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Steinitz classes of some abelian and nonabelian extensions of even degree

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Résumé. La classe de Steinitz d'une extension de corps de nombres K/k est une classe d'idéaux dans l'anneau des entiers $\mathcal{O}_k$ de k, qui avec le degré $[K:k]$ de l'extension détermine la structure de O_K comme O_k-module. Nous dénotons par $\mathrm{R}_t(k, G)$ l'ensemble des classes qui sont classes de Steinitz d'une extension modérée du k avec groupe de Galois G. Nous dirons que ces classes sont réalisables pour le groupe G; il est conjecturé que l'ensemble des classes réalisables est toujours un groupe.

Dans cet article nous allons développer des idées contenues dans [7] pour obtenir des résultats dans le cas de groupes d'ordre pair. En particulier nous allons montrer que l'étude des classes de Steinitz réalisables pour les groupes abéliens peut être réduit au cas des groupes cycliques d'ordre une puissance de 2.

Abstract. The Steinitz class of a number field extension K/k is an ideal class in the ring of integers $\mathcal{O}_k$ of k, which, together with the degree $[K:k]$ of the extension determines the $\mathcal{O}_k$-module structure of $\mathcal{O}_K$. We denote by $\mathrm{R}_t(k, G)$ the set of classes which are Steinitz classes of a tamely ramified G-extension of k. We will say that those classes are realizable for the group G; it is conjectured that the set of realizable classes is always a group.

In this paper we will develop some of the ideas contained in [7] to obtain some results in the case of groups of even order. In particular we show that to study the realizable Steinitz classes for abelian groups, it is enough to consider the case of cyclic groups of 2-power degree.

1. Introduction

Let K/k be an extension of number fields and let $\mathcal{O}_K$ and $\mathcal{O}_k$ be their rings of integers. By Theorem 1.13 in [18] we know that

$$\mathcal{O}_K \cong \mathcal{O}_k^{[K:k]-1} \oplus I$$

where I is an ideal of $\mathcal{O}_k$. By Theorem 1.14 in [18] the $\mathcal{O}_k$-module structure of $\mathcal{O}_K$ is determined by $[K:k]$ and the ideal class of I. This class is called

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the *Steinitz class* of K/k and we will indicate it by $\mathrm{st}(K/k)$. Let k be a number field and G a finite group, then we define:

$$\mathrm{R}_t(k,G) = \{x \in \mathrm{Cl}(k) : \ \exists K/k \text{ tame Galois}, \mathrm{Gal}(K/k) \cong G, \mathrm{st}(K/k) = x\}.$$

It is conjectured that $\mathrm{R}_t(k,G)$ is always a subgroup of the ideal class group, but up to now no general proof is known. The problem has been studied in a lot of particular situations and the conjecture has actually been verified for a lot of groups, including all the finite abelian groups (this is a consequence of [17]). However from [17] it is not possible to deduce an explicit description of $\mathrm{R}_t(k,G)$. If the order of G is odd, such a description can be found in Lawrence P. Endo's PhD thesis [9] (we will recall this result in Theorem 4.1). Endo has also proved the following theorem concerning the case of cyclic 2-power extensions.

Theorem 1.1. *Suppose* $\mathrm{Gal}(k(\zeta_{2^r})/k)$ *is cyclic. Then*

$$\mathrm{R}_t(k, C(2^r)) = W(k, 2^r),$$

where $W(k, 2^r)$ *is defined in the next section, unless* $k(\zeta_{2^r})/k$ *is unramified and* $\mathrm{Gal}(k(\zeta_{2^r})/k) = \langle -5^{2^t} \rangle$, $0 \le t \le r - 2$, *in which case*

$$\mathrm{R}_t(k, C(2^r)) = W(k, 2^r)^{\frac{1}{2}}.$$

If $\mathrm{Gal}(k(\zeta_{2^r})/k)$ is not cyclic, Endo was unable to provide any significant definitive result, neither the problem has been solved by other authors. In the last section of the present article we will show that we can reduce the study of realizable classes for finite abelian groups to the case of cyclic 2-power extensions, showing that the only difficulties in the abelian case are actually the ones pointed out by Endo.

Also the case of nonabelian groups has been studied by a lot of authors and the set of realizable classes $\mathrm{R}_t(k,G)$ has been described for a lot of groups G, always showing that this set is actually a subgroup of the ideal class group. In this paper we will use the notations and some results from [7], in which the author considers some nonabelian groups obtained by semidirect and direct products of abelian groups, with some restrictive hypotheses. In particular one of the results from [7] is an explicit description of realizable classes for all semidirect products $G = H \rtimes \mathcal{G}$ with $H, \mathcal{G}$ both abelian and of odd coprime order. In the present paper we apply the same techniques from class field theory to the case of groups of even order. In particular we will obtain an explicit description of $\mathrm{R}_t(k,G)$ where G is a semidirect product as above, H being an elementary abelian 2-group and $\mathcal{G}$ an abelian group of odd order (see Proposition 3.5).

This paper is a slightly shortened version of parts of the author's PhD thesis [8]. For earlier results see [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [13], [14], [15], [16], [21], [22] and [23].

2. Preliminary results

We start recalling the following two fundamental results.

Theorem 2.1. *If K/k is a finite tame Galois extension then*

$$d(K/k) = \prod_{\mathfrak{p}} \mathfrak{p}^{(e_{\mathfrak{p}}-1)\frac{[K:k]}{e_{\mathfrak{p}}}},$$

where $d(K/k)$ is the discriminant of the extension K/k and $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$.

Proof. This follows by Propositions 8 and 14 of chapter III of [12]. $\square$

Theorem 2.2. *Assume K is a finite Galois extension of a number field k.*
  (a) *If its Galois group either has odd order or has a noncyclic 2-Sylow subgroup then $d(K/k)$ is the square of an ideal and this ideal represents the Steinitz class of the extension.*
  (b) *If its Galois group is of even order with a cyclic 2-Sylow subgroup and α is any element of k whose square root generates the quadratic subextension of K/k then $d(K/k)/\alpha$ is the square of a fractional ideal and this ideal represents the Steinitz class of the extension.*

Proof. This is a corollary of Theorem I.1.1 in [9]. In particular it is shown in [9] that in case (b) K/k does have exactly one quadratic subextension. $\square$

Further, considering Steinitz classes in towers of extensions, we will need the following proposition.

Proposition 2.1. *Suppose K/E and E/k are number fields extensions. Then*

$$st(K/k) = st(E/k)^{[K:E]} N_{E/k}(st(K/E)).$$

Proof. This is Proposition I.1.2 in [9]. $\square$

We will use a lot of results proved in [7]. In this section we prove a few facts we will use later.

For any integer $n \in \mathbb{N}$ and any prime l, we denote by $n(l)$ the power of l such that $n(l)|n$ and $l \nmid n/n(l)$. We will always use the letter l only for prime numbers, even if not explicitly indicated.

Lemma 2.1. *For any $e|m$ the greatest common divisor, for $l|e$, of the integers $(l-1)\frac{m}{e(l)}$ divides $(e-1)\frac{m}{e}$.*

Proof. This is Lemma 3.16 in [7]. $\qquad\square$

Lemma 2.2. *Let m, n, x be integers. If $x \equiv 1 \pmod{m}$ and any prime q dividing n divides also m then*

$$x^n \equiv 1 \pmod{mn}.$$

Proof. Let q be a prime number dividing m. If $x \equiv 1 \pmod{m}$ then there exists $b \in \mathbb{N}$ such that

$$x^q = (1+bm)^q = 1 + \sum_{i=1}^{q-1}\binom{q}{i}(bm)^i + (bm)^q \equiv 1 \pmod{mq}.$$

Let $n = q_1 \ldots q_r$ be the prime decomposition of n (q_i and q_j with $i \neq j$ are allowed to be equal). Assuming that $x^{q_1 \cdots q_{r-1}} \equiv 1 \pmod{mq_1 \cdots q_{r-1}}$ we can conclude by the above calculation that

$$x^{q_1 \cdots q_r} = (x^{q_1 \cdots q_{r-1}})^{q_r} \equiv 1 \pmod{mq_1 \cdots q_r}$$

and the Lemma is proved by induction on the number r of prime divisors of n. $\qquad\square$

Definition. Let K/k be a finite abelian extension of number fields, let J_K and P_K be the ideal group and the group of principal ideals of K respectively. Then we define $W(k, K)$ in the following equivalent ways (the equivalence is shown in [7], Proposition 2.10):

$W(k, K) = \{x \in J_k/P_k : x \text{ contains infinitely many primes of absolute}$
$\qquad\qquad\qquad \text{degree 1 splitting completely in } K\}$

$W(k, K) = \{x \in J_k/P_k : x \text{ contains a prime splitting completely in } K\}$

$W(k, K) = \mathrm{N}_{K/k}(J_K) \cdot P_k/P_k.$

In the case of cyclotomic extensions we will also use the shorter notation $W(k, m) = W(k, k(\zeta_m))$.

Lemma 2.3. *If $q|n \Rightarrow q|m$ then $W(k, m)^n \subseteq W(k, mn)$.*

Proof. Let $x \in W(k, m)$. By the definition and by Lemma 2.11 of [7], x contains a prime ideal $\mathfrak{p}$, prime to mn and such that $\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{p}) \in P_{\mathbb{Q}}^{\mathfrak{m}}$, where $\mathfrak{m} = m \cdot p_\infty$ and $P_{\mathbb{Q}}^{\mathfrak{m}}$ is the group of all principal ideals generated by an element $a \in \mathbb{N}$ such that $a \equiv 1 \pmod{m}$. Then by Lemma 2.2, $\mathrm{N}_{k/\mathbb{Q}}(\mathfrak{p}^n) \in P_{\mathbb{Q}}^{\mathfrak{n}}$, with $\mathfrak{n} = mn \cdot p_\infty$, and it follows from Lemma 2.12 of [7] that $x^n \in W(k, mn)$. $\qquad\square$

3. Some general results

We recall the following definition, from [7].

Definition. We will call a finite group G of order m *good* if the following properties are verified:

(1) For any number field k, $\mathrm{R}_t(k, G)$ is a group.
(2) For any tame G-extension K/k of number fields there exists an element $\alpha_{K/k} \in k$ such that:
   (a) If G is of even order with a cyclic 2-Sylow subgroup, then a square root of $\alpha_{K/k}$ generates the quadratic subextension of K/k; if G either has odd order or has a noncyclic 2-Sylow subgroup, then $\alpha_{K/k} = 1$.
   (b) For any prime $\mathfrak{p}$, with ramification index $e_\mathfrak{p}$ in K/k, the ideal class of
   $$\left(\mathfrak{p}^{(e_\mathfrak{p}-1)\frac{m}{e_\mathfrak{p}} - v_\mathfrak{p}(\alpha_{K/k})} \right)^{\frac{1}{2}}$$
   is in $\mathrm{R}_t(k, G)$.
(3) For any tame G-extension K/k of number fields, for any prime ideal $\mathfrak{p}$ of k and any rational prime l dividing its ramification index $e_\mathfrak{p}$, the class of the ideal
   $$\mathfrak{p}^{(l-1)\frac{m}{e_\mathfrak{p}(l)}}$$
   is in $\mathrm{R}_t(k, G)$ and, if 2 divides $(l-1)\frac{m}{e_\mathfrak{p}(l)}$, the class of
   $$\mathfrak{p}^{\frac{l-1}{2}\frac{m}{e_\mathfrak{p}(l)}}$$
   is in $\mathrm{R}_t(k, G)$.
(4) G is such that for any number field k, for any class $x \in \mathrm{R}_t(k, G)$ and any integer n, there exists a tame G-extension K with Steinitz class x and such that every non trivial subextension of K/k is ramified at some primes which are unramified in $k(\zeta_n)/k$.

In [7] we prove that abelian groups of odd order are good and, more generally, we construct nonabelian good groups by an iteration of direct and semidirect products.

Let $\mathcal{G}$ be a finite group of order m, let $H = C(n_1) \times \cdots \times C(n_r)$ be an abelian group of order n, with generators $\tau_1, \ldots, \tau_r$ and with $n_{i+1}|n_i$. Let

$$\mu : \mathcal{G} \to \mathrm{Aut}(H)$$

be an action of $\mathcal{G}$ on H and let

$$0 \to H \xrightarrow{\varphi} G \xrightarrow{\psi} \mathcal{G} \to 0$$

be an exact sequence of groups such that the induced action of $\mathcal{G}$ on H is μ. We assume that the group G is determined, up to isomorphism, by

the above exact sequence and by the action μ. The following well-known proposition shows a class of situations in which our assumption is true.

Proposition 3.1 (Schur-Zassenhaus, 1937). *If the order of H is prime to the order of $\mathcal{G}$ then G is a semidirect product:*

$$G \cong H \rtimes_\mu \mathcal{G}.$$

Proof. This is Theorem 7.41 in [20]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are going to study $\mathrm{R}_t(k, G)$, considering in particular the case in which the order of H is even.

We also define

$$\eta_G = \begin{cases} 1 & \text{if } 2 \nmid n \text{ or the 2-Sylow subgroups of } G \text{ are not cyclic} \\ 2 & \text{if } 2|n \text{ and the 2-Sylow subgroups of } G \text{ are cyclic} \end{cases}$$

and in a similar way we define η_H and $\eta_\mathcal{G}$.

We say that (K, k_1, k) is of type μ if k_1/k, K/k_1 and K/k are Galois extensions with Galois groups isomorphic to $\mathcal{G}$, H and G respectively and such that the action of $\mathrm{Gal}(k_1/k) \cong \mathcal{G}$ on $\mathrm{Gal}(K/k_1) \cong H$ is given by μ. For any $\mathcal{G}$-extension k_1 of k we define $\mathrm{R}_t(k_1, k, \mu)$ as the set of those ideal classes of k_1 which are Steinitz classes of a tamely ramified extension K/k_1 for which (K, k_1, k) is of type μ.

We will repeatedly use the following generalization of the Multiplication Lemma on page 22 in [9] by Lawrence P. Endo.

Lemma 3.1. *Let (K_1, k_1, k) and (K_2, k_1, k) be extensions of type μ, such that $(\mathrm{d}(K_1/k_1), \mathrm{d}(K_2/k_1)) = 1$ and K_1/k_1 and K_2/k_1 have no nontrivial unramified subextensions. Then there exists an extension (K, k_1, k) of type μ, such that $K \subseteq K_1 K_2$ and for which*

$$\mathrm{st}(K/k_1) = \mathrm{st}(K_1/k_1)\mathrm{st}(K_2/k_1).$$

Proof. This is Lemma 3.5 in [7]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we recall some further notations introduced in [7].

For any $\tau \in H$ we define

$$\tilde{G}_{k,\mu,\tau} = \left\{ (g_1, g_2) \in \mathcal{G} \times \mathrm{Gal}(k(\zeta_{o(\tau)})/k) : \ \mu(g_1)(\tau) = \tau^{\nu_{k,\tau}(g_2)} \right\},$$

where $g_2(\zeta_{o(\tau)}) = \zeta_{o(\tau)}^{\nu_{k,\tau}(g_2)}$ for any $g_2 \in \mathrm{Gal}(k(\zeta_{o(\tau)})/k)$,

$$G_{k,\mu,\tau} = \left\{ g \in \mathrm{Gal}(k(\zeta_{o(\tau)})/k) : \ \exists g_1 \in \mathcal{G}, \ (g_1, g) \in \tilde{G}_{k,\mu,\tau} \right\}$$

and $E_{k,\mu,\tau}$ as the fixed field of $G_{k,\mu,\tau}$ in $k(\zeta_{o(\tau)})$.

Given a $\mathcal{G}$-extension k_1 of k, there is an injection of $\mathrm{Gal}(k_1(\zeta_{o(\tau)})/k)$ into $\mathcal{G} \times \mathrm{Gal}(k(\zeta_{o(\tau)})/k)$ (defined in the obvious way). We will always identify

$\mathrm{Gal}(k_1(\zeta_{o(\tau)})/k)$ with its image in $\mathcal{G} \times \mathrm{Gal}(k(\zeta_{o(\tau)})/k)$. So we may consider the subgroup

$$\tilde{G}_{k_1/k,\mu,\tau} = \tilde{G}_{k,\mu,\tau} \cap \mathrm{Gal}(k_1(\zeta_{o(\tau)})/k)$$

of $\tilde{G}_{k,\mu,\tau}$. Let $Z_{k_1/k,\mu,\tau}$ be its fixed field in $k_1(\zeta_{o(\tau)})$.

If $k_1 \cap k(\zeta_{o(\tau)}) = k$ then $\mathrm{Gal}(k_1(\zeta_{o(\tau)})/k) \cong \mathcal{G} \times \mathrm{Gal}(k(\zeta_{o(\tau)})/k)$ and hence $\tilde{G}_{k_1/k,\mu,\tau} = \tilde{G}_{k,\mu,\tau}$.

Further for any $\tau \in H$ and any prime l dividing the order $o(\tau)$ of τ we define the element

$$\tau(l) = \tau^{\frac{o(\tau)}{o(\tau)(l)}}$$

in the l-Sylow subgroup $H(l)$ of H.

Now we can state one of the principal theorems proved in [7].

Theorem 3.1. *Let k be a number field and let $\mathcal{G}$ be a good group of order m. Let $H = C(n_1) \times \cdots \times C(n_r)$ be an abelian group of odd order n prime to m and let μ be an action of $\mathcal{G}$ on H. Then*

$$\mathrm{R}_t(k, H \rtimes_\mu \mathcal{G}) = \mathrm{R}_t(k, \mathcal{G})^n \prod_{l|n} \prod_{\tau \in H(l)\setminus\{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}}.$$

Furthermore $G = H \rtimes_\mu \mathcal{G}$ is good.

Proof. This is Theorem 3.19 of [7]. □

In this paper we will obtain some results also for abelian groups H of even order, if all the other hypotheses of the above theorem continue to hold.

Lemma 3.2. *Let k_1 be a tame $\mathcal{G}$-extension of k and let $x \in W(k, k_1(\zeta_{n_1}))$. Then there exist tame extensions of k_1 of type μ, whose Steinitz classes (over k_1) are $\iota(x)^{\eta_H \alpha}$, where*

$$\alpha = \sum_{i=1}^r \frac{n_i - 1}{2} \frac{n}{n_i} + \frac{n_1 - 1}{2} \frac{n}{n_1}.$$

In particular there exist tame extensions of k_1 of type μ with trivial Steinitz class.

We can choose these extensions so that they are unramified at all infinite primes, that the discriminants are prime to a given ideal I of $\mathcal{O}_k$ and that all their proper subextensions are ramified.

Proof. If H is of odd order this is the result of Lemma 3.10 of [7]. Also without this assumption, with the same techniques as in the proof of Lemma 3.10 of [7], we can construct an extension of type μ with discriminant

$$\mathrm{d} = \left(\prod_{i=1}^r \mathfrak{q}_i^{(n_i-1)\frac{n}{n_i}}\right) \mathfrak{q}_{r+1}^{(n_1-1)\frac{n}{n_1}} \mathcal{O}_{k_1},$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_{r+1}$ are prime ideals of k in the class of x.

If H is of odd order or the 2-Sylow subgroup of H is not cyclic then the result follows immediately by Theorem 2.2 (a). If this is not the case then by Theorem 2.2 (b) we obtain extensions whose Steinitz classes have $x^{2\alpha} = x^{\eta_H \alpha}$ as their square. We may construct infinitely many such μ-extensions whose discriminants over k_1 are relatively prime and so, by the pigeonhole principle, there are two of them, which we call K_1 and K_2, with the same Steinitz class y. Then the extension K given by Lemma 3.1 has Steinitz class $y^2 = x^{2\alpha}$.

As in the proof of Lemma 3.10 in [7] we can assume that all the additional conditions are also verified. □

Lemma 3.3. *Let k_1 be a $\mathcal{G}$-extension of k, let H be a group of even order n, let $\tau \in H(2) \setminus \{1\}$ and let x be any class in $W(k, Z_{k_1/k,\mu,\tau})$. Then there exist extensions of k_1 of type μ, whose Steinitz classes (over k_1) are $\iota(x)^{\eta_H \alpha_j}$, where:*

(a)
$$\alpha_1 = \frac{n}{2},$$

(b)
$$\alpha_2 = (o(\tau) - 1)\frac{n}{o(\tau)},$$

Further there exist extensions whose Steinitz classes have $\iota(x)^{2\alpha_j}$ as their square. We can choose these extensions so that they satisfy the additional conditions of Lemma 3.2.

Proof. (a) As in the proof of Lemma 3.11 (a) in [7] we can construct an extension of type μ with discriminant

$$\mathrm{d}(K/k_1)\left((\mathfrak{q}_1\mathfrak{q}_2)^{\frac{n}{2}}\mathcal{O}_{k_1}\right),$$

where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of k in the class of x and (K, k_1, k) is a μ-extension of k_1 with trivial Steinitz class, obtained by Lemma 3.2. Its Steinitz class has $\iota(x)^{2\alpha_1}$ as its square and we conclude as in Lemma 3.2.

(b) In this case, as in the proof of Lemma 3.11 (b) of [7], we obtain an extension of type μ with discriminant

$$\mathrm{d}(K/k_1)\left((\mathfrak{q}_1\mathfrak{q}_2)^{(o(\tau)-1)\frac{n}{o(\tau)}}\mathcal{O}_{k_1}\right).$$

We conclude as in (a).

□

Lemma 3.4. *Let k_1/k be a $\mathcal{G}$-extension of number fields, let $H(2)$ be the 2-Sylow subgroup of H and let $\tilde{H}$ be such that $H = H(2) \times \tilde{H}$. Let $\mu_{\tilde{H}}$ and $\mu_{H(2)}$ be the actions of $\mathcal{G}$ induced by μ on $\tilde{H}$ and $H(2)$ respectively. Then*

$$\mathrm{R}_t(k_1, k, \mu_{\tilde{H}})^{n(2)} \subseteq \mathrm{R}_t(k_1, k, \mu).$$

Proof. Let $x \in \mathrm{R}_t(k_1, k, \mu_{\tilde{H}})$ and let $(\tilde{K}, k_1, k)$ be a $\mu_{\tilde{H}}$-extension of k_1 with Steinitz class x, which is the class of

$$\mathrm{d}(\tilde{K}/k_1)^{\frac{1}{2}}.$$

Let (K, k_1, k) be a $\mu_{H(2)}$-extension of k_1 with trivial Steinitz class and such that K/k_1 and $\tilde{K}/k_1$ are arithmetically disjoint (such an extension exists because of Lemma 3.2). The Steinitz class of K/k_1 is the class of

$$\left(\frac{\mathrm{d}(K/k_1)}{\alpha}\right)^{\frac{1}{2}}$$

for a certain $\alpha \in k_1$. Then the extension $(K\tilde{K}, k_1, k)$ is a μ-extension and its Steinitz class is the class of

$$\left(\frac{\mathrm{d}(K\tilde{K}/k_1)}{\alpha^{\frac{n}{n(2)}}}\right)^{\frac{1}{2}} = \mathrm{d}(\tilde{K}/k_1)^{\frac{n(2)}{2}} \left(\frac{\mathrm{d}(K/k_1)}{\alpha}\right)^{\frac{n}{2n(2)}}$$

which is $x^{n(2)}$. $\qquad\qquad\qquad\square$

At this point we can prove the following proposition.

Proposition 3.2. *Let $l \neq 2$ be a prime dividing n and let $\tau \in H(l) \setminus \{1\}$, then*

$$\iota\left(W\left(k, Z_{k_1/k,\mu,\tau}\right)\right)^{\frac{l-1}{2}\frac{n}{o(\tau)}} \subseteq \mathrm{R}_t(k_1, k, \mu)$$

If $2|n$ then, for any $\tau \in H(2) \setminus \{1\}$,

$$\iota\left(W\left(k, Z_{k_1/k,\mu,\tau}\right)\right)^{\eta_H \frac{n}{o(\tau)}} \subseteq \mathrm{R}_t(k_1, k, \mu)$$

and

$$\iota\left(W\left(k, Z_{k_1/k,\mu,\tau}\right)\right)^{2\frac{n}{o(\tau)}} \subseteq \mathrm{R}_t(k_1, k, \mu)^2.$$

We can choose the corresponding extensions so that they satisfy the additional conditions of Lemma 3.2.

Proof. The first inclusion follows immediately by Proposition 3.12 of [7] and by Lemma 3.4.

Now let us assume that $2|n$, let $\tau \in H(2) \setminus \{1\}$ and let $x \in W(k, Z_{k_1/k,\mu,\tau})$. It follows from Lemma 3.1 and Lemma 3.3 that $\iota(x)^{\eta_H \beta_2}$ is in $\mathrm{R}_t(k_1, k, \mu)$ and $\iota(x)^{2\beta_2}$ is in $\mathrm{R}_t(k_1, k, \mu)^2$, where

$$\beta_2 = \gcd\left(\frac{n}{2}, (o(\tau)-1)\frac{n}{o(\tau)}\right) = \frac{n}{o(\tau)}.$$

So we obtain

$$\iota(x)^{\eta_H \frac{n}{o(\tau)}} \in \mathrm{R}_t(k_1, k, \mu)$$

and

$$\iota(x)^{2\frac{n}{o(\tau)}} \in \mathrm{R}_t(k_1, k, \mu)^2.$$

To conclude we observe that applying Lemma 3.1 to extensions (K_1, k_1, k) and (K_2, k_1, k) of type μ which satisfy the additional conditions of Lemma 3.2, we obtain an extension (K, k_1, k) which still satisfies the same conditions. □

Proposition 3.3. *Let a be a multiple of a positive integer n_1. Let k be a number field and let $\mathcal{G}$ be a finite group of order m such that for any class $x \in \mathrm{R}_t(k, \mathcal{G})$ there exists a tame $\mathcal{G}$-extension k_1 with Steinitz class x and such that every subextension of k_1/k is ramified at some primes which are unramified in $k(\zeta_a)/k$.*

Let $H = C(n_1) \times \cdots \times C(n_r)$, with $n_{i+1}|n_i$, be an abelian group of order n and let μ be an action of $\mathcal{G}$ on H. We assume that the exact sequence

$$0 \to H \xrightarrow{\varphi} G \xrightarrow{\psi} \mathcal{G} \to 0,$$

in which the induced action of $\mathcal{G}$ on H is μ, determines the group G, up to isomorphism. Further we assume that H is of odd order or with noncyclic 2-Sylow subgroup, or that $\mathcal{G}$ is of odd order. Then

$$\mathrm{R}_t(k, H \rtimes_\mu \mathcal{G})$$
$$\supseteq \mathrm{R}_t(k, \mathcal{G})^n \prod_{\substack{l|n \\ l \neq 2}} \prod_{\tau \in H(l) \backslash \{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}} \prod_{\tau \in H(2) \backslash \{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{\eta_G mn}{o(\tau)}}.$$

Further we can choose tame G-extensions K/k with a given Steinitz class (of the ones considered above), such that every nontrivial subextension of K/k is ramified at some primes which are unramified in $k(\zeta_a)/k$.

Proof. Let $x \in \mathrm{R}_t(k, \mathcal{G})$ and let k_1 be a tame $\mathcal{G}$-extension of k, with Steinitz class x, and such that every subextension of k_1/k is ramified at some primes which are unramified in $k(\zeta_a)/k$. Thus, since a is a multiple of n_1, it follows also that $k_1 \cap k(\zeta_{n_1}) = k$.

By Lemma 3.1, Lemma 3.4 in [7], Proposition 3.2 and Proposition 2.1 we obtain

$$\mathrm{R}_t(k, H \rtimes_\mu \mathcal{G})$$
$$\supseteq x^n \prod_{\substack{l|n \\ l \neq 2}} \prod_{\tau \in H(l) \backslash \{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}} \prod_{\tau \in H(2) \backslash \{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{\eta_H mn}{o(\tau)}},$$

from which we obtain the result we wanted to prove, if $\eta_H = \eta_G$.

With our hypotheses $\eta_H \neq \eta_G$ implies that the order of H is odd, i.e. that there does not exist any nontrivial $\tau \in H(2)$. Hence we obtain the desired result also in this case. □

Proposition 3.4. *Let $\tau, \tilde{\tau} \in H(2) \backslash \{1\}$ be elements such that $\tau, \tilde{\tau}, \tau\tilde{\tau}$ are all of the same order. Let k_1 be a $\mathcal{G}$-extension of k. Then*

$$\iota\left(W(k, Z_{k_1/k,\mu,\tau} Z_{k_1/k,\mu,\tilde{\tau}} Z_{k_1/k,\mu,\tau\tilde{\tau}})\right)^{\frac{n}{2o(\tau)}} \subseteq \mathrm{R}_t(k_1, k, \mu).$$

In particular, if $Z_{k_1/k,\mu,\tau} = Z_{k_1/k,\mu,\tau} Z_{k_1/k,\mu,\tilde{\tau}} Z_{k_1/k,\mu,\tau\tilde{\tau}}$ and all the other hypotheses of Proposition 3.3 hold[1], then the factor

$$W(k, E_{k,\mu,\tau})^{\frac{mn}{2o(\tau)}}$$

can be added in the right hand side of the expression of that proposition, giving more realizable classes. The additional condition of Proposition 3.3 is also satisfied.

Proof. Let

$$x \in W(k, Z_{k_1/k,\mu,\tau} Z_{k_1/k,\mu,\tilde{\tau}} Z_{k_1/k,\mu,\tau\tilde{\tau}}).$$

We will use all the notations of the proof of Lemma 3.11 of [7] and we also consider prime ideals $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$ with analogous conditions.

We define $\varphi_i : \kappa_{\mathfrak{Q}_i}^* \to H$, for $i = 1, 2, 3$, posing

$$\varphi_1(g_{\mathfrak{Q}_1}) = \tau,$$
$$\varphi_2(g_{\mathfrak{Q}_2}) = \tilde{\tau},$$

and

$$\varphi_3(g_{\mathfrak{Q}_3}) = (\tau\tilde{\tau})^{-1}.$$

In the usual way we obtain an extension of type μ with discriminant

$$d(K/k_1)\left((\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3)^{(o(\tau)-1)\frac{n}{o(\tau)}}\mathcal{O}_{k_1}\right)$$

and Steinitz class $\iota(x)^{\alpha_3}$ (with the above hypotheses the 2-Sylow subgroup of H can not be cyclic), where

$$\alpha_3 = 3(o(\tau) - 1)\frac{n}{2o(\tau)}.$$

Thus by Lemma 3.1 and Proposition 3.2 we obtain that

$$\iota(W(k, Z_{k_1/k,\mu,\tau} Z_{k_1/k,\mu,\tilde{\tau}} Z_{k_1/k,\mu,\tau\tilde{\tau}}))^{\frac{n}{2o(\tau)}} \subseteq R_t(k_1, k, \mu).$$

To prove that

$$W(k, E_{k,\mu,\tau})^{\frac{mn}{2o(\tau)}}$$

can be added in the expression of Proposition 3.3, it is now enough to use Lemma 3.4 in [7], assuming that $k_1 \cap k(\zeta_{o(\tau)}) = k$ and that every subextension of k_1/k is ramified (we can make these assumptions thanks to the hypotheses of Proposition 3.3). $\qquad\square$

---

[1]If the order of τ is 2 or 4 this condition is obviously verified (possibly, after renaming τ, $\tilde{\tau}$ and $\tau\tilde{\tau}$).

Lemma 3.5. *Let (K, k_1, k) be a tame μ-extension and let $\mathfrak{P}$ be a prime in k_1 ramifying in K/k_1 and let $\mathfrak{p}$ be the corresponding prime in k. Then*

$$x \in W(k, Z_{k_1/k,\mu,\tau}) \subseteq W(k, E_{k,\mu,\tau}) \subseteq \bigcap_{l \mid e_\mathfrak{P}} W(k, E_{k,\mu,\tau(l)})$$

where x is the class of $\mathfrak{p}$ and τ generates $([U_\mathfrak{P}], K/k_1)$.

Proof. This is Lemma 3.14 in [7][2]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.6. *Let $\mathcal{G}$ be a good group of order m, let H be an abelian group of order n prime to m, with trivial or noncyclic 2-Sylow subgroup, and let μ be an action of $\mathcal{G}$ on H. Suppose (K, k_1, k) is tamely ramified and of type μ. Let $e_\mathfrak{p}$ be the ramification index of a prime $\mathfrak{p}$ in k_1/k and $e_\mathfrak{P}$ be the ramification index of a prime $\mathfrak{P}$ of k_1 dividing $\mathfrak{p}$ in K/k_1. Then the class of*

$$\left(\mathfrak{p}^{(e_\mathfrak{p} e_\mathfrak{P} - 1)\frac{mn}{e_\mathfrak{p} e_\mathfrak{P}} - v_\mathfrak{p}(\alpha_{k_1/k}^n)} \right)^{\frac{1}{2}}$$

is in

$$\mathrm{R}_t(k, \mathcal{G})^n \cdot \prod_{l \mid n} \prod_{\tau \in H(l) \setminus \{1\}} W(k, E_{k,\mu,\tau})^{\frac{l-1}{2}\frac{mn}{o(\tau)}}.$$

Proof. If the order of H is odd, then this is Lemma 3.17 in [7]. So we can assume that the order of H is even and the order of $\mathcal{G}$ is odd. By our assumption $\alpha_{k_1/k} = 1$ and, exactly as in the proof of Lemma 3.17 in [7], we obtain

$$\mathfrak{p}^{(e_\mathfrak{p} e_\mathfrak{P} - 1)\frac{mn}{e_\mathfrak{p} e_\mathfrak{P}} - v_\mathfrak{p}(\alpha_{k_1/k}^n)} = \mathfrak{p}^{(e_\mathfrak{p} e_\mathfrak{P} - 1)\frac{mn}{e_\mathfrak{p} e_\mathfrak{P}}} = \mathfrak{p}^{a_\mathfrak{p}(e_\mathfrak{p} - 1)\frac{mn}{e_\mathfrak{p}}} \prod_{l \mid e_\mathfrak{P}} \mathfrak{p}^{b_{\mathfrak{p},l}(l-1)\frac{mn}{e_\mathfrak{P}(l)}}.$$

By the hypothesis that $\mathcal{G}$ is good we have that the class of the ideal

$$\left(\mathfrak{p}^{(e_\mathfrak{p} - 1)\frac{m}{e_\mathfrak{p}}} \right)^{\frac{1}{2}} = \left(\mathfrak{p}^{(e_\mathfrak{p} - 1)\frac{m}{e_\mathfrak{p}} - v_\mathfrak{p}(\alpha_{k_1/k})} \right)^{\frac{1}{2}}$$

is in $\mathrm{R}_t(k, \mathcal{G})$. By Lemma 3.5 if $l \mid e_\mathfrak{P}$ the class of $\mathfrak{p}$ belongs to $W(k, E_{k,\mu,\tau(l)})$, where τ generates $([U_\mathfrak{P}], K/k_1)$ and, in particular, $\tau(l) \in H(l) \setminus \{1\}$. Further for any prime l dividing $e_\mathfrak{P}$, $(l-1)\frac{mn}{e_\mathfrak{P}(l)}$ is even (in the case $l = 2$ this is due to the fact that the inertia group at $\mathfrak{P}$ must be cyclic, while the 2-Sylow subgroup of H is not), i.e. $\frac{l-1}{2}\frac{mn}{e_\mathfrak{P}(l)} \in \mathbb{N}$. Hence the class of

$$\left(\mathfrak{p}^{(e_\mathfrak{p} e_\mathfrak{P} - 1)\frac{mn}{e_\mathfrak{p} e_\mathfrak{P}} - v_\mathfrak{p}(\alpha_{k_1/k}^n)} \right)^{\frac{1}{2}} = \left(\mathfrak{p}^{(e_\mathfrak{p} - 1)\frac{m}{e_\mathfrak{p}}} \right)^{\frac{n a_\mathfrak{p}}{2}} \prod_{l \mid e_\mathfrak{P}} \mathfrak{p}^{b_{\mathfrak{p},l}\frac{l-1}{2}\frac{mn}{e_\mathfrak{P}(l)}}$$

---

[2]In this case we made no assumption concerning the parity of the order of H, so the result holds also in the present setting.

is actually in

$$\mathrm{R}_t(k,\mathcal{G})^n \cdot \prod_{l|n} \prod_{\tau \in H(l)\backslash\{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}}.$$

$\square$

Lemma 3.7. *Under the same hypotheses as in the preceding lemma, if $l|e_{\mathfrak{p}}e_{\mathfrak{P}}$, the class of*

$$\mathfrak{p}^{(l-1)\frac{mn}{e_{\mathfrak{p}}(l)e_{\mathfrak{P}}(l)}}$$

is in

$$\mathrm{R}_t(k,\mathcal{G})^n \prod_{\tau \in H(l)\backslash\{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}}.$$

and, if 2 divides $(l-1)\frac{mn}{e_{\mathfrak{p}}(l)e_{\mathfrak{P}}(l)}$, the class of

$$\mathfrak{p}^{\frac{l-1}{2}\frac{mn}{e_{\mathfrak{p}}(l)e_{\mathfrak{P}}(l)}}$$

is in

$$\mathrm{R}_t(k,\mathcal{G})^n \prod_{\tau \in H(l)\backslash\{1\}} W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}}.$$

Proof. As in the previous lemma we can assume that the order of H is even, since the odd case has been proved in Lemma 3.18 of [7]. Thus if l is a prime dividing $e_{\mathfrak{p}}$, then l is odd and so 2 divides $(l-1)\frac{m}{e_{\mathfrak{p}}(l)}$ and the class of

$$\mathfrak{p}^{\frac{l-1}{2}\frac{m}{e_{\mathfrak{p}}(l)}}$$

is in $\mathrm{R}_t(k,\mathcal{G})$, by the hypothesis that $\mathcal{G}$ is good. We conclude that the class of

$$\mathfrak{p}^{\frac{l-1}{2}\frac{mn}{e_{\mathfrak{p}}(l)e_{\mathfrak{P}}(l)}} = \mathfrak{p}^{\frac{l-1}{2}\frac{mn}{e_{\mathfrak{p}}(l)}}$$

is in $\mathrm{R}_t(k,\mathcal{G})^n$.

If l divides $e_{\mathfrak{P}}$, then $(l-1)\frac{mn}{e_{\mathfrak{p}}(l)e_{\mathfrak{P}}(l)}$ is even (by hypothesis the 2-Sylow subgroup of H is not cyclic and thus $\frac{n}{e_{\mathfrak{P}}(2)}$ is even). We conclude by Lemma 3.5 that the class of

$$\mathfrak{p}^{\frac{l-1}{2}\frac{mn}{e_{\mathfrak{p}}(l)e_{\mathfrak{P}}(l)}} = \mathfrak{p}^{\frac{l-1}{2}\frac{mn}{e_{\mathfrak{P}}(l)}}$$

is in $W\left(k, E_{k,\mu,\tau}\right)^{\frac{l-1}{2}\frac{mn}{o(\tau)}}$ for some $\tau \in H(l) \backslash \{1\}$. $\square$

Proposition 3.5. *Let k be a number field and let $\mathcal{G}$ be a good group of odd order.*

Let $n > 1$ be an integer, let $H = C(2)^{(n)} = C(2) \times \cdots \times C(2)$ and let μ be an action of $\mathcal{G}$ on H. Then

$$\mathrm{R}_t(k, H \rtimes_\mu \mathcal{G}) = \mathrm{R}_t(k,\mathcal{G})^{2^n} \mathrm{Cl}(k)^{m2^{n-2}}.$$

Further $G = H \rtimes_\mu \mathcal{G}$ is good.

Proof. Clearly $E_{k,\mu,\tau} = k$, i.e. $W(k, E_{k,\mu,\tau}) = \mathrm{Cl}(k)$ for any $\tau \in H(2) = H$. Thus, by Propositions 3.3 and 3.4,

$$\mathrm{R}_t(k, H \rtimes_\mu \mathcal{G}) \supseteq \mathrm{R}_t(k, \mathcal{G})^{2^n} \mathrm{Cl}(k)^{m 2^{n-2}}.$$

The opposite inclusion comes from Theorems 2.1 and 2.2 and from Lemma 3.6. So we obtain an equality and, in particular, this gives the first property of good groups. The other properties follow now respectively from Lemmas 3.6 and 3.7 and from Propositions 3.3 and 3.4.   □

If $\mathcal{G}$ is cyclic of order $2^n - 1$ and the representation μ is faithful, then the above proposition is one of the results proved by Nigel P. Byott, Cornelius Greither and Bouchaïb Sodaïgui in [3].

Example. The group A_4, which is isomorphic to a semidirect product of the form $(C(2) \times C(2)) \rtimes_\mu C(3)$, is good by Proposition 3.5. By a classical result about realizable Steinitz classes for abelian extensions of odd order, which we will recall in the next section (Theorem 4.1), we know that $\mathrm{R}_t(k, C(3)) = W(k, 3)$. Further by the third characterization of $W(k, 3) = W(k, k(\zeta_3))$ it is clear that $\mathrm{Cl}(k)^2 \subseteq W(k, 3)$, since $k(\zeta_3)/k$ is an extension of degree 2. Now we can calculate the realizable classes for A_4:

$$\mathrm{R}_t(k, A_4) = W(k, 3)^4 \mathrm{Cl}(k)^3 \supseteq \mathrm{Cl}(k)^8 \mathrm{Cl}(k)^3 = \mathrm{Cl}(k)$$

and hence

$$\mathrm{R}_t(k, A_4) = \mathrm{Cl}(k).$$

This result has been obtained by Marjory Godin and Bouchaïb Sodaïgui in [10].

4. Abelian extensions of even degree

We will conclude this paper considering the case of abelian groups of even order. To this aim we will use the preceding results and notations, with the assumption that $\mathcal{G}$ is the trivial group, i.e. that $G = H$.

It follows by the paper [17] of Leon McCulloh that $\mathrm{R}_t(k, G)$ is a group for any finite abelian group G. However, this result does not yield an explicit description of $\mathrm{R}_t(k, G)$, which is known only if the order of G is odd and in a few other cases.

Theorem 4.1. *Let k be a number field and let $G = C(n_1) \times \cdots \times C(n_r)$ with $n_{i+1} | n_i$ be an abelian group of odd order n. Then*

$$\mathrm{R}_t(k, G) = \prod_{l | n} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}.$$

Proof. This result was proved by Endo in his PhD thesis [9] (in a slightly different form), but it is also a particular case of Theorem 3.19 of [7], using also Lemma 2.3.   □

Further we will also use the following proposition proved by Endo.

Proposition 4.1. *For any number field k*

$$W(k, 2^n) \subseteq \mathrm{R}_t(k, C(2^n)).$$

Proof. This is Proposition II.2.4 in [9]. □

The equality of Theorem 4.1 is not true in general for abelian groups of even order. Nevertheless it is not difficult to prove one inclusion.

Proposition 4.2. *Let k be a number field and let $G = C(n_1) \times \cdots \times C(n_r)$ with $n_{i+1}|n_i$ be an abelian group of order n. Then*

$$\mathrm{R}_t(k, G) \subseteq \prod_{l|n} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}.$$

Proof. Let K/k be a tamely ramified extension of number fields with Galois group G. By Theorem 2.1 and by Lemma 2.1 there exist $b_{e_\mathfrak{p},l} \in \mathbb{Z}$ such that

$$\mathrm{d}(K/k) = \prod_{e_\mathfrak{p} \neq 1} \mathfrak{p}^{(e_\mathfrak{p}-1)\frac{n}{e_\mathfrak{p}}} = \prod_{e_\mathfrak{p} \neq 1} \prod_{l|e_\mathfrak{p}} \mathfrak{p}^{b_{e_\mathfrak{p},l}(l-1)\frac{n}{e_\mathfrak{p}(l)}} = \prod_{l|n} \prod_{e_\mathfrak{p}(l) \neq 1} \mathfrak{p}^{b_{e_\mathfrak{p},l}(l-1)\frac{n}{e_\mathfrak{p}(l)}}.$$

Since K/k is tame, the ramification index $e_\mathfrak{p}$ of a prime $\mathfrak{p}$ in K/k divides n_1. Thus, defining

$$J_l = \prod_{e_\mathfrak{p}(l) \neq 1} \mathfrak{p}^{b_{e_\mathfrak{p},l}\frac{n_1(l)}{e_\mathfrak{p}(l)}},$$

we obtain

$$\mathrm{d}(K/k) = \prod_{l|n} J_l^{(l-1)\frac{n}{n_1(l)}}$$

and by Lemma 2.13 of [7] and Lemma 2.3 the class of the ideal J_l belongs to $W(k, n_1(l))$. We easily conclude by Theorem 2.2. □

Proposition 4.3. *Let $l \neq 2$ be a prime dividing n, then*

$$W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}} \subseteq \mathrm{R}_t(k, G).$$

If $2|n$ then

$$W(k, n_1(2))^{\eta_G \frac{n}{n_1(2)}} \subseteq \mathrm{R}_t(k, G)$$

and

$$W(k, n_1(2))^{2\frac{n}{n_1(2)}} \subseteq \mathrm{R}_t(k, G)^2.$$

We can choose the corresponding extensions so that they satisfy the additional conditions of Lemma 3.2.

Proof. This is a particular case of Proposition 3.2. □

Lemma 4.1. *If $2|n$ and $n_2(2) \neq 1$ (in this case $\eta_G = 1$) then*

$$W(k, n_2(2))^{\frac{n}{2n_2(2)}} \subseteq \mathrm{R}_t(k, G).$$

We can choose the corresponding extensions so that they satisfy the additional conditions of Lemma 3.2.

Proof. This is a particular case of Proposition 3.4. □

Using this lemma we can easily prove a first interesting proposition, which gives a characterization of realizable classes in a particular situation.

Proposition 4.4. *Let k be a number field, let $G = C(n_1) \times \cdots \times C(n_r)$, with $n_{i+1}|n_i$, be an abelian group of order n. If $2|n$ and $n_1(2) = n_2(2)$, then*

$$\mathrm{R}_t(k, G) = \prod_{l|n} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}$$

and the group G is good.

Proof. One inclusion is Proposition 4.2.

The other inclusion follows by Proposition 4.3 and Lemma 4.1, using Lemma 3.1.

Thus, in particular, the first and the fourth property of good groups are verified. Now let K/k be a tamely ramified extension of number fields with Galois group G. By Lemma 2.1 there exist $b_{e_\mathfrak{p}, l} \in \mathbb{Z}$ such that

$$\mathfrak{p}^{(e_\mathfrak{p}-1)\frac{n}{e_\mathfrak{p}}} = \prod_{l|e_\mathfrak{p}} \mathfrak{p}^{b_{e_\mathfrak{p},l}(l-1)\frac{n}{e_\mathfrak{p}(l)}} = \prod_{l|e_\mathfrak{p}} \mathfrak{p}^{\frac{n_1(l)}{e_\mathfrak{p}(l)}b_{e_\mathfrak{p},l}(l-1)\frac{n}{n_1(l)}}.$$

By Lemma 2.13 of [7] and Lemma 2.3, the class of the ideal $\mathfrak{p}^{\frac{n_1(l)}{e_\mathfrak{p}(l)}}$ is contained in $W(k, n_1(l))$. Since $(l-1)\frac{n}{n_1(l)}$ is even for any prime l dividing $e_\mathfrak{p}$, we easily conclude that also the second and the third property of good groups hold for G. □

Corollary 4.1. *Under the assumptions of Proposition 4.4, $\mathrm{R}_t(k, G)$ is a subgroup of the ideal class group of the number field k.*

The above corollary follows also from [17], which is however much less explicit than Proposition 4.4. The above description of $\mathrm{R}_t(k, G)$ generalizes the result concerning the group $G = C(2) \times C(2)$ in [21].

Corollary 4.2. *If n is odd then D_{2n} is a good group, it is isomorphic to a semidirect product of the form*

$$C(n) \rtimes_\mu (C(2) \times C(2))$$

and

$$\mathrm{R}_t(k, D_{2n}) = \mathrm{Cl}(k)^n \cdot \prod_{l|n} \prod_{\tau \in H(l)\backslash\{1\}} W(k, E_{k,\mu,\tau})^{(l-1)\frac{2n}{o(\tau)}}.$$

Proof. It is easy to see that

$$D_{2n} \cong D_n \times C(2) \cong C(n) \rtimes_\mu (C(2) \times C(2)),$$

for a certain action $\mu : C(2) \times C(2) \to \mathrm{Aut}(C(n))$. By the above proposition $C(2) \times C(2)$ is good and

$$\mathrm{R}_t(k, C(2) \times C(2)) = \mathrm{Cl}(k).$$

Thus we conclude by Theorem 3.1 that D_{2n} is good and we obtain the desired expression for $\mathrm{R}_t(k, D_{2n})$. $\hfill\square$

An analogous result for a dihedral group D_n, where n is an odd integer, is given in Theorem 3.26 of [7].

Lemma 4.2. *Let k be a number field and let $G = C(n_1) \times \cdots \times C(n_r)$, with $n_{i+1}|n_i$, be an abelian group of even order n. Then*

$$\mathrm{R}_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}} \subseteq \mathrm{R}_t(k, G),$$

where $n_1(2)$ is the maximal power of 2 dividing n_1.

Proof. By hypothesis $G = C(n_1(2)) \times \tilde{G}$, where $\tilde{G}$ is an abelian group. Let $x \in \mathrm{R}_t(k, C(n_1(2)))$ and let L be a tame $C(n_1(2))$-extension whose Steinitz class is x. Because of Lemma 3.2 there exists a tame $\tilde{G}$-extension K of k whose discriminant is prime to that of L over k, with trivial Steinitz class and with no unramified subextensions. The composition of the two extensions is a G-extension and its discriminant is

$$\mathrm{d}(L/k)^{\frac{n}{n_1(2)}} \mathrm{d}(K/k)^{n_1(2)}.$$

If the 2-Sylow subgroup of G is not cyclic then the Steinitz class is the class of

$$\mathrm{d}(KL/k)^{\frac{1}{2}} = \mathrm{d}(L/k)^{\frac{n}{2n_1(2)}} \mathrm{d}(K/k)^{n_1(2)/2},$$

that is

$$(x^2)^{\frac{n}{2n_1(2)}} = x^{\frac{n}{n_1(2)}}.$$

Now we have to consider the case in which the 2-Sylow subgroup of G is cyclic. The subextension $k(\sqrt{\alpha})$ of L of degree 2 over k is also a subextension of KL. We have $k(\sqrt{\alpha}) = k\left(\sqrt{\alpha^{\frac{n}{n_1(2)}}}\right)$ (the exponent $\frac{n}{n_1(2)}$ is odd) and so the Steinitz class of KL/k is the class of the square root of

$$\frac{\mathrm{d}(KL/k)}{\alpha^{\frac{n}{n_1(2)}}} = \left(\frac{\mathrm{d}(L/k)}{\alpha}\right)^{\frac{n}{n_1(2)}} \mathrm{d}(K/k)^{n_1(2)},$$

that is exactly $x^{\frac{n}{n_1(2)}}$. $\hfill\square$

Lemma 4.3. *Let k be a number field and $G = C(n_1) \times \cdots \times C(n_r)$ an abelian group of even order n, with $n_{i+1}|n_i$ and $n_2(2) \neq 1$. Then*

$$\mathrm{R}_t(k, G) \subseteq \mathrm{R}_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}} \cdot W\left(k, n_2(2)\right)^{\frac{n}{2n_2(2)}} \cdot \prod_{\substack{l|n \\ l \neq 2}} W\left(k, n_1(l)\right)^{\frac{l-1}{2}\frac{n}{n_1(l)}}.$$

Proof. Let K/k be a G-Galois extension whose Steinitz class is $x \in \mathrm{R}_t(k, G)$ and let L be a subextension of K/k whose Galois group over k is the first component of the 2-Sylow subgroup $C(n_1(2)) \times \cdots \times C(n_r(2))$ of G. By Theorem 2.3 of [7] and Proposition II.3.3 in [19]

$$e_{\mathfrak{p},K} = e_{\mathfrak{p},K}(2)e'_{\mathfrak{p},K} = \#([U_\mathfrak{p}], K/k);$$
$$e_{\mathfrak{p},L} = e_{\mathfrak{p},L}(2) = \#([U_\mathfrak{p}], L/k) = \#([U_\mathfrak{p}], K/k)|_L,$$

where $e_{\mathfrak{p},L}$ and $e_{\mathfrak{p},K}$ are the ramification indices of $\mathfrak{p}$ in L and K respectively and $e'_{\mathfrak{p},K}$ is odd. By Theorem 2.1 and Theorem 2.2, x is the class of

$$\prod_{\mathfrak{p}} \mathfrak{p}^{\frac{e_{\mathfrak{p},K}-1}{2}\frac{n}{e_{\mathfrak{p},K}}}.$$

The class x_1 of the ideal

$$\prod_{\mathfrak{p}} \mathfrak{p}^{\frac{e_{\mathfrak{p},L}-1}{2}\frac{n}{e_{\mathfrak{p},L}}}$$

is the $n/n_1(2)$-th power of the Steinitz class of L/k and thus

$$x_1 \in \mathrm{R}_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}}.$$

Since $e_{\mathfrak{p},L}|e_{\mathfrak{p},K}(2)$ and $2e_{\mathfrak{p},K}(2)|n$ we can define x_2 as the class of

$$\prod_{\mathfrak{p}} \mathfrak{p}^{\left(\frac{e_{\mathfrak{p},K}(2)}{e_{\mathfrak{p},L}}-1\right)\frac{n}{2e_{\mathfrak{p},K}(2)}} = \prod_{\mathfrak{p}} \mathfrak{p}^{\left(\frac{e_{\mathfrak{p},K}(2)}{e_{\mathfrak{p},L}}-1\right)\frac{n_2(2)}{e_{\mathfrak{p},K}(2)}\frac{n}{2n_2(2)}}.$$

The only primes for which we obtain a nontrivial contribution are those for which $e_{\mathfrak{p},L} < e_{\mathfrak{p},K}(2)$ and for those we must have $e_{\mathfrak{p},K}(2)|n_2(2)$ (since $e_{\mathfrak{p},K}(2)$ must then be the order of a cyclic subgroup of $C(n_2(2)) \times \cdots \times C(n_r(2))$) and thus, recalling Lemma 2.13 of [7] and Lemma 2.3,

$$x_2 \in W(k, n_2(2))^{\frac{n}{2n_2(2)}}.$$

Let x_3 be the class of

$$\prod_{\mathfrak{p}} \mathfrak{p}^{\frac{e'_{\mathfrak{p},K}-1}{2}\frac{n}{e_{\mathfrak{p},K}}} = \prod_{\mathfrak{p}} \mathfrak{p}^{a_\mathfrak{p}\frac{e'_{\mathfrak{p},K}-1}{2}\frac{n}{e'_{\mathfrak{p},K}}} \prod_{\mathfrak{p}} \mathfrak{p}^{b_\mathfrak{p}\frac{e'_{\mathfrak{p},K}-1}{2}\frac{n}{e_{\mathfrak{p},K}(2)}},$$

where $a_\mathfrak{p}$ and $b_\mathfrak{p}$ are integers such that

$$\frac{n}{e_{\mathfrak{p},K}} = a_\mathfrak{p}\frac{n}{e'_{\mathfrak{p},K}} + b_\mathfrak{p}\frac{n}{e_{\mathfrak{p},K}(2)}.$$

By Lemma 2.1 there exist $b_{\mathfrak{p},l} \in \mathbb{Z}$ such that

$$\prod_{\mathfrak{p}} \mathfrak{p}^{a_{\mathfrak{p}} \frac{e'_{\mathfrak{p},K}-1}{2} \frac{n}{e'_{\mathfrak{p},K}}} = \prod_{\substack{l|n \\ l \neq 2}} \prod_{\mathfrak{p}} \mathfrak{p}^{b_{\mathfrak{p},l} \frac{n_1(l)}{e'_{\mathfrak{p},K}(l)} \frac{l-1}{2} \frac{n}{n_1(l)}}$$

and thus by Lemma 2.13 of [7] and Lemma 2.3 the class of this ideal is in

$$\prod_{\substack{l|n \\ l \neq 2}} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}.$$

By the same lemmas the class of

$$\prod_{\mathfrak{p}} \mathfrak{p}^{b_{\mathfrak{p}} \frac{e'_{\mathfrak{p},K}-1}{2} \frac{n}{e_{\mathfrak{p},K}(2)}}$$

is in

$$W(k, n_1(2))^{\frac{n}{n_1(2)}},$$

which is contained in

$$\mathrm{R}_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}}$$

by Proposition 4.1. Hence

$$x_3 \in \prod_{\substack{l|n \\ l \neq 2}} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}} \mathrm{R}_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}}.$$

By an easy calculation

$$\frac{e_{\mathfrak{p},K}-1}{2} \frac{n}{e_{\mathfrak{p},K}} = \frac{e_{\mathfrak{p},L}-1}{2} \frac{n}{e_{\mathfrak{p},L}} + \left(\frac{e_{\mathfrak{p},K}(2)}{e_{\mathfrak{p},L}} - 1 \right) \frac{n}{2e_{\mathfrak{p},K}(2)} + \frac{e'_{\mathfrak{p},K}-1}{2} \frac{n}{e_{\mathfrak{p},K}}$$

and we conclude that $x = x_1 x_2 x_3$, obtaining the desired inclusion. $\qquad\square$

Theorem 4.2. *Let k be a number field, let $G = C(n_1) \times \cdots \times C(n_r)$, with $n_{i+1}|n_i$, be an abelian group of order n. If $2|n$ and $n_2(2) \neq 1$ then*

$$\mathrm{R}_t(k, G) = \mathrm{R}_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}} \cdot W(k, n_2(2))^{\frac{n}{2n_2(2)}} \cdot \prod_{\substack{l|n \\ l \neq 2}} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}.$$

Proof. $\subseteq$ This is Lemma 4.3.

$\supseteq$ This follows by Proposition 4.3, by Lemma 4.1 and by Lemma 4.2, using Lemma 3.1.

$\qquad\square$

Remark. The only unknown term in the expression for $\mathrm{R}_t(k, G)$ in the above theorem is $\mathrm{R}_t(k, C(n_1(2)))$. But we really need to determine only its square, because it appears with an even exponent. This simplifies the

problem, because this allows us to consider directly the discriminants of the extensions.

In the second part of the section we consider the case in which the 2-Sylow subgroup of G is cyclic, i.e. $2|n$ and $n_2(2) = 1$.

Lemma 4.4. *If the 2-Sylow subgroup of G is cyclic, i.e. $2|n$ and $n_2(2) = 1$, then*

$$R_t(k, G) \subseteq R_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}} \cdot \prod_{\substack{l|n \\ l \neq 2}} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}.$$

Proof. Let K/k be a G-Galois extension whose Steinitz class is $x \in R_t(k, G)$ and let L be the subextension of K/k whose Galois group over k is the 2-Sylow subgroup $C(n_1(2))$ of G. By Theorem 2.3 of [7] and Proposition II.3.3 in [19]

$$e_{\mathfrak{p},K} = e_{\mathfrak{p},K}(2)e'_{\mathfrak{p},K} = \#([U_\mathfrak{p}], K/k);$$
$$e_{\mathfrak{p},L} = e_{\mathfrak{p},L}(2) = \#([U_\mathfrak{p}], L/k) = \#([U_\mathfrak{p}], K/k)|_L,$$

where $e_{\mathfrak{p},L}$ and $e_{\mathfrak{p},K}$ are the ramification indices of $\mathfrak{p}$ in L and K respectively, $e'_{\mathfrak{p},K}$ is odd and $e_{\mathfrak{p},K}(2) = e_{\mathfrak{p},L}(2)$. Let $\alpha \in k$ be such that $k \subsetneq k(\sqrt{\alpha}) \subseteq L$.

Since $k(\sqrt{\alpha}) = k\left(\sqrt{\alpha^{n/n_1(2)}}\right)$, by Theorem 2.1 and Theorem 2.2, x is the class of

$$\left(\frac{\prod_\mathfrak{p} \mathfrak{p}^{(e_{\mathfrak{p},K}-1)\frac{n}{e_{\mathfrak{p},K}}}}{\alpha^{\frac{n}{n_1(2)}}} \right)^{\frac{1}{2}}.$$

As in the proof of Lemma 4.3 we can define[3]

$$x_1 \in R_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}} \cdot \prod_{\substack{l|n \\ l \neq 2}} W(k, n_1(l))^{\frac{l-1}{2} \frac{n}{n_1(l)}}.$$

as the class of the ideal

$$\prod_\mathfrak{p} \mathfrak{p}^{\frac{e'_{\mathfrak{p},K}-1}{2} \frac{n}{e_{\mathfrak{p},K}}}.$$

By Theorem 2.1 and Theorem 2.2,

$$\left(\frac{\prod_\mathfrak{p} \mathfrak{p}^{(e_{\mathfrak{p},L}-1)\frac{n_1(2)}{e_{\mathfrak{p},L}}}}{\alpha} \right)^{\frac{n}{2n_1(2)}}$$

is an ideal, whose class x_2 is the $n/n_1(2)$-th power of the Steinitz class of L/k. Thus

$$x_2 \in R_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}}.$$

_____

[3]The analogous element in Lemma 4.3 was called x_3.

By an easy calculation

$$\left(\frac{\prod_{\mathfrak{p}} \mathfrak{p}^{(e_{\mathfrak{p},K}-1)\frac{n}{e_{\mathfrak{p},K}}}}{\alpha^{\frac{n}{n_1(2)}}}\right)^{\frac{1}{2}} = \prod_{\mathfrak{p}} \mathfrak{p}^{\frac{e'_{\mathfrak{p},K}-1}{2}\frac{n}{e_{\mathfrak{p},K}}} \left(\frac{\prod_{\mathfrak{p}} \mathfrak{p}^{(e_{\mathfrak{p},L}-1)\frac{n_1(2)}{e_{\mathfrak{p},L}}}}{\alpha}\right)^{\frac{n}{2n_1(2)}}$$

and we conclude that $x = x_1 x_2$, from which we obtain the desired inclusion.

$\square$

Theorem 4.3. *Let k be a number field, let $G = C(n_1) \times \cdots \times C(n_r)$, with $n_{i+1}|n_i$, be an abelian group of order n. If $2|n$ and $n_2(2) = 1$ then*

$$R_t(k, G) = R_t(k, C(n_1(2)))^{\frac{n}{n_1(2)}} \prod_{\substack{l|n \\ l \neq 2}} W(k, n_1(l))^{\frac{l-1}{2}\frac{n}{n_1(l)}}.$$

Proof. $\subseteq$ This is Lemma 4.4.
     $\supseteq$ This follows by Lemma 3.1, Proposition 4.3 and Lemma 4.2.

$\square$

Thus in any case we reduce the study of the realizable Steinitz classes for abelian groups to that of 2-power order cyclic groups. As a consequence of our results we also prove the following corollary.

Corollary 4.3. *Let k be a number field, let G be an abelian group of order n and let $G(l)$ be its l-Sylow subgroup for any prime $l|n$. Then*

$$R_t(k, G) = \prod_{l|n} R_t(k, G(l))^{\frac{n}{n(l)}}.$$

Proof. This is immediate by Theorem 4.1, Theorem 4.2 and Theorem 4.3.

$\square$

In [7] we prove a similar result concerning a relation between the realizable classes for two groups and for their direct product, in a quite general situation, which however does not include abelian groups of even order.

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