JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

Florian LUCA et Diego MARQUES

Perfect powers in the summatory function of the power tower

Tome 22, n° 3 (2010), p. 703-718.
<http://jtnb.cedram.org/item?id=JTNB_2010__22_3_703_0>

Journal de Théorie des Nombres
de Bordeaux **22** (2010), 703-718

Perfect powers in the summatory function of the power tower

par Florian LUCA et Diego MARQUES

Résumé. Soit $(a_n)_{n\geq 1}$ la suite donnée par $a_1 = 1$ et $a_n = n^{a_{n-1}}$ pour $n \geq 2$. Dans cet article, on montre que la seule solution de l'équation

$$a_1 + \cdots + a_n = m^l$$

avec des entiers positifs $l > 1$, m et n est $m = n = 1$.

Abstract. Let $(a_n)_{n\geq 1}$ be the sequence given by $a_1 = 1$ and $a_n = n^{a_{n-1}}$ for $n \geq 2$. In this paper, we show that the only solution of the equation

$$a_1 + \cdots + a_n = m^l$$

is in positive integers $l > 1$, m and n is $m = n = 1$.

1. Introduction

Let $(a_n)_{n\geq 1}$ be the tower given by $a_1 = 1$ and $a_n = n^{a_{n-1}}$ for $n \geq 2$. This is sometimes referred to as the *exponential factorial* sequence and appears in Sloane's [7] as A049384. Sondow [8] and [9] showed that the number

$$\sum_{n\geq 1} 1/a_n$$

is Liouville; hence, transcendental.

Here, we prove the following result:

Theorem 1.1. *The only solution of the equation*

$$a_1 + \cdots + a_n = m^l$$

is in positive integers $l > 1$, m *and* n *is* $m = n = 1$.

Before proceeding to the proof of Theorem 1.1, let us describe in a few words the method of proof. Observe that $a_n = n^{a_{n-1}}$ is a perfect power of huge exponent. Moreover,

$$m^l - a_n = a_{n-1} + \cdots + a_1$$

and the right hand side is logarithmically small compared to the order of magnitude of the two terms of the difference from the left hand side. This

Manuscrit reçu le 7 août 2009.

makes it possible to apply classical techniques from the theory of effective resolution of exponential Diophantine equations, like linear forms both in archimedian and non-archimendian logarithms. While these techniques have the draw back that the resulting bounds are huge (doubly or triply exponential), the tower exponential growth of our sequence works to our advantage and, in fact, as we will see, the "huge bound" is already surpassed by the time we reached $n = 9$.

Now, let's proceed to the details.

2. The proof

Assume that $n > 1$. We shall assume of course that the exponent l is prime. Observe that if we put $b_n := \sum_{1\le k\le n} a_n$, then

$$\begin{aligned} b_1 &= 1,\\ b_2 &= 3,\\ b_3 &= 2^2 \times 3,\\ b_4 &= 2^2 \times 65539,\\ b_5 &\equiv 17 \times 5 \pmod{17^2},\\ b_6 &\equiv 7 \times 2 \pmod{7^2},\\ b_7 &\equiv 2 \pmod{4},\\ b_8 &\equiv 2 \pmod{4}. \end{aligned}$$

In particular, $n \ge 9$ in our equation.

Observe next that $a_n = n^{a_{n-1}} > e^{a_{n-1}}$ for $n \ge 3$, so that $a_{n-1} < \log a_n$. Furthermore, $a_n \ge 2a_{n-1}$ holds for all $n \ge 2$. Thus, for $n \ge 3$, we have that

$$0 < m^l - n^{a_{n-1}} \le a_{n-1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) < 2a_{n-1} < 2\log a_n. \tag{2.1}$$

The above relation (2.1) will be very important throughout the rest of the proof.

2.1. The case $l = 2$. If n is odd, then a_{n-1} is even, so a_n is a perfect square. Thus, estimate (2.1) with $l = 2$ leads to

$$0 < m - \sqrt{a_n} < \frac{2\log a_n}{\sqrt{a_n}} < 1,$$

so we get a contradiction. The same contradiction is obtained when n is even and a perfect square, since then a_n is also a perfect square.

From now on, we assume that n is even and not a perfect square. Thus, $n \ge 10$. We then have that

$$0 < m - \sqrt{n} \times n^{(a_{n-1}-1)/2} < \frac{2\log a_n}{n^{a_{n-1}/2}} = \frac{2a_{n-1}\log n}{n^{a_{n-1}/2}}.$$

Since $n \geq 10$, the right hand side above is < 1, so n is a not a perfect square. Now

$$\left|\sqrt{n} - \frac{m}{n^{(a_{n-1}-1)/2}}\right| < \frac{2a_{n-1}\log n}{n^{a_{n-1}-0.5}}.$$

A result of Worley [11] says that if α is real irrational and

$$\left|\alpha - \frac{p}{q}\right| < \frac{\kappa}{q^2},$$

then there exist integers k, r, s with $|r| < 2\kappa$, $|s| < 2\kappa$ and $p = rp_k + sp_{k-1}$, $q = rq_k + sq_{k-1}$, where p_k/q_k is the kth convergent of α. Furthermore, k is chosen in such a way as to be maximal subject to the condition that $q_k \leq q$. So,

$$n^{(a_{n-1}-1)/2} = rq_k + sq_{k-1},$$

where

$$\max\{|r|, |s|\} < \frac{4a_{n-1}\log n}{\sqrt{n}}, \tag{2.2}$$

and k is the largest positive integer such that $q_k < n^{(a_{n-1}-1)/2}$. In particular, since

$$q_k \geq F_k \geq \left(\frac{1+\sqrt{5}}{2}\right)^{k-2},$$

where F_k is the kth Fibonacci number, we have that

$$k \leq \frac{(a_{n-1}-1)\log n}{2\log\left(\frac{1+\sqrt{5}}{2}\right)} + 2 < 2a_{n-1}\log n. \tag{2.3}$$

We now look at the sequence $(q_k)_{k\geq 0}$. Let h be the minimal even period of the continued fraction of $\sqrt{n}$. For every fixed $\ell \in \{0, \ldots, h-1\}$, the sequence $(q_{h\lambda+\ell})_{\lambda\geq 0}$ is binary recurrent. Its two initial values are $q_\ell \leq q_h$ and $q_{h+\ell} \leq q_{2h}$. Its characteristic equation has roots

$$\zeta = p_h + \sqrt{n}q_h \qquad \text{and} \qquad \zeta^{-1} = p_h - \sqrt{n}q_h.$$

Here, $(X, Y) := (p_h, q_h)$ is the minimal solution of the Pell equation $X^2 - nY^2 = 1$. Then we can write

$$q_{h\lambda+\ell} = c_1\zeta^{\lambda} + c_2\zeta^{-\lambda},$$

where

$$q_\ell = c_1 + c_2 \qquad \text{and} \qquad q_{h+\ell} = c_1\zeta + c_2\zeta^{-1}.$$

Solving for c_1 and c_2, we get that

$$c_1 = \frac{q_{h+\ell} - \zeta^{-1}q_\ell}{\zeta - \zeta^{-1}}, \qquad c_2 = \frac{\zeta q_\ell - q_{h+\ell}}{\zeta - \zeta^{-1}}. \tag{2.4}$$

Now write $k = h\lambda + \ell$ for some $\ell \in \{1, \ldots, h\}$. Then

$$q = (rc_1 + sd_1)\zeta^{\lambda} + (rc_2 + sd_2)\zeta^{-\lambda},$$

where $c_1,\ c_2$ are given by (2.4) and $d_1,\ d_2$ are given by the same formulae as c_1, c_2 except that with ℓ replaced by $\ell - 1$. Thus, we have arrived at the relation

$$n^{(a_{n-1}-1)/2} = \alpha_1\zeta^{\lambda} + \alpha_2\zeta^{-\lambda}, \qquad \text{where } \alpha_i = rc_i + sd_i \text{ for } i = 1, 2.$$

Since n is even, it follows that $2^{(a_{n-1}-1)/2}$ divides the left hand side above. It remains to study the exponent of 2 on the right hand side above. Observe first that $\beta_i := (\zeta - \zeta^{-1})\alpha_i$ is an algebraic integer for $i = 1, 2$. Let $\beta_1 = 2^t\gamma_1$, where $t \geq 0$ and γ_1 is not a multiple of 2, meaning that $\gamma_1/2$ is not an algebraic integer. Then

$$t\log 2 \leq \log(q_{2h} + \zeta q_h) < \log(\zeta^2 + \zeta^2) = \log(2\zeta^2),$$

giving that

$$t < 1 + \frac{2\log(\zeta)}{\log 2}.$$

For the above inequalities, we used the fact that

$$q_{uh} = \frac{\zeta^u - \zeta^{-u}}{2\sqrt{n}} < \zeta^u \qquad \text{with} \quad u = 1, 2.$$

We shall use the fact that

$$\zeta < e^{3\sqrt{n}\log n} \tag{2.5}$$

(see, for example, Theorem 13.5 on page 329 in [3]). Then,

$$t < 1 + \frac{6\sqrt{n}\log n}{\log 2} < 9\sqrt{n}\log n. \tag{2.6}$$

We then have that

$$2^{(a_{n-1}-1)/2-t} \text{ divides } \gamma_1\zeta^{\lambda} + \gamma_2\zeta^{-\lambda} = -\gamma_2\zeta^{-\lambda}\left(\left(\frac{-\gamma_1}{\gamma_2}\right)\zeta^{2\lambda} - 1\right). \tag{2.7}$$

We now estimate the order at which 2 can appear in the expression

$$\Lambda := \left(\frac{-\gamma_1}{\gamma_2}\right)\zeta^{2\lambda} - 1 \tag{2.8}$$

via the following lower bound for linear forms in p-adic logarithms due to Bugeaud and Laurent [1].

Let η_1 and η_2 be real algebraic numbers. Put $\mathbb{K} := \mathbb{Q}[\eta_1, \eta_2]$ and let D be the degree of $\mathbb{K}$ over $\mathbb{Q}$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_{\mathbb{K}}$, p be the rational prime such that $\mathfrak{p} \mid p$, and let f be such that $|\mathcal{O}_{\mathbb{K}}/\mathfrak{p}| = p^f$. Assume that A_1, A_2 are numbers such that

$$\log A_i \geq \max\left\{h(\eta_i), \frac{f\log p}{D}\right\}, \tag{2.9}$$

where $h(\eta_i)$ is the logarithmic height of the number η_i for $i = 1, 2$. Let

$$\Lambda = \eta_1^{b_1}\eta_2^{b_2} - 1 \tag{2.10}$$

be nonzero, where b_1 and b_2 are nonzero integers. Put

$$b' = \frac{|b_1|}{D \log A_2} + \frac{|b_2|}{D \log A_1}. \tag{2.11}$$

For an algebraic number $\eta \in \mathbb{K}$, we write $\mathrm{ord}_{\mathfrak{p}}(\eta)$ for the exponent with which $\mathfrak{p}$ appears in the prime factorization of the principal fractional ideal generated by η in $\mathbb{K}$. Then the result from [1] which we will use is the following:

Lemma 2.1. *With the previous notations and conventions and assuming that η_1 and η_2 are multiplicatively independent, we have that*

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}}(\Lambda) \le & \frac{24p(p^f-1)D^5}{f^5(p-1)(\log p)^4} \max\left\{\log b' + \log\log p + 0.4, \frac{10 f \log p}{D}, 10\right\}^2 \\ & \times \log A_1 \log A_2.\end{aligned}$$

Indeed, the above result is Corollary 1 in [1] except that in [1] the expression (D/f) appears with the exponent 4 whereas in our case it appears with exponent 5. This is because the p-adic valuation in [1] is normalized, whereas ours is the exponent of a prime ideal so it is not normalized. This explains the extra factor of (D/f).

We take $\eta_1 = -\gamma_1/\gamma_2$, $\eta_2 = \zeta$, $b_1 = 1$, $b_2 = 2m$. Then $\mathbb{K} = \mathbb{Q}[\sqrt{n}]$ and $D = 2$. We take $p = 2$ and $\mathfrak{p}$ be some prime factor of 2 in $\mathcal{O}_{\mathbb{K}}$. Clearly, $f \le 2$, so that $(p^f - 1)/f^5 \le 1$. On the one hand, by estimates (2.6) and (2.7), we have that

$$\mathrm{ord}_{\mathfrak{p}}(\Lambda) \ge \frac{a_{n-1} - 1}{2} - 9\sqrt{n}\log n > \frac{a_{n-1}}{4}. \tag{2.12}$$

In order to get an upper bound on $\mathrm{ord}_{\mathfrak{p}}(\Lambda)$, we use Lemma 2.1 in the case when η_1 and η_2 are multiplicatively independent. It remains to estimate A_1 and A_2. The conjugate of γ_1 is γ_2 and they are both algebraic integers. Thus, only one of γ_1/γ_2 and its conjugate γ_2/γ_1 exceeds 1 in absolute value. Hence, assuming say that $|\gamma_1| \ge |\gamma_2|$ and using also the fact that $|\gamma_1\gamma_2| \ge 1$, we get that

$$\begin{aligned} h(\eta_1) \le & \frac{\log|\gamma_1/\gamma_2|}{2} \le \log|\gamma_1| \le \log\left((|r| + |s|)(q_{h+\ell} + \zeta q_\ell)\right) \\ & < \log\left(2(|r| + |s|)\zeta^2\right). \end{aligned} \tag{2.13}$$

Now using estimates (2.2) and (2.5), we get that

$$2(|r| + |s|)\zeta^2 < \frac{16 a_{n-1} \log n}{\sqrt{n}} e^{6\sqrt{n}\log n} < 16 a_{n-1} e^{6\sqrt{n}\log n}, \tag{2.14}$$

since $\sqrt{n} > \log n$ for $n \ge 10$. Thus, we can take

$$\log A_1 = 2\log a_{n-1} > \log(16 a_{n-1}) + 6\sqrt{n}\log n. \tag{2.15}$$

In the same way, we can take

$$\log A_2 = \log a_{n-1} > 1.5\sqrt{n}\log n > \frac{\log\zeta}{2}.$$

Observe that $2\lambda \le 2k < 4a_{n-1}\log n$ by estimate (2.3). Finally, we can take

$$b' = \frac{1}{2} + \frac{4a_{n-1}\log n}{\log A_1} < \frac{a_{n-1}}{2}.$$

Now Lemma 2.1 gives us that

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda) \le \frac{24\cdot 2\cdot 2^5}{(\log 2)^4}(\log a_{n-1})^2\log A_1\log A_2. \tag{2.16}$$

Comparing the above bound (2.16) with (2.12), we get

$$\frac{a_{n-1}}{4} < 7000(\log a_{n-1})^2(2\log a_{n-1})(\log a_{n-1}),$$

so

$$a_{n-1} < 56000(\log a_{n-1})^4,$$

giving $a_{n-1} < 10^{11}$, which is false for $n \ge 10$. So, there are no solutions with $n > 1$ to the given equation when $l = 2$ in case when η_1 and η_2 are multiplicatively independent.

It remains to deal with the easier case when η_1 and η_2 are multiplicatively dependent. Note that ζ is either the generator of the torsion free part of the group of units of $\mathcal{O}_{\mathbb{K}}$, or $\zeta = \zeta_1^2$, where $\zeta_1 > 1$ is a generator of the torsion free part of the group of units of $\mathcal{O}_{\mathbb{K}}$ and it has norm -1. To deal with both cases at once, we shall write $\zeta = \zeta_1^{\delta}$, where $\delta \in \{1,2\}$. Write $\gamma_1/\gamma_2 = \varepsilon\zeta_1^{\sigma}$, where $\varepsilon = \pm 1$. To bound $|\sigma|$, we use the height calculation (2.13), (2.14) and (2.15) to get that

$$2\log a_{n-1} > h(\eta_1) = \frac{|\sigma|\log\zeta_1}{2} > |\sigma|\left(\frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{2}\right),$$

giving $|\sigma| < 9\log a_{n-1}$. Observe that

$$\Lambda = -\varepsilon\zeta_1^{2\delta\lambda+\sigma} - 1 \quad \text{divides} \quad \zeta_1^{4\delta\lambda+2\sigma} - 1 \quad \text{divides} \quad \zeta^{4\delta\lambda+2\sigma} - 1.$$

Applying the obvious inequality

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda) \le \operatorname{ord}_{\mathfrak{p}}(\zeta^f - 1) + \frac{2\log(4\delta\lambda + 2|\sigma|)}{\log p},$$

with $p = 2$ and $\mathfrak{p}$ a prime ideal dividing 2 in $\mathbb{K}$, we get that

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda) \le \frac{1}{\log 2}\Big(\log(|N_{\mathbb{K}/\mathbb{Q}}(\zeta^3 - 1)|) + 2\log(8\lambda + 2|\sigma|)\Big).$$

Here, $N_{\mathbb{K}/\mathbb{Q}}$ is the norm function from $\mathbb{K}$ to $\mathbb{Q}$. Now

$$|N_{\mathbb{K}/\mathbb{Q}}(\zeta^3 - 1)| = |(\zeta^3 - 1)(\zeta^{-3} - 1)| < \zeta^3 + \zeta^{-3} + 2 < \zeta^4,$$

so, by estimate (2.5), we have

$$\log(|N_{\mathbb{K}/\mathbb{Q}}(\zeta^3 - 1)|) < 4\log\zeta < 12\sqrt{n}\log n < a_{n-1}.$$

Since

$$8\lambda + 2|\sigma| \leq 16a_{n-1}\log n + 18\log a_{n-1} < a_{n-1}^2,$$

we get that

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda) < \frac{5}{\log 2}\log a_{n-1},$$

which together with the inequality (2.12) gives

$$a_{n-1} < \frac{20}{\log 2}\log a_{n-1} < 30\log a_{n-1},$$

yielding $a_{n-1} < 200$, which is again false for $n \geq 10$.

2.2. Bounding *l*. Relation (2.1) gives

$$0 < \left|m^l n^{-a_{n-1}} - 1\right| < \frac{2a_{n-1}}{n^{a_{n-1}}} < \frac{1}{n^{(a_{n-1}-1)/2}}, \tag{2.17}$$

where we used the obvious inequalities $n^{a_{n-1}/2} > 2^{a_{n-1}} > 2a_{n-1}$ and $\log n < n^{1/2}$ for $n \geq 9$. In order to bound the left hand side of inequality (2.17), we use the following result of Laurent, Mignotte and Nesterenko [5].

Lemma 2.2. *Assume that η_1 and η_2 are real, positive and multiplicatively independent algebraic numbers and let Λ be given by* (2.10)*. Then, assuming that $\Lambda \neq 0$, we have*

$$\log|\Lambda| \geq -24.34D^4\left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log A_1 \log A_2,$$

where A_1, A_2 satisfy inequalities (2.9) *(without the term $f\log p/D$ to the right), and b' is given by* (2.11)*.*

For us, we take $\eta_1 = m$, $\eta_2 = n$, $b_1 = l$, $b_2 = -a_{n-1}$ and we apply Lemma 2.2 above to bound the expression Λ appearing in the right hand side of inequality (2.17). We first need to verify that η_1 and η_2 are multiplicatively independent. Well, if they were not, then there exist positive integers $\rho > 1$, a and b such that $m = \rho^a$ and $n = \rho^b$. Thus,

$$m^l - n^{a_{n-1}} = \rho^{al} - \rho^{ba_{n-1}}.$$

Since the above expression is positive, it follows that $al > ba_{n-1}$, therefore

$$m^l - n^{a_{n-1}} \geq \rho^{ba_{n-1}}(\rho - 1) \geq \rho^{ba_{n-1}} = n^{a_{n-1}}.$$

Comparing with estimate (2.1), we get

$$n^{a_{n-1}} < 2\log a_n = 2a_{n-1}\log n,$$

which is of course false for $n \geq 9$. Thus, η_1 and η_2 are multiplicatively independent, so we can apply Lemma 2.2. We have $D = 1$ and we can take $\log A_1 = \log m$ and $\log A_2 = \log n$. Hence, we take

$$b' = \frac{a_{n-1}}{\log m} + \frac{l}{\log n} < \frac{2l}{\log n},$$

where the last inequality follows because $m^l > n^{a_{n-1}}$. Lemma 2.2 tells us that

$$\log|\Lambda| \geq -23.34\left(\max\{\log b' + 0.14, 21\}\right)^2 \log m \log n.$$

Comparing this with inequality (2.17), we get

$$a_{n-1} - 1 \leq 46.68\left(\max\{\log b' + 0.14, 21\}\right)^2 \log m. \tag{2.18}$$

Now clearly

$$\begin{aligned} l \log m = \log m^l < \log(a_n + 2\log a_n) &= \log a_n + \log\left(1 + \frac{2\log a_n}{a_n}\right) \\ &< a_{n-1}\log n + 1, \end{aligned}$$

so

$$\log m < \frac{a_{n-1}\log n + 1}{l} < \frac{(a_{n-1}+1)\log n}{l}. \tag{2.19}$$

Inserting bound (2.19) into bound (2.18), we get

$$\begin{aligned} \frac{l}{\log n} &< 46.68\left(\frac{a_{n-1}+1}{a_{n-1}-1}\right)\left(\max\{\log b' + 0.14, 21\}\right)^2 \\ &< 46.68\left(\max\{\log b' + 0.14, 21\}\right)^2. \end{aligned}$$

If the maximum on the right above is 21, then $l < 21000\log n$. Otherwise, we get that

$$\frac{l}{\log n} < 46.68\left(\log\left(\frac{l}{\log n}\right) + \log 2 + 0.14\right)^2,$$

giving $l < 4000\log n$. Thus, in both cases, the inequality

$$l < 21000\log n \tag{2.20}$$

holds.

2.3. The case $l \geq 3$. Here, we assume that $l \geq 3$. Recall that we have already made the convention that l is prime. Relation (2.1) tells us that

$$0 < m - n^{a_{n-1}/l} < \frac{2\log a_n}{n^{(l-1)a_{n-1}/l}} \leq \frac{2a_{n-1}}{n^{2a_{n-1}/3}}.$$

The right hand side above is obviously < 1 for $n \geq 9$. This shows in particular that a_{n-1} is not a multiple of l and that n is not an lth power either. Let us put $a_{n-1} = bl + r$, where $r \in \{1, \ldots, l-1\}$. We work in the field $\mathbb{K} := \mathbb{Q}[n^{1/l}, \zeta_l]$, where $\zeta_l = e^{2\pi i/l}$ is a primitive lth root of unity.

This is a Kummerian extension of degree $l(l-1)$ of $\mathbb{Q}$. We will need some statistics on the field $\mathbb{K}$.

We put $d = [\mathbb{K} : \mathbb{Q}]$ for the degree of $\mathbb{K}$ over $\mathbb{Q}$ and note that

$$d = l(l-1). \tag{2.21}$$

We write $\Delta_{\mathbb{K}}$ for the discriminant of $\mathbb{K}$. Let $\mathbb{L}_1 := \mathbb{Q}[n^{1/l}]$ and $\mathbb{L}_2 := \mathbb{Q}[\zeta_l]$. The minimal polynomial of $n^{1/l}$ over $\mathbb{Z}$ is $x^l - n$, therefore

$$|\Delta_{\mathbb{L}_1}| \le \left| \prod_{0 \le i < j \le l-1} (n^{1/l}\zeta_l^i - n^{1/l}\zeta_l^j)^2 \right| = n^{(l-1)} l^{l-2} < (nl)^{l-1}.$$

The discriminant of $\mathbb{L}_2$ satisfies $|\Delta_{\mathbb{L}_2}| = l^{l-2}$. Since $\mathbb{K}$ is the compositum of $\mathbb{L}_1$ and $\mathbb{L}_2$, we get that

$$\begin{aligned} |\Delta_{\mathbb{K}}| &\le |\Delta_{\mathbb{L}_1}|^{[\mathbb{L}_2:\mathbb{Q}]} |\Delta_{\mathbb{L}_2}|^{[\mathbb{L}_1:\mathbb{Q}]} \le ((nl)^{l-1})^{l-1} (l^{l-2})^l \\ &= (nl)^{l^2} l^{l^2} = (nl^2)^{l^2}. \end{aligned} \tag{2.22}$$

We next put $R_{\mathbb{K}}$ for the regulator of $\mathbb{K}$. We recall a result of Landau [4].

Lemma 2.3. *Let $\mathbb{K}$ be a number field of degree $d = r + 2s$, where r and $2s$ are the number of real and complex embeddings of $\mathbb{K}$, respectively. Let w be the number of roots of unity in $\mathbb{K}$. Let L be a real number such that $|\Delta_{\mathbb{K}}| \le L$. Let*

$$a = 2^{-r}\pi^{-d/2}\sqrt{L},$$

and define the function $f_{\mathbb{K}}(L, \sigma)$ given by

$$f_{\mathbb{K}}(L, \sigma) := 2^{-r} w a^{\sigma} \Gamma(\sigma/2)^r \Gamma(\sigma)^s \sigma^{d+1} (\sigma - 1)^{1-d}.$$

Then $R_{\mathbb{K}} \le \min\{f_{\mathbb{K}}(L, 2 - t/1000) : t = 0, 1, \ldots, 999\}$.

In the above Lemma 2.3, we put $t = 0$ (so, $\sigma = 2$), and $L = |\sqrt{\Delta_{\mathbb{K}}}|$ and get

$$R_{\mathbb{K}} \le 2^{-r} w a^2 2^{d+1} \le 2^{-r} w (2^{-2s}\pi^{-d}|\Delta_{\mathbb{K}}|) 2^{d+1} = \frac{2w|\Delta_{\mathbb{K}}|}{\pi^d}.$$

Since $l \ge 3$, it follows that $d \ge 6$. Observe that since the group of roots of unity in $\mathbb{K}$ is cyclic, it follows that w is at most the largest positive integer satisfying $\phi(w) \le d$, where ϕ is the Euler function. Since $\phi(p) \ge \sqrt{p}$ holds for $p = 4$ and when p is an odd prime, it follows that $\phi(w) \ge \sqrt{w/2}$. Thus, $w \le 2d^2$, so that

$$\frac{2w}{\pi^d} \le \frac{4d^2}{\pi^d} < 1 \qquad \text{for } d \ge 6.$$

We thus conclude that

$$R_{\mathbb{K}} < |\Delta_{\mathbb{K}}| < (nl^2)^{l^2}. \tag{2.23}$$

Next, we go back to equation (2.1), write it as

$$m^l - n^{a_{n-1}} = K, \quad \text{where } 0 < K < 2a_{n-1},$$

and decompose the left hand side of it in $\mathbb{K}$ as

$$\prod_{j=0}^{l-1}(m - n^{r/l}n^{(a_{n-1}-r)/l}\zeta_l^j) = K.$$

Let $k_1 = m - n^{r/l}n^{(a_{n-1}-r)/l}$. Its norm in $\mathbb{K}$ is

$$N_{\mathbb{K}/\mathbb{Q}}(k_1) = N_{\mathbb{L}_1/\mathbb{Q}}(k_1)^{[\mathbb{L}_2:\mathbb{Q}]} = K^{l-1} < (2a_{n-1})^{l-1}.$$

Next we shall need a result of Poulakis (see Lemma 1 in [6]). In what follows, we use the standard notation that for an algebraic number γ of degree d we write $\gamma^{(1)}, \dots, \gamma^{(d)}$ for its conjugates.

Lemma 2.4. *Let $\mathbb{K}$ be an algebraic number field of degree d and α and algebraic integer in $\mathbb{K}$. Then there exists an algebraic integer β in $\mathbb{K}$ and unit ε in $\mathcal{O}_{\mathbb{K}}$ such that*

$$\alpha = \beta\varepsilon,$$

where

$$\max\{|\beta^{(j)}| : j = 1, \dots, d\} \leq |N_{\mathbb{K}/\mathbb{Q}}(\alpha)|^{1/d}\exp(c_1(d)R_{\mathbb{K}}),$$

where $c_1(d) = d(6d^3/\log d)^d$.

Since $6 \leq d < l^2$ and the function $t \mapsto t^3/\log t$ is increasing for $t \geq 3$, it follows, using also (2.23), that we have the bound

$$c_1(d)R_{\mathbb{K}} < \exp\left(l^2\log(3l^6/\log l) + (2l^2+2)\log l + l^2\log n\right).$$

We now use also the fact that $l < 21000\log n$ above (see (2.20)) we get a function of n as an upper bound on the expression $c_1(d)R_{\mathbb{K}}$. With Maple, we checked that this is at most $\exp(2\cdot 10^{10}(\log n)^3)$ for all $n \geq 3$. Thus,

$$c_1(d)R_{\mathbb{K}} < \exp(2\cdot 10^{10}(\log n)^3). \tag{2.24}$$

In conclusion, there exists a number $\beta \in \mathcal{O}_{\mathbb{K}}$ and a unit ε in $\mathcal{O}_{\mathbb{K}}$ such that

$$m - n^{r/l}n^{(a_{n-1}-r)/l} = \beta\varepsilon, \tag{2.25}$$

and

$$\max\{|\beta^{(j)}| : j = 1, \dots, d\} \leq (2a_{n-1})^{1/l}\exp(\exp(2\cdot 10^{10}(\log n)^3)). \tag{2.26}$$

Let us simplify this bound. For this, we show that

$$(2a_{n-1})^{1/2} > \exp(\exp(2\cdot 10^{13}(\log n)^4)). \tag{2.27}$$

Indeed, since $a_{n-1} = (n-1)^{a_{n-2}} \geq e^{2a_{n-2}}$, it suffices that

$$a_{n-2} > \exp(2\cdot 10^{10}(\log n)^3),$$

and since $a_{n-2} > e^{a_{n-3}}$, it suffices that

$$a_{n-3} > 2\cdot 10^{10}(\log n)^3.$$

Since $a_{n-3} = (n-3)^{a_{n-4}}$ and $a_{n-4} \geq a_5 > 20$, it follows that it suffices that

$$(n-3)^{20} > 2 \cdot 10^{10} (\log n)^3,$$

and this last inequality is true for all $n > 6$. From estimates (2.26) and (2.27), we get that

$$\max\{|\beta^{(j)}| : j = 1, \ldots, d\} \leq (2a_{n-1})^{1/3+1/2} < a_{n-1}. \tag{2.28}$$

Next we discuss the units of $\mathbb{K}$. Let $r_1 := r + s - 1$ be the rank of the free part of the group of units of $\mathbb{K}$. We need the following result which is Lemma 9.6 in [2].

Lemma 2.5. *There exists in $\mathbb{K}$ a fundamental system $\zeta_1, \ldots, \zeta_{r_1}$ of units such that*

$$\prod_{i=1}^{r_1} h(\varepsilon_i) \leq 2^{1-r_1} r_1!^2 d^{-r_1} R_{\mathbb{K}},$$

and such that the absolute values of the entries of the inverse matrix of $(\log|\varepsilon_i^{(j)}|)_{1 \leq i,j \leq r_1}$ do not exceed $r_1!^2 2^{-r_1} (\log(3d))^3$.

Here is how we apply this lemma. We go back to (2.25) and write $\varepsilon = \zeta \prod_{i=1}^{r_1} \varepsilon_i^{m_i}$, where ζ is some root of unity and $\{\varepsilon_1, \ldots, \varepsilon_{r_1}\}$ is a system of units as in Lemma 2.5. Taking the j'th conjugate, and absolute values, we get

$$\frac{|m - n^{r/l} n^{(a_{n-1} r)/l} \zeta_l^{(j)}|}{|\beta^{(j)}|} = \prod_{i=1}^{r_1} |\varepsilon_i^{(j)}|^{m_i}. \tag{2.29}$$

Note that since $0 < m - n^{r/l} n^{(a_{n-1}-r)/l}$, it follows that the complex numbers

$$m - n^{r/l} n^{(a_{n-1}-r)/l} \zeta_l^{(j)}$$

have real part at most $2m$ in absolute value and imaginary part at most m in absolute value, so themselves have absolute value at most $\sqrt{5}m$. Furthermore, from

$$1 < N_{\mathbb{K}/\mathbb{Q}}(\beta) = \prod_{i=1}^{d} |\beta^{(i)}| = |\beta^{(j)}| \prod_{\substack{1 \leq i \leq d \\ i \neq j}} |\beta^{(i)}| \leq |\beta^{(j)}| a_{n-1}^{l^2-1},$$

it follows that

$$\frac{1}{|\beta^{(j)}|} \leq a_{n-1}^{l^2-1} \tag{2.30}$$

holds for all $j = 1, \ldots, r$. Thus, putting

$$x_j := \frac{m - n^{r/l} n^{(a_{n-1}-r)/l} \zeta_l^{(j)}}{\beta^{(j)}},$$

we get that

$$|x_j| < \sqrt{5} m a_{n-1}^{l^2-1} < 2\sqrt{5} a_n^{1/l} a_{n-1}^{l^2-1} < a_n^{l^2}. \tag{2.31}$$

Now, writing $\mathbf{x} = (\log|x_1|, \ldots, \log|x_{r_1}|)^T$, $\mathbf{m} = (m_1, \ldots, m_{r_1})$ and M for the inverse matrix of $(\log|\varepsilon_i^{(j)}|)_{1\le i,j\le r_1}$, we see that by taking logarithms in formulae (2.29) and solving for $m_1, \ldots, m_{r_1}$, we get that

$$M\mathbf{x} = \mathbf{m}.$$

Combining this with Lemma 2.5, we get immediately that

$$\max\{|m_j| : j = 1, \ldots, r_1\} \le l^2 (\log a_n) (r_1 - 1)!(r_1!^2 2^{-r_1} (\log(3d))^3)^{r_1-1}.$$

The factor $l^2 \log a_n = \log(a_n^{l^2})$ on the right hand side above is a bound on $\log|x_j|$ according to inequality (2.31). The remaining factor of the left is a bound on the absolute value of any $(r_1 - 1) \times (r_1 - 1)$ minor of the matrix M according to the last part of Lemma 2.5. The function $t \mapsto t!^2/2^t$ is increasing for integer $t \ge 2$, so

$$\frac{r_1!^2}{2^{-r_1}} < \frac{(l^2)!}{2^{l^2}} < l^{2l^2} 2^{-l^2}.$$

Since the function $t \mapsto (\log(3t))^3/2^t$ is increasing for $t \ge 5$, and $l^2 \ge 9 > 5$, we get that

$$r_1!^2 2^{-r_1} (\log(3d))^3 < l^{2l^2}.$$

Thus,

$$(r_1 - 1)!(r_1!^2 2^{-r_1} (\log(3d))^3)^{r_1-1} < l^{2l^2} (l^{2l^2})^{r_1-1} = l^{2l^2 r_1} < l^{2l^4}.$$

Hence,

$$\max\{|m_j| : j = 1, \ldots, r_1\} \le a_{n-1} (\log n) l^{2l^2+4}.$$

To keep things easy, we show that $a_{n-1} > (\log n) l^{2l^2+4}$. Since $a_{n-1} > (n-1)^{a_{n-2}} > e^{a_{n-2}}$, it suffices to show that

$$a_{n-2} > (2l^2 + 4)\log l + \log\log n.$$

Now $a_{n-2} = (n-2)^{a_{n-3}}$ and $a_{n-3} > 20$, so it suffices to show in light of (2.20) that

$$(n-2)^{20} > (2(21000\log n)^2 + 4)\log(21000\log n) + \log\log n,$$

and the above inequality is true for all $n > 6$. Thus, we record that

$$\max\{|m_j| : j = 1, \ldots, r_1\} < a_{n-1}^2. \tag{2.32}$$

We now take j_1 and j_2 to be two different conjugations, apply them to equation (2.25) and subtract the resulting equations getting

$$n^{r/l} n^{(a_{n-1}-r)/l} (\zeta_l^{(j_1)} - \zeta_l^{(j_2)}) = \zeta^{(j_1)} \beta^{(j_1)} \prod_{i=1}^{r_1} (\varepsilon_i^{(j_1)})^{m_i} - \zeta^{(j_2)} \beta^{(j_2)} \prod_{i=1}^{r_1} (\varepsilon_i^{(j_2)})^{m_i}.$$

Now let $\mathfrak{p}$ be some prime ideal of $\mathbb{K}$ dividing n. We look at the $\mathfrak{p}$-adic valuation of the above formula. In the left hand side, it is at least

$$\frac{a_{n-1}-r}{l}.$$

In the right hand side, it is

(2.33)

$$\operatorname{ord}_{\mathfrak{p}}\left(\zeta^{(j_2)}\beta^{(j_2)}\prod_{i=1}^{r}(\varepsilon_i^{(j_2)})^{m_i}\right)+\operatorname{ord}_{\mathfrak{p}}\left(\left(\frac{\zeta^{(j_1)}}{\zeta^{(j_2)}}\right)\left(\frac{\beta^{(j_1)}}{\beta^{(j_2)}}\right)\prod_{i=1}^{r}\left(\frac{\varepsilon_i^{(j_1)}}{\varepsilon_i^{(j_2)}}\right)^{m_i}-1\right).$$

Since ζ and $\varepsilon_1,\ldots,\varepsilon_{r_1}$ are units, it follows that the first valuation above is $\operatorname{ord}_{\mathfrak{p}}(\beta^{(j_2)})$. Assume this is c. Taking norms in $\mathbb{K}$, we get that $p^c\le N_{\mathbb{K}}(k_1)<(2a_{n-1})^l$, where p is the integer prime such that $\mathfrak{p}$ divides p. Thus,

$$c=\operatorname{ord}_{\mathfrak{p}}(\beta^{(j_2)})\le\frac{l\log(2a_{n-1})}{\log p}\le\frac{l\log(2a_{n-1})}{\log 2}<2l\log a_{n-1}.$$

For the second valuation appearing in (2.33), we use the following linear form in p-adic logarithms due to Kun Rui Yu [12].

Lemma 2.6. *Let $\alpha_1,\ldots,\alpha_k$ be algebraic numbers contained in a field $\mathbb{K}$ of degree d and $b_1,\ldots,b_k$ be integers such that $\Lambda:=\alpha_1^{b_1}\cdots\alpha_k^{b_k}-1$ is nonzero. Let*

$$B=\max\{|b_1|,\ldots,|b_k|,3\}.$$

Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_{\mathbb{K}}$ sitting above an integer prime p. Let $A_1,\ldots,A_k$ be real numbers such that

$$\log A_i\ge\max\{h(\alpha_i),\log p\},\qquad \text{for } i=1,\ldots,k. \tag{2.34}$$

Then

(2.35)

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda)\le 12\left(\frac{6(k+1)}{\sqrt{\log p}}\right)^{2(k+1)}p^d\log(e^5kd)\log A_1\log A_2\cdots\log A_k\log B.$$

We take $k:=r_1+2$,

$$\alpha_1=\frac{\zeta^{(j_1)}}{\zeta^{(j_2)}},\qquad \alpha_2=\frac{\beta^{(j_1)}}{\beta^{(j_2)}},\qquad \alpha_i=\frac{\varepsilon_{i-2}^{(j_1)}}{\varepsilon_{i-2}^{(j_2)}}\qquad i=3,\ldots,k.$$

We take $b_1=b_2=1$ and $b_i=m_{i-2}$ for $i=3,\ldots,k$. Observe that by (2.32) it follows that we can take $B=a_{n-1}^2$. Clearly $p\le n$. Observe that α_1 is a root of 1 so it has a zero logarithmic height, and we can take $\log A_1=\log n$. As for α_2, any conjugate of it has absolute value, by estimates (2.28) and (2.30), at most

$$|\alpha_2^{(s)}|<a_{n-1}^{l^2},$$

therefore $h(\alpha_2) < l^2 \log a_{n-1}$. So, we take $\log A_2 = l^2 \log a_{n-1}$ and observe that it fulfills the condition (2.34) for $i = 2$. Finally, note that

$$|\alpha_i^{(s)}| = |\varepsilon_{i-2}^{(s_1)}|^2 \prod_{j \neq s_1, s_2} |\varepsilon_{i-2}^{(j)}| \qquad \text{for } i = 3, \ldots, k$$

for some two conjugations s_1 and s_2 depending on s, so it follows that

$$h(\alpha_i) \leq dh(\varepsilon_{i-2}) \qquad \text{for } i = 3, \ldots, k.$$

We claim that we can take $\log A_i = L^5 h(\varepsilon_{i-2})$, where $L = 21000 \log n$. Note that with this choice $\log A_i \geq d^3 h(\alpha_i) > h(\alpha_i)$. Furthermore, by a result of Voutier [10], we have that

$$d^2 h(\varepsilon_{i-2}) \geq \frac{d(\log \log d)^3}{4(\log d)^2},$$

and the function appearing on the right is > 0.1 when $d = l(l-1) \geq 6$. Thus,

$$\log A_i > L(d^2 h(\varepsilon_{i-2})) > 0.1L > \log n,$$

so condition (2.34) is fulfilled for $i = 3, \ldots, k$. Note that $k = r_1 + 2 \leq l(l-1) + 2 \leq l^2 - 1$. Finally, it is clear that our form Λ is nonzero. Lemma 2.6 now tells us that

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda) < 12 \left(\frac{6l^2}{\sqrt{\log 2}} \right)^{2l^2} n^{2l^2} \log(e^5 l^4) L^{5l^2} \log(a_{n-1}^2)(\log n)(l^2 \log a_{n-1}) \times \prod_{i=1}^{r_1} h(\varepsilon_i).$$

The last product is estimated by Lemma 2.5 as

$$\prod_{i=1}^{r_1} h(\varepsilon_i) < r_1!^2 R_{\mathbb{K}} < ((l^2)!)^2 R_{\mathbb{K}} < (l^2)^{2l^2} R_{\mathbb{K}} < (nl^6)^{l^2},$$

where for the last inequality we used inequality (2.23). Using the fact that $l < L$ and collecting alike terms we get

$$\operatorname{ord}_{\mathfrak{p}}(\Lambda) < 24(49n^3 L^{15})^{L^2} (\log n)(\log(e^5 L^4)) L^2 (\log a_{n-1})^2.$$

Thus, comparing the $\mathfrak{p}$-adic valuations we get the master inequality

$$\frac{a_{n-1} - L}{L} \leq 2L \log a_{n-1} + 24(49n^3 L^{15})^{L^2} (\log n)(\log(e^5 L^4)) L^2 (\log a_{n-1})^2,$$

which leads to

$$a_{n-1} < 26(49n^3 L^{15})^{L^2} (\log n)(\log(e^5 L^4)) L^2 (\log a_{n-1})^2.$$

Since a_{n-1} is very large, it follows that $\sqrt{a_{n-1}} > (\log a_{n-1})^2$, yielding

$$a_{n-1} < 26^2 (49n^3 L^{15})^{2L^2} (\log n)^2 \log(e^5 L^4)^2 L^4.$$

To see why this wrong observe the following easy estimates:

$$n^5 \geq 9^5 > 21000 \qquad \text{and} \qquad n > \log n.$$

Thus, $L < n^6$. Also, $49 < n^2$ and

$$\log(e^5 L^4) < 5 + 4\log L < 5 + 24\log n < 29n < n^3.$$

Thus,

$$a_{n-1} < n^4 \cdot (n^{2+3+6\cdot 15})^{2L^2} n^{2+6+4\cdot 6} = n^{190L^2+36},$$

and since $a_{n-1} = (n-1)^{a_{n-2}}$, we get that

$$a_{n-2} < (190L^2+36)\frac{\log n}{\log(n-1)} < 201L^2+40 < 201n^{12}+40 < 202n^{12} < n^{15},$$

and since $a_{n-2} \geq (n-2)^{a_{n-3}}$, we get that

$$a_{n-3} \leq 15\frac{\log n}{\log(n-2)} < 17,$$

which is of course false for $n \geq 9$. This finishes the proof of the theorem.

Acknowledgements.

The authors thank the anonymous referee for a careful reading of the manuscript and for pointing out some inaccuracies. Work on this paper started when F. L. visited the Max Plank Institute for Mathematics in Bonn, Germany in March of 2009, and continued during a visit of F. L. at the Mathematics Department of the University of Brasilia in Brasil in June of 2009. This author thanks these institutions for their hospitality and Professors Pieter Moree and Hemar Godinho Teixeira for useful conversations. F. L. was also supported in part by grants SEP-CONACyT 79685 and PAPIIT 100508.

References

[1] Y. Bugeaud and M. Laurent, *Minoration effective de la distance p-adique entre puissances de nombres algébriques*. J. Number Theory **61** (1996), 31–42.

[2] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers*. Ann. of Math. (2) **163** (2006), 969–1018.

[3] L. K. Hua, *Introduction to number theory*. Springer-Verlag, 1982.

[4] E. Landau, *Verallgemeinerung eines Polyaschen Satzes auf algebraische Zahlkörper*. Nachr. Kgl. Ges. Wiss. Göttingen, Math.-Phys. Kl. (1918), 478–488.

[5] M. Laurent, M. Mignotte and Yu. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*. J. Number Theory **55** (1995), 285–321.

[6] D. Poulakis, *Solutions entières de l'équation $Y^m = f(X)$*. Sém. Théor. Nombres Bordeaux (2) **3** (1991), 187–199.

[7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. http://www.research.att.com/ njas/sequences/.

[8] J. Sondow, *An irrationality measure for Liouville numbers and conditional measures for Euler's constant*. Preprint, 2003, http://arXiv.org/abs/math.NT/0307308.

[9] J. Sondow, *Irrationality measures, irrationality bases, and a theorem of Jarnik*. Preprint, 2004, http://arXiv.org/abs/math.NT/0406300.

[10] P. VOUTIER, *An effective lower bound for the height of algebraic numbers.* Acta Arith. **74** (1996), 81–95.
[11] R. T. WORLEY, *Estimating* $|\alpha - p/q|$. J. Austral. Math. Soc. Ser. A **31** (1981), 202–206.
[12] K. YU, *p-adic logarithmic forms and group varieties I.* J. Reine Angew. Math. **502** (1998), 29–92.

Florian LUCA
Instituto de Matemáticas
Universidad Nacional Autónoma de México
C.P. 58089, Morelia, Michoacán, México
E-mail: fluca@matmor.unam.mx
URL: http://www.matem.unam.mx

Diego MARQUES
Departamento de Matemática
Universidade de Brasília
Brasília, DF, Brazil
E-mail: diego@mat.unb.br
URL: http://www.mat.unb.br