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## Torsors under tori and Néron models

par MARTIN BRIGHT

RÉSUMÉ. Soit  $R$  un anneau de valuation discrète hensélien et  $K$  son corps des fractions. Si  $X$  est une variété lisse sur  $K$  et  $G$  un tore sur  $K$ , on considère les toreseurs sur  $X$  sous  $G$ . Soit  $\mathcal{X}/R$  un modèle de  $X$ ; en utilisant un résultat de Brahm, on montre que les toreseurs sur  $X$  sous  $G$  se prolongent aux toreseurs sur  $\mathcal{X}$  sous un modèle de Néron de  $G$  si  $G$  est scindé par une extension modérément ramifiée de  $K$ . On déduit que l'application d'évaluation associée à un tel toseur se factorise par la réduction à la fibre spéciale. On peut ainsi étudier l'arithmétique de  $X$  en utilisant la géométrie de la fibre spéciale.

ABSTRACT. Let  $R$  be a Henselian discrete valuation ring with field of fractions  $K$ . If  $X$  is a smooth variety over  $K$  and  $G$  a torus over  $K$ , then we consider  $X$ -torsors under  $G$ . If  $\mathcal{X}/R$  is a model of  $X$  then, using a result of Brahm, we show that  $X$ -torsors under  $G$  extend to  $\mathcal{X}$ -torsors under a Néron model of  $G$  if  $G$  is split by a tamely ramified extension of  $K$ . It follows that the evaluation map associated to such a torsor factors through reduction to the special fibre. In this way we can use the geometry of the special fibre to study the arithmetic of  $X$ .

### 1. Introduction

Let  $R$  be a Henselian discrete valuation ring, with field of fractions  $K$  and perfect residue field  $k$ . Let  $s: \text{Spec } k \rightarrow \text{Spec } R$  denote the inclusion of the special point.

If  $X$  is a smooth variety<sup>1</sup> over  $K$ , and  $Y$  an  $X$ -torsor under an algebraic  $K$ -group  $G$ , there is an “evaluation” map  $X(K) \rightarrow H^1(K, G)$  which associates to each point  $P$  of  $X$  the isomorphism class of the fibre  $Y_P$ . The main result of this article is to show that, when  $G$  is a torus split by a tamely ramified extension of  $K$ , this map depends only on the image of  $P$  in the special fibre of a model of  $X$ .

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<sup>1</sup>By a variety over  $K$  we mean a separated, integral scheme of finite type over  $K$ .

**Theorem 1.1.** *Let  $X$  be a smooth variety over  $K$ , and let  $\mathcal{X}/R$  be a model of  $X$ ; we write  $X(K)_{\text{sm}}$  for the set of  $K$ -points of  $X$  which extend to smooth  $R$ -points of  $\mathcal{X}$ . Let  $G$  be a torus over  $K$  with a Néron model  $\mathcal{G}$  over  $R$ , and suppose that  $G$  is split by a tamely ramified extension of  $K$ . Let  $Y \rightarrow X$  be an  $X$ -torsor under  $G$ . Then the natural maps*

$$H^1(K, G) \leftarrow H^1(R, \mathcal{G}) \rightarrow H^1(k, \mathcal{G}_s)$$

are isomorphisms, and the evaluation map  $X(K)_{\text{sm}} \rightarrow H^1(K, G)$  coming from  $Y$  factors through the  $k$ -points of the special fibre  $\mathcal{X}_s$ , as follows:

$$\begin{array}{ccc} X(K)_{\text{sm}} & \longrightarrow & \mathcal{X}_s(k) \\ Y \downarrow & & \downarrow f \\ H^1(K, G) & \xrightarrow{\cong} & H^1(k, \mathcal{G}_s). \end{array}$$

Here  $\mathcal{G}_s$  is the special fibre of  $\mathcal{G}$ , and the map  $f$  comes from an element of  $H^1(\mathcal{X}_s, \mathcal{G}_s)$ .

A main step in proving Theorem 1.1 is the following technical result, in which the principal ingredient is due to Brahm.

**Theorem 1.2.** *Let  $\mathcal{X}$  be a smooth scheme over  $R$ . Let  $G$  be a torus over  $K$  with a Néron model  $\mathcal{G}$  over  $R$ , and suppose that  $G$  is split by a tamely ramified extension of  $K$ . Then the natural map  $H^1(\mathcal{X}, \mathcal{G}) \rightarrow H^1(X, G)$  is an isomorphism.*

If the residue field  $k$  is in fact finite, then we can easily deduce the following refinement of Theorem 1.1, which is more useful in applications.

**Corollary 1.3.** *Suppose in addition that  $k$  is finite. Let  $\Phi(\mathcal{G})$  denote the  $k$ -group of connected components of  $\mathcal{G}$ . Then we can replace  $\mathcal{G}_s$  with  $\Phi(\mathcal{G})$  in the conclusion of Theorem 1.1: the evaluation map  $X(K)_{\text{sm}} \rightarrow H^1(K, G)$  coming from  $Y$  factors as follows:*

$$\begin{array}{ccc} X(K)_{\text{sm}} & \longrightarrow & \mathcal{X}_s(k) \\ Y \downarrow & & \downarrow g \\ H^1(K, G) & \xrightarrow{\cong} & H^1(k, \Phi(\mathcal{G})). \end{array}$$

Here the map  $g$  comes from an element of  $H^1(\mathcal{X}_s, \Phi(\mathcal{G}))$ .

This form of the theorem is particularly useful because  $H^1(\mathcal{X}_s, \Phi(\mathcal{G}))$  is often easy to calculate, if the geometry of  $\mathcal{X}_s$  is known.

*Remark.* If the model  $\mathcal{X}$  is proper over  $R$  and regular, then in fact every  $K$ -point of  $X$  extends to a smooth  $R$ -point of  $\mathcal{X}$ : see [1, Chapter 3, Proposition 2].

Our proof of Theorem 1.2, and hence Theorem 1.1, will in fact use only the following properties of the torus  $G$ :

- (1)  $G$  is a smooth, commutative algebraic group over  $K$  admitting a Néron model over  $R$ ;
- (2) if  $\eta: \text{Spec } K \rightarrow \text{Spec } R$  denotes the inclusion of the generic point, then  $R^1\eta_*G = 0$ , considered as a sheaf on the smooth site over  $R$ .

That (2) holds for any torus split by a tamely ramified extension was proved by Brahm [2, Chapter 4], and relies on our assumption that the residue field is perfect. Brahm also studied the structure of  $R^1\eta_*G$  when this assumption is removed.

**1.1. Motivation.** Recall that a  $K$ -torus is an affine group scheme over  $K$  which becomes isomorphic to a product of some number of copies of  $\mathbb{G}_m$  after a finite base extension; we say that such an extension *splits* the torus. As described in [1, Chapter 10], tori have Néron models which are locally of finite type, but may not be of finite type. The group of components of the Néron model is a finitely generated Abelian group with an action of  $\text{Gal}(\bar{k}/k)$ ; it has been described by Xarles [17].

For example, let  $K = \mathbb{Q}_p$  with  $p$  odd, let  $L = \mathbb{Q}_p(\sqrt{p})$ , a tamely ramified quadratic extension of  $K$ , and let  $T = \mathbb{R}_{L/K}^1\mathbb{G}_m$  be the norm torus associated to this extension. The component group of the Néron model of  $T$  is of order 2. It is well known that  $H^1(K, T) = K^\times/NL^\times$ , and that there is an isomorphism of this group with  $k^\times/(k^\times)^2 = H^1(k, \mathbb{Z}/2\mathbb{Z})$ ; see [13, Chapter V, §3]. For units in  $K$ , the isomorphism is simply reduction modulo  $p$ :  $\mathcal{O}_K^\times/N_{L/K}\mathcal{O}_L^\times \cong k^\times/(k^\times)^2$ .

Let  $X$  be a smooth variety over  $K = \mathbb{Q}_p$ , and  $Y$  an  $X$ -torsor under  $T$ . On a sufficiently small open subset  $U \subset X$ ,  $Y$  looks like

$$\{x^2 - py^2 = f\} \subset \mathbb{A}_U^2$$

for some  $f \in K[U]^\times$ . For any  $P \in U(K)$ , the isomorphism class of the fibre  $Y_P$  is thus given by the class of  $f(P)$  in  $K^\times/NL^\times$ . If  $f$  takes values in  $\mathbb{Z}_p^\times$  on a neighbourhood of  $P$ , then we see that in fact the isomorphism class of  $Y_P$  depends only on whether the reduction modulo  $p$  of  $f(P)$  is a square, and hence depends only on the reduction of  $P$  modulo  $p$ .

Given a model  $\mathcal{X}/\mathbb{Z}_p$  of  $X$ , it is not too hard to show that, whenever  $P$  extends to a smooth  $\mathbb{Z}_p$ -point of  $\mathcal{X}$ , we can arrange for  $f$  to take values in  $\mathbb{Z}_p^\times$  on a neighbourhood of  $P$ . We have therefore shown that there is a commutative diagram

$$\begin{array}{ccc} X(K)_{\text{sm}} & \longrightarrow & \mathcal{X}_s(k) \\ Y \downarrow & & \downarrow \\ H^1(K, T) = K^\times/NL^\times & \xrightarrow{\cong} & k^\times/(k^\times)^2 = H^1(k, \mathbb{Z}/2\mathbb{Z}). \end{array}$$

The aim of this article is to generalise this example. In doing so, there are two principal questions:

- What other algebraic groups may replace the torus  $T$ ?
- Given a suitable algebraic group, what should replace  $\mathbb{Z}/2\mathbb{Z}$  in this example?

Theorem 1.1 states that  $T$  may be replaced by any  $K$ -torus which is split by a tamely ramified extension, and that  $\mathbb{Z}/2\mathbb{Z}$  should be replaced by the component group of the corresponding Néron model.

*Remark.* It is not reasonable to expect Theorem 1.1 to hold for all torsors under tori. Indeed, consider the norm torus  $T = \mathbb{R}_{L/\mathbb{Q}_2}^1 \mathbb{G}_m$  for a quadratic, ramified extension  $L/\mathbb{Q}_2$ . The question of whether  $a \in \mathbb{Z}_2^\times$  is a norm from  $L$  does not depend only on  $a$  modulo 2: rather, it depends on  $a$  modulo 4 or 8. We cannot therefore expect the evaluation map coming from a torsor under  $T$  to factor through reduction modulo 2.

**1.2. Background.** The use of torsors to study the rational points of varieties has been an important part of the theory of Diophantine equations since the middle of the 20th century; an excellent reference for torsors and their applications to the study of rational points is [16]. In the study of rational varieties, Colliot-Thélène and Sansuc showed that the Brauer–Manin obstruction is naturally studied using torsors under tori [6]; indeed, they conjectured that the obstructions coming from torsors under tori are the only obstructions, both to the existence of rational points and to weak approximation, on rational varieties. It is therefore important to understand the arithmetic and geometry of torsors under tori for two reasons: to approach this conjecture; and to develop ways of calculating the obstructions. The object of this article is to describe one way in which the arithmetic of torsors on a variety  $X$  is linked to the geometry of models of  $X$ , thus giving a new way to study them.

In this article, we concern ourselves entirely with the local picture: we fix a local field  $K$ , a smooth variety  $X$  over  $K$ , and a torsor  $Y \rightarrow X$  under some  $K$ -group  $G$ . We show that, under certain hypotheses on  $G$ , the evaluation map  $X(K) \rightarrow H^1(K, G)$  factors through reduction to the special fibre: that is, for a suitable model  $\mathcal{X}$  of  $X$ , the evaluation map comes from a map  $\mathcal{X}(k) \rightarrow H^1(k, G)$ , where  $k$  is the residue field of  $K$  and the map comes from a torsor under a  $k$ -group. In this way, the arithmetic of the torsor  $Y$  is related to the geometry of the special fibre of  $\mathcal{X}$ ; we give examples of how this can usefully be used to deduce facts about  $Y$ .

Although our principal application is to torsors under tori, many of the results collected will apply equally well to torsors under more general algebraic groups, and we will state them in this generality where possible.

The method used to obtain these results is to look at a Néron model of the group  $G$ , and try to extend  $X$ -torsors under  $G$  to torsors over  $\mathcal{X}$  under a Néron model of  $G$ , if one exists. In Section 2.1, we collect some results about constant torsors: how do  $K$ -torsors under a group  $G$  extend to torsors under a Néron model? In Section 2.2, we ask the same question for  $X$ -torsors, and deduce the main result (Theorem 1.1).

**1.3. Notation.** Throughout the rest of this article,  $R$  denotes a Henselian discrete valuation ring, with field of fractions  $K$  and perfect (but not necessarily finite) residue field  $k$ . Let  $K^s$  denote a separable closure of  $K$ , and  $\bar{k}$  the corresponding algebraic closure of  $k$ . Let  $K^{\text{ur}}$  denote the maximal unramified extension of  $K$ . Let  $\eta: \text{Spec } K \rightarrow \text{Spec } R$  be the inclusion of the generic point, and let  $s: \text{Spec } k \rightarrow \text{Spec } R$  be the inclusion of the special point.

If  $S$  is a scheme, then the site  $S_{\text{ét}}$  is the small étale site, the category of schemes étale and of finite type over  $S$ , endowed with the étale topology, for which the coverings are surjective families of étale morphisms of finite type. The site  $S_{\text{sm}}$  is the smooth site, the category of schemes smooth over  $S$  with the topology in which coverings are surjective families of smooth morphisms.

Suppose that  $X$  is a smooth variety over  $K$ . By *model* of  $X$  over  $R$  we mean a scheme  $\mathcal{X}$ , separated, flat and locally of finite type over  $R$ , with generic fibre  $\mathcal{X}_\eta$  isomorphic to  $X$ . Suppose that  $\mathcal{X}/R$  is a model of  $X$ . If  $\mathcal{X}$  is not proper, then  $K$ -points of  $X$  do not necessarily extend to  $R$ -points of  $\mathcal{X}$ ; even if  $\mathcal{X}$  is proper, then  $K$ -points of  $X$  cannot be expected to extend to *smooth*  $R$ -points of  $\mathcal{X}$ . In either situation, define  $X(K)_{\text{sm}}$  to be the set of  $K$ -points of  $X$  which do extend to smooth points of  $\mathcal{X}$ .

If  $G$  is a smooth group scheme over  $K$ , then a *Néron model* of  $G$  is a smooth and separated group scheme, locally of finite type, over  $R$  which represents the sheaf  $\eta_*G$  on the smooth site  $R_{\text{sm}}$ . Néron's original construction applied to Abelian varieties, but Raynaud showed that Néron models also exist for some affine algebraic groups, and in particular for tori [1, Chapter 10]. Given a group scheme  $\mathcal{G}$  over  $R$ , let  $\Phi(\mathcal{G})$  denote the group of connected components of  $\mathcal{G}$ , considered either as a group scheme over  $k$  or as a  $\text{Gal}(\bar{k}/k)$ -module.

Cohomology groups, unless otherwise indicated, are étale cohomology groups. Since we will be dealing only with smooth group schemes, the étale topology is fine enough to classify torsors – see [12, Chapter III, Remark 4.8].

If  $S$  is any scheme and  $\mathcal{G}$  a smooth, commutative group scheme over  $S$ , then  $\mathcal{G}$  gives rise to a sheaf on the étale site  $X_{\text{ét}}$ , where  $X$  is any scheme locally of finite type over  $S$ . (Indeed,  $\mathcal{G}$  defines a sheaf on the big étale site over  $S$ .) If  $Y \rightarrow X$  is any morphism of  $S$ -schemes, then

there is a natural morphism  $\mathcal{G}(X) \rightarrow \mathcal{G}(Y)$ , functorial in  $\mathcal{G}$ ; by the universal  $\delta$ -functor property of cohomology groups, this gives rise to a natural morphism  $H^i(X, \mathcal{G}) \rightarrow H^i(Y, \mathcal{G})$  for each  $i \geq 0$ , again functorial in  $\mathcal{G}$ . In particular, let  $\alpha \in H^1(X, \mathcal{G})$  be fixed, and let the morphism  $Y \rightarrow X$  vary; we obtain a map  $X(Y) \rightarrow H^1(Y, \mathcal{G})$ , which we will denote again by  $\alpha$ , and call it the *evaluation map* associated to  $\alpha$ .

Let  $P \rightarrow X$  be an  $X$ -torsor under  $\mathcal{G}$ ; then we can associate to  $P$  a class  $\alpha$  in  $H^1(X, \mathcal{G})$ . The pull-back  $P \times_X Y$  is a  $Y$ -torsor under  $\mathcal{G}$ , and the corresponding class in  $H^1(Y, \mathcal{G})$  is the image of  $\alpha$  under the natural morphism just defined. See [8, Chapitre III, §2.4.6.2]. In this way the evaluation map  $X(Y) \rightarrow H^1(Y, \mathcal{G})$  is seen to be the map which associates to each point  $y \in X(Y)$  the isomorphism class of the fibre  $P_y$ .

## 2. Extending torsors

Let  $G$  be a smooth algebraic group over  $K$ , with a Néron model  $\mathcal{G}$ . Our aim is to understand how torsors under  $G$  over some  $K$ -scheme extend to torsors under  $\mathcal{G}$ . In this section we collect some known results and show how they answer this question.

**2.1. Constant torsors.** If we are to have any hope of understanding torsors on varieties, we should first look at the question for constant torsors: how do  $K$ -torsors under  $G$  extend to  $R$ -torsors under  $\mathcal{G}$ ? This is essentially a question in Galois cohomology, but we present it using étale cohomology for consistency with the next section.

In fact we will take a purely cohomological view of this problem, and ask about extending classes in  $H^1(K, G)$  to classes in  $H^1(R, \mathcal{G})$ . Whether the resulting cohomology classes are actually represented by torsors is unimportant for our purposes.

The facts presented in this section are no more difficult to state for non-commutative groups  $G$ , and so we remind the reader of the definitions of non-commutative  $H^1$  in Galois cohomology [15, Chapitre 1, §5] and in étale cohomology [8, Chapitre III] or [12, III.4]. We will also need the definition of the non-commutative higher direct image  $R^1$ : see [8, Chapitre V, §2].

We begin by recalling that, for any connected linear algebraic group  $G$  over  $K$ , we have  $H^1(K^{\text{nr}}, G) = 0$  by Théorème 1' of [15, Chapitre III, §2.3]. (This result depends on the fact that  $K^{\text{nr}}$  is a field of dimension  $\leq 1$ , which in turn relies on our hypothesis that the residue field  $k$  be perfect.)

**Lemma 2.1.** *Let  $G$  be a smooth algebraic group over  $K$ . Suppose that  $H^1(K^{\text{nr}}, G) = 0$ . Then  $R^1\eta_*G = 0$  as a sheaf on  $R_{\text{ét}}$ .*

*Proof.* The étale sheaf  $R^1\eta_*G$  has two stalks: that at the generic point is  $H^1(K^s, G)$ , which is trivial since  $K^s$  is separably closed; that at the special

point is  $H^1(K^{nr}, G)$ , which is trivial by assumption. Therefore  $R^1\eta_*G = 0$  on  $R_{\text{ét}}$ .  $\square$

**Lemma 2.2.** *Let  $G$  be a smooth algebraic group over  $K$ ; suppose that  $G$  admits a Néron model  $\mathcal{G}$ , and suppose that  $H^1(K^{nr}, G) = 0$ . Then there is an isomorphism (of pointed sets, or of groups if  $G$  is commutative)*

$$H^1(K, G) \cong H^1(R, \mathcal{G}).$$

*Proof.* If  $G$  is commutative, this follows from the Leray spectral sequence for  $\eta$ . More generally, there is an exact sequence [8, Chapitre V, Proposition 3.1.3] as follows

$$1 \rightarrow H^1(R, \eta_*G) \xrightarrow{\alpha} H^1(K, G) \rightarrow H^0(R, R^1\eta_*G).$$

Although this is only an exact sequence of pointed sets, twisting shows that  $\alpha$  is injective. Since  $\mathcal{G}$  represents the sheaf  $\eta_*G$  on  $R_{\text{sm}}$  and hence on  $R_{\text{ét}}$ , Lemma 2.1 shows that  $\alpha$  gives the desired isomorphism.  $\square$

*Remark.* This is really the same as using the inflation-restriction sequence in Galois cohomology: from the exact sequence

$$0 \rightarrow H^1(\text{Gal}(K^{nr}/K), G(K^{nr})) \xrightarrow{\text{inf}} H^1(\text{Gal}(K^s/K), G(K^s)) \xrightarrow{\text{res}} H^1(\text{Gal}(K^s/K^{nr}), G(K^{nr}))$$

we deduce an isomorphism

$$H^1(\text{Gal}(K^s/K), G(K^s)) \cong H^1(\text{Gal}(K^{nr}/K), G(K^{nr})).$$

The first of these groups is the same as  $H^1(K, G)$ , using the usual correspondence between sheaves on  $K_{\text{ét}}$  and  $\text{Gal}(K^s/K)$ -sets. The second is isomorphic to  $H^1(R, \mathcal{G})$  since  $\mathcal{G}(R^{nr}) = G(K^{nr})$ , where  $R^{nr}$  is the integral closure of  $R$  in  $K^{nr}$ , and étale cohomology of a Henselian local ring may be computed using finite étale covers.

**Lemma 2.3.** *Let  $\mathcal{G}$  be a smooth group scheme over  $R$ , not necessarily of finite type. Then there is an isomorphism*

$$H^1(R, \mathcal{G}) \cong H^1(k, \mathcal{G}_s).$$

*Proof.* That the natural map  $H^1(R, \mathcal{G}) \rightarrow H^1(k, \mathcal{G}_s)$  is an isomorphism is well known, and essentially due to Hensel’s Lemma, but there are some issues around representability to be dealt with before Hensel’s Lemma can be applied. A very general proof of this fact, which does not require that  $\mathcal{G}$  be of finite type over  $R$ , can be found at [9, Théorème 11.7 and Remarque 11.8(3)].  $\square$

When the residue field  $k$  is finite, we can go one step further.



**Lemma 2.4.** *Suppose further that  $k$  is finite, and that  $\mathcal{G}$  is either commutative or of finite type over  $k$ . Then there is an isomorphism*

$$H^1(k, \mathcal{G}_s) \rightarrow H^1(k, \Phi(\mathcal{G})).$$

*Proof.* Consider the exact sequence of group schemes over  $k$ :

$$0 \rightarrow (\mathcal{G}_s)_0 \rightarrow \mathcal{G}_s \rightarrow \Phi(\mathcal{G}) \rightarrow 0$$

where  $(\mathcal{G}_s)_0$  is the connected component of  $\mathcal{G}_s$  containing the identity. If  $G$  is commutative, the theorem of Lang [14, Chapter VI, Proposition 5] shows that  $H^i(k, (\mathcal{G}_s)_0) = 0$  for  $i > 0$ , and so  $H^1(k, \mathcal{G}_s) \rightarrow H^1(k, \Phi(\mathcal{G}))$  is an isomorphism. If  $G$  is not commutative, the same theorem of Lang still gives  $H^1(k, (\mathcal{G}_s)_0) = 0$ , showing that our map is injective. When  $\mathcal{G}$  is of finite type, then  $\mathcal{G}_s$  is an algebraic group over  $k$  and Corollary 3 of [15, Chapitre III, §2.4] shows that the map is also surjective.  $\square$

**2.2. Torsors over varieties.** Let  $X$  be a variety over  $K$ . In this theorem, we describe when a torsor  $Y \rightarrow X$  under a group  $G$  extends to a (sheaf) torsor under the Néron model of  $G$  over a smooth model of  $X$ . When this does happen, we can deduce that the evaluation map associated to  $Y$  factors through the  $k$ -points of the model, resulting in Theorem 1.1.

**Lemma 2.5.** *Let  $\mathcal{X}$  be a smooth scheme over  $R$ . Let  $G$  be a smooth algebraic group over  $K$  with a Néron model  $\mathcal{G}$  over  $R$ , and suppose that  $R^1\eta_*G = 0$  as a sheaf on the smooth site  $R_{\text{sm}}$ . Then the natural map  $H_{\text{ét}}^1(\mathcal{X}, \mathcal{G}) \rightarrow H_{\text{ét}}^1(\mathcal{X}_\eta, G)$  is an isomorphism.*

*Proof.* Firstly note that, since  $\mathcal{G}$  is smooth over  $R$ , both of  $H_{\text{ét}}^1(\mathcal{X}, \mathcal{G})$  and  $H_{\text{ét}}^1(\mathcal{X}_\eta, G)$  are isomorphic to the corresponding cohomology groups or sets on the respective smooth sites. (For a proof, see [7, XXIV, Proposition 8.1] or [9, Remarque 11.8(3)].)

Now apply [8, Chapitre V, Proposition 3.1.3] to the inclusion of the generic fibre  $j: (\mathcal{X}_\eta)_{\text{sm}} \rightarrow \mathcal{X}_{\text{sm}}$ , to obtain an exact sequence of pointed sets

$$1 \rightarrow H_{\text{sm}}^1(\mathcal{X}, j_*G) \xrightarrow{\alpha} H_{\text{sm}}^1(\mathcal{X}_\eta, G) \rightarrow H_{\text{sm}}^0(\mathcal{X}, R^1j_*G).$$

Since  $\mathcal{X}$  is smooth over  $R$ , the sheaf  $j_*G$  is represented by  $\mathcal{G}$ , and the sheaf  $R^1j_*G$  on  $\mathcal{X}_{\text{sm}}$  is simply the restriction to  $\mathcal{X}_{\text{sm}}$  of the sheaf  $R^1\eta_*G$  on  $R_{\text{sm}}$ . The hypothesis implies that  $R^1j_*G = 0$  on  $\mathcal{X}_{\text{sm}}$ , and so  $\alpha$  is surjective. From the exact sequence above, a twisting argument shows that  $\alpha$  is also injective.  $\square$

In particular, we can deduce Theorem 1.2.

*Proof of Theorem 1.2.* Brahm [2, Chapter 4] has shown that a torus split by a tamely ramified extension satisfies the hypotheses of Lemma 2.5.  $\square$

Note that, in contrast to the situation for constant torsors, we can in general say nothing interesting about the natural map from  $H^1(\mathcal{X}, \mathcal{G})$  to  $H^1(\mathcal{X}_s, \mathcal{G}_s)$ .

Let us now derive Theorem 1.1 by putting Theorem 1.2 together with the facts collected in Section 2.1.

**Lemma 2.6.** *Let  $\mathcal{X}$  be a smooth scheme over  $R$ ,  $\mathcal{G}$  a group scheme over  $R$ , and  $P \in \mathcal{X}(R)$  a point of  $\mathcal{X}$ . Then the following diagram of evaluation maps commutes:*

$$\begin{array}{ccccccc}
 H^1(\mathcal{X}_s, \Phi(\mathcal{G})) & \longleftarrow & H^1(\mathcal{X}_s, \mathcal{G}_s) & \longleftarrow & H^1(\mathcal{X}, \mathcal{G}) & \longrightarrow & H^1(\mathcal{X}_\eta, \mathcal{G}_\eta) \\
 \downarrow P_s & & \downarrow P_s & & \downarrow P & & \downarrow P_\eta \\
 H^1(k, \Phi(\mathcal{G})) & \longleftarrow & H^1(k, \mathcal{G}_s) & \longleftarrow & H^1(R, \mathcal{G}) & \longrightarrow & H^1(K, \mathcal{G}_\eta)
 \end{array}$$

where the vertical maps are those corresponding to evaluation at  $P_s \in \mathcal{X}_s(k)$ ,  $P \in \mathcal{X}(R)$  and  $P_\eta \in \mathcal{X}_\eta(K)$  respectively.

*Proof.* The left-hand square commutes because, as observed above, pulling back by the morphism  $P_s: \text{Spec } k \rightarrow \mathcal{X}_s$  gives functorial maps on cohomology.

We now treat the right-hand two squares. Observe that, if  $S$  is any scheme over  $k$ , then every  $R$ -morphism from  $S$  to  $\mathcal{G}$  factors through the special fibre  $\mathcal{G}_s$ , by the universal property of fibre products. So the sheaves on  $\mathcal{X}_s$  defined by  $\mathcal{G}$  and  $\mathcal{G}_s$  are the same. Similarly, the sheaves on  $\mathcal{X}_\eta$  defined by  $\mathcal{G}$  and  $\mathcal{G}_\eta$  are the same. We may therefore replace  $\mathcal{G}_s$  or  $\mathcal{G}_\eta$  in the diagram by  $\mathcal{G}$  wherever they occur. The maps in the diagram are then all seen to be natural morphisms coming from morphisms of schemes, given by the following diagram.

$$\begin{array}{ccccc}
 \mathcal{X}_s & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_\eta \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{Spec } k & \longrightarrow & \text{Spec } R & \longleftarrow & \text{Spec } K
 \end{array}$$

Since this diagram commutes, so does the one above. □

*Proof of Theorem 1.1.* We begin by replacing the model  $\mathcal{X}$  with its maximal smooth subscheme, which means simply removing the singular locus of the special fibre. Now  $X(K)_{\text{sm}}$  consists of all those points of  $X(K)$  which still extend to  $\mathcal{X}(R)$ .

By Theorem 1.2, the cohomology class of the torsor  $Y \rightarrow X$  extends to a cohomology class  $\alpha \in H^1(\mathcal{X}, \mathcal{G})$ . There is therefore a diagram

$$\begin{array}{ccccc} \mathcal{X}_s(k) & \longleftarrow & \mathcal{X}(R) & \longrightarrow & X(K)_{\text{sm}} \\ \downarrow & & \alpha \downarrow & & \downarrow Y \\ H^1(k, \mathcal{G}_s) & \longleftarrow & H^1(R, \mathcal{G}) & \longrightarrow & H^1(K, G) \end{array}$$

in which the right-hand vertical map comes from the torsor  $Y$ ; the middle one from the class  $\alpha$ ; and the left-hand one from the image of  $\alpha$  in  $H^1(\mathcal{X}_s, \mathcal{G}_s)$ . This diagram commutes by Lemma 2.6. By construction, the map  $\mathcal{X}(R) \rightarrow X(K)_{\text{sm}}$  is surjective; since  $\mathcal{X}$  is separated, it is bijective. The maps in the bottom row are both isomorphisms, by Lemmas 2.2 and 2.3, hence the result.  $\square$

*Proof of Corollary 1.3.* If  $k$  is finite, let  $\beta$  be the class in  $H^1(\mathcal{X}_s, \Phi(\mathcal{G}))$  which is the image of  $\alpha$  under the map  $H^1(\mathcal{X}_s, \mathcal{G}_s) \rightarrow H^1(\mathcal{X}_s, \Phi(\mathcal{G}))$ . Then we can extend the above diagram:

$$\begin{array}{ccccccc} \mathcal{X}_s(k) & \xlongequal{\quad} & \mathcal{X}_s(k) & \longleftarrow & \mathcal{X}(R) & \longrightarrow & X(K)_{\text{sm}} \\ \beta \downarrow & & \downarrow & & \alpha \downarrow & & \downarrow Y \\ H^1(k, \Phi(\mathcal{G})) & \longleftarrow & H^1(k, \mathcal{G}_s) & \longleftarrow & H^1(R, \mathcal{G}) & \longrightarrow & H^1(K, G) \end{array}$$

The diagram still commutes, by Lemma 2.6. The maps in the bottom row are all isomorphisms (by Lemma 2.4), and the result again follows.  $\square$

### 3. Examples and applications

The use of Theorem 1.1 is that it allows us to reduce questions about the arithmetic of  $X$  to questions about the geometry of the special fibre  $\mathcal{X}$ . In this section, we demonstrate this with some results on torsors under tori. For these examples, we take  $K$  to be a  $p$ -adic field, so that the residue field  $k$  is always finite.

**Proposition 3.1.** *Let  $X$  be a smooth projective variety over  $K$ , and suppose that  $X$  has a model  $\mathcal{X}$  such that the smooth locus of the special fibre  $\mathcal{X}_s$  is geometrically simply connected. Then, for any  $K$ -torus  $T$  split by a tame extension of  $K$ , and any  $X$ -torsor under  $T$ , the evaluation map  $X(K) \rightarrow H^1(K, T)$  is constant on  $X(K)_{\text{sm}}$ .*

*Proof.* Let  $U$  be the smooth subscheme of the special fibre  $\mathcal{X}_s$ . Let  $\Phi$  be the component group of the Néron model of  $T$ ; in view of Theorem 1.1, it suffices to show that every class in  $H^1(U, \Phi)$  is constant. After base change to  $\bar{k}$ ,  $\Phi$  becomes isomorphic to a direct sum of groups each of the form  $\mathbb{Z}/n\mathbb{Z}$  or  $\mathbb{Z}$ . But, since  $\bar{U}$  is simply connected,  $H^1(\bar{U}, \mathbb{Z}) = H^1(\bar{U}, \mathbb{Z}/n\mathbb{Z}) = 0$ , and therefore  $H^1(\bar{U}, \Phi) = 0$ . Now the Hochschild–Serre spectral sequence gives an isomorphism  $H^1(k, \Phi) \rightarrow H^1(U, \Phi)$ , completing the proof.  $\square$

In particular, let  $X$  be a smooth del Pezzo surface; then the special fibre of  $\mathcal{X}$  may be a del Pezzo surface with isolated singularities. Proposition 3.1 applies if the singularities are of certain types, as listed in [11, §3]. These are precisely the singularity types such that  $\text{Pic } \bar{U}$  is free; this can easily be verified by computing  $\text{Pic } \bar{U}$  as the quotient of the relevant root lattice by the sublattice generated by the exceptional curves of a resolution.

*Remark.* We can also make conclusions about del Pezzo surfaces with other reduction types. For example, suppose that  $X$  is a smooth del Pezzo surface, and  $\mathcal{X}_s$  is the cone over a smooth curve  $C$  of genus 1. After removing the vertex to obtain the smooth locus  $U$ , there is an isomorphism  $H^1(U, \Phi) \cong H^1(C, \Phi)$ . Evaluating  $X$ -torsors under  $T$  becomes a question about  $C$ -torsors under  $\Phi$ . For an application of this in terms of the Brauer–Manin obstruction, see [3].

*Remark.* Applying the techniques of this article can only give information about the points of  $X$  reducing to the smooth locus of  $\mathcal{X}_s$ . For more complete information, we should first construct a weak Néron model for  $X$ , which is a model such that any  $K$ -point of  $X$  extends to a smooth  $R$ -point of (some component of) the model.

Following the work of Colliot-Thélène and Sansuc [6] on torsors under tori, we can also use information about the Galois action on the Picard group of  $X$  to deduce results about  $X$ -torsors under tori.

**Theorem 3.2.** *Let  $X$  be a smooth, geometrically integral, projective variety over a  $p$ -adic field  $K$ , and suppose that the Galois action on the geometric Picard group of  $X$  is tame. Let  $\mathcal{X}/R$  be a model of  $X$ . Let  $T$  be any torus over  $K$ , and let  $Y \rightarrow X$  be an  $X$ -torsor under  $T$ . Let  $\mathcal{T}$  denote the Néron model of  $T$ . Then the conclusion of Corollary 1.3 holds, i.e. the evaluation map  $X(K)_{\text{sm}} \rightarrow H^1(K, T)$  coming from  $Y$  factors through the  $k$ -points of the special fibre  $\mathcal{X}_s$ , as follows:*

$$\begin{array}{ccc} X(K)_{\text{sm}} & \longrightarrow & \mathcal{X}_s(k) \\ Y \downarrow & & \downarrow g \\ H^1(K, T) & \xrightarrow{\cong} & H^1(k, \Phi(\mathcal{T})) \end{array}$$

where the map  $g$  comes from an element of  $H^1(\mathcal{X}_s, \Phi(\mathcal{T}))$ . The bottom isomorphism is the composite of the isomorphisms

$$H^1(K, T) \leftarrow H^1(R, \mathcal{T}) \rightarrow H^1(k, \mathcal{T}_s) \rightarrow H^1(k, \Phi(\mathcal{T}))$$

of Section 2.1.

*Proof.* As in the proof of Theorem 1.1, we first turn  $\mathcal{X}$  into a smooth model by removing the singular subscheme of the special fibre. Now, although it is not necessarily true that  $R_{\text{sm}}^1 \eta_* T = 0$ , we will show that  $Y$  nevertheless

extends to a cohomology class in  $H^1(\mathcal{X}, T)$ ; the remainder of the proof is exactly as for Theorem 1.1.

Let us first define a sub-torus  $S \subseteq T$ , as follows. Let  $M = \text{Hom}_{K^s}(T, \mathbb{G}_m)$  be the module of characters of  $T$ ; it is a finitely generated, free Abelian group with an action of  $\Gamma = \text{Gal}(K^s/K)$ . Let  $\Gamma_1 \subseteq \Gamma$  be the wild inertia group. Let  $N$  be the largest free quotient of  $M$  on which  $\Gamma_1$  acts trivially. Let  $S$  be the sub-torus of  $T$  dual to  $N$ ; then  $S$  is the largest sub-torus of  $T$  which is split by a tamely ramified extension of  $K$ .

Now we will show that  $H^1(X, T)$  is generated by the images of  $H^1(X, S)$  and  $H^1(K, T)$ . We may assume that  $X(K)$  is non-empty, since otherwise the conclusion of the theorem is vacuous. Write  $\bar{X} = X \times K^s$ ; Colliot-Thélène and Sansuc [5, Lemme] have shown that there is an exact sequence

$$0 \rightarrow H^1(K, T) \rightarrow H^1(X, T) \rightarrow \text{Hom}_K(M, \text{Pic } \bar{X}) \rightarrow 0$$

which is functorial in  $T$ . There is therefore a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(K, S) & \longrightarrow & H^1(X, S) & \longrightarrow & \text{Hom}_K(N, \text{Pic } \bar{X}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & H^1(K, T) & \longrightarrow & H^1(X, T) & \longrightarrow & \text{Hom}_K(M, \text{Pic } \bar{X}) \longrightarrow 0 \end{array}$$

where, since  $\Gamma_1$  acts trivially on  $\text{Pic } \bar{X}$ , the right-hand vertical arrow is an isomorphism. It follows that  $H^1(K, T)$  surjects onto the cokernel of  $H^1(X, S) \rightarrow H^1(X, T)$ ; therefore  $H^1(X, T)$  is generated by the images of  $H^1(X, S)$  and  $H^1(K, T)$ .

The image of  $H^1(\mathcal{X}, T) \rightarrow H^1(X, T)$  contains both  $H^1(X, S)$  (since  $S$  is split by a tamely ramified extension) and  $H^1(K, T)$  (by Lemma 2.2). It therefore is the whole of  $H^1(X, T)$ . □

As an application, we deduce the following corollary. The result is already known: for example, it follows from the result of Colliot-Thélène [4] that the Chow group of 0-cycles on such a surface is trivial.

**Corollary 3.3.** *Let  $X$  be a smooth, projective, geometrically rational surface over  $K$  and suppose that there exists a smooth, proper model of  $X$  over  $R$ . Let  $T$  be any torus over  $K$ , and let  $Y \rightarrow X$  be an  $X$ -torsor under  $T$ . Then the evaluation map  $X(K) \rightarrow H^1(K, T)$  defined by  $Y$  is constant.*

*Proof.* In this case of good reduction,  $\text{Pic } \bar{X}$  is unramified as a Galois module, as follows. Let  $\mathcal{X}$  be a smooth, proper model of  $X$  over  $R$ . Write  $\bar{\mathcal{X}}_s = \mathcal{X} \times_{\bar{k}}$ , and recall that there is a reduction map from  $\text{Pic } \bar{X}$  to  $\text{Pic } \bar{\mathcal{X}}_s$ , which under our hypotheses is an isomorphism [10, Proposition 3.4.2] and clearly respects the Galois action. Since the inertia group  $I$  acts trivially on  $\text{Pic } \bar{\mathcal{X}}_s$ , we deduce that  $I$  also acts trivially on  $\text{Pic } \bar{X}$ . So Theorem 3.2 applies.

The same argument as in the proof of Proposition 3.1 now shows that the evaluation map defined by  $Y$  is constant.  $\square$

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