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Cohomologial dimension of Laumon 1-motives up to isogenies

par Nicola MAZZARI

Résumé. Nous prouvons que la dimension cohomologique de la catégorie des 1-motifs de Laumon à isogénie près sur un corps de caractéristique nulle est ≤ 1. En conséquence, cela implique le même résultat pour la catégorie des structures de Hodge formelles de niveau ≤ 1 (sur $\mathbb{Q}$).

Abstract. We prove that the category of Laumon 1-motives up to isogenies over a field of characteristic zero is of cohomological dimension ≤ 1. As a consequence this implies the same result for the category of formal Hodge structures of level ≤ 1 (over $\mathbb{Q}$).

1. Introduction

In [6] P. Deligne defined a *1-motive* over a field k as $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-equivariant morphism $[u : \boldsymbol{X} \to \boldsymbol{G}(k^{\mathrm{sep}})]$ where $\boldsymbol{X}$ is a free $\mathrm{Gal}(k^{\mathrm{sep}}|k)$-module and $\boldsymbol{G}$ is a semi-abelian algebraic group over k. They form a category that we shall denote by $\mathcal{M}_{1,k}$ or $\mathcal{M}_1$.

Deligne's definition was motivated by Hodge theory. In fact the category of 1-motives over the complex numbers is equivalent, via the so called *Hodge realization* functor, to the category MHS_1 of mixed Hodge structures of level ≤ 1. It is known the the category MHS_1 is of cohomological dimension 1 (see [5]) and the same holds for $\mathcal{M}_{1,\mathbb{C}}$.

F. Orgogozo proved more generally that for any field k, the category $\mathcal{M}_{1,k} \otimes \mathbb{Q}$ is of cohomological dimension ≤ 1 (see [14, Prop. 3.2.4]).

Over a field of characteristic 0 it is possible to define the category $\mathcal{M}^{\mathrm{a}}_{1,k}$ of Laumon 1-motives generalizing that of Deligne 1-motives (See [11]). In [3] L. Barbieri-Viale generalized the Hodge realization functor to Laumon 1-motives. He defined the category FHS_1 of formal Hodge structures of level ≤ 1 containing MHS_1 and proved that FHS_1 is equivalent to the category of Laumon 1-motives over $\mathbb{C}$ (compatibly with the Hodge realization).

In this paper we prove that the category of Laumon 1-motives up to isogenies is of cohomological dimension 1.

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2. Laumon 1-motives

Let k be a (fixed) field of characteristic zero. Let Sch_k be the category of schemes over k and Aff_k be the full sub-category of affine schemes. According to [1, Exp. IV §6.3] the fppf topology on Sch_k is the one generated by: the families of jointly surjective open immersions in Sch_k; the finite families of jointly surjective, flat, of finite presentation and quasi-finite morphisms in Aff_k.

Let Ab_k be the category of abelian sheaves on Aff_k w.r.t. the fppf topology. We will consider both the category of commutative group schemes and that of formal group schemes (over k) as full sub-categories of Ab_k. We denote by $\bar{k}$ the algebraic closure of k.

Definition. A *Laumon 1-motive* over k (or an effective free 1-motive over k, *cf.* [2, 1.4.1]) is the data of

(1) a (commutative) formal group $\boldsymbol{F}$ over k, such that $\operatorname{Lie}\boldsymbol{F}$ is a finite dimensional k-vector space and $\boldsymbol{F}(\bar{k}) = \lim_{[k':k]<\infty} \boldsymbol{F}(k')$ is a finitely generated and torsion-free $\operatorname{Gal}(\bar{k}/k)$-module;

(2) a connected commutative algebraic group scheme $\boldsymbol{G}$ over k;

(3) a morphism $u : \boldsymbol{F} \to \boldsymbol{G}$ in the category Ab_k.

Note that we can consider a Laumon 1-motive (over k) $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ as a complex of sheaves in Ab_k concentrated in degree $0, 1$.

It is known that any formal k-group $\boldsymbol{F}$ splits canonically as product $\boldsymbol{F}^o \times \boldsymbol{F}_{\text{ét}}$ where $\boldsymbol{F}^o$ is the identity component of $\boldsymbol{F}$ and is a connected formal k-group, and $\boldsymbol{F}_{\text{ét}} = \boldsymbol{F}/\boldsymbol{F}^o$ is étale. Moreover, $\boldsymbol{F}_{\text{ét}}$ admits a maximal subgroup scheme $\boldsymbol{F}_{\text{tor}}$, étale and finite, such that the quotient $\boldsymbol{F}_{\text{ét}}/\boldsymbol{F}_{\text{tor}} = \boldsymbol{F}_{\text{fr}}$ is constant of the type $\mathbb{Z}^r$ over $\bar{k}$. One says that $\boldsymbol{F}$ is torsion-free if $\boldsymbol{F}_{\text{tor}} = 0$.

By a theorem of Chevalley any connected algebraic group scheme $\boldsymbol{G}$ is the extension of an abelian variety $\boldsymbol{A}$ by a linear k-group scheme $\boldsymbol{L}$ that is product of its maximal sub-torus $\boldsymbol{T}$ with a vector k-group scheme $\boldsymbol{V}$. See [7] for more details on algebraic and formal groups.

Definition. A *morphism* of Laumon 1-motives is a commutative square in the category Ab_k. We denote by $\mathcal{M}_1^{\mathrm{a}} = \mathcal{M}_{1,k}^{\mathrm{a}}$ the category of Laumon k-1-motives, *i.e.* the full sub-category of $C^b(\mathsf{Ab}_k)$ whose objects are Laumon 1-motives. We denote by $\mathcal{M}_1$ the full sub-category of $\mathcal{M}_1^{\mathrm{a}}$ whose objects are Deligne 1-motives (over k) [6, §10.1.2].

Proposition 2.1. *The category $\mathcal{M}_1^{\mathrm{a}}$ of Laumon 1-motives (over k) is an additive category with kernels and co-kernels. In particular let (f, g) be a* *morphism from* $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ *to* $M' = [u' : \boldsymbol{F}' \to \boldsymbol{G}']$ *(i.e.* $u'f = gu$*), then*

$$\mathrm{Ker}(f,g) = [u^* \mathrm{Ker}(g)^o \to \mathrm{Ker}(g)^o] \tag{2.1}$$

and

$$\mathrm{Coker}(f,g) = [\mathrm{Coker}(f)_{fr} \to \mathrm{Coker}(g)] \tag{2.2}$$

Proof. See [11, Prop. 5.1.3]. □

Remark. The category of connected algebraic groups is fully embedded in $\mathcal{M}_1^{\mathrm{a}}$ and it is not abelian. So the category of Laumon 1-motives is not abelian too. In fact consider a surjective morphism of connected algebraic groups $g : \boldsymbol{G} \to \boldsymbol{G}'$. Then $\mathrm{Ker}(g)$ is not necessarily connected and the canonical map (in the category of connected algebraic groups)

$$\mathrm{Coim}(g) = \boldsymbol{G}/\mathrm{Ker}(g)^o \to \mathrm{Im}(g) = \boldsymbol{G}'$$

is not an isomorphism in general.

According to [14] we define the category $\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q}$ of Laumon 1-motives up to isogenies: the objects are the same of $\mathcal{M}_1^{\mathrm{a}}$; the Hom groups are $\mathrm{Hom}_{\mathcal{M}_1^{\mathrm{a}}}(M, M') \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark. Note that a morphism $(f,g) : M \to M'$ is an isogeny (*i.e.* an isomorphism in $\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q}$) if and only if f is injective with finite co-kernel and g is surjective with finite kernel.

Proposition 2.2. *The category of Laumon 1-motives up to isogenies is abelian.*

Proof. By construction $\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q}$ is an additive category. Let $(f,g) : M \to M'$ be a morphism of Laumon 1-motives. We know that the group $\pi_0(\mathrm{Ker}(g)) = \mathrm{Ker}(g)/\mathrm{Ker}(g)^o$ is a finite group scheme, hence there exists an integer n such that the following diagram commutes in Ab_k

$$\begin{array}{ccccc} & & \mathrm{Ker}(f) & & \\ & & \downarrow{\scriptstyle n\cdot u} & \searrow^{0} & \\ \mathrm{Ker}(g)^o & \longrightarrow & \mathrm{Ker}(g) & \longrightarrow & \pi_0(\mathrm{Ker}(g)) \end{array}$$

Then $n \cdot u$ factors through $\mathrm{Ker}(g)^o$ and it is easy to check that $\mathrm{Ker}((f,g)) = [(u^* \mathrm{Ker}(g)^o) \to \mathrm{Ker}(g)^o]$ is isogenous to $[\mathrm{Ker}(f) \to \mathrm{Ker}(g)^0]$.

It follows that $\mathrm{Coim}(f,g)$ is isogenous to $[(\boldsymbol{F}/\mathrm{Ker}(f))_{\mathrm{fr}} \to \boldsymbol{G}/\mathrm{Ker}(g)]$. As $\boldsymbol{G}/\mathrm{Ker}(g)^o \to \boldsymbol{G}/\mathrm{Ker}(g)$ is an isogeny we get that the canonical map $\mathrm{Coim}(f,g) \to \mathrm{Im}(f,g)$ is an isogeny too.

This is enough to prove that the category $\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q}$ is abelian. □

Remark. One can also define the category ${}^t\mathcal{M}_1^{\mathrm{a}}$ of *1-motives with torsion* over k (See [2, Def. 1.4.4]). We note that using the same arguments as in [4, C.7.3] it is easy to show that there is an equivalence of categories

$$\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q} \xrightarrow{\sim} {}^t\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q} \ .$$

2.1. Weights. A Deligne 1-motive is endowed with an increasing filtration (of sub-1-motives) called the weight filtration ([6, §10.1.4]) defined as follows

$$W_i = W_i M := \begin{cases} [\boldsymbol{X} \to \boldsymbol{G}] & i \geq 0 \\ [0 \to \boldsymbol{G}] & i = -1 \\ [0 \to \boldsymbol{T}] & i = -2 \\ [0 \to 0] & i \leq -3 \end{cases} ; \quad \mathrm{gr}_i^W M = \begin{cases} [\boldsymbol{X} \to 0] & i = 0 \\ [0 \to \boldsymbol{A}] & i = -1 \\ [0 \to \boldsymbol{T}] & i = -2 \\ [0 \to 0] & \text{otherwise} \end{cases} .$$

According to [4, C.11.1] we extend the weight filtration to Laumon 1-motives.

Definition. Let $M = [u : \boldsymbol{F} \to \boldsymbol{G}]$ be a Laumon 1-motive. The *weight filtration* of M is

$$W_{-3} = 0 \ \subset W_{-2} = [0 \to \boldsymbol{L}] \ \subset W_{-1} = [0 \to \boldsymbol{G}] \ \subset W_0 = M \ .$$

Remark.

(1) The morphisms of Laumon 1-motives are compatible w.r.t. the weight filtration. Also the weight filtration extends to a filtration on the objects of $\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q}$.

(2) Let Mod_k^f be the category of finite dimensional k-vector spaces. The full sub-category of $\mathcal{M}_1^{\mathrm{a}} \otimes \mathbb{Q}$ of Laumon 1-motives of weight 0 is equivalent to the category $\mathsf{Mod}_k^f \times \mathsf{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\bar{k}/k))$ via the functor $\boldsymbol{F} \mapsto (\mathrm{Lie}(\boldsymbol{F}), \boldsymbol{F}(k) \otimes \mathbb{Q})$.

3. Cohomological dimension

3.1. Extensions. Let A be any abelian category (we don't suppose it has enough injective objects), then we can define its derived category $D(\mathsf{A})$ and the group of n-fold extension classes $\mathrm{Ext}^n_{\mathsf{A}}(A, B) := \mathrm{Hom}_{D(\mathsf{A})}(A, B[n])$, $A, B \in \mathsf{A}$. As usual we identify this group with the group of classes of *Yoneda extensions*, *i.e.* the set of exact sequences

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

modulo congruences (See [10] or [9]).

We will use the two following well-known facts about extensions and filtrations.

(1) Let $W_{-2} \subset W_{-1} \subset W_0 = W$ be a filtration of $W \in \mathsf{A}$. We have the following exact sequences

$$\gamma : \quad 0 \to W_{-2} \to W_{-1} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$

$$\gamma_1 : \qquad 0 \to W_{-2} \to W_{-1} \to W_{-1}/W_{-2} \to 0$$
$$\gamma_2 : \qquad 0 \to W_{-1}/W_{-2} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$

and $\gamma = \gamma_1 \cdot \gamma_2 \in \mathrm{Ext}^2_{\mathsf{A}}(W_0/W_{-1}, W_{-2})$ by definition. This class is trivial, *i.e.* $\gamma = 0$ in $\mathrm{Ext}^2_{\mathsf{A}}(W_0/W_{-1}, W_{-2})$. (See [14, Lemma 3.2.5], or [9, p. 184])

(2) Assume that the objects of A are filtered (by a separated and exhaustive filtration W) and that the morphisms in A are compatible w.r.t. this filtration. If $\mathrm{Ext}^n_{\mathsf{A}}(\mathrm{gr}^W_i A, \mathrm{gr}^W_j B) = 0$ for any i, j, then $\mathrm{Ext}^n_{\mathsf{A}}(A, B) = 0$. In fact assume for instance that B has a 3 steps filtration $0 \subset W_{-2} \subset W_{-1} \subset W_0 = B$: then we have the canonical exact sequences

$$0 \to W_{-1}M' \to M' \to \mathrm{gr}^W_0 M' \to 0$$
$$0 \to W_{-2}M' \to W_{-1}M' \to \mathrm{gr}^W_{-1} M' \to 0$$

By applying $\mathrm{Hom}(A, -)$ we get two long exact sequences

$$\cdots \mathrm{Ext}^2(A, W_{-1}B') \to \mathrm{Ext}^2(A, B) \to \mathrm{Ext}^2(M, \mathrm{gr}^W_0 B) \cdots$$
$$\cdots \mathrm{Ext}^2(A, \mathrm{gr}^W_{-2} B) \to \mathrm{Ext}^2(A, W_{-1}B) \to \mathrm{Ext}^2(A, \mathrm{gr}^W_{-1} B) \cdots$$

from this follows that we can reduce to prove $\mathrm{Ext}^2(A, \mathrm{gr}^W_i B) = 0$. This process can be easily adapted to the general case.

Now we can give a sketch of the proof of the main theorem: one first checks that $\mathrm{Ext}^1_{\mathbb{Q}}(M, M') = 0$ if M, M' are pure of weights w, w' and $w \leq w'$ (a 1-motive is *pure* if it is isomorphic to one of its graded pieces w.r.t. the weight filtration). By point (2) above this formally reduces the problem to checking that if M, M', M'' are pure respectively of weights $0, -1, -2$, then the Yoneda product of two classes $(\gamma_1, \gamma_2) \in \mathrm{Ext}^1_{\mathbb{Q}}(M', M'') \times \mathrm{Ext}^1_{\mathbb{Q}}(M, M')$ is 0. Of course we may assume γ_1 and γ_2 integral. Then the point is that γ_1 and γ_2 *glue* into a 1-motive and we can conclude by (1) above.

3.2. Main result. From now on we call 1-motive a Laumon 1-motive (over k) and $\mathrm{Ext}^i_{\mathbb{Q}}(M, M')$ is the group of classes of i-fold extensions in $\mathcal{M}^{\mathrm{a}}_1 \otimes \mathbb{Q}$.

Theorem 3.1. *The category $\mathcal{M}^{\mathrm{a}}_1 \otimes \mathbb{Q}$ (and in particular $\mathsf{FHS}_1 \otimes \mathbb{Q}$) is of cohomological dimension 1.*

Proof. By the general facts on extensions (§3.1 (2)) we can restrict to consider only pure motives $M = \mathrm{gr}^W_w M$ and $M' = \mathrm{gr}^W_{w'} M'$ of weight w and w', respectively.

(*Equal weights*) If $w = w'$ we can show that $\mathrm{Ext}^1_{\mathbb{Q}}(M, M') = 0$. Let $0 \to M' \to E \to M \to 0$ be an exact sequence in $\mathcal{M}^{\mathrm{a}}_1 \otimes \mathbb{Q}$, then also E is pure of weight w. We have to consider 3 cases: first note the category $\mathsf{Mod}^f_k \times \mathsf{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\bar{k}/k))$ is semi-simple by Maschke's lemma [15, p. 47] and

so the claim holds for the weight zero case by point (2) of the remark in §2.1: also the category of abelian varieties up to isogenies (*i.e.* $w = -1$) is semi-simple by [13, p. 173]; the third case (weight -2) can be reduced to the first by Cartier duality (See [11, §5]) or proved explicitly.
(*Different weights*) Fix a 2-fold extension $\gamma \in \mathrm{Ext}^2_{\mathbb{Q}}(M, M')$ represented by

$$0 \to M' \to E_1 \to E_2 \to M \to 0$$

and let $E = \mathrm{Ker}(E_2 \to M)$. Then we can write $\gamma = \gamma_1 \cdot \gamma_2$, where $\gamma_2 \in \mathrm{Ext}^1_{\mathbb{Q}}(M, E)$, $\gamma_1 \in \mathrm{Ext}^1_{\mathbb{Q}}(E, M')$. Using the canonical exact sequence induced by weights and the first part of the proof it is easy to reduce to the case $E = \mathrm{gr}_{-1} E$, *i.e.* E is an abelian variety.

If $w < w'$ then γ_1 is an extension of an abelian variety E by M' which is a formal group or an abelian variety. Then $\gamma_1 = 0$ (if M' is a formal group we refer to [2, Lemma A.4.5]).

It remains to study what happens if $w > w'$. If w or w' is equal to -1 there is nothing to prove because E is an abelian variety too. So the only case left is when $w = 0$ and $w' = -2$, *i.e.* $M = \boldsymbol{F}[1]$, $M' = \boldsymbol{L}[0]$. We want to reduce to the situation considered in § 3.1 (1). Thus we have to show that there exists a 1-motive N such that $\gamma_1 \in \mathrm{Ext}^1_{\mathbb{Q}}(E, \boldsymbol{L})$ is represented by $0 \to W_{-2}N \to W_{-1}N \to \mathrm{gr}_{-1} N \to 0$; $\gamma_2 \in \mathrm{Ext}^1_{\mathbb{Q}}(\boldsymbol{F}[1], E)$ is represented by $0 \to \mathrm{gr}_{-1} N \to W_0 N/W_{-2} \to \mathrm{gr}_0 N \to 0$.

We claim that γ_1 and γ_2 can be represented by extensions in the category Laumon-1-motives. In fact let

$$\gamma_1 : \quad 0 \to \boldsymbol{L} \xrightarrow{f \otimes n^{-1}} \boldsymbol{G} \xrightarrow{g \otimes m^{-1}} E \to 0$$

be an extension in the category of 1-motives modulo isogenies: f, g are morphism of algebraic groups, $n, m \in \mathbb{Z}$. Then consider the push-forward by n^{-1} and the pull-back by m^{-1}, we get the following commutative diagram with exact rows in $\mathcal{M}_1^{\mathrm{a,fr}} \otimes \mathbb{Q}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{L} & \xrightarrow{f/n} & \boldsymbol{G} & \xrightarrow{g/m} & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle n^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \boldsymbol{L} & \xrightarrow{f} & \boldsymbol{G} & \xrightarrow{g/m} & E & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle m^{-1}} & & \\
0 & \longrightarrow & \boldsymbol{L} & \xrightarrow{f} & \boldsymbol{G} & \xrightarrow{g} & E & \longrightarrow & 0
\end{array}$$

The exactness of the last row is equivalent to the following: $\mathrm{Ker}\, f$ is finite; let $(\mathrm{Ker}\, g)^0$ be the connected component of $\mathrm{Ker}\, g$, then $\mathrm{Im} f \to (\mathrm{Ker}\, g)^0$ is surjective with finite kernel K; g is surjective. So after replacing $\boldsymbol{L}, E$ with isogenous groups we have an exact sequence in $\mathcal{M}_1^{\mathrm{a,fr}}$

$$0 \to \boldsymbol{L} \to \boldsymbol{G} \to E \to 0$$

Explicitly

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{L} & \xrightarrow{f} & \boldsymbol{G} & \xrightarrow{g} & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \boldsymbol{L}/\operatorname{Ker} f & \longrightarrow & \boldsymbol{G} & \xrightarrow{g} & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Im} f/K & \longrightarrow & \boldsymbol{G} & \xrightarrow{g} & E & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Im} f/K & \longrightarrow & \boldsymbol{G} & \longrightarrow & \boldsymbol{G}/(\operatorname{Ker} g)^0 & \longrightarrow & 0
\end{array}$$

With similar arguments we can prove that γ_2 is represented by an extension in the category $\mathcal{M}_1^{\mathrm{a,fr}}$

$$0 \to E \to N \to \boldsymbol{F}[1] \to 0$$

with $N = [u : \boldsymbol{F} \to E]$.

To conclude we have to prove that there is lifting $u' : \boldsymbol{F} \to \boldsymbol{G}$. First suppose $\boldsymbol{F} = \boldsymbol{F}_{\text{ét}}$: consider the long exact sequence

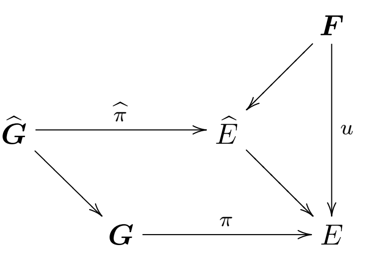
$$\operatorname{Hom}_{\mathsf{Ab}_k}(\boldsymbol{F}, \boldsymbol{G}) \to \operatorname{Hom}_{\mathsf{Ab}_k}(\boldsymbol{F}, E) \xrightarrow{\partial} \operatorname{Ext}^1_{\mathsf{Ab}_k}(\boldsymbol{F}, \boldsymbol{L}) \ .$$

We can consider a (Galois) extension k'/k of finite degree d trivializing $\boldsymbol{F}$. By [12, Theorem 3.9] we get the vanishing of $\operatorname{Ext}^1_{\mathsf{Ab}_{k'}}(\boldsymbol{F}_{k'}, \boldsymbol{L}_{k'})$. Recall that the multiplication by d on $\boldsymbol{F}$ can be written as the composition

$$\boldsymbol{F} \overset{can.}{\to} \Pi_{k'/k} \boldsymbol{F}_{k'} \overset{\mathrm{tr}}{\to} \boldsymbol{F}$$

where $\Pi_{k'/k}$ is Weil restriction functor and tr is the trace map. This implies that $\operatorname{Ext}^1_{\mathsf{Ab}_k}(\boldsymbol{F}, \boldsymbol{L})$ is torsion, hence $\partial u = 0$ and the lift exists (up to isogeny).

In case $\boldsymbol{F} = \boldsymbol{F}^o$ is a connected formal group we have a commutative diagram in Ab_k



where $\widehat{?}$ is the formal completion at the origin of $? = \boldsymbol{G}, E$. The formal completion is an exact functor so $\widehat{\pi}$ is an epimorphism. The category of connected formal groups is equivalent to Mod_k^f, thus we can choose a section σ of $\widehat{\pi}$. Then we can easily construct a (non canonical) lifting of u. □

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