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Explicit Hecke series for symplectic group of genus 4

par Kirill VANKOV

Résumé. Shimura a conjecturé la rationalité de la série de Hecke des groupes symplectiques de genre n. La conjecture a été prouvée par Andrianov pour un genre arbitraire mais une forme explicite n'était connue que pour les cas des genres 1, 2 et 3. Dans l'article, la forme explicite des polynômes rationnels pour la somme de la série génératrice de Hecke dans le groupe symplectiques de genre 4 a été présentée. Le calcul est basé sur l'isomorphisme de Satake, qui permet de réaliser toutes les opérations dans l'algèbre des polynômes à plusieurs variables. Nous avons aussi calculé les séries génératrices dans le cas spécial du choix des paramètres de Satake.

Abstract. Shimura conjectured the rationality of the generating series for Hecke operators for the symplectic group of genus n. This conjecture was proved by Andrianov for arbitrary genus n, but the explicit expression was out of reach for genus higher than 3. For genus $n = 4$, we explicitly compute the rational fraction in this conjecture. Using formulas for images of double cosets under the Satake spherical map, we first compute the sum of the generating series, which is a rational fraction with polynomial coefficients. Then we recover the coefficients of this fraction as elements of the Hecke algebra using polynomial representation of basis Hecke operators under the spherical map. Numerical examples of these fractions for special choice of Satake parameters are given.

1. Introduction

Let p be a prime. We consider the symplectic group $\mathrm{Sp}_n(\mathbb{Z}) \subset \mathrm{GL}_{2n}(\mathbb{Z})$ of genus n, and let

$$\mathbf{T}(p), \mathbf{T}_1(p^2), \ldots, \mathbf{T}_{n-1}(p^2), [\mathbf{p}]_n$$

be $n+1$ generators of the Hecke ring over $\mathbb{Z}$ for Sp_n. Let $\mathbf{D}_p(X)$ denote the generating power series of Hecke operators

$$\mathbf{D}_p(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) X^\delta \ . \tag{1.1}$$

The result presented in this article provides a complement to the solution of Shimura's conjecture of rationality of generating Hecke power series stated in [14] at p. 825 as follows:

> "In general, it is plausible that $\mathbf{D}_p(X) = \mathbf{E}(X)/\mathbf{F}(X)$ with polynomials $\mathbf{E}(X)$ and $\mathbf{F}(X)$ in X with integral coefficients of degree $2^n - 2$ and 2^n, respectively"

(*i.e. with coefficients in Hecke algebra*

$$\mathcal{L}_{\mathbb{Z}} = \mathbb{Z}[\mathbf{T}(p), \mathbf{T}_1(p^2), \ldots, \mathbf{T}_{n-1}(p^2), [\mathbf{p}]_n].)$$

The existence of a rational representation $\mathbf{E}(X)/\mathbf{F}(X)$ was proved by Andrianov in [1, 2, 3] for arbitrary genus n. For genus 1 and genus 2 the results were given by Hecke and Shimura ([8], Theorem 3.21 in [15], and Theorem 2 in [14]):

$$\mathbf{D}_p^{(1)}(X) = \frac{1}{1 - \mathbf{T}(p)X + p[\mathbf{p}]_1 X^2},$$

$$\mathbf{D}_p^{(2)}(X) = \frac{1 - p^2[\mathbf{p}]_2 X^2}{1 - \mathbf{T}(p)X + p\left(\mathbf{T}_1(p^2) + (p^2+1)[\mathbf{p}]_2\right)X^2 - p^3[\mathbf{p}]_2\mathbf{T}(p)X^3 + p^6[\mathbf{p}]_2^2 X^4}.$$

Andrianov was the first who obtained the expression for genus 3 using the multiplication table of Hecke operators in [1]. Later, this result was produced by Miyawaki (see [10]), where he computed a few local factors of some Siegel modular forms of degree 3 and made an interesting conjectures related to modular liftings. No explicit results for higher genus were known due to an enormous complexity of the Hecke algebra manipulations. Recently the author together with Panchishkin developed a formal calculus approach using a computer. We were able to compute more directly the generating series in Shimura's conjecture for genus 3 (see [12]), and then to explore the case of genus 4. Below is the result for genus 3, where coefficients in p are factorized into irreducible polynomials:

$$\mathbf{D}_p^{(3)}(X) = \frac{\mathbf{E}_3(X)}{\mathbf{F}_3(X)}, \tag{1.2}$$

where $\mathbf{E}_3(X), \mathbf{F}_3(X) \in \mathcal{L}_{\mathbb{Z}[X]}$ and

$$\begin{aligned}\mathbf{E}_3(X) &= 1 - p^2\left(\mathbf{T}_2(p^2) + (p^2-p+1)(p^2+p+1)[\mathbf{p}]_3\right)X^2\\ &\quad + p^4(p+1)[\mathbf{p}]_3\mathbf{T}(p)X^3\\ &\quad - p^7[\mathbf{p}]_3\left(\mathbf{T}_2(p^2) + (p^2-p+1)(p^2+p+1)[\mathbf{p}]_3\right)X^4 + p^{15}[\mathbf{p}]_3^3X^6\,,\end{aligned}$$

$$\begin{aligned}\mathbf{F}_3(X) &= 1 - \mathbf{T}(p)X\\ &\quad + p\left(\mathbf{T}_1(p^2) + (p^2+1)\mathbf{T}_2(p^2) + (p^2+1)^2[\mathbf{p}]_3\right)X^2\\ &\quad - p^3\left(\mathbf{T}_2(p^2) + [\mathbf{p}]_3\right)\mathbf{T}(p)X^3\\ &\quad + p^6\Big(\mathbf{T}_2(p^2) + [\mathbf{p}]_3(\mathbf{T}(p)^2 - 2p\mathbf{T}_1(p^2) - 2(p-1)\mathbf{T}_2(p^2)\\ &\qquad - (p^2+2p-1)(p^2-p+1)(p^2+p+1)[\mathbf{p}]_3)\Big)X^4\\ &\quad - p^9[\mathbf{p}]_3\left(\mathbf{T}_2(p^2) + [\mathbf{p}]_3\right)\mathbf{T}(p)X^5\\ &\quad + p^{13}[\mathbf{p}]_3^2\left(\mathbf{T}_1(p^2) + (p^2+1)\mathbf{T}_2(p^2) + (p^2+1)^2[\mathbf{p}]_3\right)X^6\\ &\quad - p^{18}[\mathbf{p}]_3^3\mathbf{T}(p)X^7 + p^{24}[\mathbf{p}]_3^4X^8\,.\end{aligned}$$

In this article we describe this symbolic computation approach for genus 4 (i.e. the case of Sp_4). We present both numerator and denominator polynomials expressed in the terms of Hecke operators in the section 2. Notations of the article and some necessary facts about the Hecke algebras are given in section 3. In the section 4 we define the Satake mapping of the Hecke algebra to the symmetric polynomial ring, which is referred by Andrianov and Zhuravlev in [7] as the spherical map. Then we discuss in details the method of obtaining the main result in the section 5. Finally we give some interesting properties of obtained polynomials in the section 6.

Generating series of a type

$$\sum_{m=1}^{\infty}\lambda_f(m)\,m^{-s} = \prod_{p\,\text{primes}}\sum_{\delta=0}^{\infty}\lambda_f(p^\delta)\,p^{-\delta s}$$

are used as a classical method to produce L-functions for an algebraic group G over $\mathbb{Q}$, where $\lambda_f(m)$ are the eigenvalues of Hecke operators on an automorphic form f on G. Hence these series and related congruences are of number-theoretic interest. Particularly, we study here the generating series of Hecke operators $\mathbf{T}(m)$ for the symplectic group Sp_4, where $\lambda_f(m) = \lambda_f(\mathbf{T}(m))$. If we want to get Hecke series for a given Siegel modular forms concretely, this kind of calculation is necessary.

Many examples of Hecke series as rational functions for some classical groups over p-adic fields are given by Hina and Sugano in [9]. The explicit knowledge of the sum of the generating series of Hecke operators

$$\mathbf{D}_p(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) X^\delta = \mathbf{E}(X)/\mathbf{F}(X)$$

gives a relation between the Hecke eigenvalues and the Fourier coefficients of a Hecke eigenform f. This link is needed for constructing an analytic continuation of L-function on Sp_n, which was done by Andrianov for Sp_2 in [5]. An approach for constructing an analytic continuation of the spinor L-function on Sp_3 was indicated by Panchishkin at the talk on seminar Groupes Réductifs et Formes Automorphes in the Institut de Mathématiques de Jussieu ([11]).

Similar technique of a symbolic computation can be used to discover other interesting identities between Hecke operators, between their eigenvalues, relations to Fourier coefficients of modular forms of higher degree. In [13] the author together with Panchishkin study the analogue of Rankin's Lemma of higher genus and formulate a modularity lifting conjecture for convolutions of L-functions attached to Siegel modular forms.

2. The explicit formula for Sp_4

Theorem 1 (Explicit Shimura's conjecture for genus 4)**.**
For genus $g = 4$ the summation of Hecke series $\mathbf{D}_p(X)$ resolves explicitly to the rational polynomial presentation:

$$\mathbf{D}_p^{(4)}(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) X^\delta = \frac{\mathbf{E}_4(X)}{\mathbf{F}_4(X)},$$

where

$$\mathbf{E}_4(X) = \sum_{k=0}^{14} \mathbf{e}_k X^k$$

is the polynomial of degree 14 *and*

$$\mathbf{F}_4(X) = \sum_{k=0}^{16} \mathbf{f}_k X^k$$

is the polynomial of degree 16 *with the coefficients $\mathbf{e}_k$ and $\mathbf{f}_k$ listed in the Appendix A.*

The proof of this result is based on application of the spherical map in order to carry out all calculations in the ordinary polynomial ring instead of Hecke algebra. Using formal calculus on a computer it is possible to find the explicit solution for the image $\Omega(\mathbf{D}_p(X))$ in terms of symmetric polynomials. Similarly we find the images of basis Hecke operators, which we use to compose and resolve a linear system of undetermined coefficients and discover the desired expression in terms of Hecke operators.

3. The Hecke algebras

In this section we briefly define the notations used in the article. We use definitions from [7], where the detailed theory of Hecke rings is given in Chapter 3 (see also [6]).

3.1. Hecke algebra for $\mathbf{Sp}_n$. Consider the group of positive symplectic similitudes

$$\mathrm{S} = \mathrm{S}^n = \mathrm{GSp}_n^+(\mathbb{Q}) = \{M \in \mathrm{M}_{2n}(\mathbb{Q}) \mid {}^tMJ_nM = \mu(M)J_n, \mu(M) > 0\}\,,$$

where $J_n = \begin{pmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0}_n \end{pmatrix}$.

For the Siegel modular group $\Gamma = \mathrm{Sp}_n(\mathbb{Z}) \subset \mathrm{SL}_{2n}(\mathbb{Z})$ of genus n consider the double cosets

$$(M) = \Gamma M \Gamma \subset \mathrm{S}\,,$$

and the Hecke operators

$$\mathbf{T}(\mu) = \sum_{M \in \mathrm{SD}_n(\mu)} (M)\,,$$

where M runs through the following integer matrices

$$\mathrm{SD}_n(\mu) = \{\mathrm{diag}(d_1, \ldots, d_n; e_1, \ldots, e_n)\}\,,$$

where $d_1|\cdots|d_n|e_n|\cdots|e_1$, $d_i, e_j > 0$, $d_ie_i = \mu = \mu(M)$. Let us use the notation for the Hecke operators

$$\mathbf{T}(d_1, \ldots, d_n; e_1, \ldots, e_n) = \big(\mathrm{diag}(d_1, \ldots, d_n; e_1, \ldots, e_n)\big)\,.$$

In particular we have the following $n+1$ basis Hecke operators

$$\begin{aligned} &\mathbf{T}(p) = \mathbf{T}(\underbrace{1, \ldots, 1}_{n}, \underbrace{p, \ldots, p}_{n})\,, \\ (3.1) \qquad &\mathbf{T}_i(p^2) = \mathbf{T}(\underbrace{1, \ldots, 1}_{n-i}, \underbrace{p, \ldots, p}_{i}, \underbrace{p^2, \ldots, p^2}_{n-i}, \underbrace{p, \ldots, p}_{i})\,,\ i = 1, \ldots, n\,, \end{aligned}$$

generating the Hecke algebra over $\mathbb{Z}$:

$$\mathcal{L}_{n,\mathbb{Z}} = \mathbb{Z}[\mathbf{T}(p), \mathbf{T}_1(p^2), \ldots, \mathbf{T}_n(p^2)]\,.$$

We denote as $[\mathbf{p}]_n$ (or just $[\mathbf{p}]$ if the context of n is declared) the scalar matrix Hecke operator $[\mathbf{p}] = \mathbf{T}_n(p^2) = \mathbf{T}(\underbrace{p, \ldots, p}_{2n}) = p\mathbf{I}_{2n}$.

3.2. Operation of multiplication in Hecke algebras. In order to define an operation of multiplication (in the abstract Hecke algebra) we consider without loss of generality for any subgroup Γ of semigroup S a vector space over $\mathbb{Q}$ generated by all left cosets ΓM (just as a formal base)

$$L_{\mathbb{Q}}(\Gamma, S) = \Big\{ \sum_j a_j(\Gamma M_j) \mid a_j \in \mathbb{Q} \Big\} .$$

For double cosets $(M) = \Gamma M \Gamma \subset \mathrm{S}$ that can be presented as a finite union of disjoint left cosets

$$(M) = \bigcup_{j=1}^{K} \Gamma M_j \ ,$$

we denote

$$(M) = \sum_{M_j \in \Gamma \backslash \Gamma M \Gamma} (\Gamma M_j)$$

and consider an abstract Hecke algebra $\mathcal{L}_{\mathbb{Q}}(\Gamma, S) = L_{\mathbb{Q}}(\Gamma, S)^{\Gamma}$ as a vector space $L_{\mathbb{Q}}(\Gamma, S)^{\Gamma}$ for fixed Γ. Any nonzero element $t \in \mathcal{L}$ can be written in the form $t = \sum_{j=1}^{K} a_j(\Gamma M_j)$. Hence, the multiplication is well defined as

$$\Big(\sum_j a_j(\Gamma M_j) \Big) \Big(\sum_{j'} a'_{j'}(\Gamma M_{j'}) \Big) = \sum_{j,j'} a_j a'_{j'} (\Gamma M_j M_{j'}),\ a_j, a'_{j'} \in \mathbb{Q} \ .$$

3.3. Hecke algebra for $\mathbf{GL}_n$. Further, in order to define a mapping to the polynomial ring, we need to introduce a Hecke algebra for general linear group. Let $G = \mathrm{GL}_n(\mathbb{Q})$ and $\Lambda = \mathrm{GL}_n(\mathbb{Z})$. We note corresponding Hecke algebra by $\mathcal{H}_{\mathbb{Q}}(\Lambda, G) = L_{\mathbb{Q}}(\Lambda, G)^{\Lambda}$. This algebra is generated by n basis operators

$$\pi_i(p) = \big(\mathrm{diag}(\underbrace{1, \ldots, 1}_{n-i}, \underbrace{p, \ldots, p}_{i})\big)\ ,\ 1 \leqslant i \leqslant n \ . \tag{3.2}$$

Recall that every left coset Λg $(g \in G)$ has a representative of the form

$$\begin{pmatrix} p^{\delta_1} & c_{12} & \cdots & c_{1n} \\ 0 & p^{\delta_2} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p^{\delta_n} \end{pmatrix} \ , \text{ where } \delta_1, \ldots, \delta_n \in \mathbb{Z} \, . \tag{3.3}$$

The arbitrary element $t \in \mathcal{H}_{\mathbb{Q}}(\Lambda, G)$ is composed as a finite linear combination of left cosets Λg_j:

$$t = \sum_j a_j (\Lambda g_j) \ . \tag{3.4}$$

We need to introduce another element related to the product of generators $\pi_i(p)$, which will be later used for spherical map definition. Let $\pi_{\alpha\beta}$

be a double coset defined by

$$\pi_{\alpha\beta} = \pi^n_{\alpha\beta}(p) = \left(\begin{pmatrix} \mathbf{I}_{n-\alpha-\beta} & 0 & 0 \\ 0 & p\,\mathbf{I}_\alpha & 0 \\ 0 & 0 & p^2\mathbf{I}_\beta \end{pmatrix}\right) . \tag{3.5}$$

The double coset expansion of the product in the Hecke algebra $\mathcal{H}_\mathbb{Q}$ of two generators π_i and π_j, where $1 \leqslant i, j \leqslant n$, has the form

$$\pi_i\pi_j = \sum_{\substack{0\leqslant a\leqslant n-j\\ 0\leqslant b\leqslant j\\ a+b=i}} \frac{\varphi_{a+j-b}}{\varphi_a\varphi_{j-b}}\pi_{a+j-b,b} \ ,$$

where

$$\begin{aligned} \varphi_i = \varphi_i(x) = (x-1)(x^2-1)\cdots(x^i-1) \text{ for } i \geqslant 1 \\ \text{and } \varphi_0(x) = 1 \ . \end{aligned} \tag{3.6}$$

4. The spherical map

There are several methods to construct a mapping from a Hecke algebra to a polynomial ring. We use the constuction by Andrianov and Zhuravlev [7], where the description for general linear and symplectic groups is given in terms of the right cosets of the double cosets, which generate the Hecke algebra. This isomorphism plays a key role in our calculations. It allows to carry out all computation in the polynomial ring where the multiplication operation is much more straightforward than the product of double cosets.

4.1. The spherical map in general linear group case. The spherical map for the Hecke algebra of general linear group is defined for fixed representative of a left coset of the form (3.3) as

$$\omega((\Lambda g)) = \prod_{i=1}^{n}(x_i p^{-i})^{\delta_i}$$

and for an arbitrary element (3.4) we have

$$\omega(t) = \sum_j a_j \omega((\Lambda g_j)) \,.$$

This definition is unique due to the fact that the diagonal $(p^{\delta_1}, \dots, p^{\delta_n})$ in (3.3) is uniquely determined by the left coset.

Lemma 2.21 of chapter 3 in [7] gives the images of the basis elements (3.2) of Hecke algebra for general linear group:

$$\omega(\pi_i(p)) = p^{-i\,(i+1)/2} e_i(x_1, \dots, x_n) \quad (1 \leqslant i \leqslant n),$$

where

$$e_i(x_1, \dots, x_n) = \sum_{1\leqslant\alpha_1<\cdots<\alpha_i\leqslant n} x_{\alpha_1}\cdots x_{\alpha_i}$$

is the i–th elementary symmetric polynomial.

4.2. The spherical map in symplectic group case. The definition of the spherical map in case of symplectic group is more complicated and based on the case of general linear group. Consider an arbitrary element (double coset) $T \in \mathcal{L}_{n,\mathbb{Z}}$ as a finite linear combination of left cosets:

$$T = \sum_j b_j (\Gamma M_j)\,, \text{ with } \mu(M_j) = p^{\delta_j}\,.$$

We choose the representative of a class in the form

$$M_j = \begin{pmatrix} p^{\delta_j}\, {}^tD_j^{-1} & * \\ 0 & D_j \end{pmatrix},$$

where D_j is a triangular matrix of the form

$$D_j = \begin{pmatrix} p^{\gamma_{1j}} & * & \cdots & \cdots \\ 0 & p^{\gamma_{2j}} & * & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & p^{\gamma_{nj}} \end{pmatrix}.$$

We define the mapping as

$$\Omega(T) = \sum_j b_j x_0^{\delta_j} \omega(\Lambda D_j)\,.$$

In particular on page 146 of [7] we have the following formulas for basis Hecke operators:

$$\begin{aligned} \Omega(\mathbf{T}(p)) &= \sum_{a=0}^{n} x_0\, s_a(x_1, \ldots, x_n) = x_0 \prod_{i=1}^{n} (1 + x_i)\,, \\ \Omega(\mathbf{T}_i(p^2)) &= \sum_{\substack{a+b\leqslant n \\ a\geqslant i}} p^{b(a+b+1)}\, l_p(a-i, a)\, x_0^2\, \omega(\pi_{a,b}(p))\,, \end{aligned} \tag{4.1}$$

where the coefficient $l_p(r,a)$ is the number of $a \times a$ symmetric matrices of rank r over the field of p elements. This coefficient is explicitly given by the recurrent formula (6.79) in [7, Chapter 3, §6] on page 214:

$$l_p(r,a) = l_p(r,r) \frac{\varphi_a(p)}{\varphi_r(p)\, \varphi_{a-r}(p)}\,,$$

where the function $\varphi_i(x)$ was already defined by (3.6).

Now we apply the above formulas to the power series (1.1) and obtain the expression for the image of $\mathbf{D}_p(X)$:

$$\begin{aligned}\Omega(\mathbf{D}_p(X)) &= \sum_{\delta=0}^{\infty}\Omega(\mathbf{T}(p^\delta))X^\delta = \\ \sum_{\delta=0}^{\infty}\ &\sum_{0\leqslant\delta_1\leqslant\cdots\leqslant\delta_n\leqslant\delta} p^{n\delta_1+(n-1)\delta_2+\cdots+\delta_n}\,\omega(t(p^{\delta_1},\ldots,p^{\delta_n}))\,(x_0X)^\delta\,,\end{aligned} \tag{4.2}$$

where

$$t(p^{\delta_1},\ldots,p^{\delta_n}) = \big(\operatorname{diag}(p^{\delta_1},\ldots,p^{\delta_n})\big)\in\mathcal{H}_{\mathbb{Q}} \tag{4.3}$$

is an element of the Hecke algebra for general linear group.

4.3. Practical computation. The algorithm was programmed and the results were computed using Maple system. We found more practical and suitable for direct programming the formulas for spherical mapping in the article [4]. Remark that the notation Ω belongs in that article to the spherical mapping of the Hecke algebra for general linear group. It corresponds to our mapping ω defined above with substitution of all x_i by x_i/p for $i = 1,\ldots,n$. Therefore we used the formula (1.7) on page 432 of [4] and then performed the substitution. This formula gives direct expression for images of the elements t of a type (4.3) including images of $\pi_{\alpha\beta}(p)$ defined by (3.5). In our notation it can be written as

$$\omega(t(p^{(\delta)}) = p^{-\sum_i(n-i)\delta_i}\,\frac{Q(x)}{P^{(k)}(\frac{1}{p})}\,, \tag{4.4}$$

where

$$\begin{aligned}P^{(k)}(x) &= \frac{\varphi_{k_1}(x)\ldots\varphi_{k_t}(x)}{\varphi_1(x)^n}\,,\\ Q(x) &= \sum_{w\in\mathrm{S}_n}(wx)^{(\delta)}c(wx)\,,\\ c(x) &= \prod_{\alpha\in\Sigma}\frac{1-\frac{1}{p}(x)^{(\alpha)}}{1-(x)^{(\alpha)}}\,,\end{aligned}$$

function $\varphi(x)$ was defined by (3.6), the notation (x) is used for n-tuple $(x_1,x_2,\ldots,x_n)$, then $(x)^{(\alpha)} = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, $(wx)^{(\delta)} = x_{w(1)}^{\delta_1}x_{w(2)}^{\delta_2}\cdots x_{w(n)}^{\delta_n}$. The set $\Sigma = \{(\alpha)\} = \{(\alpha_1,\alpha_2,\ldots,\alpha_n)\} = \{\alpha_{ij}, 1\leqslant i<j\leqslant n\}$, where α_{ij} is defined by placing of 1 and -1 within the set of n zeros $\alpha_{ij} = (\ldots,0,1_i,0,\ldots,0,-1_j,0,\ldots)\in\mathbb{Z}_n$. The element of Hecke algebra for the general linear group noted as $t(p^{(\delta)})$ is $t(p^{\delta_1},\ldots,p^{\delta_n})$. Numbers $(k) = (k_1,\ldots,k_t)$ denote the quantities of t distinct elements in the set of integers $(\delta) = (\delta_1,\ldots,\delta_n)$, that is the number δ_1 occurs in (δ) exactly k_1

times, the next number following δ_1 in the ordering of (δ) and distinct from δ_1 appears t here k_2 times, etc. Note, that all $k_i > 0$ and $k_1 + \cdots + k_t = n$.

Now we consider $n = 4$. The set Σ consists of 6 elements

$$\Sigma = \{(1,-1,0,0),(1,0,-1,0),(1,0,0,-1),\\ (0,1,-1,0),(0,1,0,-1),(0,0,1,-1)\}\,.$$

Then the expression for $c(x)$ takes the explicit form

$$c(x) = \frac{(px_2-x_1)(px_3-x_1)(px_4-x_1)(px_3-x_2)(px_4-x_2)(px_4-x_3)}{p^6(x_2-x_1)(x_3-x_1)(x_4-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3)}\,.$$

In order to find images of Hecke operators (4.1) we need to apply the formula (4.4) to all $t(p^{(\delta)})$ of a type (3.5), that is, for the following set

$$\{(\delta)\} = \{(0,0,0,1),(0,0,1,2),(0,1,2,2),(1,2,2,2),(0,0,1,1),\\ (0,1,1,2),(1,1,2,2),(0,1,1,1),(1,1,1,2),(1,1,1,1)\}.$$

Further we will see that in order to compute the series (1.1) we need an expanded set $\{(\delta)\}$ with components up to 14. We can dramatically reduce this set by taking the common degree of p from the operator $t(p^{\delta_1},\ldots,p^{\delta_n})$ outside of the double coset matrix (multiplying the element by the corresponding degree of p). Therefore, we need to compute just 680 primitive elements of the form $t(1,p^{\delta_2},p^{\delta_3},p^{\delta_4})$ reducing the variable δ_1, and where $0 \leqslant \delta_2 \leqslant \delta_3 \leqslant \delta_4 \leqslant 14$.

Below are some examples of the values of these images:

$$\begin{aligned}
&\omega(t(1,1,1,1)) = 1\,,\\
&\omega(t(1,1,1,p)) = p^{-1}(x_1+x_2+x_3+x_4)\,,\\
&\omega(t(1,1,p,p)) = p^{-3}(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\,,\\
&\omega(t(1,p,p,p)) = p^{-3}(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)\,,\\
&\omega(t(1,p,p,p^3)) = p^{-9}(px_1^3x_2x_3+px_1^3x_2x_4+px_1^3x_3x_4-x_1^2x_2^2x_3+px_1^2x_2^2x_3\\
&\quad -x_1^2x_2^2x_4+px_1^2x_2^2x_4+px_1^2x_2x_3^2-x_1^2x_2x_3^2-3x_1^2x_2x_3x_4+3px_1^2x_2x_3x_4\\
&\quad +px_1^2x_2x_4^2-x_1^2x_2x_4^2-x_1^2x_3^2x_4+px_1^2x_3^2x_4+px_1^2x_3x_4^2-x_1^2x_3x_4^2\\
&\quad +px_1x_2^3x_3+px_1x_2^3x_4-x_1x_2^2x_3^2+px_1x_2^2x_3^2+3px_1x_2^2x_3x_4-3x_1x_2^2x_3x_4\\
&\quad +px_1x_2^2x_4^2-x_1x_2^2x_4^2+px_1x_2x_3^3-3x_1x_2x_3^2x_4+3px_1x_2x_3^2x_4\\
&\quad +3px_1x_2x_3x_4^2-3x_1x_2x_3x_4^2+px_1x_2x_4^3+px_1x_3^3x_4+px_1x_3^2x_4^2-x_1x_3^2x_4^2\\
&\quad +px_1x_3x_4^3+px_2^3x_3x_4+px_2^2x_3^2x_4-x_2^2x_3^2x_4-x_2^2x_3x_4^2+px_2^2x_3x_4^2\\
&\quad +px_2x_3^3x_4+px_2x_3^2x_4^2-x_2x_3^2x_4^2+px_2x_3x_4^3)\,.
\end{aligned}$$

These expressions are symmetric polynomials as expected. The written form becomes very long for higher degree (δ). In order to be able to present intermediate results preserving the structure of these polynomials we use the symmetric monomials of four variables $m_{i_1i_2i_3i_4}$, more precisely

$$m_{i_1i_2i_3i_4} = \sum_{w\in S_4/\operatorname{Stab}(x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4})} w(x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4})\,,$$

where the sum is normalized by the stabilizer $\operatorname{Stab}(x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4})$ so the resulting coefficient is equal to 1 and $i_1 \geqslant i_2 \geqslant i_3 \geqslant i_4 \geqslant 0$. For example,

$$\begin{aligned}
m_{1000} &= x_1 + x_2 + x_3 + x_4\,,\\
m_{1100} &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\,,\\
m_{2110} &= x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_2x_3^2\\
&\quad + x_1x_2x_4^2 + x_1x_3^2x_4 + x_1x_3x_4^2 + x_2^2x_3x_4 + x_2x_3^2x_4 + x_2x_3x_4^2\,,\\
m_{3333} &= x_1^3x_2^3x_3^3x_4^3\,.
\end{aligned}$$

Using this notation the previously listed examples of $\omega(\cdot)$ images become the short expressions:

$$\begin{aligned}
&\omega(t(1,1,1,1)) = 1 = m_{0000}\,,\\
&\omega(t(1,1,1,p)) = p^{-1}m_{1000}\,,\\
&\omega(t(1,1,p,p)) = p^{-3}m_{1100}\,,\\
&\omega(t(1,p,p,p)) = p^{-3}m_{1110}\,,\\
&\omega(t(1,p,p,p^3)) = p^{-9}(p\,m_{3110} + (p-1)m_{2210} + 3(p-1)\,m_{2111})\,.
\end{aligned}$$

Finally, using formulas (4.1) and (4.4) we obtain the images of basis Hecke operators for the symplectic group (3.1), which we present here in terms of $m_{i_1i_2i_3i_4}$:

$$\begin{aligned}
&\Omega(\mathbf{T}(p)) = x_0\,(m_{1111} + m_{1110} + m_{1100} + m_{1000} + 1)\,,\\
&\Omega(\mathbf{T}_1(p^2)) = x_0^2p^{-8}((p-1)^2(p+1)(4p^4+3p^3+3p^2+p+1)\,m_{1111}\\
&\qquad + p^4(p-1)(3p^2+2p+1)(m_{2111}+m_{1110})\\
&\qquad + p^5(p-1)(p+1)(m_{2211}+m_{2110}+m_{1100})\\
&\qquad + p^7(m_{2221}+m_{2210}+m_{2100}+m_{1000}))\,,\\
&\Omega(\mathbf{T}_2(p^2)) = x_0^2p^{-8}((p-1)(4p^4+3p^3+3p^2+p+1)\,m_{1111}\\
&\qquad + p^2(p-1)(p^2+p+1)(m_{2111}+m_{1110})\\
&\qquad + p^5(m_{2211}+m_{2110}+m_{1100}))\,,\\
&\Omega(\mathbf{T}_3(p^2)) = x_0^2p^{-10}((p-1)(p+1)(p^2+1)\,m_{1111}\\
&\qquad + p^4(m_{2111}+m_{1110}))\,,\\
&\Omega([\mathbf{p}]) = x_0^2p^{-10}\,m_{1111}\,.
\end{aligned}\tag{4.5}$$

5. The proof of the formula

First we find the image under spherical map of the Hecke series. Then using formulas (4.5) we construct an equation with undetermined coefficients in order to write the spherical image in terms of Hecke operators.

5.1. Spherical image of Hecke power series. Following the manipulations with the series in the book [7] on page 150 we introduce the following substitutions:

$$\begin{aligned}\delta_2 &= \delta_1 + \delta_2', \\ \delta_3 &= \delta_1 + \delta_3', \\ \delta_4 &= \delta_1 + \delta_4', \\ \delta &= \delta_1 + \delta_4' + \beta,\end{aligned}$$

where $0 \leqslant \delta_2' \leqslant \delta_3' \leqslant \delta_4' \leqslant \delta'$ and $\beta \geqslant 0$. Continuing the formula (4.2) using the above substitutions we obtain for $n = 4$

$$\begin{aligned}\Omega(\mathbf{D}_p(X)) &= \sum_{\delta=0}^{\infty} \Omega(\mathbf{T}(p^\delta))X^\delta = \\ &= \sum_{\delta=0}^{\infty} \quad \sum_{0\leqslant\delta_1\leqslant\delta_2\leqslant\delta_3\leqslant\delta_4\leqslant\delta} p^{4\delta_1+3\delta_2+2\delta_3+\delta_4}\, \omega(t(p^{\delta_1}, p^{\delta_2}, p^{\delta_3}, p^{\delta_4}))\, (x_0X)^\delta \\ &= \sum_{\substack{\delta_1\geqslant 0,\, \beta\geqslant 0 \\ 0\leqslant\delta_2'\leqslant\delta_3'\leqslant\delta_4'}} \Big((x_0X)^{\delta_1+\beta+\delta_4'} p^{10\delta_1+3\delta_2'+2\delta_3'+\delta_4'} \left(\frac{x_1x_2x_3x_4}{p^{10}}\right)^{\delta_1} \\ &\qquad \times \omega(t(1, p^{\delta_2'}, p^{\delta_3'}, p^{\delta_4'}))\Big) \\ &= \sum_{\substack{\delta_1\geqslant 0,\, \beta\geqslant 0 \\ 0\leqslant\delta_2'\leqslant\delta_3'\leqslant\delta_4'}} \Big((x_0x_1x_2x_3x_4X)^{\delta_1}(x_0X)^\beta \omega(t(1, p^{\delta_2'}, p^{\delta_3'}, p^{\delta_4'}))p^{3\delta_2'+2\delta_3'+\delta_4'} \\ &\qquad \times (x_0X)^{\delta_4'}\Big).\end{aligned}$$

In the last formula we separate and perform an independent summation on δ_1 and β variables. These two series result in

$$\sum_{\delta_1\geqslant 0}(x_0x_1x_2x_3x_4X)^{\delta_1} = \frac{1}{1 - x_0x_1x_2x_3x_4X}$$

and

$$\sum_{\beta\geqslant 0}(x_0X)^\beta = \frac{1}{1 - x_0X}.$$

In the rational representation of the series $\mathbf{D}_p(X) = \mathbf{E}(X)/\mathbf{F}(X)$ the degree of the numerator $\mathbf{E}(X)$ for $n = 4$ is equal to 14. Moreover, the spherical image of the denominator $\mathbf{F}(X)$ is explicitly known

$$\begin{aligned}\Omega(\mathbf{F}(X)) &= (1-x_0X)(1-x_0x_1X)(1-x_0x_2X)(1-x_0x_3X)(1-x_0x_4X)\\ &\times(1-x_0x_1x_2X)(1-x_0x_1x_3X)(1-x_0x_1x_4X)(1-x_0x_2x_3X)\\ &\times(1-x_0x_2x_4X)(1-x_0x_3x_4X)(1-x_0x_1x_2x_3X)(1-x_0x_1x_2x_4X)\\ &\times(1-x_0x_1x_3x_4X)(1-x_0x_2x_3x_4X)(1-x_0x_1x_2x_3x_4X)\,.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\Omega(\mathbf{E}(X)) &= \left(\sum_{0\leqslant\delta_2'\leqslant\delta_3'\leqslant\delta_4'}\omega(t(1,p^{\delta_2'},p^{\delta_3'},p^{\delta_4'}))p^{3\delta_2'+2\delta_3'+\delta_4'}(x_0X)^{\delta_4'}\right)\\ &\times(1-x_0x_1X)(1-x_0x_2X)(1-x_0x_3X)(1-x_0x_4X)\\ &\times(1-x_0x_1x_2X)(1-x_0x_1x_3X)(1-x_0x_1x_4X)(1-x_0x_2x_3X)\\ &\times(1-x_0x_2x_4X)(1-x_0x_3x_4X)(1-x_0x_1x_2x_3X)(1-x_0x_1x_2x_4X)\\ &\times(1-x_0x_1x_3x_4X)(1-x_0x_2x_3x_4X)\,.\end{aligned}$$

In order to obtain an explicit expression for the image of the numerator $\mathbf{E}(X)$ we compute all $\left(\omega(t(1,p^{\delta_2'},p^{\delta_3'},p^{\delta_4'}))p^{3\delta_2'+2\delta_3'+\delta_4'}(x_0X)^{\delta_4'}\right)$ up to $\delta_4'\leqslant 14$, add them together and multiply considering only resulting powers of X up to 14. These expressions are very long, it took hours of processor time to compute all sums and products. Intermediate results would fill hundreds of pages of paper. However, the final result is quite short (using symmetric polynomial notation) and it is published in [16], showing some interesting properties of this polynomial (e.g. a functional equation).

5.2. Inverting the spherical image. In order to obtain the result of the theorem 1 we applied the method of undetermined coefficients to each coefficient of $\Omega(\mathbf{E}(X))$ and $\Omega(\mathbf{F}(X))$. Let us take as a reference the variable x_0. In expressions for $\Omega(\mathbf{E}(X))$ and $\Omega(\mathbf{F}(X))$ this variable has the same degree as X for each summand. The expression for $\Omega(\mathbf{T}(p))$ (see (4.5)) includes the variable x_0 in degree 1, other images of basis Hecke operators $\Omega(\mathbf{T}_i(p))$ include x_0 in degree 2. Therefore, to reconstruct the particularly given coefficient of degree k of the polynomial $\mathbf{E}(X)$ or $\mathbf{F}(X)$ we need to construct all possible products of $\mathbf{T}(p)$, $\mathbf{T}_1(p^2)$, $\mathbf{T}_2(p^2)$, $\mathbf{T}_3(p^2)$ and $\mathbf{T}(p)$ so the resulting degree of x_0 in the spherical image will be equal to k. For example, consider the coefficient of the degree 3 in polynomial $\mathbf{E}(X)$. We computed before its image (see [16])

$$\begin{aligned}\Omega(\mathbf{e}_3) &= x_0^3p^{-3}(p+1)\big(p\,(m_{3222}+m_{3221}+m_{3211}\\ &+m_{3111}+m_{2220}+m_{2210}+m_{2110}+m_{1110})\\ &+(p^2+4p+1)(m_{2222}+m_{2221}+m_{2211}+m_{2111}+m_{1111})\big)\,.\end{aligned}$$

All possible products of generators of the Hecke ring having the degree 3 of x_0 under the spherical mapping are: $\mathbf{T}(p)\mathbf{T}_1(p^2)$, $\mathbf{T}(p)\mathbf{T}_2(p^2)$, $\mathbf{T}(p)\mathbf{T}_3(p^2)$ and $[\mathbf{p}]\mathbf{T}(p)$. Then

$$\Omega(\mathbf{e}_3) = K_1\Omega(\mathbf{T}(p))\Omega(\mathbf{T}_1(p^2)) + K_2\Omega(\mathbf{T}(p))\Omega(\mathbf{T}_2(p^2)) \\ + K_3\Omega(\mathbf{T}(p))\Omega(\mathbf{T}_3(p^2)) + K_4\Omega(\mathbf{T}(p))\Omega([\mathbf{p}])\,.$$

Expanding these products we construct a linear system of K_j variables by comparing the coefficients of appropriate monomial symmetric functions. This system resolves uniquely due to the fact that the spherical mapping constructed on basis Hecke operators is an isomorphism. In the example above we find that $K_1 = 0$, $K_2 = 0$, $K_3 = p^4(p+1)$ and $K_4 = p^4(p+1)(p^2+1)(p^3-p^2+1)$. In practice for higher degree there exist many choices of products of generators and the expansion of them becomes a not trivial task even for a computer. Fortunately, there is a functional equation for coefficients of the denominator $\mathbf{F}(X)$ due to the symmetric structure of the spherical image polynomial:

$$\mathbf{f}_i = \mathbf{f}_{16-i} \cdot (p^{10}[\mathbf{p}])^{i-8}\,, \quad i = 0, \ldots, 16\,.$$

Therefore we used the approach of undetermined coefficient for only lower degree $\mathbf{f}_i$, where $i = 0, \ldots, 8$. The same computational problem exists for the higher degree coefficient of the numerator. To overcome the unnecessary manipulations and blind guessing of the $\mathbf{T}$–product combination we noticed that it is possible to lower the degree of the equation for the particular coefficient $\mathbf{e}_i$ for $i > 7$ by dividing this equation on factoring $\Omega([\mathbf{p}]) = x_0^2 p^{-10} x_1x_2x_3x_4$ in appropriate degree and using the same products (with non zero coefficients) of $\mathbf{T}(p)^{i_1}\mathbf{T}(p)^{i_2}\mathbf{T}(p)^{i_3}\mathbf{T}(p)^{i_4}\mathbf{T}(p)^{i_5}$ as for already computed $\mathbf{e}_{14-i}$.

6. Remarks

Remark 2. *The result of the Theorem 1 is compatible with the result of the earlier work* [12]*, where the same method was applied for the case of genus* 3*. Considering the projection from genus* 4 *to* 3 *corresponding to Siegel operator acting from* Sp_4 *to* Sp_3 *in Hecke algebra by taking* $[\mathbf{p}]_4$ *to zero,* $\mathbf{T}^{(4)}(p)$ *to* $\mathbf{T}^{(3)}(p)$*, and* $\mathbf{T}_i^{(4)}(p^2)$ *to* $\mathbf{T}_i^{(3)}(p^2)$ *for* $i = 1, 2, 3$*, we obtain the exact formula of generating power series (1.2). All formulas (3.1) for the images of basis Hecke operators transform to the exact formulas for lower genus as well. The spherical image* $\Omega(\mathbf{D}_p^{(4)})$ *under a projection* $x_4 = 0$ *transforms into* $\Omega(\mathbf{D}_p^{(3)})$*. This genus lowering procedure is valid for* $g = 2$ *and* $g = 1$ *as well.*

We noticed a very interesting symmetry property within the coefficients of the spherical image of the numerator. Knowing this relation in advance would let us to limit computation of coefficients almost in half just up to degree 7, reproducing the most time consuming higher degree coefficients using this property.

Proposition 3. *Polynomial* $\Omega(\mathbf{E}(X))$ *has the following functional relation between its coefficients* $\Omega(\mathbf{e}_k)$*,* $k = 0, \ldots, 14$*:*

$$\Omega(\mathbf{e}_{14-k})(p, x_0, x_1, x_2, x_3, x_4) = \\ - p^{-6}(x_0^2 x_1 x_2 x_3 x_4)^{7-k}\, \Omega(\mathbf{e}_k)\left(\frac{1}{p}, x_0 x_1 x_2 x_3 x_4, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}\right).$$

Remark 4. *It is suggested that this functional relation is true for all* n *in the following form:*

$$\Omega(\mathbf{E})(x_0, \ldots, x_n, X) \\ = (-1)^{n-1} \frac{(x_0^2 x_1 \ldots x_n X^2)^{2^{n-1}-1}}{p^{\frac{n(n-1)}{2}}} \Omega(\mathbf{E})\left(\frac{1}{x_0}, \ldots, \frac{1}{x_n}, \frac{p}{X}\right).$$

For a special case of a choice Satake parameters x_i the spherical image of the numerator $\mathbf{E}(X)$ can be considerably simplified.

Proposition 5. *Consider the degree homomorphism* ν *corresponding Satake parameters* $(x_0, x_1, x_2, x_3, x_4) = (1, p, p^2, p^3, p^4)$*. Then the polynomial* $\Omega(\mathbf{E})$ *takes the form*

$$\Omega_\nu(\mathbf{E}(X)) = (1 - pX)(1 - p^2X)(1 - p^3X)^2(1 - p^4X) \\ \times (1 + p^5X)(1 - p^5X)^2(1 - p^6X)^2(1 - p^7X)(1 - p^8X) \\ \times (1 + pX + p^2X + 2p^3X + p^4X + p^5X + 2p^6X + p^7X + p^8X + p^9X^2)$$

(compare to the similar result in genus 3 [12]*).*

The explicit result of the Theorem 1 makes it possible to compare the spinor Hecke series of genus 4 with the Rankin product of two Hecke series of genus 2 computed in [13]. We formulated there a conjecture on a holomorphic lifting from $\mathrm{GSp}_2 \times \mathrm{GSp}_2$ to GSp_4.

The result presented in this article was put into thesis of the author's dissertation [17], which was successfully defended in November 2008. The author is very grateful to Alexei Panchishkin for posing the problem and active discussions.

Appendix A. Coefficients of the theorem on explicit Shimura's conjecture for genus 4

$$
\begin{aligned}
\mathbf{e}_0 &= 1\,,\\
\mathbf{e}_1 &= 0\,,\\
\mathbf{e}_2 &= -p^2((p^8+p^6+2p^4+2p^2+1)[\mathbf{p}] + (p^2+p+1)(p^2-p+1)\mathbf{T}_3(p^2)\\
&\quad + \mathbf{T}_2(p^2))\,,\\
\mathbf{e}_3 &= p^4(p+1)((p^2+1)(p^3-p^2+1)[\mathbf{p}] + \mathbf{T}_3(p^2))\mathbf{T}(p)\,,\\
\mathbf{e}_4 &= p^7((p^2+p+1)(p^2-p+1)(p^8+3p^7+p^5+2p^3+p-1)[\mathbf{p}]^2\\
&\quad + (p^2+p+1)(p^2-p+1)(2p^3+p-2)\mathbf{T}_3(p^2)[\mathbf{p}]\\
&\quad - (p^2+p+1)(p^2-p+1)\mathbf{T}_3(p^2)^2\\
&\quad + (2p^5+2p^3+p^2+p-1)\mathbf{T}_2(p^2)[\mathbf{p}]\\
&\quad - \mathbf{T}_2(p^2)\mathbf{T}_3(p^2) + p(p^2+p+1)\mathbf{T}_1(p^2)[\mathbf{p}] - (p^2+p+1)\mathbf{T}(p)^2[\mathbf{p}])\,,\\
\mathbf{e}_5 &= -p^{10}(p+1)((p^2+1)(p^7-p^6-p^2-1)[\mathbf{p}] - (p^2+1)\mathbf{T}_3(p^2)\\
&\quad - \mathbf{T}_2(p^2))\mathbf{T}(p)[\mathbf{p}]\,,\\
\mathbf{e}_6 &= p^{14}((p^{16}-p^{15}-2p^{14}-3p^{12}-5p^{10}-8p^8+p^7\\
&\quad - 8p^6-5p^4-4p^2-1)[\mathbf{p}]^3\\
&\quad + (p^{12}-p^{10}-p^9-5p^8+2p^7-7p^6-6p^4-8p^2+p-2)\mathbf{T}_3(p^2)[\mathbf{p}]^2\\
&\quad + (p^7-p^4-4p^2+2p-1)\mathbf{T}_3(p^2)^2[\mathbf{p}] + p\mathbf{T}_3(p^2)^3\\
&\quad - (2p^8+3p^6+p^4-p^3+3p^2+p+1)\mathbf{T}_2(p^2)[\mathbf{p}]^2\\
&\quad - (3p^2+p+1)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] - (p^6-p^3+1)\mathbf{T}_1(p^2)[\mathbf{p}]^2\\
&\quad - \mathbf{T}_1(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] + p^2(p^3+p-1)\mathbf{T}(p)^2[\mathbf{p}]^2)\,,\\
\mathbf{e}_7 &= -p^{19}(p-1)(p+1)((p^2+p+1)(p^2-p+1)(p^2+1)[\mathbf{p}]\\
&\quad + (p^2+p+1)(p^2-p+1)\mathbf{T}_3(p^2) + \mathbf{T}_2(p^2))\mathbf{T}(p)[\mathbf{p}]^2\,,\\
\mathbf{e}_8 &= -p^{24}((p^{16}-3p^{12}-3p^{10}-p^9-9p^8-8p^6-7p^4-5p^2+p-1)[\mathbf{p}]^3\\
&\quad + (p^{10}-p^9-4p^8-6p^6-8p^4-9p^2+3p-2)\mathbf{T}_3(p^2)[\mathbf{p}]^2\\
&\quad - (p^4+4p^2-3p+1)\mathbf{T}_3(p^2)^2[\mathbf{p}] + p\mathbf{T}_3(p^2)^3\\
&\quad - (p^8+2p^6-p^5+2p^4+p^3+4p^2+1)\mathbf{T}_2(p^2)[\mathbf{p}]^2\\
&\quad - (p^3+3p^2+1)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] + (p^5-p^2-1)\mathbf{T}_1(p^2)[\mathbf{p}]^2\\
&\quad - \mathbf{T}_1(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] - p(p^3-p^2-1)\mathbf{T}(p)^2[\mathbf{p}]^2)[\mathbf{p}]\,,
\end{aligned}
$$

$$\begin{aligned}
\mathbf{e}_9 &= p^{29}(p+1)((p^2+1)(p^5-2p^4-1)[\mathbf{p}]-(p^4+1)\mathbf{T}_3(p^2)\\
&\quad -\mathbf{T}_2(p^2))\mathbf{T}(p)[\mathbf{p}]^3\,,\\
\mathbf{e}_{10} &= -p^{35}((p^2-p+1)(p^2+p+1)(p^8+2p^7+p^5+3p^3+p-1)[\mathbf{p}]^2\\
&\quad -(p^2-p+1)(p^2+p+1)(p^5-3p^3-p+2)\mathbf{T}_3(p^2)[\mathbf{p}]\\
&\quad -(p^2-p+1)(p^2+p+1)\mathbf{T}_3(p^2)^2\\
&\quad +(p^5+3p^3+p^2+p-1)\mathbf{T}_2(p^2)[\mathbf{p}]\\
&\quad -\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)+p(p^2+p+1)\mathbf{T}_1(p^2)[\mathbf{p}]\\
&\quad -(p^2+p+1)\mathbf{T}(p)^2[\mathbf{p}])[\mathbf{p}]^3\,,\\
\mathbf{e}_{11} &= -p^{41}(p+1)((p^2+1)(p^3-p^2+1)[\mathbf{p}]+\mathbf{T}_3(p^2))\mathbf{T}(p)[\mathbf{p}]^4\,,\\
\mathbf{e}_{12} &= p^{48}((2p^6+2p^4+2p^2+1)[\mathbf{p}]+(p^2-p+1)(p^2+p+1)\mathbf{T}_3(p^2)\\
&\quad +\mathbf{T}_2(p^2))[\mathbf{p}]^5\,,\\
\mathbf{e}_{13} &= 0\,,\\
\mathbf{e}_{14} &= -p^{64}[\mathbf{p}]^7\,,
\end{aligned}$$

$$\begin{aligned}
\mathbf{f}_0 &= 1\,,\\
\mathbf{f}_1 &= -\mathbf{T}(p)\,,\\
\mathbf{f}_2 &= p((p^8+2p^6+2p^4+2p^2+1)[\mathbf{p}]+(p^4+2p^2+1)\mathbf{T}_3(p^2)\\
&\quad +(p^2+1)\mathbf{T}_2(p^2)+\mathbf{T}_1(p^2))\,,\\
\mathbf{f}_3 &= p^3((p^7-p^6-p^4-p^2-1)[\mathbf{p}]-\mathbf{T}_3(p^2)-\mathbf{T}_2(p^2))\mathbf{T}(p)\,,\\
\mathbf{f}_4 &= -p^6((p^4+1)^2(p^6-p^4+2p^3-2p^2+2p-1)[\mathbf{p}]^2\\
&\quad +2(p^2-p+1)(p^4+1)(p^4+2p^3+p^2+p-1)\mathbf{T}_3(p^2)[\mathbf{p}]\\
&\quad +(p^2-p+1)(p^2+p+1)(p^2+2p-1)\mathbf{T}_3(p^2)^2\\
&\quad -2(p^2-p+1)(p^4+1)\mathbf{T}_2(p^2)[\mathbf{p}]+2(p-1)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)\\
&\quad -\mathbf{T}_2(p^2)^2+2p(p^4+1)\mathbf{T}_1(p^2)[\mathbf{p}]+2p\mathbf{T}_1(p^2)\mathbf{T}_3(p^2)-\mathbf{T}(p)^2[\mathbf{p}]\\
&\quad -\mathbf{T}(p)^2\mathbf{T}_3(p^2))\,,\\
\mathbf{f}_5 &= p^9((p^{11}+p^{10}+4p^8+2p^7+3p^6+3p^4+2p^2-1)[\mathbf{p}]^2\\
&\quad +(2p^7+2p^6+3p^4+2p^2-2)\mathbf{T}_3(p^2)[\mathbf{p}]\\
&\quad -\mathbf{T}_3(p^2)^2+(p^4+3p^2-1)\mathbf{T}_2(p^2)[\mathbf{p}]\\
&\quad -\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)+3p^2\mathbf{T}_1(p^2)[\mathbf{p}]-p\mathbf{T}(p)^2[\mathbf{p}])\mathbf{T}(p)\,,
\end{aligned}$$

$$\begin{aligned}
\mathbf{f}_6 = {} & -p^{13}((p^4+1)(p^2+1)^2(2p^8+2p^7+2p^5+2p^3+2p-1)[\mathbf{p}]^3 \\
& + (p^2+1)^2(2p^8+2p^7+4p^5-2p^4+2p^3+4p-3)\mathbf{T}_3(p^2)[\mathbf{p}]^2 \\
& - (p^2+1)^2(p^4-2p+3)\mathbf{T}_3(p^2)^2[\mathbf{p}] - (p^2+1)^2\mathbf{T}_3(p^2)^3 \\
& + (p^2+1)(2p^8+4p^7+4p^5+4p^3+4p-1)\mathbf{T}_2(p^2)[\mathbf{p}]^2 \\
& + 2(p^2+1)(p^3+2p-1)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] - (p^2+1)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)^2 \\
& + 2p(p^2+1)\mathbf{T}_2(p^2)^2[\mathbf{p}] + (2p^8+2p^7+2p^5+2p^3+2p-1)\mathbf{T}_1(p^2)[\mathbf{p}]^2 \\
& + 2(p-1)\mathbf{T}_1(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] - \mathbf{T}_1(p^2)\mathbf{T}_3(p^2)^2 + 2p\mathbf{T}_1(p^2)\mathbf{T}_2(p^2)[\mathbf{p}] \\
& - (p^2+1)(p^5+p^4-p^3+1)\mathbf{T}(p)^2[\mathbf{p}]^2 \\
& + (p-1)(p^2+p+1)\mathbf{T}(p)^2\mathbf{T}_3(p^2)[\mathbf{p}] - \mathbf{T}(p)^2\mathbf{T}_2(p^2)[\mathbf{p}])\,, \\
\mathbf{f}_7 = {} & -p^{17}((2p^{13}+p^{12}+3p^{10}+p^9+p^8+2p^6-p^5+p^4-p^2+p+1)[\mathbf{p}]^3 \\
& + (p^9+2p^6-2p^5+2p^4-2p^2+3p+2)\mathbf{T}_3(p^2)[\mathbf{p}]^2 \\
& - (p^5-p^4+p^2-3p-1)\mathbf{T}_3(p^2)^2[\mathbf{p}] + p\mathbf{T}_3(p^2)^3 \\
& + (p^6+2p^4-2p^2+1)\mathbf{T}_2(p^2)[\mathbf{p}]^2 - (2p^2-1)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] \\
& + (2p^4+1)\mathbf{T}_1(p^2)[\mathbf{p}]^2 + \mathbf{T}_1(p^2)\mathbf{T}_3(p^2)[\mathbf{p}] - p^3\mathbf{T}(p)^2[\mathbf{p}]^2)\mathbf{T}(p)\,, \\
\mathbf{f}_8 = {} & p^{22}((p^{18}+4p^{17}+3p^{16}+8p^{15}+12p^{14}+8p^{13}+14p^{12}+12p^{11} \\
& + 20p^{10}+4p^9+20p^8+16p^6+10p^4-4p^3+5p^2+1)[\mathbf{p}]^4 \\
& + 2(2p^{10}+2p^9+p^8+6p^7+4p^6+8p^4-6p^3+6p^2+1)(p^4+1) \\
& \times \mathbf{T}_3(p^2)[\mathbf{p}]^3 + (p^8+4p^6+8p^4-12p^3+10p^2+1)\mathbf{T}_3(p^2)^2[\mathbf{p}]^2 \\
& - 4p^2(p-1)\mathbf{T}_3(p^2)^3[\mathbf{p}] + p^2\mathbf{T}_3(p^2)^4 \\
& + 2(2p^7+3p^6+2p^5+5p^4-2p^3+3p^2+1)(p^4+1)\mathbf{T}_2(p^2)[\mathbf{p}]^3 \\
& + (2p^6+4p^5+10p^4-8p^3+6p^2+2)\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)[\mathbf{p}]^2 \\
& - 4p^3\mathbf{T}_2(p^2)\mathbf{T}_3(p^2)^2[\mathbf{p}] + (3p^4+2p^2+1)\mathbf{T}_2(p^2)^2[\mathbf{p}]^2 \\
& + 2(2p^5+p^4+2p^2+1)(p^4+1)\mathbf{T}_1(p^2)[\mathbf{p}]^3 \\
& + (4p^5+2p^4+4p^2+2)\mathbf{T}_1(p^2)\mathbf{T}_3(p^2)[\mathbf{p}]^2 \\
& + 2(p^2+1)\mathbf{T}_1(p^2)\mathbf{T}_2(p^2)[\mathbf{p}]^2 + \mathbf{T}_1(p^2)^2[\mathbf{p}]^2 \\
& - (p^8+2p^7+2p^5+2p^4+2p^3+2p-1)\mathbf{T}(p)^2[\mathbf{p}]^3 \\
& - 2(p^4+p-1)\mathbf{T}(p)^2\mathbf{T}_3(p^2)[\mathbf{p}]^2 + \mathbf{T}(p)^2\mathbf{T}_3(p^2)^2[\mathbf{p}] \\
& - 2p\mathbf{T}(p)^2\mathbf{T}_2(p^2)[\mathbf{p}]^2)
\end{aligned}$$

and the higher degree coefficients $\mathbf{f}_i$ are obtained from the following relations:

$$\mathbf{f}_9 = \mathbf{f}_7 \cdot p^{10}[\mathbf{p}], \quad \mathbf{f}_{10} = \mathbf{f}_6 \cdot p^{20}[\mathbf{p}]^2, \quad \mathbf{f}_{11} = \mathbf{f}_5 \cdot p^{30}[\mathbf{p}]^3, \quad \mathbf{f}_{12} = \mathbf{f}_4 \cdot p^{40}[\mathbf{p}]^4,$$

$$\mathbf{f}_{13} = \mathbf{f}_3 \cdot p^{50}[\mathbf{p}]^5, \quad \mathbf{f}_{14} = \mathbf{f}_2 \cdot p^{60}[\mathbf{p}]^6, \quad \mathbf{f}_{15} = \mathbf{f}_1 \cdot p^{70}[\mathbf{p}]^7, \quad \mathbf{f}_{16} = \mathbf{f}_0 \cdot p^{80}[\mathbf{p}]^8.$$

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