasing function  $\Pi(x)$ , or in terms of Mellin transforms;  $\log \hat{N}(s) = \hat{\Pi}(s)$ . However, we shall not assume this here.

On the above assumptions  $\hat{N}(s)$  has an analytic continuation to  $H_0 \setminus \{1\}$  with a simple pole at  $s = 1$ . In fact, using the Fourier development of  $N(x) - x$ , we shall show (Theorem 1) that there is an analytic continuation to the rest of the complex plane as well, and furthermore  $\hat{N}(s)$  satisfies a 'functional relationship' akin to the functional equation for  $\zeta(s)$ . As a corollary (Corollary 2) it follows that the associated Lindelöf function (see below for the definition) is at least  $\frac{1}{2} - \sigma$  for  $\sigma < \frac{1}{2}$ , except in the case when  $N(x) = x$ . Denoting by  $\Theta$  the supremum of the real parts of the zeros of  $\hat{N}$ , this further implies that  $\Theta \geq \frac{1}{2}$ .

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<sup>1</sup>For  $\theta \in \mathbb{R}$ , we denote by  $H_\theta$  the half plane  $\{s \in \mathbb{C} : \Re s > \theta\}$ .

In particular, this shows it is impossible to have a flow of such Mellin transforms from  $\frac{s}{s-1}$  to  $\zeta(s)$  in which the zeros gradually move to the right (unless RH is false).

In the final section, we discuss the zeros of a particular flow of such Mellin transforms  $\{\hat{N}_\lambda\}_{\lambda \geq 1}$  whose integrator  $N_\lambda$  has Fourier coefficients proportional to  $n^{-\lambda}$ .

### 1. Some preliminaries and notation

Let  $S$  denote the space of functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  which are zero on  $(-\infty, 1)$ , right-continuous, and of local bounded variation. (See e.g. [2], pp.50-70.) For  $\alpha \in \mathbb{R}$ , let  $S_\alpha = \{f \in S : f(1) = \alpha\}$ .

Let  $f \in S$ . If  $f(x) = O(x^A)$  for some  $A$ , then we define the *Mellin transform* by

$$\hat{f}(s) = \int_{1-}^{\infty} x^{-s} df(x).$$

This is well-defined for  $\sigma = \Re s > \alpha$ , where  $\alpha$  is the infimum of  $A$  for which  $f(x) = O(x^A)$ . Indeed, in this half-plane,  $\hat{f}$  is holomorphic. Integrating by parts gives

$$\hat{f}(s) = s \int_1^{\infty} \frac{f(x)}{x^{s+1}} dx.$$

A function  $F$  holomorphic in a vertical strip (except possibly at a finite number of isolated singularities) is said to be of *finite order* if

$$F(\sigma + it) = O(|t|^A) \quad (|t| \geq t_0, \text{ some } t_0),$$

for each  $\sigma$  in the interval of the strip. As such, we may define the *Lindelöf function*  $\mu(\sigma)$  to be the infimum of those  $A$  for which the above holds. It is well-known that  $\mu$  is a convex function. In our case (with  $F = \hat{N}$  and  $N \in S_1$ ),  $\mu$  will be decreasing and eventually zero since

$$|\hat{N}(s) - 1| \leq \int_1^{\infty} x^{-\sigma} d|N|(x) \rightarrow 0$$

as  $\sigma \rightarrow \infty$ .

Knowledge of the positivity of  $\mu$  can be used for locating zeros because of the following result: *if  $f$  is of finite order in  $H_\beta$  and has at most finitely many zeros here and  $\mu(\sigma) = 0$  for  $\sigma$  sufficiently large, then  $\mu(\sigma) = 0$  for  $\sigma > \beta$* . This was shown to hold for Beurling zeta functions in [4] (Theorem 2.3 and the Remarks following it) but actually the proof readily extends to general functions.

Thus, for example, if  $\mu(\sigma) > 0$  for  $\sigma < \frac{1}{2}$ , then  $f(s)$  has infinitely many zeros in each half-plane  $H_{\frac{1}{2}-\delta}$  for every  $\delta > 0$ .

### 2. Main results and proofs

Suppose  $N \in S_1$  and  $N(x) = x - R(x)$  where  $R(x)$  has period 1. Extend  $R$  to the whole real line by periodicity. Thus  $R$  is right continuous, locally of bounded variation, and  $R(1) = 0$ . Since  $R$  is of bounded variation, it possesses a Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx + \sum_{n=1}^{\infty} b_n \sin 2\pi nx$$

which converges to  $\frac{1}{2}(R(x + 0) + R(x - 0))$ , and the series is boundedly convergent (see [5], p.408). Also  $a_n, b_n = O(\frac{1}{n})$ .

**Theorem 1.** *Suppose that  $N(x) = x - R(x) \in S_1$  where  $R$  is periodic with period 1. Then  $\hat{N}(s)$  has an analytic continuation to  $\mathbb{C} \setminus \{1\}$  with a simple pole at  $s = 1$  with residue 1. Furthermore  $\hat{N}(s)$  is of finite order and for  $\sigma < 0$  satisfies the relation*

$$(2.1) \quad \hat{N}(s) = \frac{s}{s-1} + \int_0^1 x^{-s} dR(x) + (2\pi)^s \Gamma(1-s) \left( \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} a_n n^s - \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} b_n n^s \right).$$

The proof of Theorem 1 shows that the Lindelöf function of  $\hat{N}$  satisfies  $\mu(\sigma) \leq \frac{1}{2} - \sigma$  for  $\sigma \leq 0$ , while of course  $\mu(\sigma) = 0$  for  $\sigma \geq 1$ . By convexity one obtains upper bounds for all  $\sigma$ . We can get equality for  $\sigma \leq 0$  if we know that  $a_n$  and  $b_n$  are not identically zero. (Equivalently, since  $R$  is right-continuous, if  $R$  is not constant; i.e. non-zero.)

**Corollary 2.** *Under the assumptions of Theorem 1, if  $R \not\equiv 0$  then  $\mu(\sigma) = \frac{1}{2} - \sigma$  for  $\sigma \leq 0$  and  $\mu(\sigma) \geq \mu_0(\sigma)$  for all  $\sigma$ , where*

$$\mu_0(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq \frac{1}{2} \\ \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \end{cases}.$$

*It follows that  $\hat{N}$  has infinitely many zeros in  $H_{\frac{1}{2}-\delta}$  for any  $\delta > 0$ .*

*In particular, if we let  $\Theta$  denote the supremum of the real parts of the zeros of  $\hat{N}$ , then  $\Theta \geq \frac{1}{2}$ .*

*Proof of Theorem 1.* We have for  $\sigma > 1$ ,

$$\hat{N}(s) = s \int_1^{\infty} \frac{N(x)}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^{\infty} \frac{R(x)}{x^{s+1}} dx.$$

The integral on the right converges for  $\sigma > 0$ , and so  $\hat{N}(s)$  has an analytic continuation to  $H_0$  except for a simple pole at  $s = 1$  with residue 1.

We can extend further to the left by noting that  $a_0 = \int_0^1 R(x)dx$  so that  $\int_0^X (R(x) - a_0)dx = O(1)$ . Hence for  $\sigma > 0$ ,

$$\begin{aligned} \hat{N}(s) &= \frac{s}{s-1} - s \int_1^\infty \frac{a_0}{x^{s+1}} dx - s \int_1^\infty \frac{R(x) - a_0}{x^{s+1}} dx \\ &= \frac{s}{s-1} - a_0 - s \int_1^\infty \frac{R(x) - a_0}{x^{s+1}} dx. \end{aligned}$$

The final integral converges and is holomorphic for  $\sigma > -1$  and so this extends  $\hat{N}(s)$  holomorphically to  $H_{-1}$ . Thus  $\hat{N}(0) = -a_0$ . Note that  $\hat{N}(s)$  has finite order for  $\sigma > -1$  since in this range, writing  $V(x) = \int_1^x (R(y) - a_0)dy = O(1)$ , we have

$$s \int_1^\infty \frac{R(x) - a_0}{x^{s+1}} dx = s(s+1) \int_1^\infty \frac{V(x)}{x^{s+2}} dx = O(|t|^2).$$

Also  $s \int_0^1 \frac{R(x)-a_0}{x^{s+1}} dx$  converges for  $\sigma < 0$  and equals  $s \int_0^1 \frac{R(x)}{x^{s+1}} dx + a_0 = \int_0^1 x^{-s} dR(x) + a_0$ . Thus,

$$\hat{N}(s) = \frac{s}{s-1} + \int_0^1 x^{-s} dR(x) - s \int_0^\infty \frac{R(x) - a_0}{x^{s+1}} dx \quad \text{for } -1 < \sigma < 0.$$

Now we insert the Fourier series for  $R(x) - a_0$ . If we ignore all problems of convergence for the moment, the final integral of (2.2) becomes

$$\begin{aligned} s \int_0^\infty \frac{R(x) - a_0}{x^{s+1}} dx &= s \int_0^\infty \frac{1}{x^{s+1}} \left( \sum_{n=1}^\infty a_n \cos 2\pi nx + \sum_{n=1}^\infty b_n \sin 2\pi nx \right) dx \\ &= s \sum_{n=1}^\infty \left( a_n \int_0^\infty \frac{\cos 2\pi nx}{x^{s+1}} dx + b_n \int_0^\infty \frac{\sin 2\pi nx}{x^{s+1}} dx \right) \\ &= s \sum_{n=1}^\infty (2\pi n)^s \left( a_n \Gamma(-s) \cos \frac{\pi s}{2} - b_n \Gamma(-s) \sin \frac{\pi s}{2} \right) \\ (2.3) \qquad &= -\Gamma(1-s)(2\pi)^s \left( \cos \frac{\pi s}{2} \sum_{n=1}^\infty a_n n^s - \sin \frac{\pi s}{2} \sum_{n=1}^\infty b_n n^s \right), \end{aligned}$$

and the result follows formally. However, the term-by-term integration is permissible since the Fourier series is boundedly convergent and  $a_n$  and  $b_n$  are both  $O(1/n)$  (the argument is identical to the special case  $b_n = \frac{1}{n}$  as in [6], p.15).

Thus (2.3) holds for  $-1 < \sigma < 0$ . But the RHS of (2.3) is holomorphic for  $\sigma < 0$ . Hence this provides the analytic continuation of  $\hat{N}(s)$  to  $\mathbb{C} \setminus \{1\}$  and (2.3) holds for  $\sigma \leq -1$  also.

That  $\hat{N}(s)$  is of finite order follows directly from (2.1). For  $|\Gamma(1-s)(2\pi)^s \cos \frac{\pi s}{2}| = O(|t|^{1/2-\sigma})$  and similarly for the term involving

sin, while  $|\sum a_n n^s| \leq \sum |a_n| n^\sigma = O(1)$  for  $\sigma < 0$  and also for  $\sum b_n n^s$ . Since  $|\int_0^1 x^{-s} dR(x)| \leq \int_0^1 1 d|R(x)| = O(1)$ , (2.1) gives, for  $\sigma < 0$ ,

$$|\hat{N}(\sigma + it)| = O(1) + O(|t|^{1/2-\sigma}).$$

□

*Proof of Corollary 2.* Consider the final term in (2.1), which can be written as

$$(2.4) \quad \Gamma(1-s)(2\pi)^s \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} n^s \left( a_n - b_n \tan \frac{\pi s}{2} \right),$$

( $\sigma < 0$ ,  $s$  not an odd integer) and use the asymptotic bounds

$$\begin{aligned} |\Gamma(1-s)| &= |\Gamma(1-\sigma-it)| \sim \sqrt{2\pi} |t|^{1/2-\sigma} e^{-\frac{\pi}{2}|t|}, \\ \left| \cos \frac{\pi s}{2} \right| &\sim \frac{1}{2} e^{\frac{\pi}{2}|t|}, \end{aligned}$$

and

$$\tan \frac{\pi s}{2} = \tan \left( \frac{\pi\sigma}{2} + i \frac{\pi t}{2} \right) = \operatorname{sgn}(t)i + O(e^{-\pi|t|}).$$

(These hold as  $|t| \rightarrow \infty$ , uniformly for  $\sigma$  in bounded intervals.) Thus the term in (2.4) is, in modulus, asymptotic to

$$(2\pi)^\sigma \sqrt{\frac{\pi}{2}} |t|^{1/2-\sigma} \left( \left| \sum_{n=1}^{\infty} (a_n \pm ib_n) n^s \right| + O(e^{-\pi|t|}) \right).$$

Since the coefficients  $a_n$  and  $b_n$  are not identically zero and, furthermore, are real, there is a least integer  $n_0$  for which  $a_{n_0} \pm ib_{n_0} \neq 0$ . It follows that for  $\sigma$  sufficiently large and negative,

$$\left| \sum_{n=1}^{\infty} (a_n \pm ib_n) n^s \right| \geq \frac{1}{2} n_0^\sigma |a_{n_0} + ib_{n_0}|.$$

This implies that  $\mu(\sigma) = \frac{1}{2} - \sigma$  for  $\sigma$  sufficiently large and negative. By convexity,  $\mu(\sigma) \geq \mu_0(\sigma)$  for all  $\sigma$ . But for  $\sigma \leq 0$ , we already know that  $\mu(\sigma) \leq \frac{1}{2} - \sigma$ , so we have equality here. □

**Remarks.**

- (a) Theorem 1 and Corollary 2 extend immediately to the case where  $N(x) - cx$  is periodic for some constant  $c$ .
- (b) Similar results can be obtained more generally if  $R(x) = N(x) - x$  is almost-periodic under some extra assumptions. For example, suppose that

$$R(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi\lambda_n x + \sum_{n=1}^{\infty} b_n \sin 2\pi\lambda_n x,$$

and that the series is boundedly convergent with  $a_n$  and  $b_n$  both  $O(1/n)$ . Here suppose  $\lambda_n > 0$  increases strictly and without bound. If we assume that  $\sum \frac{\lambda_n^\sigma}{n}$  converges for every  $\sigma < 0$ , then the same method as in Theorem 1 shows that  $\hat{N}$  has an analytic continuation to  $\mathbb{C} \setminus \{1\}$ , is of finite order and satisfies

$$\hat{N}(s) = \frac{s}{s-1} + \int_0^1 x^{-s} dR(x) + (2\pi)^s \Gamma(1-s) \left( \cos \frac{\pi s}{2} \sum_{n=1}^\infty a_n \lambda_n^s - \sin \frac{\pi s}{2} \sum_{n=1}^\infty b_n \lambda_n^s \right),$$

for  $\sigma < 0$ . Corollary 2 also holds in this case if the  $a_n$  and  $b_n$  are not identically zero (i.e.  $R(x)$  not constant).

- (c) The inequality  $\mu \geq \mu_0$  seems quite robust. It holds for the Beurling zeta function associated to discrete g-prime systems (see [3]) but also for those Mellin transforms contained in (a) and (b) above. What is a natural setting for which this inequality is true?

### 3. A particular flow of Mellin transforms to $\zeta(s)$

As Corollary 2 shows, it is impossible to construct a flow of Mellin transforms with ‘periodic’ integrator converging to  $\zeta(s)$  such that the supremum of the real parts of the zeros converges to  $\frac{1}{2}$  from below. Nevertheless, it might still be of interest to investigate a particular flow of such systems with  $N(x) - x$  periodic.

Here we consider a particular flow of Mellin transforms  $\{\hat{N}_\lambda(s)\}_{\lambda \geq 1}$  converging uniformly to  $\zeta(s)$  as  $\lambda \rightarrow 1$ , and for which  $N_\lambda(x) - x$  has period 1 with Fourier coefficients proportional to  $\frac{1}{n^\lambda}$ . We shall see that for  $\lambda > 1$ ,  $\hat{N}_\lambda(s)$  shares a number of characteristics of  $\hat{N}_1(s) = \zeta(s)$ . Thus  $\hat{N}_\lambda(s)$  has roughly  $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$  zeros in  $H_0$  up to height  $T$  and an infinite number of negative zeros, roughly at the points  $\lambda - 1 - 2n$  ( $n \in \mathbb{N}$ ).

The Hurwitz zeta function  $\zeta(s, a)$ , defined for  $\Re s > 1$  and  $0 < a \leq 1$  by the series  $\sum_{n=0}^\infty (n+a)^{-s}$  has (as a function of  $s$ ) an analytic continuation to  $\mathbb{C} \setminus \{1\}$  and a simple pole at  $s = 1$  with residue 1 (see for example [1], Chapter 12). Its analytic continuation is given by  $\zeta(s, a) = \Gamma(1-s)I(s, a)$ , where  $I(s, a)$  is the entire function

$$I(s, a) = \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^{az}}{1 - e^z} dz,$$

where  $C$  is the contour which starts at  $-\infty$ , goes along the negative real axis (on the lower side) to  $-c$  where  $0 < c < 2\pi$ , encircles the origin back to  $-c$  and returns to  $-\infty$  on the upper side of the negative real axis. Note that  $\zeta(s, 1) = \zeta(s)$ . The definition actually makes sense whenever  $\Re a > 0$

(any  $s$ ). As a function of  $a$  (for any given  $s$ ),  $I(s, a)$  is holomorphic for  $\Re a > 0$ .

**Definition.** For  $\lambda \geq 1$ , let  $N_\lambda(x) = x - R_\lambda(x)$  for  $x \geq 1$  and zero otherwise, where  $R_\lambda(x)$  is periodic with period 1 and is defined for  $0 \leq x < 1$  by

$$(3.1) \quad R_\lambda(x) = \rho_\lambda(\zeta(1-\lambda, 1-x) - \zeta(1-\lambda)) = \frac{\rho_\lambda \Gamma(\lambda)}{2\pi i} \int_C \frac{z^{-\lambda}(e^{-xz} - 1)}{e^{-z} - 1} dz.$$

Here  $\rho_\lambda$  is a continuous function of  $\lambda$  (to be determined) and we set  $\rho_1 = 1$  so that  $R_1(x) = \{x\}$ .

**3.1. Some properties.**

- (a) For  $m \in \mathbb{N}$ ,  $R_m$  is a polynomial in  $[0, 1)$  since  $\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}$  where  $B_n(\cdot)$  is the  $n^{\text{th}}$  Bernoulli polynomial; i.e. for  $0 \leq x < 1$

$$R_m(x) = \frac{\rho_m}{m}(B_m(1) - B_m(1-x)) = \frac{(-1)^{m-1} \rho_m}{m}(B_m(x) - B_m(0)).$$

- (b) For  $\lambda > 1$ ,  $R_\lambda$  is continuous, while  $R_1$  is right continuous but has jump continuities at the integers. Further,  $R_\lambda$  can be holomorphically continued to a neighbourhood of the interval  $[0, 1)$ , since the function

$$R_\lambda^*(z) = \rho_\lambda(\zeta(1-\lambda, 1-z) - \zeta(1-\lambda)),$$

which agrees with  $R_\lambda$  on  $[0, 1)$ , is holomorphic for  $\Re z < 1$ . Hence we have an expansion

$$R_\lambda(x) = \sum_{n=1}^{\infty} a_n(\lambda)x^n \quad (0 \leq x < 1)$$

for some coefficients  $a_n(\lambda)$ . Expanding the integrand in (3.1) gives a formula for the coefficients.

$$\begin{aligned} R_\lambda(x) &= \frac{\rho_\lambda \Gamma(\lambda)}{2\pi i} \int_C \frac{z^{-\lambda}}{e^{-z} - 1} \sum_{n=1}^{\infty} (-1)^n \frac{x^n z^n}{n!} dz \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{\rho_\lambda \Gamma(\lambda)}{2\pi i} \int_C \frac{z^{n-\lambda}}{e^{-z} - 1} dz \right) x^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\rho_\lambda \Gamma(\lambda) \zeta(n-\lambda+1)}{\Gamma(\lambda-n)} x^n. \end{aligned}$$

Hence

$$(3.2) \quad a_n(\lambda) = (-1)^n \rho_\lambda \binom{\lambda-1}{n} \zeta(n+1-\lambda).$$

For  $\lambda > 1$  the expansion is also valid for  $x = 1$ , since  $a_n(\lambda) = O(n^{-\lambda})$ . For  $\lambda = m \in \mathbb{N}$  and  $n = m$ , (3.2) should be interpreted as  $\lim_{\lambda \rightarrow m} a_m(\lambda) = (-1)^{m-1} \rho_m/m$ . Of course in this case the expansion is finite and is a polynomial of degree  $m$ .

(c) Fourier expansion: We have

$$R_\lambda(x) = -\frac{2\rho_\lambda\Gamma(\lambda)}{(2\pi)^\lambda} \left( \cos \frac{\pi\lambda}{2} \sum_{n=1}^\infty \frac{1 - \cos 2\pi nx}{n^\lambda} + \sin \frac{\pi\lambda}{2} \sum_{n=1}^\infty \frac{\sin 2\pi nx}{n^\lambda} \right)$$

which holds for all  $x \in \mathbb{R}$  if  $\lambda > 1$  and for  $x \in \mathbb{R} \setminus \mathbb{Z}$  if  $\lambda = 1$  ([1], p. 257).

By Theorem 1,  $\hat{N}_\lambda$  extends analytically to the complex plane except for a simple pole at 1 and (after some calculation)

$$(3.3) \quad \hat{N}_\lambda(s) = \frac{s}{s-1} + \int_0^1 x^{-s} dR_\lambda(x) + 2\rho_\lambda(2\pi)^{s-\lambda}\Gamma(\lambda)\Gamma(1-s) \cos \frac{\pi(s-\lambda)}{2} \zeta(\lambda-s).$$

Using the functional equation for  $\zeta(\cdot)$  this becomes

$$(3.4) \quad \hat{N}_\lambda(s) = \frac{s}{s-1} + \int_0^1 x^{-s} dR_\lambda(x) + \rho_\lambda \frac{\Gamma(\lambda)\Gamma(1-s)}{\Gamma(\lambda-s)} \zeta(s-\lambda+1).$$

For  $\lambda > 1$  we have for  $\sigma < 1$ ,

$$(3.5) \quad \begin{aligned} \int_0^1 x^{-s} dR_\lambda(x) &= \int_0^1 x^{-s} R'_\lambda(x) dx \\ &= \int_0^1 \sum_{n=1}^\infty n a_n(\lambda) x^{n-s-1} dx \\ &= \sum_{n=1}^\infty \frac{n a_n(\lambda)}{n-s}. \end{aligned}$$

This series converges for all  $s \notin \mathbb{N}$  and provides the meromorphic continuation of the LHS to  $\mathbb{C}$  with (at most) simple poles at the positive integers. Thus (3.3)-(3.5) hold for all  $s \in \mathbb{C} \setminus \mathbb{N}$ .

**Theorem 3.** *With  $N_\lambda$  as defined above, we have  $\hat{N}_\lambda(s) \rightarrow \zeta(s)$  as  $\lambda \rightarrow 1$  uniformly on compact subsets of  $\mathbb{C} \setminus \{1\}$ .*

*Proof.* This basically follows from the fact that  $R_\lambda \rightarrow R_1$  uniformly on  $[0, a]$  for every  $a < 1$ , but we need to be a little careful near 1 since  $R_1$  is not continuous here. First consider  $\sigma > 0$ . Let  $K$  be a compact subset of  $H_0 \setminus \{1\}$ . We have for  $s \in K$

$$|\hat{N}_\lambda(s) - \hat{N}_1(s)| = \left| s \int_1^\infty \frac{R_\lambda(x) - R_1(x)}{x^{s+1}} dx \right| \leq A \int_1^\infty \frac{|R_\lambda(x) - R_1(x)|}{x^{\sigma_0+1}} dx$$

for some constants  $A, \sigma_0 > 0$ . Let  $\eta > 0$ . Then for all  $\varepsilon > 0$ , there exists  $\lambda_0$  such that for  $1 < \lambda < \lambda_0$ ,  $|R_1(x) - R_\lambda(x)| < \varepsilon$  for  $n \leq x \leq n + 1 - \eta$  (any  $n \in \mathbb{Z}$ ) while in any case  $|R_1(x) - R_\lambda(x)| \leq C$  for some absolute constant  $C$  (for all  $x$ ). Hence

$$\begin{aligned} |\hat{N}_\lambda(s) - \hat{N}_1(s)| &\leq A\varepsilon \int_1^\infty \frac{1}{x^{\sigma_0+1}} dx + A \sum_{n=1}^\infty \int_{n+1-\eta}^{n+1} \frac{C}{x^{\sigma_0+1}} dx \\ &\leq A_1\varepsilon + AC\eta \sum_{n=1}^\infty \frac{1}{n^{\sigma_0+1}}, \end{aligned}$$

which can be made as small as we please. Hence  $\hat{N}_\lambda(s) \rightarrow \hat{N}_1(s)$  uniformly on compact subsets of  $H_0 \setminus \{1\}$ .

In fact the same argument works for compact subsets of  $H_{-1} \setminus \{1\}$  if we use the expression

$$\hat{N}_\lambda(s) = \frac{s}{s-1} - a_0 + s(s+1) \int_1^\infty \frac{V_\lambda(x)}{x^{s+2}} dx,$$

where  $V_\lambda(x) = \int_1^x (R_\lambda(\cdot) - a_0)$ , and noting that  $V_\lambda \rightarrow V_1$  uniformly.

For  $\sigma < 0$  we can use (3.4). The final term tends locally uniformly to  $\zeta(s)$ , while

$$\int_0^1 x^{-s} dR_\lambda(x) = s \int_0^1 \frac{R_\lambda(x)}{x^{s+1}} dx \rightarrow s \int_0^1 \frac{R_1(x)}{x^{s+1}} dx = -\frac{s}{s-1},$$

the convergence again being uniform. The result now follows. □

**3.2. Zeros.** Since  $\hat{N}_\lambda(s) \rightarrow \zeta(s)$  locally uniformly, the Riemann Hypothesis will follow if we can show that for all  $\lambda$  close to 1 (with some particular choice of  $\rho_\lambda$ ),  $\hat{N}_\lambda(s)$  has no zeros with  $\sigma > \frac{1}{2}$ . Slightly less restrictively, RH is true if the following conjecture is true:

**Conjecture.** *Given  $\theta > \frac{1}{2}$ , there exists  $\lambda_\theta > 1$  such that for  $1 < \lambda < \lambda_\theta$  and some suitable choice of  $\rho_\lambda$ ,  $\hat{N}_\lambda$  has no zeros in  $H_\theta$ .*

It may even be the case that this conjecture is equivalent to RH. The hope is of course that it is easier to show that for  $\lambda > 1$ ,  $\hat{N}_\lambda$  has no zeros in  $H_\theta$  than it is for  $\lambda = 1$ .

Now we show that for  $\lambda \geq \frac{3}{2}$ ,  $\hat{N}_\lambda$  has only *finitely* many zeros in  $H_{\frac{1}{2}+\delta}$  (any  $\delta > 0$ ). As  $\lambda$  gets closer to 1 however, we can only be certain of having finitely many zeros in half-planes further to the right, since we do not have the strong bounds on  $\zeta$  in vertical strips. If we assume the Lindelöf Hypothesis (LH), then  $\hat{N}_\lambda$  has only finitely many zeros in  $H_{\frac{1}{2}+\delta}$  for *every*  $\lambda > 1$ .

**Theorem 4.** (i) Let  $\lambda \geq \frac{3}{2}$ . Then for every  $\delta > 0$ ,  $\hat{N}_\lambda(s)$  has at most finitely many zeros in  $H_{\frac{1}{2}+\delta}$  and in every strip where  $\sigma \in [-A, \frac{1}{2} - \delta]$  (any  $A$ ).

(ii) Let  $1 < \lambda < \frac{3}{2}$ . Then for every  $\delta > 0$ ,  $\hat{N}_\lambda(s)$  has at most finitely many zeros in  $H_{2-\lambda+\delta}$  ( $H_{\frac{1}{2}+\delta}$  on LH) and in every strip where  $\sigma \in [-A, \lambda - 1 - \delta]$  ( $\sigma \in [-A, \frac{1}{2} - \delta]$  on RH).

*Proof.* For  $\lambda > 1$ ,  $\int_0^1 x^{-s} dR_\lambda(x) = \sum_{n=1}^\infty \frac{na_n(\lambda)}{n-s} \rightarrow 0$  as  $|t| \rightarrow \infty$  for every  $\sigma$ . Hence from (3.4),

$$\hat{N}_\lambda(\sigma + it) = 1 + o(1) + \rho_\lambda \frac{\Gamma(\lambda)\Gamma(1 - \sigma - it)}{\Gamma(\lambda - \sigma - it)} \zeta(\sigma - \lambda + 1 + it).$$

The term on the right is, in modulus, asymptotic to

$$(3.6) \quad |\rho_\lambda \Gamma(\lambda) \frac{|\zeta(\sigma - \lambda + 1 + it)|}{|t|^{\lambda-1}}| = O(|t|^{\mu(\sigma-\lambda+1)-\lambda+1+\varepsilon}),$$

for every  $\varepsilon > 0$ , where  $\mu(\cdot)$  is the Lindelöf function for  $\zeta$ . Note that the implied constant is independent of  $\sigma$  for  $a \leq \sigma \leq b$ , any  $a, b$ .

Let  $\lambda \geq \frac{3}{2}$ . Consider  $\sigma \leq \lambda - 1$  and  $\sigma > \lambda - 1$  separately. If  $\sigma \leq \lambda - 1$ , then  $\mu(\sigma - \lambda + 1) = \lambda - \sigma - \frac{1}{2}$ , and the exponent of  $|t|$  in (3.6) is  $\frac{1}{2} - \sigma + \varepsilon$ . This is negative (for sufficiently small  $\varepsilon$ ) if  $\sigma > \frac{1}{2}$ . If  $\sigma > \lambda - 1$ ,  $\mu(\sigma - \lambda + 1) < \frac{1}{2}$ , so the exponent is also negative for  $\varepsilon$  small enough. Since the bound is uniform in  $\sigma$ , and there are no zeros in  $H_A$  for  $A$  sufficiently large, this implies that for  $\lambda \geq \frac{3}{2}$ ,  $\hat{N}_\lambda$  has only finitely many zeros in  $H_{\frac{1}{2}+\delta}$  for each  $\delta > 0$ .

If  $\sigma < \frac{1}{2}$ , then  $\sigma < \lambda - 1$  and the expression in (3.6) is at least<sup>2</sup>

$$c|t|^{\frac{1}{2}-\sigma},$$

for some  $c > 0$ , depending continuously on  $\lambda$  and  $\sigma$ . Hence for  $-A \leq \sigma \leq \frac{1}{2} - \delta$ , this is at least  $c_1|t|^\delta$  (some constant  $c_1 > 0$ ) which tends to infinity. Thus there are no zeros with  $|t|$  sufficiently large in such a strip, proving assertion (i).

Now consider  $1 < \lambda < \frac{3}{2}$ . If  $\sigma \geq \lambda$ , then  $\mu(\sigma - \lambda + 1) = 0$  and the exponent in (3.6) is negative. For  $\lambda - 1 \leq \sigma < \lambda$ ,  $\mu(\sigma - \lambda + 1) \leq \frac{\lambda - \sigma}{2}$  (using  $\mu(\alpha) \leq \frac{1-\alpha}{2}$  for  $0 \leq \alpha \leq 1$ ) and the exponent in (3.6) is  $1 - \frac{\lambda + \sigma}{2} + \varepsilon$ . This is negative for  $\sigma > 2 - \lambda$ , and the result follows.

If LH holds, then  $\mu(\sigma - \lambda + 1) = 0$  for  $\sigma > \lambda - \frac{1}{2}$  and  $\mu(\sigma - \lambda + 1) = \lambda - \sigma - \frac{1}{2}$  for  $\sigma \leq \lambda - \frac{1}{2}$ . Hence the exponent in (3.6) is now

$$\begin{aligned} 1 - \lambda + \varepsilon & \text{ if } \sigma > \lambda - \frac{1}{2} \\ \frac{1}{2} - \sigma + \varepsilon & \text{ if } \sigma \leq \lambda - \frac{1}{2} \end{aligned} .$$

<sup>2</sup>Assuming  $\rho_\lambda \neq 0$ . If  $\rho_\lambda = 0$ , the result is trivially true.

Both are negative if  $\sigma > \frac{1}{2}$  for sufficiently small  $\varepsilon$ .

As in part(i), if  $\sigma < \lambda - 1$ , then  $\sigma - \lambda + 1 < 0$  and the expression in (3.6) is at least  $c|t|^{\frac{1}{2}-\sigma} \rightarrow \infty$ . For  $\sigma \geq \lambda - 1$  we cannot deduce anything about (3.6) for large  $|t|$  unless we know that  $\zeta$  has no zeros in certain strips inside the critical strip. On RH, the above argument applies for  $\sigma - \lambda + 1 < \frac{1}{2}$ , and (ii) follows.  $\square$

**Remark.** For  $\lambda > \frac{3}{2}$ , the zeros in any right half-plane (apart from at most a finite number of exceptions) actually lie in a region

$$\left\{ \sigma + it : -\frac{A}{\log |t|} \leq \sigma - \frac{1}{2} \leq \frac{B}{\log |t|}, |t| \geq 2 \right\},$$

for some constants  $A, B$ . For  $\hat{N}_\lambda(s) = 0$  if and only if

$$(3.7) \quad \frac{s}{s-1} + \int_0^1 x^{-s} dR_\lambda(x) = -2\rho_\lambda(2\pi)^{s-\lambda}\Gamma(\lambda)\Gamma(1-s)\cos\frac{\pi(s-\lambda)}{2}\zeta(\lambda-s).$$

Take  $\sigma$  such that  $|\sigma - \frac{1}{2}| \leq \lambda - \frac{3}{2} - \delta$  for some  $\delta > 0$ , and  $|t| \geq 2$ . The LHS of (3.7) is  $1 + o(1)$ , while the RHS is, in modulus,

$$\sim \frac{|\rho_\lambda\Gamma(\lambda)}{(2\pi)^{\lambda-\sigma-\frac{1}{2}}}|t|^{\frac{1}{2}-\sigma}|\zeta(\lambda-\sigma-it)|.$$

Since  $\lambda - \sigma \geq 1 + \delta$ , this is  $\asymp |t|^{\frac{1}{2}-\sigma}$ , uniformly in  $\sigma$ . In particular, for  $\frac{1}{2} - \sigma > A/\log |t|$  and  $A$  sufficiently large, the LHS of (3.7) is less than the RHS in modulus, and hence there are no zeros for  $|t|$  sufficiently large in this range. Similarly, for  $\sigma - \frac{1}{2} > B/\log |t|$  and  $B$  sufficiently large, the LHS is greater than the RHS in modulus.

We can be more precise. Let  $\sigma = \frac{1}{2} + \frac{\theta_t}{\log |t|}$  where  $\theta_t = O(1)$ . Then for a zero  $\sigma + it$  with large  $|t|$ , we need

$$\frac{|\rho_\lambda\Gamma(\lambda)}{(2\pi)^{\lambda-1}}e^{-\theta_t}|\zeta(\lambda-\sigma-it)| \sim 1.$$

Since  $|\zeta(\lambda - \sigma - it)| \sim |\zeta(\lambda - \frac{1}{2} - it)|$ , this requires

$$\theta_t = \log\left(\frac{|\rho_\lambda\Gamma(\lambda)}{(2\pi)^{\lambda-1}}|\zeta\left(\lambda - \frac{1}{2} - it\right)\right) + o(1).$$

As such and taking  $t \geq 2$ , the RHS of (3.7) is, using Stirling’s formula, asymptotically

$$\begin{aligned} & -\frac{\rho_\lambda\Gamma(\lambda)}{(2\pi)^{\lambda-1}}e^{-\theta_t+\frac{i\pi}{2}(\lambda-\frac{1}{2})}e^{-i(t\log t-t-t\log 2\pi)}\zeta\left(\lambda-\frac{1}{2}-it\right) \\ & \sim -\frac{\rho_\lambda\zeta\left(\lambda-\frac{1}{2}-it\right)}{|\rho_\lambda\zeta\left(\lambda-\frac{1}{2}-it\right)|}e^{-i(t\log t-t-t\log 2\pi-\frac{\pi}{2}(\lambda-\frac{1}{2}))}. \end{aligned}$$

At a zero, we want this to be asymptotic to the LHS of (3.7); i.e. to 1. Since  $\arg\zeta(\lambda - \frac{1}{2} - it)$  is bounded (as  $\lambda > \frac{3}{2}$ ), we therefore want  $t \log t - t \log 2\pi = 2\pi k + O(1)$  for  $k \in \mathbb{Z}$ ; i.e.

$$f(t) := \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} = k + O(1).$$

Since  $f(t)$  is continuous we should expect a zero  $\sigma_k + it_k$  for each  $k$  sufficiently large. The number of such zeros with  $t_k \leq T$  is therefore roughly  $f(T)$ ; i.e. we should expect, for  $\lambda > \frac{3}{2}$ ,

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(1)$$

zeros up to height  $T$ .

**Theorem 5.** *Let  $\lambda > 1$ . Then  $\hat{N}_\lambda$  has*

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

*zeros in the rectangular strip  $\{\sigma + it : 0 \leq \sigma \leq 1, 0 \leq t \leq T\}$ .*

*Proof.* Choose  $\sigma_0$  sufficiently large so that  $|\Re \hat{N}_\lambda(\sigma_0 + it)| \geq c > 0$  for all  $t$ . (This is possible since  $|\Re \hat{N}_\lambda(\sigma + it)| \geq 1 - \int_1^\infty x^{-\sigma} dN_\lambda(x) = 2 - \hat{N}_\lambda(\sigma) \rightarrow 1$  as  $\sigma \rightarrow \infty$ .)

Denote by  $n(T)$  the number of zeros in the rectangular strip

$$\{\sigma + it : 0 \leq \sigma \leq \sigma_0, 1 \leq t \leq T\}.$$

This differs from the required number by  $O(1)$ . Let  $\gamma$  denote the (anti-clockwise) boundary path of this strip. We may assume without loss of generality that there are no zeros of  $\hat{N}_\lambda$  on  $\gamma$ . Then

$$n(T) = \frac{1}{2\pi} \Delta_\gamma \arg \hat{N}_\lambda,$$

where  $\Delta_\gamma \arg \hat{N}_\lambda$  is the continuous variation of the argument of  $\hat{N}_\lambda$  around  $\gamma$ .

On the right-hand vertical,  $\hat{N}_\lambda(\sigma_0 + it) \rightarrow 1$  as  $t \rightarrow \infty$ . Hence the variation of the argument along this vertical line segment is  $O(1)$ . For the top horizontal, we use Lemma 9.4 of [6] (with ‘2’ replaced by ‘ $\sigma_0$ ’). Since  $\hat{N}_\lambda$  has finite order, this Lemma implies that the variation along here is at most  $O(\log T)$ . The variation along the bottom horizontal is trivially  $O(1)$ . Finally on the left vertical, we have

$$\begin{aligned} \hat{N}_\lambda(it) &= 2\rho_\lambda \Gamma(\lambda) (2\pi)^{it-\lambda} \Gamma(1-it) \cos \frac{\pi(it-\lambda)}{2} \zeta(\lambda-it) + 1 + o(1) \\ &\sim \frac{\rho_\lambda \Gamma(\lambda)}{(2\pi)^{\lambda-\frac{1}{2}}} t^{\frac{1}{2}} e^{-i(t \log t - t \log 2\pi)} e^{\frac{i\pi}{2}(\lambda-\frac{1}{2})} \zeta(\lambda-it). \end{aligned}$$

Since  $\zeta(\lambda - it)$  is bounded and bounded away from zero,  $\arg \hat{N}_\lambda(it) = -(t \log t - t - t \log 2\pi) + O(1)$ , and the variation of the argument along the (downward) left hand vertical is  $T \log T - T - T \log 2\pi + O(1)$ .  $\square$

**Remark.** It seems plausible that the  $O(\log T)$ -term can be replaced by  $O((\log T)^\kappa)$ , with  $\kappa$  decreasing steadily from 1 to 0 as  $\lambda$  varies from 1 to  $\frac{3}{2}$ .

**3.3. Zeros on the negative real axis.** For  $\lambda = 1$ ,  $\hat{N}_\lambda(s) = \zeta(s)$  has zeros on the negative real axis at  $-2k$  for each positive integer  $k$  — the so-called trivial zeros. Very similar behaviour occurs for  $\lambda > 1$ .

We require the following elementary result.

**Lemma 6.** *Suppose  $f$  is holomorphic and real valued on  $[0, \infty)$ . Suppose further that, as  $x \rightarrow \infty$ ,*

$$f(x) = \cos \frac{\pi x}{2} + o(1) \quad \text{and} \quad f'(x) = -\frac{\pi}{2} \sin \frac{\pi x}{2} + o(1).$$

*Then for every sufficiently large integer  $n$ , the interval  $(2n, 2n + 2)$  contains exactly one zero, say  $x_n$ , and  $x_n = 2n + 1 + o(1)$ .*

*Proof.* For  $n \in \mathbb{N}$ ,  $f(2n) - (-1)^n \rightarrow 0$ , so for  $n$  sufficiently large, the sign of  $f(2n)$  is  $(-1)^n$ . Hence there is at least one zero in each interval  $(2n, 2n + 2)$  (for  $n$  large). In fact the zero(s) must be close to  $2n + 1$  since for  $|h| \leq 1$ ,

$$f(2n + h) - (-1)^n \cos \frac{\pi h}{2} \rightarrow 0,$$

uniformly in  $h$ , and  $\cos \frac{\pi h}{2}$  is bounded away from zero if  $|h| \leq h_0 < 1$ .

Now for  $x = 2n + y$ ,  $f'(x) = (-1)^{n-1} \frac{\pi}{2} \sin \frac{\pi y}{2} + o(1)$ , so for  $x \in [2n + h, 2n + 2 - h]$  (any fixed  $h > 0$ ),  $(-1)^{n-1} f'(x) > 0$  for  $n$  large enough; i.e.  $f$  is monotonic in this interval. Thus can be at most one zero, say  $x_n$ . This must satisfy  $x_n = 2n + 1 + o(1)$ .  $\square$

**Theorem 7.** *Let  $\lambda > 1$ . For every sufficiently large positive integer  $n$ ,  $\hat{N}_\lambda(\lambda - x)$  has exactly one zero  $x_n$  in each interval  $(2n, 2n + 2)$  ( $n \in \mathbb{N}$ ). Furthermore  $x_n = 2n + 1 + o(1)$  as  $n \rightarrow \infty$ .*

*Proof.* Apply Lemma 6 with

$$\begin{aligned} f(x) &= \frac{(2\pi)^x \hat{N}_\lambda(\lambda - x)}{2\rho_\lambda \Gamma(\lambda) \Gamma(x + 1 - \lambda)} \\ &= \zeta(x) \cos \frac{\pi x}{2} + \frac{(2\pi)^x}{2\rho_\lambda \Gamma(\lambda) \Gamma(x + 1 - \lambda)} \left( \frac{x - \lambda}{x - \lambda - 1} + \sum_{m=1}^{\infty} \frac{m a_m(\lambda)}{m - \lambda + x} \right) \end{aligned}$$

(using (3.3) and (3.5)). The final term and its derivative tend to 0 as  $x \rightarrow \infty$ , while  $\zeta(x) \rightarrow 1$ ,  $\zeta'(x) \rightarrow 0$ , so  $f$  satisfies the conditions of Lemma 6 and the result follows.  $\square$

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### References

- [1] T. M. APOSTOL, *Introduction to Analytic Number Theory*. Springer, 1976.
- [2] P. T. BATEMAN and H. G. DIAMOND, *Analytic Number Theory*. World Scientific Publishing, 2004.
- [3] T. W. HILBERDINK, *A lower bound for the Lindelöf function associated to generalised integers*. J. Number Theory **122** (2007), 336–341.
- [4] T. W. HILBERDINK and M. L. LAPIDUS, *Beurling zeta functions, Generalised Primes, and Fractal Membranes*. Acta Appl Math **94** (2006), 21–48.
- [5] E. C. TITCHMARSH, *The Theory of Functions*. Second edition, Oxford University Press, 1986.
- [6] E. C. TITCHMARSH, *The Theory of the Riemann Zeta-function*. Second edition, Oxford University Press, 1986.

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