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Résumé. On démontre la conjecture de Manin pour une surface de del Pezzo de degré six qui a une singularité de type $\mathbf{A}_2$. De plus, on établit un prolongement méromorphe et une expression explicite de la fonction zêta des hauteurs associées.

Abstract. We prove Manin's conjecture for a del Pezzo surface of degree six which has one singularity of type $\mathbf{A}_2$. Moreover, we achieve a meromorphic continuation and explicit expression of the associated height zeta function.

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1. Introduction

In this paper, our aim is to count the number of rational points of bounded height on the surface $S \subset \mathbb{P}^6$ given by

$$\begin{aligned} &x_3^2 + x_0x_5 + x_1x_6 = x_2x_3 - x_0x_6 = x_1x_2 + x_0x_3 + x_0x_4 = 0, \\ &x_3x_5 + x_4x_5 + x_6^2 = x_2x_5 - x_4x_6 = x_1x_5 - x_3x_6 = 0, \\ &x_4^2 + x_0x_5 + x_2x_6 = x_3x_4 - x_0x_5 = x_1x_4 - x_0x_6 = 0. \end{aligned} \tag{1.1}$$

This surface is an example of a singular del Pezzo surface of degree 6. A priori, it might not be clear why this is a natural diophantine problem. However in 1989, Manin and his collaborators [FMT89] formulated a general conjecture on the number of rational points of bounded height on Fano varieties. There is a programme (see [BB07] or [DT07] for example) to try to prove this conjecture for Fano surfaces, namely *del Pezzo surfaces* and

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their singular counterparts. Such surfaces have a well-known classification in terms of their singularity type and degree. See [Man86] and [CT88] for more information on smooth and singular del Pezzo surfaces respectively, and [Bro07] for a general overview of Manin's conjecture for del Pezzo surfaces.

The surface S has one singularity of type $\mathbf{A}_2$, which we can resolve using blow-ups to create two exceptional curves on the minimal desingularisation $\widetilde{S}$ of S. The set of equations (1.1) correspond to the embedding induced by a divisor in the *anticanonical divisor class*. Since S is singular normal with only rational double points, by [CT88, Prop. 0.1] an anticanonical divisor of S can be taken to be any divisor on S which pulls back to an anticanonical divisor on the minimal desingularisation $\widetilde{S}$. The anticanonical embedding is a natural choice, for example in this embedding the lines are exactly the (-1)-curves and Manin's conjecture takes a simpler form. The height function associated to the chosen embedding is the usual height on projective space, namely given $x \in S(\mathbb{Q})$, we have $H(x) = \max_{0\leq i\leq 6}|x_i|$, where $(x_0,\dots,x_6)$ is a primitive integer vector in the affine cone above x. Further details about the geometry of S can be found in Lemma 2.1.

Now, S contains the two lines

$$L_1 : x_1 = x_3 = x_4 = x_5 = x_6 = 0,$$
$$L_2 : x_2 = x_3 = x_4 = x_5 = x_6 = 0,$$

which both contain "many" rational points whose contribution will dominate the counting problem. Hence, it is natural to let $U = S \setminus \{L_1 \cup L_2\}$ and take

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) : H(x) \leq B\}$$

to be the associated counting function. In this context, Manin's conjecture predicts an asymptotic formula of the shape

$$N_{U,H}(B) \sim c_{\widetilde{S},H} B(\log B)^{\rho-1}$$

as $B \to \infty$, where $\rho = \operatorname{rank}(\operatorname{Pic}(\widetilde{S})) = 4$ and $c_{\widetilde{S},H}$ is some constant. In this paper, we establish a significantly sharper version of this estimate.

Theorem 1.1. *Let $\varepsilon > 0$. Then there is a monic cubic polynomial $P \in \mathbb{R}[x]$ such that*

$$N_{U,H}(B) = c_{\widetilde{S},H} BP(\log B) + O_\varepsilon(B^{7/8+\varepsilon})$$

where $c_{\widetilde{S},H} = \alpha(\widetilde{S})\tau_\infty(\widetilde{S})\prod_p \tau_p(\widetilde{S})$ and

$$\alpha(\widetilde{S}) = 1/432, \quad \tau_p(\widetilde{S}) = \left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right),$$

$$\tau_\infty(\widetilde{S}) = 6\int_{\{t,v,u\in\mathbb{R}:0<|t(ut+v^2)|,|uvt|,|uvt+v^3|,|u^2t|,|u^2t+uv^2|,u^3,u^2v\leq 1\}} \mathrm{d}u\mathrm{d}v\mathrm{d}t.$$

The leading constant in this expression agrees with the prediction of Peyre [Pey95], which we shall verify in Section 2.2. The calculation of the real density $\tau_\infty(\widetilde{S})$ poses something of a challenge, since in our case S is not given by a complete intersection, so standard methods for calculating this constant do not apply. In Section 2.2 we also prove a general result which assists in the calculation of the p-adic densities $\tau_p(\widetilde{S})$ (See Lemma 2.3).

The second theorem of this paper is intimately related to the above asymptotic formula. We give an explicit expression and meromorphic continuation of the associated *height zeta function*

$$Z_{U,H}(s) = \sum_{x \in U(\mathbb{Q})} \frac{1}{H(x)^s}. \tag{1.2}$$

To state the result, let $\mathrm{Re}(s) > 0$ and define

$$\begin{aligned} E_1(s+1) &= \zeta(4s+1)\zeta(3s+1)^2\zeta(2s+1), \\ E_2(s+1) &= \frac{\zeta(7s+3)^4\zeta(8s+3)^2}{\zeta(4s+2)^3\zeta(5s+2)^2\zeta(6s+2)\zeta(10s+4)}. \end{aligned} \tag{1.3}$$

It is clear that $E_1(s)$ and $E_2(s)$ have a meromorphic continuation to the whole complex plane. Also $E_1(s)$ has a single pole of order 4 at $s = 1$ and $E_2(s)$ is holomorphic on $\mathrm{Re}(s) > 3/4$. We then prove the following.

Theorem 1.2. *Let $\varepsilon > 0$, then*

$$Z_{U,H}(s) = E_1(s)E_2(s)G_1(s) + \frac{12/\pi^2 + 2\lambda}{s-1} + G_2(s).$$

Here, $\lambda \in \mathbb{R}$ is a constant and $G_1(s)$ and $G_2(s)$ are complex functions that are holomorphic on $\mathrm{Re}(s) > 5/6$ and $\mathrm{Re}(s) \geq 3/4 + \varepsilon$ respectively and satisfy $G_1(s) \ll_\varepsilon 1$ and $G_2(s) \ll_\varepsilon (1 + |\mathrm{Im}(s)|)$ on these half-planes.

In particular, $(s-1)^4 Z_{U,H}(s)$ has a holomorphic continuation to the half-plane $\mathrm{Re}(s) > 5/6$.

Expressions for $G_1(s)$ and $G_2(s)$ can be found in (3.17),(3.19),(3.20) and Lemma 3.8. Here $E_1(s)E_2(s)G_1(s)$ and $G_2(s)$ correspond to the main term and error term in the counting argument respectively and $12/\pi^2$ corresponds to an isolated conic in the surface. We only prove the existence of λ, however a keen reader can build an explicit (and complicated) expression for it using the work in Section 3.5. We shall only say that λ arises naturally in the proof as an error term created by approximating a sum by an integral and has appeared in some form in other works (e.g. [BB07]), however it is currently severely lacking in geometric interpretation.

We will show in Lemma 2.1 that the surface S is an equivariant compactification of $\mathbb{G}_a^2$, so that the work of Chambert-Loir and Tschinkel [CT02] applies, where they have already achieved an analytic continuation of the associated height zeta function and an asymptotic formula for the counting problem. However, our results are stronger for a number of reasons. Firstly, we do not use the fact that S is an equivariant compactification of $\mathbb{G}_a^2$, so our methods seem applicable to more general situations. We also get an explicit expression for the height zeta function in terms of the Riemann zeta function, which gives a better insight into how these zeta functions look and behave for a concrete example. Furthermore, whereas [CT02] only gives a holomorphic continuation of $(s-1)^4 Z_{U,H}(s)$ to an unspecified half-plane $\mathrm{Re}(s) > 1-\delta$, we are able to show that $\delta = 1/6$ is acceptable, and that $\delta = 1/4$ appears to be a natural boundary under the assumption of the Riemann hypothesis. As a consequence, we get an explicit (and stronger) error term in our asymptotic formula.

The first important step in the proof of Theorem 1.2 is to relate the counting problem on S to that of counting integral points on the associated *universal torsor*. Universal torsors were introduced by Colliot-Thélène and Sansuc in [CTS87] to aid the study of the Hasse principle and weak approximation. However, Salberger [Sal98] showed that they could be a valuable tool in counting problems on varieties. In general a variety may have more than one universal torsor, however in our case there is only one universal torsor (see Section 3.1 for further details). It can be visualised as a certain open subset $\mathcal{T}$ of the affine variety in $\mathbb{A}^7$ given by the following equation

$$\eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3 = 0.$$

For our purposes, the universal torsor is a variety with a surjective morphism $\pi : \mathcal{T} \to S$ defined over $\mathbb{Q}$, and an action of $\mathbb{G}_m^4$ on $\mathcal{T}$ which preserves the fibres of π and acts freely and transitively on them. Exact definitions can be found in the above references, and a concrete realisation of the universal torsor can be found in Lemma 3.1.

To relate the two counting problems we find a suitable set-theoretic section of the map π, which corresponds to requiring that we count certain integral points satisfying the universal torsor equation and certain coprimality conditions. Previous methods for achieving this in similar problems have been the "elementary method" [BB07, Section 4] and the "blow-up method" [DT07, Section 4]. The first method involves looking for divisibility relations given by the equations of the surface, and then performing a lengthy chain of substitutions to pull out any highest common factors among the variables. The second method involves knowing which exact points of $\mathbb{P}^2$ are blown-up to create your surface, and using these to guide you through various algebraic manipulations.

Here we present a new method, which uses the action of $\mathbb{G}_m^4$ on the universal torsor. Essentially, we use this action to "rescale" each point in each fibre to a unique point. Since the universal torsor (if it exists) of a more general variety always has a free and transitive group action on its fibres, this method is more likely to generalise to other situations than the previously two mentioned methods. See Lemma 3.2 for more details.

Notation: To simplify notation, throughout this paper ε is any positive real number which all implied constants are allowed to depend upon. We use the common practice that ε can take different values at different points of the argument.

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2. Preliminary steps

2.1. Some geometry. The underlying geometry of the surface S is well understood, and we gather some facts about it in the following lemma, which also helps to fix some notation.

Lemma 2.1. *Let S be given by (1.1). Then the following holds.*

- *S is a split singular del Pezzo surface of degree 6 given by its anti-canonical embedding.*
- *It contains the singular point $(1:0:0:0:0:0:0)$ of type $\mathbf{A}_2$.*
- *The only lines in S are given by*

$$L_1 : x_1 = x_3 = x_4 = x_5 = x_6 = 0,$$
$$L_2 : x_2 = x_3 = x_4 = x_5 = x_6 = 0.$$

  In particular $U = S \setminus \{L_1 \cup L_2\} = S \setminus \{x_5 = 0\}$.
- *S is the closure of $\mathbb{P}^2$ under the rational map $\varphi : \mathbb{P}^2 \dashrightarrow S$ given by*

$$\varphi(x_3 : x_5 : x_6) = (\varphi_0(x_3, x_5, x_6) : \cdots : \varphi_6(x_3, x_5, x_6)) =$$
$$(-x_3^2x_5 - x_3x_6^2 : x_3x_5x_6 : -x_3x_5x_6 - x_6^3 : x_3x_5^2 : -x_3x_5^2 - x_5x_6^2 : x_5^3 : x_5^2x_6),$$

  where $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = \langle x_3, x_5, x_6\rangle$.
- *The group law on $\varphi(\mathbb{G}_a^2) = U$ extends to an action on S by translation. i.e. S is an equivariant compactification of $\mathbb{G}_a^2$.*

Proof. First, it is clear that φ defines an isomorphism $U \cong \mathbb{G}_a^2$. Hence the divisor class group of S is generated by the L_1 and L_2, as $\mathrm{Pic}(\mathbb{G}_a^2) = 0$. It is simple enough to check that the induced group law on U extends to an action on all of S. However as mentioned in the introduction we will not use this fact in this paper, so the proof is omitted and can be found in [DL10].

Resolving the singularity explicitly via blow-ups creates two exceptional curves E_1 and E_2 on the minimal desingularisation $\widetilde{S}$. The singularity is of type $\mathbf{A}_2$ and $\mathrm{Pic}(\widetilde{S}) = \langle E_1, E_2, E_3, E_4\rangle \cong \mathbb{Z}^4$, where E_3 and E_4 are the strict transforms of L_1 and L_2 respectively. Now, one can use the adjunction formula [Har77, Ch. V, Prop. 1.5] to show that $-K_{\widetilde{S}} = 4E_1 + 2E_2 + 3E_3 + 3E_4$, which proves that $K_{\widetilde{S}}^2 = 6$. Also, one can show that the pull back of the hyperplane section on S is $-K_{\widetilde{S}}$, thus proving that S is a singular del Pezzo surface of degree 6 given by its anticanonical embedding.

Finally, we note that the $\mathbf{A}_2$ singular del Pezzo surface of degree 6 contains only two lines by the classification of singular del Pezzo surfaces [CT88, Prop. 8.3]. These are both defined over $\mathbb{Q}$, so the surface is indeed split. □

We also include the extended Dynkin diagram of $\widetilde{S}$ in Figure 2.1, which records the intersection behaviour of relevant curves on $\widetilde{S}$. This can be derived from the proof of Lemma 2.1, or found in [Der06, Sec. 5]. Here E_1, E_2, E_3 and E_4 are as in the proof of Lemma 2.1 and

$$A_1 : S \cap \{x_1 = x_2 = x_6 = 0\}, \quad A_2 : S \cap \{x_0 = x_1 = x_3 = 0\},$$
$$A_3 : S \cap \{x_0 = x_2 = x_4 = 0\}.$$

These rational curves correspond to generators of the nef cone and will be needed in our work in section 3.1.

$$\begin{tikzcd}
A_2 \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & E_3 \arrow[dr, dash] & \\
& A_1 \arrow[r, dash] \arrow[dl, dash] & E_2 \arrow[r, dash] & E_1 \arrow[dl, dash] \\
A_3 \arrow[rr, dash] & & E_4 &
\end{tikzcd}$$

Figure 2.1. The extended Dynkin diagram for $\widetilde{S}$.

2.2. Calculating Peyre's constant. In this section we shall verify that the constant achieved in the asymptotic formula for Theorem 1.1 is in agreement with the conjectural expression as formulated by Peyre [Pey95, Sec. 2]. Since our surface is split, it is birational to $\mathbb{P}^2$ over $\mathbb{Q}$. So the constant is equal to the following three factors multiplied together:

- The volume $\alpha(\widetilde{S})$ of a certain polytope in the cone of effective divisors,
- The real density $\tau_\infty(\widetilde{S})$,
- The p-adic densities $\prod_p \tau_p(\widetilde{S})$.

By the work of [Der07, Table 3] we know that

$$\alpha(\widetilde{S}) = \frac{1}{432},$$

which is in agreement with the constant $\alpha(\widetilde{S})$ in Theorem 1.1.

We shall now calculate the real density, which corresponds to the measure of some region, where we consider $\widetilde{S}(\mathbb{R})$ as a real analytic manifold. Since removing a codimension one subset does not change this volume, we may consider the measure of the coordinate chart $U = S \setminus \{x_5 = 0\}$, with local coordinates x_3 and x_6. By Lemma 2.1, this is just a reflection of the fact that our surface is a compactification of $\mathbb{A}^2$ with φ as a local homeomorphism. Since S is given by its anticanonical embedding, we have by [Pey95, Section 2.2.1]

$$\begin{aligned}\tau_\infty(\widetilde{S}) &= \int_{\mathbb{R}^2} \frac{\mathrm{d}x_3\mathrm{d}x_6}{\max(|x_3^2 + x_3x_6^2|, |x_3x_6|, |x_3x_6 + x_6^3|, |x_3|, |x_3 + x_6^2|, 1, |x_6|)} \\ &= \int_{\mathbb{R}^2}\int_{x_5 \geq \{\max(|x_3^2+x_3x_6^2|,|x_3x_6|,|x_3x_6+x_6^3|,|x_3|,|x_3+x_6^2|,1,|x_6|)} \frac{\mathrm{d}x_3\mathrm{d}x_5\mathrm{d}x_6}{x_5^2} \\ &= 3\int_{\{t,v,u\in\mathbb{R}: 0<|t(ut+v^2)|,|uvt|,|uvt+v^3|,|u^2t|,|u^2t+uv^2|,u^3,|u^2v|\leq 1\}} \mathrm{d}u\mathrm{d}v\mathrm{d}t,\end{aligned}$$

where we have used the change of variables

$$x_3 = t/u, x_5 = u^{-3}, x_6 = v/u.$$

Then noticing that we have the obvious automorphism $v \mapsto -v$ in the above integral, this gives the required expression for the constant in Theorem 1.1. We note that more generally, the real density of any anticanonically embedded del Pezzo surface can be calculated similarly by knowing which linear system of cubics in $\mathbb{P}^2$ determines the given embedding.

The calculation of the p-adic densities for similar problems (see [BB07] for example) have normally involved a "hands-on" approach to point counting modulo p for each prime p. Here we opt for a more general method, which applies to any surface that is the blow-up of $\mathbb{P}^2$ at a sequence of (possibly infinitely near) rational points. First we recall some definitions.

Definition. Let V be a non-singular projective variety defined over $\mathbb{Q}$. A *model* for V over $\mathbb{Z}$ is a projective morphism of schemes $\mathcal{V} \to \operatorname{Spec}\mathbb{Z}$, whose generic fibre is isomorphic to V. For each prime p, we denote by $\mathcal{V}_p = \mathcal{V} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ the reduction of $\mathcal{V}$ modulo p.

We say that V has *everywhere good reduction* if there exists a model whose structure morphism is a *smooth morphism* (i.e. $\mathcal{V}_p$ is a non-singular variety for each prime p).

Lemma 2.2. *Let S be a surface over $\mathbb{Q}$ with everywhere good reduction, and $\pi : \widetilde{S} \to S$ the blow-up of S at a rational point P. Then $\widetilde{S}$ also has everywhere good reduction.*

Proof. Let $\mathcal{S}$ be the model of S with everywhere good reduction. Since $\mathcal{S}$ is projective, the rational point P extends uniquely to an integral point $\mathcal{P}$ of $\mathcal{S}$. Then the scheme $\widetilde{\mathcal{S}}$, which is defined to be the blow-up of $\mathcal{S}$ at $\mathcal{P}$, is a model for $\widetilde{S}$. For every prime p it is clear that $\widetilde{\mathcal{S}}_p$ is simply the blow-up of $\mathcal{S}_p$ at a smooth $\mathbb{F}_p$-point, so $\widetilde{\mathcal{S}}$ also has everywhere good reduction. □

Now let $S, \mathcal{S}, \widetilde{S}$ and $\widetilde{\mathcal{S}}$ be as in Lemma 2.2. Then it is clear that for every prime p we have $\#\widetilde{\mathcal{S}}_p(\mathbb{F}_p) = \#\mathcal{S}_p(\mathbb{F}_p) + p$, since blowing up a smooth $\mathbb{F}_p$-point replaces one $\mathbb{F}_p$-point by a copy of $\mathbb{P}^1_{\mathbb{F}_p}$, which has $p+1$ $\mathbb{F}_p$-points. We can use this simple fact to prove the following.

Lemma 2.3. *Let S be a surface over $\mathbb{Q}$ which is the blow-up of $\mathbb{P}^2$ at r (possibly infinitely near) rational points. Then for every prime p the local density at p is*

$$\tau_p(S) = \left(1 - \frac{1}{p}\right)^{r+1} \left(1 + \frac{r+1}{p} + \frac{1}{p^2}\right).$$

Proof. We begin by noting that the definition of $\tau_p(S)$ is independent of the choice of model, as pointed out in [Pey95, Def. 2.2]. Since $\mathbb{P}^2$ has everywhere good reduction, then so does S by Lemma 2.2. Let $\mathcal{S}$ be the corresponding model, then $\#\mathcal{S}_p(\mathbb{F}_p) = 1+(r+1)p+p^2$ since $\#\mathbb{P}^2(\mathbb{F}_p) = 1+p+p^2$. It is also clear that $\mathrm{Pic}(\mathcal{S}_p) \cong \mathbb{Z}^{r+1}$ with trivial galois action, hence the associated Artin L-function is $\zeta(s)^{r+1}$. This gives the correct "convergence factors" and the result follows. □

Applying Lemma 2.3 to $\widetilde{S}$ (which is split by Lemma 2.1) with $r = 3$, we deduce the result.

3. The proof

3.1. Passage to the universal torsor. As mentioned in the introduction, the first step in the proof is transferring the problem of counting rational points on the surface S, to counting integral points on the corresponding universal torsor $\mathcal{T}$.

A variety may in general have more than one universal torsor, however in our case there is only one. Indeed if a smooth projective variety V over a field k has a universal torsor, then the set of isomorphism classes of universal torsors is a principal homogeneous space under $H^1(k, T)$, where $T = \mathrm{Hom}(\mathrm{Pic}(V), \mathbb{G}_m)$ is the Néron-Severi torus. However in our case $T = \mathbb{G}_m^4$ since $\mathrm{Pic}(\widetilde{S}) \cong \mathbb{Z}^4$ with trivial galois action, and also $H^1(\mathbb{Q}, \mathbb{G}_m^4) = 0$ by Hilbert's theorem 90. Hence $\widetilde{S}$ can have at most one universal torsor.

However, the existence of a rational point on $\widetilde{S}$ implies the existence of a universal torsor. These facts (and more) can be found in [Sko01, Sec. 2.3]. The following lemma gives us a concrete description of the universal torsor.

Lemma 3.1. *Let*

$$\mathrm{Cox}(\widetilde{S}) = \bigoplus_{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4} H^0(\widetilde{S}, \mathcal{O}(E_1)^{\otimes n_1} \otimes \cdots \otimes \mathcal{O}(E_4)^{\otimes n_4})$$

be the Cox ring of $\widetilde{S}$. Then

- $\mathrm{Cox}(\widetilde{S}) \cong \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3, \eta_1, \eta_2, \eta_3, \eta_4]/(\eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3)$.
- *The universal torsor $\mathcal{T}$ of $\widetilde{S}$ is an open subset of* $\mathrm{Spec}(\mathrm{Cox}(\widetilde{S}))$.
- *We have a commutative diagram*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\widetilde{\pi}"] \arrow[dr, "\pi"'] & \widetilde{S} \arrow[d] \\ & S \end{tikzcd}$$

where π is the map

$$\begin{aligned}\pi(\boldsymbol{\eta}, \boldsymbol{\alpha}) \mapsto (&\alpha_2\alpha_3 : \eta_1\eta_2\eta_3\alpha_1\alpha_2 : \eta_1\eta_2\eta_4\alpha_1\alpha_3 : \eta_1^2\eta_2\eta_3^2\eta_4\alpha_2 \\ &: \eta_1^2\eta_2\eta_3\eta_4^2\alpha_3 : \eta_1^4\eta_2^2\eta_3^3\eta_4^3 : \eta_1^3\eta_2^2\eta_3^2\eta_4^2\alpha_1).\end{aligned} \tag{3.1}$$

- *The action of a point $(k_1, k_2, k_3, k_4) \in \mathbb{G}_m^4$ on the universal torsor is $\eta_i \mapsto k_i\eta_i$ for $i = 1, 2, 3, 4$, and*

$$\alpha_1 \mapsto k_1k_3k_4\alpha_1, \quad \alpha_2 \mapsto k_1^2k_2k_3k_4^2\alpha_2, \quad \alpha_3 \mapsto k_1^2k_2k_3^2k_4\alpha_3.$$

Proof. The calculation of the Cox ring, the map π and the action of the Néron-Severi torus on the Cox ring can be found in [Der06]. That the universal torsor is an open subset of $\mathrm{Spec}(\mathrm{Cox}(\widetilde{S}))$ is well-known, see [HK00, Cor. 2.16, Prop. 2.9] for example. □

In fact, everything we need to know about the universal torsor can be deduced from first principles. Firstly, it is not actually necessary for us to calculate explicitly which open subset of $\mathrm{Spec}(\mathrm{Cox}(\widetilde{S}))$ the universal torsor corresponds to. However, it is easy to check that the action given in Lemma 3.1 is well-defined and that it preserves the fibres of π. Moreover, π is surjective on its domain of definition since $\varphi^{-1} \circ \pi$ is surjective onto U, where φ and U are as in Lemma 2.1. And also, it is easy enough to see that π hits every point on $S \setminus U$ as well, hence it is surjective.

In particular, when we consider the universal torsor as being over U, it is simple to see that we get a free and transitive action on the fibres of π on the corresponding open subset where $\eta_1\eta_2\eta_3\eta_4 \neq 0$. That is, it is clear that $\mathrm{Spec}(\mathrm{Cox}(\widetilde{S})) \setminus \{\eta_1\eta_2\eta_3\eta_4 = 0\}$ is a U-torsor under $\mathbb{G}_m^4$.

Now to find a suitable section of the morphism π. Bearing in mind that we are counting points on U where $x_5 \neq 0$, we see that the rational points which have some coordinate equal to zero lie in the image of the points on the torsor where $\alpha_1\alpha_2\alpha_3 = 0$. These are exactly the curves A_1, A_2 and A_3 given in Figure 2.1. They are rational curves, and it is easy enough to show that the corresponding counting functions satisfy

$$N_{A_1}(B) = \frac{12}{\pi^2}B + O(B^{1/2}), \quad N_{A_2}(B) = N_{A_3}(B) = O(B^{2/3}).$$

Since these have been taken into account, we can now assume that each coordinate of each rational point is non-zero.

Lemma 3.2. *Above each rational point $x \in U(\mathbb{Q})$ with non-zero coordinates, there is a unique integral point $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ on the universal torsor satisfying*

$$\begin{aligned}
&(\alpha_1, \eta_1\eta_3\eta_4) = (\alpha_2, \eta_1\eta_2\eta_4) = (\alpha_3, \eta_1\eta_2\eta_3) = 1,\\
&(\eta_2, \eta_3) = (\eta_2, \eta_4) = (\eta_3, \eta_4) = 1,\\
&\eta_1, \eta_2, \eta_3, \eta_4 > 0, \alpha_1\alpha_2\alpha_3 \neq 0.
\end{aligned}$$

Proof. We should note that we are guided to the above coprimality conditions by Figure 2.1, whereby two variables are coprime if and only if the corresponding curves do not intersect each other.

First let $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ be an integral point on the universal torsor lying above a rational point with non-zero coordinates. Suppose that there is a prime $p \mid (\eta_1, \alpha_1)$. Then using the torsor action in Lemma 3.1 with $k_1 = 1/p, k_2 = p^3, k_3 = k_4 = 1$, we map

$$\begin{aligned}
&\eta_1 \mapsto \eta_1/p, \quad \eta_2 \mapsto p^3\eta_2,\\
&\alpha_1 \mapsto \alpha_1/p, \quad \alpha_2 \mapsto p\alpha_2, \quad \alpha_3 \mapsto p\alpha_3.
\end{aligned}$$

So we have successfully managed to divide η_1 and α_1 by p, and left the other variables as integers, meaning that if they have any common factor we can remove it. A very similar argument works for η_3 and η_4, so we can assume

$$(\alpha_1, \eta_1\eta_3\eta_4) = 1.$$

We now fix our choice of α_1 modulo $\{\pm 1\}$, meaning that from now on we impose the condition $|k_1k_3k_4| = 1$. This simplifies the action on α_2 and α_3 to

$$\alpha_2 \mapsto \frac{k_2}{k_3}\alpha_2, \quad \alpha_3 \mapsto \frac{k_2}{k_4}\alpha_3.$$

Carrying on with the same procedure, if $p \mid (\alpha_2, \eta_1)$, take $k_1 = 1/p, k_2 = k_4 = 1, k_3 = p$ to get $(\alpha_2, \eta_1) = 1$ and for $p \mid (\alpha_2, \eta_4)$ take $k_1 = k_2 = 1, k_3 = p, k_4 = 1/p$ to get $(\alpha_2, \eta_4) = 1$.

We have now come to interesting part of the proof, since so far we have not used the equation of the universal torsor, but now we are driven to use it since it encodes divisibility conditions. Namely, if $p \mid (\alpha_2, \eta_2)$, then p must also divide η_4 or α_3. But $(\alpha_2, \eta_4) = 1$, so we are safe to choose $k_1 = k_3 = k_4 = 1, k_2 = 1/p$ and keep α_3 as an integer. So we have successfully shown that we can choose

$$(\alpha_2, \eta_1\eta_2\eta_4) = 1.$$

We fix this choice of α_2 modulo $\{\pm 1\}$, which is equivalent to requiring $|k_2| = |k_3|$.

The reader should now be familiar with the method and can check that we can assume $(\alpha_3, \eta_1\eta_2\eta_3) = 1$ after performing the following

- If $p \mid (\alpha_3, \eta_1)$, choose $k_1 = 1/p, k_2 = k_3 = 1, k_4 = p$,
- If $p \mid (\alpha_3, \eta_3)$, choose $k_1 = p, k_2 = k_3 = 1/p, k_4 = 1$,
- If $p \mid (\alpha_3, \eta_2)$, contradiction since $(\alpha_3, \eta_3) = (\alpha_2, \eta_2) = 1$.

So fixing α_3 modulo $\{\pm 1\}$, we are restricted to

$$|k_2| = |k_3| = |k_4|.$$

But if $p \mid (\eta_2, \eta_3, \eta_4)$, choosing $k_1 = p^2, k_2 = k_3 = k_4 = 1/p$ then gives $(\eta_2, \eta_3, \eta_4) = 1$, and moreover the torsor equation implies they must also be pairwise coprime. Finally, by choosing the η_i to be positive, we have used all degrees of freedom in the torsor action and so the choice of integral point is unique. $\square$

Using this lemma, we see that counting those points $x \in U(\mathbb{Q})$ satisfying the height bound $H(x) \leq B$, is equivalent to counting the unique integral points above them on the universal torsor which satisfy the bound $H(\pi(\boldsymbol{\alpha}, \boldsymbol{\eta})) \leq B$. Naively, this corresponds to 7 separate height conditions. However, using the map φ from Lemma 2.1, we know that we actually have 3 degrees of freedom. With this in mind, we define

$$\begin{aligned} X_3 &= \left(\frac{\eta_1^2\eta_2\eta_3^2\eta_4}{X_5^2 B}\right) = \left(B\eta_1^2\eta_2\eta_4^3\right)^{-1/3}, \\ X_5 &= \left(\frac{\eta_1^4\eta_2^2\eta_3^3\eta_4^3}{B}\right)^{1/3}, \\ X_6 &= \left(\frac{\eta_1^3\eta_2^2\eta_3^2\eta_4^2}{X_5^2 B}\right) = \left(\frac{\eta_1\eta_2^2}{B}\right)^{1/3}, \end{aligned} \tag{3.2}$$

and let $\overline{\varphi_i}(\alpha_1, \alpha_2) = \varphi_i(\alpha_2 X_3, X_5, \alpha_1 X_6)$ for $i = 0, 1, 2, 4$, and $\overline{\varphi_3}(\alpha_2) = \varphi_3(\alpha_2 X_3, X_5, 1), \overline{\varphi_6}(\alpha_1) = \varphi_6(1, X_5, \alpha_1 X_6)$. Then it is clear that the height condition $H(\pi(\boldsymbol{\alpha},\boldsymbol{\eta})) \leq B$ is equivalent to the condition

$$|\overline{\varphi_i}(\alpha_1,\alpha_2)|, |\overline{\varphi_3}(\alpha_2)| \leq 1, i = 0,1,2,4, \tag{3.3}$$

$$X_5, \overline{\varphi_6}(\alpha_1) \leq 1. \tag{3.4}$$

Finally, on noticing we have the obvious automorphism $\alpha_1 \mapsto -\alpha_1$ on the torsor, we have shown the following.

Lemma 3.3. *The counting function for U satisfies*

$$N_U(B) = 2T(B) + \frac{12}{\pi^2}B + O(B^{2/3})$$

where

$$T(B) = \#\left\{ (\boldsymbol{\alpha},\boldsymbol{\eta}) \in \mathbb{Z}^7 \ : \begin{array}{l} \eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3 = 0, (3.3), (3.4), \\ (\alpha_1, \eta_1\eta_3\eta_4) = (\alpha_2, \eta_1\eta_2\eta_4) = (\eta_2,\eta_3) = 1, \\ (\alpha_3, \eta_1\eta_2\eta_3) = (\eta_2,\eta_4) = (\eta_3,\eta_4) = 1, \\ \alpha_1, \eta_1, \eta_2, \eta_3, \eta_4 > 0, \alpha_2\alpha_3 \neq 0. \end{array} \right\}.$$

We note that we have the natural upper bound $\alpha_1 \leq 1/X_5^2X_6$ given by $\overline{\varphi_6}$. However, we can actually do better than this, which will be quite important to improving our error term later on. Notice that $\overline{\varphi_4}$ and $\overline{\varphi_3}$ imply

$$-\frac{1}{X_3X_5^2} \leq \alpha_2 \leq \frac{1}{X_3X_5^2}\left(1 - \alpha_1^2X_5X_6^2\right).$$

Rearranging this in terms of α_1, we deduce the stronger bound

$$\alpha_1 \leq \frac{\sqrt{2}}{X_6\sqrt{X_5}}. \tag{3.5}$$

3.2. Möbius inversion. Now we shall use Möbius inversion to remove the coprimality conditions on the α_i's. Recalling the counting problem in Lemma 3.3 and the height conditions (3.4), it makes sense to define

$$\mathcal{N} = \left\{ \boldsymbol{\eta} \in \mathbb{Z}^4 : \begin{array}{l} \eta_1,\eta_2,\eta_3,\eta_4 > 0, X_5 \leq 1, \\ (\eta_2,\eta_3) = (\eta_2,\eta_4) = (\eta_3,\eta_4) = 1. \end{array} \right\}. \tag{3.6}$$

Then it is clear that

$$T(B) = \sum_{\boldsymbol{\eta}\in\mathcal{N}} \sum_{\substack{\alpha_1 > 0 \\ (\alpha_1,\eta_1\eta_3\eta_4)=1 \\ \overline{\varphi_6}(\alpha_1)\leq 1}} S$$

where

$$S = \#\left\{ \alpha_2,\alpha_3 \in \mathbb{Z} : \begin{array}{l} \alpha_2\alpha_3 \neq 0, (3.3) \text{ holds}, \\ (\alpha_2, \eta_1\eta_2\eta_4) = (\alpha_3,\eta_1\eta_2\eta_3) = 1, \\ \eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3 = 0. \end{array} \right\}.$$

Now using Möbius inversion on $(\alpha_3, \eta_1\eta_2\eta_3) = 1$ gives us

$$S = \sum_{k_3 | \eta_1\eta_2\eta_3} \mu(k_3) S_{k_3}$$

where

$$S_{k_3} = \#\left\{ \alpha_2, \alpha_3 \in \mathbb{Z} : \begin{array}{l} \alpha_2\alpha_3 \neq 0, (3.3) \text{ holds}, \\ (\alpha_2, \eta_1\eta_2\eta_4) = 1, \\ \eta_2\alpha_1^2 + \eta_3\alpha_2 + k_3\eta_4\alpha_3 = 0. \end{array} \right\}.$$

However $S_{k_3} \neq 0$ if and only if $(k_3, \eta_2\eta_3) = 1$, so

$$S = \sum_{\substack{k_3 | \eta_1 \\ (k_3, \eta_2\eta_3) = 1}} \mu(k_3) S_{k_3}.$$

A similar argument yields

$$T(B) = \sum_{\boldsymbol{\eta} \in \mathcal{N}} \sum_{\substack{k_3 | \eta_1 \\ (k_3, \eta_2\eta_3) = 1}} \mu(k_3) \sum_{\substack{k_2 | \eta_1\eta_2 \\ (k_2, k_3\eta_4) = 1}} \mu(k_2) \sum_{\substack{\alpha_1 > 0 \\ (\alpha_1, \eta_1\eta_3\eta_4) = 1 \\ \overline{\varphi_6}(\alpha_1) \leq 1}} S_{k_2,k_3} \tag{3.7}$$

where

$$S_{k_2,k_3} = \#\left\{ \alpha_2, \alpha_3 \in \mathbb{Z} : \begin{array}{l} \alpha_2\alpha_3 \neq 0, k_2\eta_3\alpha_2 + \eta_2\alpha_1^2 + k_3\eta_4\alpha_3 = 0, \\ |\overline{\varphi_i}(\alpha_1, k_2\alpha_2)|, |\overline{\varphi_4}(k_2\alpha_2)| \leq 1, i = 0, 1, 2, 3. \end{array} \right\}.$$

3.3. Sum over $\boldsymbol{\alpha_2}$ and $\boldsymbol{\alpha_3}$ via congruences. In this section, we shall perform the summation over α_2. We note that there are no conditions on α_3 other than the equation of the universal torsor, so we find that

$$S_{k_2,k_3} = \#\left\{ \alpha_2 \in \mathbb{Z} : \begin{array}{l} \alpha_2 \neq 0, k_2\eta_3\alpha_2 \equiv -\eta_2\alpha_1^2 \pmod{k_3\eta_4}, \\ |\overline{\varphi_i}(\alpha_1, k_2\alpha_2)|, |\overline{\varphi_4}(k_2\alpha_2)| \leq 1, i = 0, 1, 2, 3. \end{array} \right\}.$$

However since $(k_2\eta_3, k_3\eta_4) = 1$, α_2 is uniquely determined modulo $k_3\eta_4$. For any integers q, n_0, a, b with $a < b$, we have the simple estimate

$$\#\{n \in \mathbb{Z} \cap [a, b] : n \equiv n_0 \pmod{q}\} = \frac{b - a}{q} + O(1).$$

Using this and the change of variables $t \mapsto k_2 t X_3$, we see that

$$S_{k_2,k_3} = \frac{1}{k_2 k_3 \eta_4 X_3} F_1(X_5, \alpha_1 X_6) + O(1) \tag{3.8}$$

where $F_1(u, v)$ is defined by the following result.

Lemma 3.4. *Let*

$$F_1(u, v) = \int_{\{t \in \mathbb{R}:, 0 < |t(ut+v^2)|, |uvt|, |uvt+v^3|, |u^2t|, |u^2t+uv^2| \leq 1\}} \mathrm{d}t$$

for $u, v \geq 0$ and $(u, v) \neq (0, 0)$. Then

(a) *For* $u \neq 0$,

$$F_1(u,v) \leq \frac{2}{\sqrt{u}}.$$

(b) $F_1(u,v)$ *is piecewise differentiable with respect to* u *and* v.

Proof. The differentiability condition is clear, so it remains to prove the inequality. First let $M(u,v) = \mathrm{vol}\{t \in \mathbb{R} : |t(ut+v^2)| \leq 1\}$, then we have $M(u,v) = \mathrm{vol}\{t \in \mathbb{R} : v^4/4u^2 - 1/u \leq t^2 \leq 1/u + v^4/4u^2\}$ after completing the square. If $v^4/4u^2 \geq 1/u$, then using the simple fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all non-negative real numbers a and b, we deduce that

$$M(u,v) = \sqrt{v^4/4u^2 + 1/u} - \sqrt{v^4/4u^2 - 1/u} \leq \sqrt{2/u}.$$

Similarly, if $v^4/4u^2 \leq 1/u$ then $M(u,v) = \sqrt{v^4/4u^2 + 1/u} \leq 2/\sqrt{u}$. □

We now have our first error term in the counting problem (3.7). First recall that $\sum_{k|n} |\mu(k)| = 2^{\omega(n)}$ where $\omega(n)$ is the number of prime divisors of n, that we have the stronger bound on α_1 given by (3.5), and the definition (3.6) of $\mathcal{N}$. Using these, we see that the overall contribution to the error term from (3.8) is

$$\begin{aligned}
&\ll \sum_{\eta_1^4\eta_2^2\eta_3^3\eta_4^3 \leq B} \sum_{k_3|\eta_1} |\mu(k_3)| \sum_{k_2|\eta_1\eta_2} |\mu(k_2)| \sum_{|\alpha_1| \leq \frac{\sqrt{2}}{X_6\sqrt{X_5}}} 1 \\
&\ll B^{1/2} \sum_{\eta_1^4\eta_2^2\eta_3^3\eta_4^3 \leq B} \frac{2^{\omega(\eta_1)}2^{\omega(\eta_1\eta_2)}}{\eta_1\eta_2\eta_3^{1/2}\eta_4^{1/2}} \\
&\ll B^{1/2} \sum_{\eta_1^4\eta_2^2\eta_3^3 \leq B} \frac{4^{\omega(\eta_1)}2^{\omega(\eta_2)}}{\eta_1\eta_2\eta_3^{1/2}} \cdot \frac{B^{1/6}}{\eta_1^{2/3}\eta_2^{1/3}\eta_3^{1/2}} \ll B^{2/3+\varepsilon}
\end{aligned}$$

since $2^{\omega(n)} \leq d(n) \ll n^{\varepsilon}$, where $d(n)$ is the usual divisor function. This error term is clearly satisfactory for Theorem 1.1.

3.4. Sum over α_1. Recall that the main term in our counting problem is given by (3.7) and (3.8). Applying Möbius inversion to remove the coprimality condition in the sum over α_1 gives

$$\sum_{\substack{\alpha_1 > 0 \\ (\alpha_1, \eta_1\eta_3\eta_4)=1 \\ \overline{\varphi_6}(\alpha_1) \leq 1}} F_1(X_5, \alpha_1 X_6) = \sum_{k_1|\eta_1\eta_3\eta_4} \mu(k_1) \sum_{0 < \alpha_1 \leq 1/k_1 X_5^2 X_6} F_1(X_5, \alpha_1 k_1 X_6).$$

A natural step is to now apply Euler-Maclaurin summation. To simplify our notation in what follows, we shall use Stieltjes integral notation, and also use $\{\cdot\}$ to denote the fractional part of a real number.

Lemma 3.5. *We have*

$$\sum_{0<\alpha_1\le 1/k_1X_5^2X_6} F_1(X_5,\alpha_1k_1X_6)=\frac{1}{k_1X_6}F_2(X_5)+E(\boldsymbol{\eta},k_1,B),$$

where for $u>0$ *we have*

$$\begin{aligned} F_2(u) &= \int_0^{\frac{1}{u^2}} F_1\left(u,v\right)\mathrm{d}v,\\ &= \int_{\{t,v\in\mathbb{R}:0<|t(ut+v^2)|,|uvt|,|uvt+v^3|,|u^2t|,|u^2t+uv^2|,u^2v\le 1\}} \mathrm{d}v\mathrm{d}t, \end{aligned}$$

and

$$E(\boldsymbol{\eta},k_1,B)=\int_0^1\left\{\frac{v}{k_1X_5^2X_6}\right\}\mathrm{d}F_1\left(X_5,\frac{v}{X_5^2}\right)-\left\{\frac{1}{k_1X_5^2X_6}\right\}F_1\left(X_5,\frac{1}{X_5^2}\right).$$

We also have the bounds

$$|E(\boldsymbol{\eta},k_1,B)|\le\frac{6}{\sqrt{X_5}},\qquad F_2(u)\le\frac{4}{\sqrt{u}}. \tag{3.9}$$

Proof. Euler-Maclaurin summation gives

$$\begin{aligned} &\sum_{0<\alpha_1\le 1/k_1X_5^2X_6} F_1(X_5,\alpha_1k_1X_6)\\ &= \int_0^{\frac{1}{k_1X_5^2X_6}} F_1\left(X_5,vk_1X_6\right)\mathrm{d}v-\int_0^{\frac{1}{k_1X_5^2X_6}} F_1\left(X_5,vk_1X_6\right)\mathrm{d}\{v\}. \end{aligned}$$

Changing variables and applying integration by parts gives the first part of the lemma. As for the first upper bound, recall the properties of F_1 given in Lemma 3.4. Then we have

$$|E(\boldsymbol{\eta},k_1,B)|\le 2F_1\left(X_5,\frac{1}{X_5^2}\right)+F_1\left(X_5,0\right)\le\frac{6}{\sqrt{X_5}}.$$

For the second upper bound, note that $|uvt|\le 1$ and $|uvt+v^3|\le 1$ imply that $v\le 2^{1/3}$, hence

$$F_2(u)\le\int_0^{2^{1/3}} F_1\left(u,v\right)\mathrm{d}v\le\frac{4}{\sqrt{u}}.$$

□

3.5. Making a lower order term explicit. The counting problem (3.7) now stands as

$$\begin{aligned} T(B)=\sum_{\boldsymbol{\eta}\in\mathcal{N}}\frac{F_2(X_5)}{\eta_4X_3X_6}\sum_{\substack{k_3\mid\eta_1\\(k_3,\eta_2\eta_3)=1}}\frac{\mu(k_3)}{k_3}\sum_{\substack{k_2\mid\eta_1\eta_2\\(k_2,k_3\eta_4)=1}}\frac{\mu(k_2)}{k_2}\sum_{k_1\mid\eta_1\eta_3\eta_4}\frac{\mu(k_1)}{k_1}\\ +T_1(B) \end{aligned}$$

where $T_1(B)$ denotes the same expression, but with $F_2(X_5)/k_1X_6$ replaced by $E(\boldsymbol{\eta}, k_1, B)$. It turns out that there is a term of order B in T_1, which we shall handle by performing the sum over η_2 explicitly. Taking out the factors which depend on η_2 and recalling the definition of $\mathcal{N}$ in (3.6) and the height conditions (3.2), we see that

$$T_1(B) = B^{1/3} \sum_{\substack{\eta_1^4\eta_3^3\eta_4^3 \leq B \\ (\eta_3,\eta_4)=1}} \eta_1^{2/3} \sum_{k_1 \mid \eta_1\eta_3\eta_4} \mu(k_1) T_2\left(\eta_1, \eta_3, \eta_4, k_1, \widetilde{X_5}\right) \tag{3.10}$$

where we define

$$\widetilde{X_5} = \eta_2 / X_5^{3/2} = \sqrt{B/(\eta_1^4\eta_3^3\eta_4^3)} \tag{3.11}$$

and

$$T_2(\eta_1, \eta_3, \eta_4, k_1, \widetilde{X_5}) = \sum_{\substack{\eta_2 \leq \widetilde{X_5} \\ (\eta_2,\eta_3\eta_4)=1}} \eta_2^{1/3} E(\boldsymbol{\eta}, k_1, B) \sum_{\substack{k_3 \mid \eta_1 \\ (k_3,\eta_2\eta_3)=1}} \frac{\mu(k_3)}{k_3} \sum_{\substack{k_2 \mid \eta_1\eta_2 \\ (k_2,k_3\eta_4)=1}} \frac{\mu(k_2)}{k_2}.$$

This is essentially a sum involving an arithmetic function and a real valued function, so partial summation is the natural method to use. However first we need to unravel this arithmetic function to get a multiplicative function in η_2. To simplify our notation, let

$$\phi^*(a_1, \dots, a_n) = \prod_{p \mid (a_1,\dots,a_n)} \left(1 - \frac{1}{p}\right), \tag{3.12}$$

and we use the shorthand $\phi^*(a) = \phi^*(a,a)$. There will also be unfortunate 2-adic conditions we shall need to take care of, so we define

$$\mathcal{N}_0 = \{(\eta_1, \eta_3, \eta_4) \in \mathbb{N}^3 : 2 \nmid \eta_1 \text{ or } 2 \mid \eta_3\eta_4\}, \quad \mathcal{N}_1 = \mathbb{N}^3 \setminus \mathcal{N}_0.$$

Lemma 3.6. *We have*

$$T_2(\eta_1, \eta_3, \eta_4, k_1, \widetilde{X_5}) = \psi(\eta_1, \eta_3, \eta_4) \sum_{\eta_2 \leq \widetilde{X_5}} \nu_{\eta_1,\eta_3,\eta_4}(\eta_2) \eta_2^{1/3} E(\boldsymbol{\eta}, k_1, B),$$

where

$$\psi(\eta_1, \eta_3, \eta_4) = \phi^*(\eta_1, \eta_3\eta_4) \prod_{\substack{p \mid \eta_1 \\ p \nmid \eta_3\eta_4 \\ p \neq 2}} \left(1 - \frac{2}{p}\right),$$

$$\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}(\eta_2) = \begin{cases} \phi^*(\eta_2) \prod_{\substack{p \mid \eta_1,\eta_2 \\ p \neq 2}} \left(1 - \frac{2}{p}\right)^{-1}, & (\eta_2, \eta_3\eta_4) = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

and if $(\eta_1,\eta_3,\eta_4)\in\mathcal{N}_i$*, then*

$$\nu_{\eta_1,\eta_3,\eta_4}(\eta_2)=\begin{cases}\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}(\eta_2), & 2^i\mid\eta_2,\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. One can verify the following expression

$$\sum_{\substack{k_3\mid\eta_1\\(k_3,\eta_2\eta_3)=1}}\frac{\mu(k_3)}{k_3}\sum_{\substack{k_2\mid\eta_1\eta_2\\(k_2,k_3\eta_4)=1}}\frac{\mu(k_2)}{k_2}=\phi^*(\eta_1,\eta_3\eta_4)\phi^*(\eta_2)\prod_{\substack{p\mid\eta_1\\p\nmid\eta_2\eta_3\eta_4}}\left(1-\frac{2}{p}\right),$$

by checking its value at prime powers and recalling that $(\eta_2,\eta_3)=(\eta_2,\eta_4)=(\eta_3,\eta_4)=1$.

We want this to be written as a multiplication function of η_2 times some other arithmetic function independent of η_2. In order to do this, we need to split up the product over primes, but we can only safely do this if it is non-zero, i.e. if $2\nmid\eta_1$ or $2\mid\eta_2\eta_3\eta_4$. So we have defined $\nu_{\eta_1,\eta_3,\eta_4}$ be zero exactly when $2\mid\eta_1, 2\nmid\eta_2\eta_3\eta_4$ and the coprimality conditions are not satisfied, and simplified its definition in the remaining cases. $\square$

Note that $\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}$ is a multiplicative function of η_2, but $\nu_{\eta_1,\eta_3,\eta_4}$ is not. The next natural step is to find the average order of $\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}$. However to simplify our notation and argument, from now on we shall assume that $(\eta_1,\eta_3,\eta_4)\in\mathcal{N}_0$. The other case is almost exactly the same, the only difference being the condition that η_2 must be even, and it will still contribute a power of B to the main term and give the same error term. With this in mind, we have the following.

Lemma 3.7. *Let* $V(s)$ *be the Dirichlet series associated to* $\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}$ *and* $\widetilde{V}(s)=V(s)/\zeta(s)$*. Then* $\widetilde{V}(s)$ *is a holomorphic and bounded function on* $\mathrm{Re}(s)>0$ *satisfying* $0\le\widetilde{V}(1)\ll 2^{\omega(\eta_1)}$ *and*

$$\sum_{n\le X}\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}(n)=\widetilde{V}(1)X+O(2^{\omega(\eta_1)}X^{\varepsilon}).$$

Proof. For $p\neq 2$, it is easy to see that

$$\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}(p^k)=\begin{cases}\left(1-\frac{1}{p}\right), & p\nmid\eta_1\eta_3\eta_4,\\ \left(\frac{1-1/p}{1-2/p}\right), & p\mid\eta_1, p\nmid\eta_3\eta_4,\\ 0, & \text{otherwise.}\end{cases}$$

Then by considering Euler products, one can check that $V(s)$ is equal to

$$\frac{\zeta(s)V'(s)}{\zeta(s+1)}\prod_{p\mid\eta_1\eta_3\eta_4}\left(1+\frac{1-1/p}{p^s-1}\right)^{-1}\prod_{\substack{p\mid\eta_1,p\neq 2\\p\nmid\eta_3\eta_4}}\left(1+\frac{1-1/p}{(1-2/p)(p^s-1)}\right)$$

where $V'(s)$ is some function corresponding to the Euler factor at the prime 2. So $\widetilde{V}(s)$ has the properties stated in the lemma. Ignoring convergence issues for now, we have

$$\begin{aligned}\sum_{n\le X}\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}(n) &= \sum_{n\le X}((\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)*1)(n)\\ &= \sum_{n\le X}\sum_{d|n}(\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)(d)\\ &= X\sum_{d=1}^{\infty}\frac{(\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)(d)}{d} + O\left(X^{\varepsilon}\sum_{d=1}^{\infty}\frac{|(\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)(d)|}{d^{\varepsilon}}\right)\end{aligned}$$

where we have used the trivial bound $[x]=x+O(x^{\varepsilon})$. To make this rigorous, first note that

$$\lim_{s\to 1}\sum_{d=1}^{\infty}\frac{(\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)(d)}{d^s} = \lim_{s\to 1}V(s)\zeta(s)^{-1} = \widetilde{V}(1).$$

Next, we need to find an expression for the Dirichlet series $V^{+}(s)$ of $|(\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)|$. It is easy to verify that

$$(\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}*\mu)(p^k) = \begin{cases}\widetilde{\nu}_{\eta_1,\eta_3,\eta_4}(p)-1, & k=1,\\ 0, & k>1.\end{cases}$$

By considering Euler products, one can check that $V^{+}(s)$ is a holomorphic and bounded function of s on $\mathrm{Re}(s)>0$ and satisfies

$$\widetilde{V}^{+}(\varepsilon) \ll \prod_{\substack{p|\eta_1\\ p\nmid\eta_3\eta_4}}\left(1+\frac{1}{(p-2)p^{\varepsilon}}\right) \ll 2^{\omega(\eta_1)}$$

on this domain. Thus we are done. □

We shall now perform the summation over η_2, and to do this we will need a slight abuse of notation. Namely, we define

$$\widetilde{E}(t) = E(\eta_1,t,\eta_3,\eta_4,k_1,B),$$

where E is given in Lemma 3.5, and for this we also need to think of X_5 and X_6 as being functions of η_2. Recalling the expression we had for T_2 as given in Lemma 3.6 and using Lemma 3.7, by partial summation we have

$$\begin{aligned}
&\sum_{\eta_2 \le \widetilde{X_5}} \nu_{\eta_1,\eta_3,\eta_4}(\eta_2)\eta_2^{1/3}\widetilde{E}(\eta_2) \\
&= \widetilde{X_5}^{1/3}\widetilde{E}(\widetilde{X_5}) \sum_{\eta_2 \le \widetilde{X_5}} \nu_{\eta_1,\eta_3,\eta_4}(\eta_2) - \int_0^{\widetilde{X_5}} \sum_{\eta_2 \le t} \nu_{\eta_1,\eta_3,\eta_4}(\eta_2)\mathrm{d}\left(t^{1/3}\widetilde{E}(t)\right) \\
&= \widetilde{V}(1)\int_0^{\widetilde{X_5}} t^{1/3}\widetilde{E}(t)\mathrm{d}t + O\left(2^{\omega(\eta_1)}|\widetilde{E}(\widetilde{X_5})|\widetilde{X_5}^{1/3+\varepsilon}\right) \\
&= \widetilde{V}(1)\widetilde{X_5}^{4/3}\int_0^1 u^{1/3}\widetilde{E}(u\widetilde{X_5})\mathrm{d}u + O\left(B^\varepsilon|\widetilde{E}(\widetilde{X_5})|\widetilde{X_5}^{1/3+\varepsilon}\right).
\end{aligned}$$

We now note an interesting feature, namely that $\widetilde{E}(\widetilde{X_5})$ is actually independent of B. Indeed, viewing X_5 and X_6 as functions of η_2, we find that $X_5(u\widetilde{X_5}) = u^{2/3}$ and $X_6(u\widetilde{X_5}) = \eta_1\eta_3\eta_4/u^{2/3}$. Hence $\widetilde{E}(u\widetilde{X_5})$ is independent of B and moreover by (3.9) we deduce that

$$\widetilde{E}(u\widetilde{X_5}) \le \frac{6}{u^{1/3}}.$$

Hence referring back to (3.10), the overall error term contribution to $T_1(B)$ in this case is

$$\begin{aligned}
&\ll B^{1/3+\varepsilon} \sum_{\eta_1^4\eta_3^3\eta_4^3 \le B} \eta_1^{2/3} 2^{\omega(\eta_1\eta_3\eta_4)}\widetilde{X_5}^{1/3+\varepsilon} \\
&\ll B^{1/2+\varepsilon} \sum_{\eta_1^4\eta_3^3\eta_4^3 \le B} \frac{1}{\eta_3^{1/2}\eta_4^{1/2}} \ll B^{3/4+\varepsilon}
\end{aligned}$$

which is satisfactory. Now we can finally make the main term of T_1 explicit, which in the case $(\eta_1,\eta_3,\eta_4) \in \mathcal{N}_0$ is

$$B \sum_{\eta_1,\eta_3,\eta_4 \in \mathcal{N}_0} \widetilde{V}(1)\frac{\psi(\eta_1,\eta_3,\eta_4)}{\eta_1^2\eta_3^2\eta_4^2} \sum_{k_1 \mid \eta_1\eta_3\eta_4} \mu(k_1) \int_0^1 u^{1/3}\widetilde{E}(u\widetilde{X_5})\mathrm{d}u.$$

We know that $\widetilde{E}(u\widetilde{X_5}) \le 6/u^{1/3}$ is actually independent of B, so letting the sum over the η_i go to infinity, we get a main term of the form $\lambda' B$ where $\lambda' \in \mathbb{R}$ is some constant and an error term of the order

$$\ll B^{1+\varepsilon} \sum_{\eta_1^4\eta_3^3\eta_4^3 > B} \frac{1}{\eta_1^2\eta_3^2\eta_4^2} \ll B^{3/4+\varepsilon}$$

which is satisfactory. This was only for the case $(\eta_1,\eta_3,\eta_4) \in \mathcal{N}_0$, however it is clear that the sum over the case where $(\eta_1,\eta_3,\eta_4) \in \mathcal{N}_1$ is almost exactly the same and hence it is omitted. So returning to the original problem

(3.10), we have shown that there exists a constant $\lambda \in \mathbb{R}$ such that

$$T_1(B) = \lambda B + O(B^{3/4+\varepsilon}).$$

3.6. Summation over the $\boldsymbol{\eta_i}$. We now know that

$$T(B) = \sum_{\boldsymbol{\eta} \in \mathcal{N}} \frac{\vartheta(\boldsymbol{\eta}) F_2(X_5)}{\eta_4 X_3 X_6} + \lambda B + O(B^{3/4+\varepsilon}),$$

where X_3, X_5, X_6 and $\mathcal{N}$ are given by (3.2) and (3.6), F_2 is as in Lemma 3.5, and we define

$$\vartheta(\boldsymbol{\eta}) = \sum_{\substack{k_3 \mid \eta_1 \\ (k_3, \eta_2\eta_3)=1}} \frac{\mu(k_3)}{k_3} \sum_{\substack{k_2 \mid \eta_1\eta_2 \\ (k_2, k_3\eta_4)=1}} \frac{\mu(k_2)}{k_2} \sum_{k_1 \mid \eta_1\eta_3\eta_4} \frac{\mu(k_1)}{k_1}$$

when $(\eta_2, \eta_3) = (\eta_2, \eta_4) = (\eta_3, \eta_4) = 1$ and $\vartheta(\boldsymbol{\eta}) = 0$ otherwise. We have already simplified a very similar sum in Lemma 3.6, and using a similar method one can check that

$$\vartheta(\boldsymbol{\eta}) = \phi^*(\eta_1)\phi^*(\eta_2)\phi^*(\eta_3)\phi^*(\eta_4) \prod_{\substack{p \mid \eta_1 \\ p \nmid \eta_2\eta_3\eta_4}} \left(1 - \frac{2}{p}\right) \tag{3.13}$$

when $(\eta_2, \eta_3) = (\eta_2, \eta_4) = (\eta_3, \eta_4) = 1$ and $\vartheta(\boldsymbol{\eta}) = 0$ otherwise. Recalling the height conditions (3.2) it follows that

$$T(B) = B^{2/3} \sum_{n \leq B} \Delta(n) F_2\left(\left(\frac{n}{B}\right)^{1/3}\right) + \lambda B + O(B^{3/4+\varepsilon})$$

where

$$\Delta(n) = \sum_{\eta_1^4\eta_2^2\eta_3^3\eta_4^3 = n} \vartheta(\boldsymbol{\eta}) \left(\frac{\eta_1}{\eta_2}\right)^{1/3}. \tag{3.14}$$

Hence we have the expression

$$N_{U,H}(B) = 2B^{2/3} \sum_{n \leq B} \Delta(n) F_2\left(\left(\frac{n}{B}\right)^{1/3}\right) + \left(\frac{12}{\pi^2} + 2\lambda\right) B + O(B^{3/4+\varepsilon}) \tag{3.15}$$

for the counting function.

3.7. The height zeta function. In this section we shall prove Theorem 1.2 on the height zeta function $Z_{U,H}(s)$ as defined in (1.2). A standard application of Perron's formula [Tit86, Lemma 3.12] gives us an expression for the counting function $N_{U,H}(B)$ in terms of the zeta function via an inverse Mellin transform. Then performing the corresponding Mellin transform tells us that for $\mathrm{Re}(s) \gg 1$ we have

$$Z_{U,H}(s) = s \int_1^\infty u^{-s-1} N_{U,H}(u)\mathrm{d}u \tag{3.16}$$

where $s = \sigma + it$ is a complex variable. Recalling (3.15), we have $Z_{U,H}(s) = Z_1(s) + Z_2(s)$ where

$$\begin{aligned} Z_1(s) &= 2s \int_1^\infty u^{-s-1/3} \sum_{n \le u} \Delta(n) F_2\left(\left(\frac{n}{u}\right)^{1/3}\right) \mathrm{d}u, \\ Z_2(s) &= \frac{12/\pi^2 + 2\lambda}{s-1} + G_2(s), \\ G_2(s) &= s \int_1^\infty u^{-s-1} R(u)\mathrm{d}u, \end{aligned} \tag{3.17}$$

and $R(u)$ is some function such that $R(u) \ll u^{3/4+\varepsilon}$ for all $\varepsilon > 0$. From this it follows that $G_2(s)$ is holomorphic on the half-plane $\mathrm{Re}(s) \ge 3/4 + \varepsilon$, and moreover

$$G_2(s) \ll |s| \int_1^\infty u^{-\sigma-1} u^{3/4+\varepsilon} \mathrm{d}u \ll \frac{|1 + i\frac{t}{\sigma}|}{|\frac{3}{4\sigma} - 1|} \ll 1 + |t|$$

on this domain (note that here we use the common abuse of notation that ε is allowed to take different values simultaneously). In particular $Z_2(s)$ has a meromorphic continuation to the same half plane with a simple pole at $s = 1$ of residue $12/\pi^2 + 2\lambda$.

Now that $Z_2(s)$ is under control, let us turn our attention to $Z_1(s)$. Define Δ's Dirichlet series by $D(s) = \sum_{n=1}^\infty \Delta(n) n^{-s}$. Then by choosing a suitable s to make sure that change of sum and integral are valid, we can simplify Z_1 by

$$\begin{aligned} Z_1(s) &= 2s \sum_{n=1}^\infty \Delta(n) \int_n^\infty u^{-s-1/3} F_2\left(\left(\frac{n}{u}\right)^{1/3}\right) \mathrm{d}u \\ &= 2sD\left(s - \frac{2}{3}\right) \int_1^\infty u^{-s-1/3} F_2\left(\left(\frac{1}{u}\right)^{1/3}\right) \mathrm{d}u \\ &= D\left(s - \frac{2}{3}\right) G_{1,1}(s), \end{aligned} \tag{3.18}$$

where

$$G_{1,1}(s) = 6s \int_0^1 u^{3(s-1)} F_2(u)\mathrm{d}u. \tag{3.19}$$

A standard application of [Tit86, Lemma 4.3] combined with (3.9) now tells us that $G_{1,1}(s)$ is a bounded and holomorphic function on the half plane

$\mathrm{Re}(s) > 5/6$. Recalling the definition of Δ in (3.14), we find that

$$\begin{aligned} D(s+1/3) &= \sum_{\eta_1,\eta_2,\eta_3,\eta_4=1}^{\infty} \frac{\vartheta(\eta_1,\eta_2,\eta_3,\eta_4)}{\eta_1^{4s+1}\eta_2^{2s+1}\eta_3^{3s+1}\eta_4^{3s+1}} \\ &= \prod_p \sum_{k_i=0}^{\infty} \frac{\vartheta(p^{k_1},p^{k_2},p^{k_3},p^{k_4})}{p^{(4s+1)k_1+(2s+1)k_2+(3s+1)(k_3+k_4)}}. \end{aligned}$$

After recalling the expression for $\vartheta(\boldsymbol{\eta})$ in (3.13) and using the fact that $\vartheta(\boldsymbol{\eta}) \neq 0$ if and only if $(n_2,n_3) = (n_2,n_4) = (n_3,n_4) = 1$, this sum greatly simplifies and it is easy to see that $D(s+1/3) = \prod_p D_p(s+1/3)$ where

$$\begin{aligned} D_p(s+1/3) = 1 &+ \left(1-\frac{1}{p}\right)\left(\frac{1}{p^{2s+1}-1} + \frac{2}{p^{3s+1}-1} + \frac{1-2/p}{p^{4s+1}-1}\right) \\ &+ \left(1-\frac{1}{p}\right)^2 \left(\frac{1}{p^{4s+1}-1}\right)\left(\frac{1}{p^{2s+1}-1} + \frac{2}{p^{3s+1}-1}\right). \end{aligned}$$

Recalling the definition of $E_1(s)$ and $E_2(s)$ given by (1.3), we can prove the following.

Lemma 3.8. *We have*

$$D(s+1/3) = E_1(s+1)E_2(s+1)G_{1,2}(s+1)$$

where $G_{1,2}(s+1)$ is holomorphic and bounded on the half plane $\mathcal{H} = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq -1/3+\varepsilon\}$.

Proof. Defining $G_{1,2}(s+1) = D(s+1/3)/(E_1(s+1)E_2(s+1))$, it is clear that it will be enough to show that $G_{1,2}(s+1) = \prod_p (1 + O(1/p^{1+\varepsilon}))$ on $\mathcal{H}$. A routine calculation tells us that

$$\begin{aligned} D_p(s+1/3)\left(1-\frac{1}{p^{4s+1}}\right) &= 1 - \frac{3}{p^{4s+2}} + \frac{2}{p^{4s+3}} \\ &+ \left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{4s+2}}\right)\left(\frac{1}{p^{2s+1}-1} + \frac{2}{p^{3s+1}-1}\right). \end{aligned}$$

Now on $\mathcal{H}$ we have the following estimates

$$\begin{aligned} \frac{1}{p^{4s+2}} &= O\left(\frac{1}{p^{2/3+\varepsilon}}\right), & \frac{1}{p^{2s+1}-1} &= O\left(\frac{1}{p^{1/3+\varepsilon}}\right), \\ \frac{1}{p^{4s+3}} &= O\left(\frac{1}{p^{5/3+\varepsilon}}\right), & \frac{1}{p^{3s+1}-1} &= O\left(\frac{1}{p^{\varepsilon}}\right). \end{aligned}$$

So on $\mathcal{H}$ we have

$$\begin{aligned} D_p(s+1/3)\left(1-\frac{1}{p^{4s+1}}\right) = 1 &- \frac{3}{p^{4s+2}} + \frac{1}{p^{2s+1}-1} \\ &+ \frac{2}{p^{3s+1}-1}\left(1-\frac{1}{p^{4s+2}}\right) + O\left(\frac{1}{p^{1+\varepsilon}}\right). \end{aligned}$$

And finally an easy calculation gives us

$$\begin{aligned}\frac{D_p(s+1/3)}{E_{1,p}(s+1)} =&1-\frac{3}{p^{4s+2}}-\frac{2}{p^{5s+2}}-\frac{1}{p^{6s+2}}+\frac{4}{p^{7s+3}}\\ &+\frac{2}{p^{8s+3}}-\frac{1}{p^{10s+4}}+O\left(\frac{1}{p^{1+\varepsilon}}\right)\end{aligned}$$

where $E_{1,p}(s+1)$ is the corresponding Euler factor of $E_1(s+1)$, thus proving the claim. □

Thus letting

$$G_1(s) = G_{1,1}(s)G_{1,2}(s) \tag{3.20}$$

and combining (3.17),(3.18) and (3.19) with Lemma 3.8, we have proved Theorem 1.2.

3.8. The asymptotic formula. In this section we shall prove Theorem 1.1. Our starting point is the expression for the counting function given by (3.15), which we shall simplify using partial summation and the properties of the Dirichlet series $D(s)$ deduced in Lemma 3.8. In what follows let $M(B) = \sum_{n\le B} \Delta(B)$.

Lemma 3.9. *We have*

$$M(B) = \frac{E_2(1)G_{1,2}(1)}{144}B^{1/3}Q(\log B) + O(B^{7/8-2/3+\varepsilon})$$

where $Q \in \mathbb{R}[x]$ is some monic cubic polynomial.

Proof. Letting $T \in [1, B]$, Perron's formula [Tit86, Theorem 3.12] tells us that for non-integral B we have

$$M(B) = \frac{1}{2\pi i}\int_{1/3+\varepsilon-iT}^{1/3+\varepsilon+iT} D(s)\frac{B^s}{s}ds + O\left(\frac{B^{1+\varepsilon}}{T}\right).$$

Changing variables and using Lemma 3.8 we deduce that

$$M(B) = \frac{1}{2\pi i B^{2/3}}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} E_1(s)E_2(s)G_{1,2}(s)\frac{B^s}{s-2/3}ds + O\left(\frac{B^{1+\varepsilon}}{T}\right).$$

Now let $a \in [7/8, 1)$ and let Γ be the rectangular contour through the points $a-iT, a+iT, 1+\varepsilon-iT, 1+\varepsilon+iT$. Then, as we have already shown, $E_2(s)$ and $G_{1,2}(s)$ are holomorphic and bounded inside this contour, and $E_1(s)$ has a pole of order 4 at $s = 1$. Recalling that $\zeta(s)$ has a simple pole of order 1 at $s = 1$ with residue 1, we have $\lim_{s\to 1} E_1(s)(s-1)^4 = 4\cdot 3^2 \cdot 2 = 72$. Also we have the following Taylor series

$$B^s = B\sum_{n=1}^{\infty}\frac{(\log B)^n(s-1)^n}{n!}$$

which gives us the residue

$$\mathrm{Res}_{s=1}\left\{E_1(s)E_2(s)G_{1,2}(s)\frac{B^s}{s-2/3}\right\} = \frac{E_2(1)G_{1,2}(1)}{144}BQ(\log B)$$

where $Q \in \mathbb{R}[x]$ is some monic cubic polynomial. So letting

$$\mathcal{E}(s) = \sum_{n\leq B}\Delta(n) - \frac{E_2(1)G_{1,2}(1)}{144}B^{1/3}Q(\log B)$$

and applying Cauchy's residue theorem to the contour Γ, we deduce that

$$\mathcal{E}(s) \ll B^{-2/3}\left(\int_{a-iT}^{a+iT} + \int_{a-iT}^{1+\varepsilon-iT} + \int_{1+\varepsilon+iT}^{a+iT}\right)\left|E_1(s)\frac{B^s}{s}\right| ds + \frac{B^{1+\varepsilon}}{T}.$$

From [Ten95, Ch. II.3.4, Theorem 6] we have the bound

$$\zeta(\sigma+it) \ll |t|^{(1-\sigma)/3+\varepsilon}, \qquad \text{if } \sigma\in[1/2,1].$$

Note that our choice of a implies that in the strip $a < \mathrm{Re}(s) < 1$, we have $4\sigma-3, 3\sigma-2, 2\sigma-1 > 1/2$, so $|E_1(s)| \ll |t|^{4(1-\sigma)+\varepsilon}$. Then the contribution from the first horizontal contour is

$$\begin{aligned}\int_{a-iT}^{1+\varepsilon-iT}\left|E_1(s)\frac{B^s}{s}\right| ds &\ll \int_a^{1+\varepsilon} T^{3-4\sigma+\varepsilon}B^\sigma d\sigma\\ &\ll \frac{B^{1+\varepsilon}T^\varepsilon}{T} + B^aT^{3-4a+\varepsilon},\end{aligned}$$

and the same bound is obtained for the other horizontal contour. For the vertical contour we will use well-known estimates for the fourth moment of the zeta function. First note that

$$\int_{a-iT}^{a+iT}\left|E_1(s)\frac{B^s}{s}\right| ds \ll B^a\int_{-T}^{T}\frac{|E_1(a+it)|}{1+|t|}\mathrm{d}t.$$

Now let $0 < U \ll T$ and consider the following dyadic interval

$$\int_U^{2U}\frac{|E_1(a+it)|}{1+|t|}\mathrm{d}t \ll \frac{1}{U}\int_U^{2U}|E_1(a+it)|\mathrm{d}t = \frac{J(U)}{U},$$

say. Hölder's inequality now tells us that

$$J(U) \leq J_4(U)^{1/4}J_3(U)^{1/2}J_2(U)^{1/4}$$

where $J_k(U) = \int_U^{2U}|\zeta(k(a-1)+1+kit)|^4\mathrm{d}t$. Now by convexity [Tit86, Ch. VII.8] and the fact that we have $\int_0^T|\zeta(1/2+it)|^4 \ll T\log^4 T$ by [HB79, Th. 1], we see that for $\sigma\in[1/2,1]$ we have

$$\int_U^{2U}|\zeta(\sigma+it)|^4\mathrm{d}t \ll U^{1+\varepsilon}.$$

Hence we deduce that $J(U) \ll U^{1+\varepsilon}$. Now summing over these dyadic intervals we find

$$\int_0^T \frac{|E_1(a+it)|}{1+|t|}\mathrm{d}t \ll T^{\varepsilon}.$$

The same estimate holds over the interval $[-T, 0]$, and so putting everything together we find an overall error of

$$\mathcal{E}(s) \ll \frac{B^{1+\varepsilon}}{T} + B^{a-2/3+\varepsilon}.$$

Taking $T = B, a = 7/8 + \varepsilon$, the error we obtain is satisfactory for the lemma. □

Using this lemma we can deduce the following.

Lemma 3.10. *We have*

$$\sum_{n \le B} \Delta(n) F_2\left(\left(\frac{n}{B}\right)^{1/3}\right)$$
$$= \frac{E_2(1)G_{1,2}(1)}{144}\left(\int_0^1 F_2(u)\mathrm{d}u\right) B^{1/3} P(\log B) + O(B^{7/8-2/3+\varepsilon})$$

where $P \in \mathbb{R}[x]$ is some monic cubic polynomial.

Proof. For ease of notation let $C = E_2(1)G_{1,2}(1)/144$. Applying partial summation, using (3.9) and Lemma 3.9 we deduce that

$$\sum_{n \le B} \Delta(n) F_2\left(\left(\frac{n}{B}\right)^{1/3}\right)$$
$$= F_2(1)M(B) - \int_1^B M(t)\mathrm{d}F_2\left(\left(\frac{t}{B}\right)^{1/3}\right)$$
$$= C\int_1^B F_2\left(\left(\frac{t}{B}\right)^{1/3}\right)\mathrm{d}\left(t^{1/3}Q(\log t)\right) + O\left(B^{7/8-2/3+\varepsilon}\right)$$

It remains to simplify the main term. In what follows we focus on the leading term of the polynomial Q, the lower order terms being dealt with similarly. After changing variables we deduce that it equals

$$CB^{1/3}\int_{1/B^{1/3}}^1 F_2(u)\mathrm{d}\left(u(\log u^3 B)^3\right)$$
$$= CB^{1/3}(\log B)^3 \int_{B^{-1/3}}^1 F_2(u)\mathrm{d}u + \cdots,$$

where all the implied lower order terms are easily seen to be of the order $O(B^{1/3}(\log B)^2 \int_{B^{-1/3}}^1 F_2(u)u^\varepsilon \mathrm{d}u)$. On using (3.9) to deduce that

$$\int_0^{B^{-1/3}} F_2(u)u^\varepsilon \mathrm{d}u \ll B^{-1/6+\varepsilon},$$

the result follows. □

Hence, combing Lemma 3.10 with (3.15), we deduce the asymptotic formula given in Theorem 1.1. One can also verify the leading constant, after noticing that $\tau_\infty(\widetilde{S}) = 6\int_0^1 F_2(u)\mathrm{d}u$ and using Lemma 3.8 to deduce that $\prod_p \tau_p(\widetilde{S}) = E_2(1)G_{1,2}(1)$.

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