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Résumé. Soit F un corps local non archimédien et localement compact, et soit B/F un corps de quaternions. La correspondance de Jacquet-Langlands fournit une bijection entre les représentations lisses et irréductibles de $B^\times$ de dimension > 1 et les représentations cuspidales et irréductibles de $\mathrm{GL}_2(F)$. Nous présentons une nouvelle construction de cette bijection pour laquelle la préservation des facteurs epsilon est automatique. Nous construisons une famille de paires $(\mathcal{L}, \rho)$, ou $\mathcal{L} \subset M_2(F) \times B$ est un ordre et ρ est une représentation d'une certaine sous-groupe de $\mathrm{GL}_2(F) \times B^\times$ qui contient $\mathcal{L}^\times$. Soit $\pi \otimes \pi'$ une représentation irréductible de $\mathrm{GL}_2(F) \times B^\times$; nous prouvons que $\pi \otimes \pi'$ contient une telle ρ si et seulement si π est cuspidale et correspond à $\check{\pi}'$ sous la correspondence de Jacquet-Langlands. On y voit tous les π et les π'. L'égalité des facteurs epsilon est reduite à un calcul Fourier-analytique sur un anneau quotient de $\mathcal{L}$.

Abstract. Let F be a locally compact non-Archimedean field, and let B/F be a division algebra of dimension 4. The Jacquet-Langlands correspondence provides a bijection between smooth irreducible representations π' of $B^\times$ of dimension > 1 and irreducible cuspidal representations of $\mathrm{GL}_2(F)$. We present a new construction of this bijection in which the preservation of epsilon factors is automatic. This is done by constructing a family of pairs $(\mathcal{L}, \rho)$, where $\mathcal{L} \subset M_2(F) \times B$ is an order and ρ is a finite-dimensional representation of a certain subgroup of $\mathrm{GL}_2(F) \times B^\times$ containing $\mathcal{L}^\times$. Let $\pi \otimes \pi'$ be an irreducible representation of $\mathrm{GL}_2(F) \times B^\times$; we show that $\pi \otimes \pi'$ contains such a ρ if and only if π is cuspidal and corresponds to $\check{\pi}'$ under Jacquet-Langlands, and also that every π and π' arises this way. The agreement of epsilon factors is reduced to a Fourier-analytic calculation on a finite ring quotient of $\mathcal{L}$.

1. Introduction

Let F be a non-Archimedean local field, *i.e.* a finite extension either of $\mathbf{Q}_p$ or of the field of Laurent series over the finite field $\mathbf{F}_p$. Let B/F be a central

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division algebra of dimension n^2. The Jacquet-Langlands correspondence assigns to each irreducible admissible representation π' of $B^\times$ a square-integrable representation π of $\mathrm{GL}_n(F)$. The passage $\pi' \mapsto \pi$ is characterized by a character relation. It also manifests as a relationship between epsilon factors, see for instance [DKV84]. When $n = 2$, the collection of epsilon factors of the twists of π by characters determines π up to isomorphism, so that the Jacquet-Langlands correspondence is characterized completely by its preservation of epsilon factors. In this case the reciprocity between $\mathrm{GL}_n(F)$ and $B^\times$ was proved by Jacquet and Langlands [JL70] in both the local and global settings. In the case of a division algebra B in characteristic 0 it was established for all n by [Rog83]. The case of a general inner form of $\mathrm{GL}_n(F)$ was carried out by Deligne, Kazhdan and Vignéras in [DKV84] in characteristic 0 and Badulescu [Bad02] in characteristic p. Each of these cases was accomplished by embedding the local problem into a global one and then applying trace formula methods.

There has also been a great deal of effort to construct the Jacquet-Langlands correspondence in an explicit manner using purely local techniques. The simplest case is when π' and π are both associated to a so-called "admissible pair" (E, θ), where E/F is a field extension of degree n and θ is a character of $E^\times$. (All supercuspidal π will arise this way if $p \nmid n$.) In this case the corresponding π was constructed explicitly by Howe [How77]; Gérardin [Gér79] constructed the representation π' and proved that the epsilon factors of π and π' agree. Henniart [Hen93b] showed that if n is a prime distinct from p the representations π and π' so constructed have the correct character identity. Using the technology of types laid down by Bushnell and Kutzko in [BK93], Henniart and Bushnell construct the explicit correspondence in the case of $n = p$ in [BH00]. The case of n a power of p with p odd and π totally ramified is carried out in [BH05].

In this paper we present a novel approach to the passage $\pi' \mapsto \pi$ in the case $n = 2$ in such a way that the preservation of epsilon factors is manifest in the construction. Our approach is entirely Fourier-analytic, and there is no special treatment needed for the case $p = 2$. In that sense it is similar to Gérardin-Li [GL85]. Unlike that paper, however, our method is linked to the theory of strata developed for GL_n in [BK93]. That theory is summarized in Section 2.2. Roughly speaking, a stratum for GL_2 is a certain sort of character of a compact open subgroup of $\mathrm{GL}_2(F)$. Then irreducible representations of $\mathrm{GL}_2(F)$ can be conveniently classified according to which strata they contain. There is a notion of simple stratum: these are parametrized by certain regular elliptic elements $\beta \in \mathrm{GL}_2(F)$. It can be shown that an admissible representation of $\mathrm{GL}_2(F)$ contains a simple stratum if and only if it is supercuspidal. A similar notion of stratum exists for $B^\times$, and strata for $B^\times$ are easily seen to be more or less the same objects as simple strata for $\mathrm{GL}_2(F)$. It is therefore natural to try to define the correspondence $\pi' \mapsto \pi$ relative to each stratum.

Let S be a simple stratum associated to the regular elliptic element $\beta \in \mathrm{GL}_2(F)$, and let S' be the stratum in $B^\times$ corresponding to S. We choose an embedding of the field $E = F(\beta)$ into B. Let $\Delta\colon E \to M_2(F) \times B$ be the diagonal map. We construct what we have called a "linking order" $\mathcal{L}_S$ inside $M_2(F) \times B$; this is a $\Delta(\mathcal{O}_E)$-order defined by certain congruence conditions. We then define an irreducible (and thus finite-dimensional) representation ρ_S of the unit group $\mathcal{L}_S^\times$ which is trivial on $\Delta(\mathcal{O}_F^\times)$. In the case where E/F is unramified, the construction of ρ_S comes from the Weil representation of SL_2 over a finite field. However, we also give a geometric construction using ℓ-adic cohomology which is well-suited to our purposes. Then loosely speaking, the induction of ρ_S to $\mathrm{GL}_2(F) \times B^\times$ will realize the Jacquet-Langlands correspondence for those representations π which contain S.

To make this precise, we must pay careful attention to the role of the center $Z = F^\times \times F^\times$ of $\mathrm{GL}_2(F) \times B^\times$. Choose a character ω of $F^\times = F^\times \times 1$ which extends $\rho_S|_{(F^\times \times 1) \cap \mathcal{L}_S^\times}$. We will give a recipe for an extension of ρ_S to the group $\mathcal{K}_S = \Delta(E^\times) Z \mathcal{L}_S^\times \subset \mathrm{GL}_2(F) \times B^\times$ whose restriction to Z is $(g,h) \mapsto \omega(gh^{-1})$. Call this representation $\rho_{S,\omega}$.

Let $\Pi_{S,\omega}$ be the compactly supported induction of $\rho_{S,\omega}$ up to $\mathrm{GL}_2(F) \times B^\times$. Then $\Pi_{S,\omega}$ is the direct sum of irreducible representations $\pi \otimes \pi'$ of $\mathrm{GL}_2(F) \times B^\times$; here π must have central character ω and π' must have central character ω^{-1}. We show that Π_S realizes the Jacquet-Langlands correspondence relative to the stratum S and the character ω in the following sense. First, we show that a representation π of $\mathrm{GL}_2(F)$ (resp., $B^\times$) of central character ω (resp., ω^{-1}) appears in Π_S if and only if π (resp., the contragredient $\check{\pi}$) contains S (resp., S'). Then, we show that an irreducible admissible representation $\pi \otimes \check{\pi}'$ of $\mathrm{GL}_2(F) \times B^\times$ appears inside of $\Pi_{S,\omega}$ if and only if the epsilon factors of twists of π and π' agree up to a minus sign:

$$\varepsilon(\pi\chi, s, \psi) = -\varepsilon(\pi'\chi, s, \psi). \tag{1.0.1}$$

Here χ runs through sufficiently many characters of $F^\times$ to determine π and π' uniquely. Therefore if π is a given supercuspidal irreducible representation of $\mathrm{GL}_2(F)$ which contains the stratum S, then $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, \Pi_S)$ is a sum of copies of a single supercuspidal representation π' of $B^\times$. Then the contragredient representation of π' is the one corresponding to π under the Jacquet-Langlands correspondence. It must be stressed that our approach does not yield a proof of the Jacquet-Langlands correspondence *de novo*. One must be able to deduce the correct character identity from Eq. 1.0.1. For this, we refer the reader to [BH06], §56, where a proof of the correspondences is sketched in a series of exercises. Our approach may nonetheless be of interest because it avoids the computation of any particular epsilon factors.

The linking orders $\mathcal{L}_S$ are constructed in Section 4. We also define corresponding additive characters ψ_S of the ring $M_2(F) \times B$ for which the $\mathcal{O}_F$-module

$$\mathcal{L}_S^* = \left\{ x \in M_2(F) \times B \;\middle|\; \psi_S(x\mathcal{L}_S) = 1 \right\}$$

happens to be a two-sided ideal in $\mathcal{L}_S$. The required representation ρ_S of $\mathcal{L}_S^\times$ is inflated from a representation of the unit group of the finite k-algebra $\mathcal{R}_S = \mathcal{L}_S/\mathcal{L}_S^*$. The additive character ψ_S descends to a nondegenerate additive character of this ring, so that we have a theory of Fourier transforms $f \mapsto \mathcal{F}_S f$ for functions f on $\mathcal{R}_S$. The characteristic property of ρ_S is that its matrix coefficients f, considered as functions on $\mathcal{R}_S$ supported on $\mathcal{R}_S^\times$, satisfy the functional equation

$$\mathcal{F}_S f(y) = \pm f(y^{-1}) \tag{1.0.2}$$

for $y \in \mathcal{R}_S^\times$; see Prop. 5.2.1 and Theorem 5.0.3. (The sign in this equation depends only on S.) The functional equation in Eq. 1.0.2 on the level of finite rings is used in Section 6 to deduce the functional equation in Eq. 1.0.1 concerning constituents of the induced representation of ρ_S up to $\mathrm{GL}_2(F) \times B^\times$.

The reader may be wondering if this sort of strategy may be extended to the general case of GL_n, where one still lacks a complete local proof of the existence of the correspondences. It will not be difficult to extend the definitions of $\mathcal{L}_S$, ρ_S, and $\Pi_{S,\omega}$ to this context. In doing so one would produce a recipe for some sort of correspondence $\pi' \mapsto \pi$ for π supercuspidal which satisfies Eq. 1.0.1 for a certain collection of characters χ. For $n = 3$, we do not know if this collection of characters is enough to characterize the Jacquet-Langlands correspondence. And for $n > 4$, the establishment of Eq. 1.0.1 for *all* characters is not enough to characterize the correspondence. Indeed, the epsilon factors of pairs of representations are required to characterize the isomorphism class of a given representation π, see [Hen93a]. One would have to work harder to obtain access to the characters of the representations π and π' so constructed in order to prove the right identity.

The present effort fits into a larger program concerning the geometry of Lubin-Tate curves. Suppose F has uniformizer π_F and residue field k. Let $\mathcal{F}_0$ be a formal $\mathcal{O}_F$-module of height 2 over the algebraic closure of the residue field k of F. For each $m \geq 0$, consider the functor that assigns to each complete local Noetherian $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$-algebra A having residue field $\overline{k}$ the set of one-dimensional formal $\mathcal{O}_F$-modules $\mathcal{F}$ over A equipped with an isomorphism $\mathcal{F}_0 \to \mathcal{F}_{\overline{k}}$ and a Drinfeld π_F^m-level structure. This functor is represented by a formal curve X_m over $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$. The inverse system of curves

$(X_m)_{m\geq 1}$ admits an action by a subgroup $\mathcal{G}$ of the triple product group $\mathrm{GL}_2(F) \times B^\times \times W_F$ of "index $\mathbf{Z}$". It is known by the theorems of Deligne and Carayol, see [Car86], that the ℓ-adic étale cohomology of this curve realizes (up to some benign modifications) both the Jacquet-Langlands correspondence $\pi' \mapsto \pi$ and the local Langlands correspondence $\sigma \mapsto \pi(\sigma)$ for the discrete series of $\mathrm{GL}_2(F)$.

It would be very interesting to compute a system of semistable models of the curves X_m over a ramified extension of $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$; then the special fiber of the system ought to realize the supercuspidal parts of the correspondences in its cohomology. This has already been done in the case of $m = 1$ by Bouw-Wewers [BW04]; the generalization of this case for GL_n was carried out by Yoshida [Yos09]. But for $m \geq 2$ the structure of this special fiber is still unknown. Ignore the Weil group for the moment and consider the action of $(\mathrm{GL}_2(F) \times B^\times) \cap \mathcal{G}$ on the semi-stable reduction of the system $(X_m)_{m\geq 1}$. We conjecture that for a simple stratum S arising from an elliptic element $\beta \in \mathrm{GL}_2(F)$, the special fiber contains a smooth component X_S whose stabilizer is exactly $\Delta(E^\times)\mathcal{L}_S^\times$, such that for primes $\ell \neq p$, the ℓ-adic versions of the representations ρ_S appear in the action of this group on $H^1(X_S, \overline{\mathbf{Q}}_\ell)$. In light of the preceding paragraphs this would be consistent with the theorems of Deligne-Carayol. In future work we intend to give a description of the structure of the special fiber of the stable reduction of X_m which includes the action of the Weil group W_F.

A different approach to the Jacquet-Langlands correspondence for GL_2 has been advanced in A. Snowden's thesis [Sno09].

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2. Preparations: The representation theory of $\mathbf{GL_2}(F)$ and $B^\times$

2.1. Basic Notations. In this paper, F will be a finite extension of $\mathbf{Q}_p$, or else a finite extension of $\mathbf{F}_p((T))$. For a finite extension E of F (possibly F itself), we use the notation $\mathcal{O}_E$, $\mathfrak{p}_E$, and k_E for the ring of integers, maximal ideal, and quotient field of E. Let $q_E = \#k_E$, and let $q = q_F$. We fix a uniformizer π_F for F. Let $|\ |_F$ be the absolute value on F^* for which $|\pi_F|_F = q^{-1}$.

We also fix a character ψ_F of F of level 1; this means that ψ_F vanishes on $\mathfrak{p}_F$ but not on $\mathcal{O}_F$.

Let B/F be a division algebra of dimension 4; this is unique up to isomorphism. Let $\mathcal{O}_B$ be its unique maximal order. We use $N_{B/F}$ and $\mathrm{Tr}_{B/F}$ to denote the reduced norm and trace, respectively, from B to F; sometimes we will omit the "B/F" from this notation. If G is the group $\mathrm{GL}_2(F)$ or $B^\times$, and $g \in G$, we will use the notation $\|g\|$ to mean $|\det g|_F$ or $|\mathrm{N}\, g|_F$ as appropriate.

Let A be the algebra $M_2(F)$ or B. For any additive character ψ of F, let ψ_A be the character of A defined by $\psi_A(x) = \psi(\mathrm{Tr}_{A/F}\, x)$. Let μ_{ψ_A} (or just μ_ψ) be the measure on A which is self-dual with respect to ψ.

Let $\mu_\psi^\times$ be the corresponding Haar measure on $A^\times$: $\mu_\psi^\times(g) = \|g\|_{A^\times}^{-2}\, \mu_\psi(g)$.

2.2. Chain Orders and Strata. In this subsection, A is the algebra $M_2(F)$ or B. We will closely follow the notation of [BH06] concerning chain orders and strata for GL_2, where the situation is somewhat simpler than the general case of GL_n.

First consider the case $A = M_2(F)$. A *lattice chain* is an F-stable family of lattices $\Lambda = \{L_i\}$ with each $L_i \subset F \oplus F$ an $\mathcal{O}_F$-lattice and $L_{i+1} \subset L_i$, all integers i. Let $e(\Lambda)$ be the unique integer for which $\pi_F L_i = L_{i+e(\Lambda)}$. Let $\mathfrak{A}_\Lambda$ be the stabilizer in A of Λ; that is, $\mathfrak{A}_\mathcal{L} = \{a \in A | aL_i \subset L_i,\ \text{all } i\}$. A *chain order* in A is an $\mathcal{O}_F$-order $\mathfrak{A} \subset A$ equal to $\mathfrak{A}_\mathcal{L}$ for some lattice chain Λ. We set $e_\mathfrak{A} = e_\Lambda$.

For example, suppose E/F is a quadratic field extension of ramification index e. Identify E with $F \oplus F$ as F-vector spaces. Then $\Lambda = \{\mathfrak{p}_E^i\}$ is a lattice chain with $e_\mathcal{L} = e$. Up to conjugation by an element of $A^\times$, every lattice chain arises in this manner. We have the following description of $\mathfrak{A}$, again only up to $A^\times$-conjugation:

$$\mathfrak{A} = \begin{cases} M_2(\mathcal{O}_F), & e = 1, \\ \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}, & e = 2. \end{cases}$$

Note also that $\mathfrak{A}^\times \subset A^\times$ is normalized by $E^\times \subset \mathrm{GL}_2(F)$, and that $\mathcal{O}_E \subset \mathfrak{A}$.

For a chain order $\mathfrak{A} \subset M_2(F)$, let $\mathcal{K}_\mathfrak{A}$ be its normalizer in $\mathrm{GL}_2(F)$. This equals $F^* M_2(\mathcal{O}_F)$ if $e_\mathfrak{A} = 1$. If $e_\mathfrak{A} = 2$ then $\mathcal{K}_\mathfrak{A}$ is the semidirect product of $\mathfrak{A}^\times$ with the cyclic group generated by a prime element of $\mathfrak{A}$.

Let $\mathfrak{P}_\mathfrak{A}$ be the Jacobson radical of $\mathfrak{A}$: this equals $\pi_F M_2(\mathcal{O}_F)$ for $\mathfrak{A} = M_2(\mathcal{O}_F)$ and $\begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$ in the case that $\mathfrak{A} = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}$. We have a filtration of $\mathfrak{A}^\times$ by the subgroups $U_\mathfrak{A}^n = 1 + \mathfrak{P}_\mathfrak{A}^n$. This filtration is normalized by $\mathcal{K}_\mathfrak{A}$.

All of the above constructions have obvious (and simpler) analogues in the quaternion algebra B: If $\mathfrak{A} = \mathcal{O}_B$ is the maximal order in B, then the normalizer of $\mathfrak{A}^\times$ in $B^\times$ is all of $B^\times$. The Jacobson radical $\mathfrak{P}_\mathfrak{A}$ is the unique maximal two-sided ideal of $\mathfrak{A}$, generated by a prime element π_B; we let $U_\mathfrak{A}^n = 1 + \mathfrak{P}_\mathfrak{A}$ and $e_\mathfrak{A} = 2$.

Definition 2.2.1. Let A be the matrix algebra $M_2(F)$ or the quaternion algebra B. A *stratum* in A is a triple $(\mathfrak{A}, n, \alpha)$, where $\mathfrak{A}$ is a chain order if $A = M_2(F)$ (resp. $\mathcal{O}_B$ if $A = B$), n is an integer, and $\alpha \in \mathfrak{P}_\mathfrak{A}^{-n}$. Two strata $(\mathfrak{A}, n, \alpha)$ and $(\mathfrak{A}, n, \alpha')$ are equivalent if $\alpha \equiv \alpha' \pmod{\mathfrak{P}^{1-n}}$. The stratum $(\mathfrak{A}, n, \alpha)$ is *ramified simple* if $E = F(\alpha)$ is a ramified quadratic extension of F, n is odd, and $\alpha \in E$ has valuation exactly $-n$. The stratum is *unramified simple* if E is an unramified quadratic extension of F, $\alpha \in E$ has valuation exactly $-n$, and the minimal polynomial of $\pi_F^n \alpha$ is irreducible mod $\mathfrak{p}_F$. Finally, the stratum is *simple* if it is ramified simple or unramified simple.

There is a correspondence $\mathfrak{S}' \mapsto \mathfrak{S}$ between simple strata in B and simple strata in $M_2(F)$. Given the simple stratum $\mathfrak{S}' = (\mathfrak{A}', n', \alpha')$, let $E = F(\alpha')$. Choose an embedding $E \hookrightarrow M_2(F)$, and let α be the image of α'. Finally, let $\mathfrak{A} \subset M_2(F)$ be a chain order associated to E. Then $\mathfrak{S} = (\mathfrak{A}, n, \alpha)$. The correspondence $\mathfrak{S}' \to \mathfrak{S}$ is a bijection between conjugacy classes of simple strata in B and in $M_2(F)$, respectively. The relationship between n' and n is as follows: $n' = n$ if E/F is ramified and $n' = 2n$ if E/F is unramified.

Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$. The level $\ell(\pi)$ is defined to be the least value of n/e, where (n, e) runs over pairs of integers for which there exists a chain order $\mathfrak{A}$ of ramification index e such that π contains the trivial character of $U_{\mathfrak{A}}^{n+1}$. If π is a representation of $B^\times$, we define $\ell(\pi)$ to be $n/2$, where n is the least integer for which π contains the trivial character of $U_{\mathcal{O}_B}^{n+1}$.

We shall call π *minimal* if its level cannot be lowered by twisting by one-dimensional characters of $F^\times$.

When $n \geq 1$, a stratum $S = (\mathfrak{A}, n, \alpha)$ of $M_2(F)$ or B determines a nontrivial character ψ_α of $U_{\mathfrak{A}}^n / U_{\mathfrak{A}}^{n+1}$ by $\psi_\alpha(1 + x) = \psi_F(\mathrm{Tr}_{A/F}(\alpha x))$. This character only depends on the equivalence class of S.

If S is a stratum, we say that π *contains the stratum* S if $\pi|_{U_{\mathfrak{A}}^n}$ contains the character ψ_α. From [BH06], 14.5 Theorem, we have the following classification of supercuspidal representations of $\mathrm{GL}_2(F)$:

Theorem 2.2.2. *A minimal irreducible representation π of $\mathrm{GL}_2(F)$ is supercuspidal if and only if exactly one of the following conditions holds:*

(1) *π has level 0, and π is contains a representation of $\mathrm{GL}_2(\mathcal{O}_F)$ inflated from an irreducible cuspidal representations of $\mathrm{GL}_2(k_F)$.*
(2) *π has level $\ell > 0$, and π contains a simple stratum.*

The classification of representations of $B^\times$ is analogous:

Theorem 2.2.3. *A minimal irreducible representation π of $B^\times$ of dimension greater than one satisfies exactly one of the following properties:*

(1) *π has level 0, and π contains a representation of $\mathcal{O}_B^\times$ inflated from a character χ of $k_B^\times$ not factoring through the norm map $k_B^\times \to k^\times$.*
(2) *π has level $\ell > 0$, and π contains a simple stratum.*

By k_B we mean the finite field $\mathcal{O}_B/\mathfrak{P}_B$: this is a quadratic extension of k.

The supercuspidal representations of $\mathrm{GL}_2(F)$ and $B^\times$ are all induced from irreducible representations of open compact-mod-center subgroups in a manner which can be made explicit. Suppose $S = (\mathfrak{A}, n, \alpha)$ is a simple stratum in $M_2(F)$ or B. Let $E \subset \mathrm{GL}_2(F)$ be the subfield $F(\alpha)$. The definition of ψ_α given above is well-defined on the subgroup $U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}$. Let $J_S \subset \mathrm{GL}_2(F)$ denote the group $E^\times U_{\mathfrak{A}}^{\lfloor (n+1)/2 \rfloor}$ and let $C(\psi_\alpha, \mathfrak{A})$ denote the set of isomorphism classes of irreducible representations $\Lambda \in \hat{J}_S$ for which $\Lambda|_{U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}}$ is a multiple of ψ_α.

Definition 2.2.4. A *cuspidal inducing datum* in $A^\times$ is a pair $(\mathfrak{A}, \Xi)$, where $\mathfrak{A}$ is a chain order in A and Ξ is a representation of $\mathcal{K}_{\mathfrak{A}}$ of one of the following types:

1. $A = M_2(F)$, $\mathfrak{A} \cong M_2(\mathcal{O}_F)$, and the restriction of Ξ to $\mathrm{GL}_2(\mathcal{O}_F)$ is inflated from a cuspidal representation of $\mathrm{GL}_2(k)$.
2. $A = B$, and the restriction of Ξ to $\mathcal{O}_B^\times$ contains a character of inflated from a character of $k_B^\times$ not factoring through the norm map $k_B^\times \to k^\times$.
3. There is a simple stratum $(\mathfrak{A}, n, \alpha)$ and a representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ for which $\Xi = \mathrm{Ind}_{J_S}^{\mathcal{K}_{\mathfrak{A}}} \Lambda$.
4. The representation Ξ is the twist of a representation of one of the above types by a character of $F^\times$.

In the first two cases we will say that $(\mathfrak{A}, \Xi)$ has level zero.

The following construction of supercuspidal representations is found in Section 15.5 of [BH06] in the case of $A = M_2(F)$:

Theorem 2.2.5. *If $(\mathfrak{A}, \Xi)$ is a cuspidal inducing datum then $\pi_\Xi = \mathrm{Ind}_{\mathcal{K}_{\mathfrak{A}}}^{A^\times} \Xi$ is an irreducible supercuspidal representation of $A^\times$. Conversely, every supercuspidal representation of $A^\times$ arises in this manner. The cuspidal inducing datum $(\mathfrak{A}, \Xi)$ has level zero if and only if π_Ξ has level zero. Furthermore, $(\mathfrak{A}, \Xi)$ arises from the simple stratum S if and only if π_Ξ contains S.*

2.3. Zeta functions and local constants. In this section we follow Godement and Jacquet [GJ72], §3. Let A be the algebra B or $M_2(F)$, and let $G = A^\times$. Let $\psi \in \hat{F}$ be an additive character of F. Let π be a supercuspidal (not necessarily irreducible) representation of G, realized on the space W. Let $\check{\pi}$ be the contragredient representation, with underlying space $\check{W}$. When $w \in W$, $\check{w} \in \check{W}$, we let $\gamma_{\check{w},w} \colon G \to \mathbf{C}$ denote the function

$$g \mapsto \langle \check{w}, \pi(g) w \rangle .$$

Let $\mathcal{C}(\pi)$ denote the $\mathbf{C}$-span of the functions $\gamma_{\check{w},w}$ for $w \in W$, $\check{w} \in \check{W}$. These functions are compactly supported modulo the center Z of G.

Let $C_c^\infty(A)$ be the space of locally constant compactly supported complex-valued functions on A. For $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$, define the zeta function

$$\zeta(\Phi, f, s) = \int_G \Phi(g) f(g) \left\|g\right\|^s \; d\mu_\psi^\times(g).$$

When π is irreducible (and still cuspidal), there is a rational function $\varepsilon(\pi, s, \psi) \in \mathbf{C}(q^{-s})$ satisfying

$$\zeta\left(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s\right) = \varepsilon(\pi, s, \psi)\zeta\left(\Phi, f, \tfrac{1}{2} + s\right),$$

where $\hat{\Phi}$ is the Fourier transform of Φ with respect to ψ. (Since π is cuspidal, its L-function vanishes.) See [GJ72], Thm. 3.3.

The local constant further satisfies

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1 - s, \psi) = \omega_\pi(-1) \tag{2.3.1}$$

where ω_π is the central character of π ([GJ72], p. 33).

2.4. Converse Theory. By the converse theorem, a supercuspidal representation of $\mathrm{GL}_2(F)$ of $B^\times$ is determined by the epsilon factors of all of its twists by one-dimensional characters. We need an effective version of this theorem, which states that a supercuspidal representation is determined up to isomorphism by the data of its level together with the epsilon factors of twists of π by a collection of characters of $F^\times$ of bounded level.

Next, we observe that epsilon factors have the "stability" property. If χ is a character of F^*, let the level $\ell(\chi)$ be the least integer n such that χ vanishes on $1 + \mathfrak{p}_F^{n+1}$. Then if π is an irreducible representation of $\mathrm{GL}_2(F)$ or $B^\times$, and χ is a character of $F^\times$ with $\ell(\chi) > \ell(\pi)$, then $\varepsilon(\pi\chi, s, \psi)$ only depends on χ and the central character of π (and of course ψ). This is Prop. 3.8 of [JL70] in the case of $\mathrm{GL}_2(F)$ and Prop. 2.2.5 of [GL85] in the case of $B^\times$.

As χ varies through all characters of $F^\times$, the quantities $\varepsilon(\chi\pi, s, \psi)$ determine π up to isomorphism. We may therefore conclude the following explicit converse theorem:

Theorem 2.4.1. *Let π_1 and π_2 be two minimal supercuspidal representations of $\mathrm{GL}_2(F)$ or $B^\times$ having the same central character and equal level ℓ. Then $\pi_1 \cong \pi_2$ if and only if*

$$\varepsilon(\pi_1\chi, s, \psi) = \varepsilon(\pi_2\chi, s, \psi) \tag{2.4.1}$$

for all characters $\chi \in \hat{F}^\times$ for which $\ell(\chi) \leq \ell$.

Definition 2.4.2. For minimal supercuspidal representations π' and π of $B^\times$ and $\mathrm{GL}_2(F)$ having the same central character, we say that π' and π *correspond* if the following conditions hold:

(1) π and π' have the same level ℓ.

(2) The equation

$$\varepsilon(\pi\chi, s, \psi) = -\varepsilon(\pi'\chi, s, \psi)$$

holds for all characters χ with $\ell(\chi) \leq \ell$.

In view of Theorem 2.4.1, at most one π can correspond to a given π', and vice versa.

3. Zeta functions for $GL_2(F) \times B^\times$.

In this section we adopt the abbreviations $A_1 = M_2(F)$, $A_2 = B$, $G_1 = GL_2(F)$, $G_2 = B^\times$.

Let $\mathbf{A} = A_1 \times A_2$. Let $\mathbf{G} = \mathbf{A}^\times = \mathrm{GL}_2(F) \times B^\times$. We will define zeta functions for representations of $\mathbf{G}$ and use them to give a criterion for when such a representation "realizes the Jacquet-Langlands correspondence." We will adopt the convention that if $g \in \mathbf{G}$, then g_1 and g_2 are its projections in $\mathrm{GL}_2(F)$ and $B^\times$ respectively. Let Π be an admissible cuspidal representation of $\mathbf{G}$. For $\Phi \in C_c^\infty(\mathbf{A})$ and $f \in \mathcal{C}(\Pi)$, define the zeta function

$$\zeta(\Phi, f, s) = \int_{\mathbf{G}} \Phi(g) f(g) \left\|g_1\right\|^s \left\|g_2\right\|^{2-s} d\mu^\times(g),$$

where $\mu^\times$ is a Haar measure on $\mathbf{G}$.

Let ψ be an additive character of F, and let $\mu_\psi^\times = \mu_{\mathbf{A},\psi}^\times = \mu_{A_1,\psi}^\times \times \mu_{A_2,\psi}^\times$; this is a Haar measure on $\mathbf{G}$. Let $\psi_{\mathbf{A}}$ be the additive character $(x_1, x_2) \mapsto \psi_{A_1}(x_1)\psi_{A_2}(-y_1)$. The Fourier transform of a decomposable test function $\Phi = \Phi_1 \otimes \Phi_2 \in C_c^\infty(\mathbf{A})$ is $\hat{\Phi}(x_1, x_2) = \hat{\Phi}_1(x_1)\hat{\Phi}_2(-x_2)$. Consequently if $f = f_1 \otimes f_2 \in \mathcal{C}(\pi_1 \otimes \pi_2)$ is a decomposable matrix coefficient for a tensor product representation $\pi_1 \otimes \pi_2$ of $\mathbf{G}$, then

$$\zeta(\hat{\Phi}, f, s) = \omega_{\pi_2}(-1)\zeta(\hat{\Phi}_1, f_1, s)\zeta(\hat{\Phi}_2, f_2, 2-s), \tag{3.0.1}$$

where ω_{π_2} is the central character of π_2.

Proposition 3.0.1. *Let* Π *be an admissible cuspidal semisimple (not necessarily irreducible) representation of* $\mathrm{GL}_2(F) \times B^\times$*. The following are equivalent:*

(1) *For every irreducible representation* $\pi_1 \otimes \pi_2$ *of* $\mathrm{GL}_2(F) \times B^\times$ *appearing in* Π*, we have*

$$\varepsilon(\pi_1, s, \psi) = -\varepsilon(\check{\pi}_2, s, \psi).$$

(2) *The functional equation*

$$\zeta(\Phi, f, s) = -\zeta(\hat{\Phi}, \check{f}, 2-s) \tag{3.0.2}$$

holds for all $\Phi \in C_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi)$*. (Here the integral is taken with respect to the measure* $\mu_{\mathbf{A},\psi}^\times$*, and the Fourier transform is taken with respect to the character* $\psi_{\mathbf{A}}$*.)*

Proof. It will simplify our notation if we set $s_1 = s$, $s_2 = 2 - s$. Let $\pi_1 \otimes \pi_2$ be any irreducible representation of $G_1 \times G_2$ appearing in Π. For $i = 1, 2$, let $\Phi_i \in C_c^\infty(G_i)$ and $f_i \in \mathcal{C}(\pi_i)$ be such that $\zeta(\Phi_i, f_i, s_i) \neq 0$. Let $\Phi = \Phi_1 \otimes \Phi_2$ and $f = f_1 \otimes f_2$. The respective functional equations for π_1 and π_2 are

$$\zeta(\hat{\Phi}_i, \check{f}_i, 2 - s_i) = \varepsilon\left(\pi_i, s_i - \tfrac{1}{2}, \psi\right)\zeta(\Phi_i, f_i, s_i),\ i = 1, 2.$$

Multiplying these together and applying Eq. 3.0.1 yields

$$\omega_{\pi_2}(-1)\zeta(\hat{\Phi}, \check{f}, 2 - s) = \varepsilon\left(\pi_1, s - \tfrac{1}{2}, \psi\right)\varepsilon\left(\pi_2, \tfrac{3}{2} - s, \psi\right)\zeta(\Phi, f, s).$$

Therefore Eq. 3.0.2 holds if and only if

$$\varepsilon\left(\pi_1, s - \tfrac{1}{2}, \psi\right)\varepsilon\left(\pi_2, \tfrac{3}{2} - s, \psi\right) = -\omega_{\pi_2}(-1).$$

Combining this with the standard relation

$$\varepsilon\left(\pi_2, \tfrac{3}{2} - s, \psi\right)\varepsilon\left(\check{\pi}_2, s - \tfrac{1}{2}, \psi\right) = \omega_{\pi_2}(-1)$$

yields

$$\varepsilon\left(\pi_1, s - \tfrac{1}{2}, \psi\right) = -\varepsilon\left(\check{\pi}_2, s - \tfrac{1}{2}, \psi\right).$$

We see now that (2) $\implies$ (1): Apply Eq. 3.0.2 to an arbitrary matrix coefficient $f = f_1 \otimes f_2$ belonging to $\pi_1 \otimes \pi_2 \subset \Pi$. For the converse, one need only note that every $\Phi \in C_c^\infty(\mathbf{A})$ and $f \in \mathcal{C}(\Pi)$ is a finite sum of pure tensors, and $\zeta(\Phi, f, s)$ is linear in Φ and f. □

Combining Prop. 3.0.1 with the Converse Theorem 2.4.1 gives a necessary and sufficient condition for a representation Π of $\mathrm{GL}_2(F) \times B^\times$ to realize the Jacquet-Langlands correspondence. When $f \in C_c^\infty(\mathbf{G})$ and $\chi \in \hat{F}^\times$, we let χf be the function $g \mapsto \chi(\det(g_1)\,\mathrm{N}(g_2)^{-1})f(g)$.

Corollary 3.0.2. *Let Π be an admissible cuspidal semisimple representation of $\mathrm{GL}_2(F) \times B^\times$ on which the diagonally-embedded group $\Delta(F^\times)$ acts trivially. Assume either that every irreducible representation of $\mathrm{GL}_2(F)$ (resp., $B^\times$) appearing in Π is minimal of the same level ℓ. Then the following are equivalent:*

1. *Π is the direct sum of irreducible representations of $\mathbf{G}$ of the form $\pi_1 \otimes \check{\pi}_2$, where π_1 and π_2 correspond.*
2. *The functional equation*

$$\zeta(\Phi, \chi f, s) = -\zeta(\hat{\Phi}, \chi^{-1}\check{f}, 2 - s) \tag{3.0.3}$$

holds for all $\Phi \in C_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi)$, and for all characters $\chi \in \hat{F}^\times$ for which $\ell(\chi) \leq \ell$.

Proof. That (1) $\implies$ (2) is clear from Prop. 3.0.1. Therefore assume (2). Suppose $\pi_1 \otimes \check{\pi}_2$ appears in Π. Since Π vanishes on $\Delta(F^\times)$, the central characters of π_1 and π_2 agree. By Prop. 3.0.1 we find that $\varepsilon(\pi_1\chi, s, \psi) =$

$-\varepsilon(\pi_2\chi, s, \psi)$ for all characters χ of level no greater than ℓ, so π_1 and π_2 correspond. □

4. Linking orders and congruence subgroups of $\mathbf{GL_2}(\boldsymbol{F}) \times \boldsymbol{B}^\times$

Our goal now is to produce, for each simple stratum S in $M_2(F)$, a certain semisimple representation Π_S of $\mathrm{GL}_2(F) \times B^\times$ having the following properties:

1. Π_S vanishes on the diagonal subgroup $\Delta(F^\times) \subset \mathrm{GL}_2(F) \times B^\times$.
2. The restriction of Π_S to the first factor $\mathrm{GL}_2(F)$ is a sum of exactly those irreducible representations which contain S. Similarly, the restriction of Π_S to the second factor $B^\times$ is a sum of exactly those irreducible representations of $B^\times$ which contain the corresponding stratum S' in B.
3. Matrix coefficients for Π_S satisfy the functional equation in Eq. 3.0.3 for sufficiently many χ.

We will present a similar construction for representations of level zero. In light of Cor. 3.0.2, such a family $\{\Pi_S\}$ is sufficient to establish the Jacquet-Langlands correspondence.

The strategy for producing Π_S is as follows: We will first define an order $\mathcal{L}_S \subset M_2(F) \times B$. The required representation Π_S will be induced from a certain representation of $\mathcal{L}_S^\times$. In this section we construct the orders $\mathcal{L}_S$ and gather some geometric properties in preparation for proving the properties listed above.

4.1. Geometric preparations: $\boldsymbol{M_2(F)}$ and $\boldsymbol{B}$. Let E/F be a separable quadratic extension field of ramification degree e. Let $\mathcal{O}_E$ be its ring of integers, $\mathfrak{p}_E$ its maximal ideal, k_E its quotient field and σ the nontrivial element of $\mathrm{Gal}(E/F)$.

Let A be the ring $M_2(F)$ or B. Define an order $\mathfrak{A} \subset A$ as follows: if $A = M_2(F)$, let $\mathfrak{A}$ be the chain order equal to the endomorphism ring of the lattice chain $\{\mathfrak{p}_E^i\}$, as in Section 2.2. If $A = B$, let $\mathfrak{A} = \mathcal{O}_B$. Either way, we may identify $\mathcal{O}_E$ with an $\mathcal{O}_F$-subalgebra of $\mathfrak{A}$ in such a way that $\mathfrak{A} \cap E = \mathcal{O}_E$.

There is a nondegenerate pairing $A \times A \to F$ given by $(x, y) \mapsto \mathrm{Tr}_{A/F}(xy)$. Let C be the complement of E in A with respect to this pairing, so that $A = E \oplus C$. Let $s_A \colon A \to E$ be the projection onto the first factor. Note that both the space C and the map s_A are stable under multiplication by E on either side. C is a (left and right!) E-vector space of dimension 1. It satisfies the property that $\alpha v = v\alpha^\sigma$ for all $v \in C$, $\alpha \in E$. Let $\mathfrak{C} = \mathfrak{A} \cap C$.

Lemma 4.1.1. *We have*

$$\mathfrak{C}\mathfrak{C} = \begin{cases} \mathfrak{p}_E, & E/F \text{ unramified and } A = B \\ \mathcal{O}_E, & \text{all other cases.} \end{cases}$$

Proof. Since elements of E commute with $\mathfrak{C}\mathfrak{C}$, we must have $\mathfrak{C}\mathfrak{C} \subset E$; since $\mathfrak{C} \subset \mathfrak{A}$ this implies $\mathfrak{C}\mathfrak{C} \subset E \cap \mathfrak{A} = \mathcal{O}_E$. Thus $\mathfrak{C}\mathfrak{C}$ is an $\mathcal{O}_E$-submodule of $\mathcal{O}_E$; *i.e.* it is an ideal of $\mathcal{O}_E$.

If $A = M_2(F)$ then $\mathfrak{A}$ is the endomorphism ring of the lattice chain $\{\mathfrak{p}_E^i\}$. Consider the element $\sigma \in \mathrm{Gal}(E/F)$: this certainly preserves each $\mathfrak{p}_E^i$ and therefore belongs to $\mathfrak{A}$. For any $\alpha \in E$, we have that $(\alpha\sigma)^2 = \mathrm{N}_{E/F}(\alpha)$ belongs to the center $F \subset M_2(F)$, but $\alpha\sigma$ does not itself belong to F, implying that $\mathrm{Tr}_{A/F}(\alpha\sigma) = 0$ and therefore that $\sigma \in C$. So $\sigma \in C \cap \mathfrak{A} = \mathfrak{C}$. Consequently $\mathfrak{C}\mathfrak{C}$ contains $\sigma^2 = 1$, whence it is the unit ideal.

Now suppose $A = B$. Let $v_B \colon B^\times \to \mathbf{Z}$ denote the valuation on B. If E/F is ramified, then a uniformizer π_E of E has $v_B(\pi_E) = 1$, so that if $x \in \mathfrak{C}$ has valuation n, then $\pi_E^{-n} x \in \mathfrak{C}$ is a unit. This implies that $\mathfrak{C}\mathfrak{C}$ is the unit ideal.

On the other hand if E/F is unramified, then every element of E has even valuation in B. Considering that $A = E \oplus C$, this means that $\mathfrak{C}$ contains an element π_B of valuation 1, so that $\mathfrak{C} = \mathcal{O}_E \pi_B$. Then $\mathfrak{C}\mathfrak{C} = \mathcal{O}_E \pi_B^2 = \mathfrak{p}_E$ as required. □

Now suppose that $S = (\mathfrak{A}, n, \alpha)$ is a simple stratum in A with $E = F(\alpha)$. By replacing α with a sufficiently nearby element of $\mathfrak{A}$, it may be assumed that E/F is a separable field extension. This may be done without changing the character ψ_α of $U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}$. Choose an additive character ν of E vanishing on $\mathfrak{p}_E^{n+1}$ but not on $\mathfrak{p}_E^n$. Assume that $\nu = \nu^\sigma$ if $e = 1$. Then define a character ν_S of A by $\nu_S(x) = \nu(s_A(x))$.

Whenever W is an $\mathcal{O}_E$-stable lattice of A, we may define the annihilator of W with respect to ν_S:

$$W^* = \{x \in A \mid \nu_S(xW) = 1\};$$

then W^* is also an $\mathcal{O}_E$-module. Note that $(\mathfrak{p}_E^k W)^* = \mathfrak{p}_E^{-k} W^*$.

Lemma 4.1.2. *The $\mathcal{O}_E$-module $\mathfrak{C}^*$ equals $E \oplus \mathfrak{p}_E^n \mathfrak{C}$ if E/F is unramified and $A = B$. It equals $E \oplus \mathfrak{p}_E^{n+1} \mathfrak{C}$ in all other cases.*

Proof. Certainly we have $E \subset \mathfrak{C}^*$; all that remains is to find $\mathfrak{C}^* \cap \mathfrak{C}$. This last is an $\mathcal{O}_E$-submodule of the free rank-one $\mathcal{O}_E$-module $\mathfrak{C}$, so that it equals $I\mathfrak{C}$ for an ideal $I \subset \mathcal{O}_E$. For an element $x \in \mathcal{O}_E$ to belong to I the condition is $\nu_S(s_A(x\mathfrak{C}\mathfrak{C})) = \nu(I\mathfrak{C}\mathfrak{C}) = 1$. The lemma now follows from Lemma 4.1.1 and the definition of ν. □

For an integer $m \geq 1$, we define an $\mathcal{O}_E$-submodule $V_A^m \subset \mathfrak{C}$ as follows:

$$V_A^m = \begin{cases} \mathfrak{p}_E^{\lfloor m/2 \rfloor} \mathfrak{C}, & A = B \text{ and } E/F \text{ unramified} \\ \mathfrak{p}_E^{\lfloor (m+1)/2 \rfloor} \mathfrak{C}, & \text{all other cases.} \end{cases}$$

The next proposition shows that $V_A^n \subset \mathfrak{C}$ is nearly a "square root" of the ideal $\mathfrak{p}_E^n$:

Proposition 4.1.3. *The module V_A^n has the following properties:*

(1) $V_A^n V_A^n \subset \mathfrak{p}_E^n$. *More precisely, if E/F is unramified then the value of $V_A^n V_A^n$ is given by the following table:*

	n *even*	n *odd*
$A = M_2(F)$	$\mathfrak{p}_E^n$	$\mathfrak{p}_E^{n+1}$
$A = B$	$\mathfrak{p}_E^{n+1}$	$\mathfrak{p}_E^n$

(2) *If E/F is ramified, then $V_A^n = V_A^{n+1}$.*

(3) *If E/F is unramified, then the dimension of V_A^n/V_A^{n+1} as a k_E-vector space is given by the following table:*

	n *even*	n *odd*
$A = M_2(F)$	1	0
$A = B$	0	1

(4) *With respect to the character ν_S, we have $(V_A^n)^* = E \oplus V_A^{n+1}$.*

Proof. Claim (1) follows from Lemma 4.1.1. For claim (2): Since E/F is ramified, n must be odd by definition of simple stratum; then $\lfloor (n+1)/2 \rfloor = \lfloor ((n+1)+1)/2 \rfloor$. For claim (3), assume E/F is unramified. When $A = M_2(F)$ we have $V_A^n = \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor}\mathfrak{C}$, so that there is an isomorphism of k_E-vector spaces $V_A^n/V_A^{n+1} \approx \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor}/\mathfrak{p}_E^{\lfloor (n+2)/2 \rfloor}$, and this has dimension 1 or 0 as n is even or odd, respectively. When $A = M_2(F)$ we have $V_A^n = \mathfrak{p}_E^{\lfloor n/2 \rfloor}\mathfrak{C}$, so that there is an isomorphism of k_E-vector spaces $V_A^n/V_A^{n+1} = \mathfrak{p}_E^{\lfloor n/2 \rfloor}/\mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor}$, and this has dimension 0 or 1 as n is even or odd, respectively.

Claim (4) follows directly from Lemma 4.1.2. ☐

4.2. Congruence subgroups and cuspidal representations. Keeping the notations from the previous subsection, we let

$$H_S = 1 + \mathfrak{p}_E^n + V_A^n$$
$$H_S^1 = 1 + \mathfrak{p}_E^n + V_A^{n+1}.$$

These are subgroups of $\mathfrak{A}^\times$ because V_A^n is an $\mathcal{O}_E$-module and because $V_A^n V_A^n \subset \mathfrak{p}_E^n$ by Prop. 4.1.3. Note the inclusions $U_{\mathfrak{A}}^n \subset H_S^1 \subset H_S \subset J_S$ and $H_S^1 \subset U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}$.

Proposition 4.2.1. *For a representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$, we have that $\Lambda|_{H_S}$ is irreducible. Further, $\Lambda|_{H_S}$ is the unique irreducible representation of H_S whose restriction to H_S^1 is a sum of copies of $\psi_\alpha|_{H_S^1}$.*

Proof. If E/F is ramified, the claims in the proposition are trivial, because $H_S = H_S^1$ and Λ is a one-dimensional character. If E/F is unramified, then the same is true in the case that $A = M_2(F)$ and n is odd, and as well in the case that $A = B$ and n is even.

Therefore assume that E/F is unramified, and that $A = M_2(F)$ and n is even, or else that $A = B$ and and n is odd. Then V_A^n/V_A^{n+1} is a k_E-module of dimension 1. Let ψ_α^1 denote the restriction of ψ_α to H_S^1. We have an exact sequence

$$1 \to H_S^1/\ker\psi_\alpha^1 \to H_S/\ker\psi_\alpha^1 \to V_A^n/V_A^{n+1} \to 1$$

in which $H_S^1/\ker\psi_\alpha^1$ is the center. Thus $H_S/\ker\psi_\alpha^1$ is a discrete Heisenberg group. By the discrete Stone-von Neumann Theorem, there is a unique irreducible representation $\tilde{\psi}_\alpha$ of H_S lying over ψ_α^1.

If $\Lambda \in C(\psi_\alpha, \mathfrak{A})$, then $\Lambda|_{H_S}$ is a q-dimensional representation of H_S whose restriction to H_S^1 is a multiple of ψ_α^1. By the uniqueness property of $\tilde{\psi}_\alpha$, we must have $\Lambda|_{H_S} = \tilde{\psi}_\alpha$. The proposition follows. □

4.3. Linking Orders. It is time to investigate the geometry of the product algebra $M_2(F) \times B$. It will be helpful to use the abbreviations $A_1 = M_2(F)$, $A_2 = B$, $\mathbf{A} = M_2(F) \times B$. Suppose $S = S_1 = (\mathfrak{A}_1, n_1, \alpha_1)$ is a simple stratum in $M_2(F)$. Choose an embedding $E = F(\alpha_1) \hookrightarrow B$ and let $\alpha_2 \in B^\times$ be the image of α_1 so that $S_2 = (\mathfrak{A}_2, n_2, \alpha_2)$ is the simple stratum in B which corresponds to S. Here $\mathfrak{A}_2 = \mathcal{O}_B$. For convenience of notation we set $n = n_1$. Let $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$ and let $\Delta\colon E \to \mathbf{A}$ be the diagonal map $\Delta(a) = (a, a)$. We denote by s_1 and s_2 the projections $A_1 \to E$, $A_2 \to E$, respectively. Let C_i be the complement of E in A_i.

Let ν be an additive character of E as in Section 4.1. We define a character ν_S of $\mathbf{A}$ by

$$\nu_S(x_1, x_2) = \nu(s_1(x_1) - s_2(x_2)).$$

Lemma 4.3.1. *With respect to ν_S, the annihilator of the diagonally embedded subring $\Delta(\mathcal{O}_E) \subset \mathfrak{A}$ is*

$$(\Delta(\mathcal{O}_E))^* = \Delta(E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + C_1 \times C_2.$$

Proof. Suppose $(x_1, x_2) \in (\Delta(\mathcal{O}_E))^*$; then for all $\beta \in \mathcal{O}_E$, $v(\beta(s_1(x_1) - s_2(x_2))) = 1$. This means exactly that $s(x_1) \equiv s(x_2) \pmod{\mathfrak{p}_E^{n+1}}$, so that the pair $(s(x_1), s(x_2))$, being equal to $(s(x_1), s(x_1)) + (0, s(x_2) - s(x_1))$, lies in $\Delta(E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1}$ as required. □

Let $\mathbf{V}^n = V_A^n \times V_B^n \subset \mathfrak{A}$. The following properties of $\mathbf{V}^n$ follow directly from Prop. 4.1.3:

Proposition 4.3.2. *The module $\mathbf{V}^n$ has the following properties:*

1. $\mathbf{V}^n\mathbf{V}^n \subset \mathfrak{p}_E^n \times \mathfrak{p}_E^n$. *Furthermore, if E/F is unramified then $\mathbf{V}^n\mathbf{V}^n$ equals $\mathfrak{p}_E^n \times \mathfrak{p}_E^{n+1}$ or $\mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^n$ as n is even or odd, respectively.*

(2) *If E/F is unramified, then $\mathbf{V}^n/\mathbf{V}^{n+1}$ is a left and right k_E-vector space of dimension 1, with the property that $\alpha v = v\alpha^q$ for $\alpha \in k_E$, $v \in \mathbf{V}^n/\mathbf{V}^{n+1}$.*
(3) *If E/F is ramified, then $\mathbf{V}^n = \mathbf{V}^{n+1}$.*
(4) *With respect to ψ_S, the annihilator of $\mathbf{V}^n$ is $(E \times E) \oplus \mathbf{V}^{n+1}$.*

Definition 4.3.3. The *linking order* $\mathcal{L}_S$ is defined by

$$\mathcal{L}_S = \Delta(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n + \mathbf{V}^n.$$

Then $\mathcal{L}_S$ is a (left and right) $\mathcal{O}_E$-submodule of $\mathfrak{A}$. It is easy to check that $\mathcal{L}_S$ is indeed an order; this is a consequence of item (1) of the previous paragraph. We will also have use for a smaller subspace $\mathcal{L}_S^\circ \subset \mathcal{L}_S$, defined by

$$\mathcal{L}_S^\circ = \Delta(\mathfrak{p}_E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + \mathbf{V}^{n+1}.$$

Proposition 4.3.4. *The linking order $\mathcal{L}_S$ has the following properties:*

(1) *The group $\mathcal{L}_S^\times$ is normalized by $\Delta(E^\times)$.*
(2) *With respect to ν_S, the annihilator of $\mathcal{L}_S$ is $\mathcal{L}_S^\circ$.*
(3) *$\mathcal{L}_S^\circ$ is a double-sided ideal of $\mathcal{L}_S$.*
(4) *If E/F is ramified, then $\mathcal{L}_S/\mathcal{L}_S^\circ$ is a commutative ring of order q^2, isomorphic to $k[X]/(X^2)$.*
(5) *If E/F is unramified, then $\mathcal{L}_S/\mathcal{L}_S^\circ$ is a noncommutative ring of order q^6 whose isomorphism class depends only on q (and not n).*
(6) *$\mathcal{L}_S^\times \cap \mathrm{GL}_2(F) = H_{S_1}$, and $\mathcal{L}_S^\times \cap B^\times = H_{S_2}$.*

Proof. Claim (1) is easy to check. For claim (2), we calculate the annihilator of $\mathcal{L}_S$ as follows:

$$\begin{aligned}\mathcal{L}_S^* &= [\Delta(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n + \mathbf{V}^n]^* \\ &= \Delta(\mathcal{O}_E)^* \cap (\mathfrak{p}_E^n \times \mathfrak{p}_E^n)^* \cap (\mathbf{V}^n)^*\end{aligned}$$

The three terms to be intersected are

$$\begin{aligned}\Delta(\mathcal{O}_E)^* &= \Delta(E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + C_1 \times C_2, \text{ by Lemma 4.3.1} \\ (\mathfrak{p}_E^n \times \mathfrak{p}_E^n)^* &= \mathfrak{p}_E \times \mathfrak{p}_E + C_1 \times C_2 \\ (\mathbf{V}^n)^* &= (E \times E) \oplus \mathbf{V}^{n+1}, \text{ by Lemma 4.3.2}\end{aligned}$$

We claim the intersection is $\mathcal{L}_S^\circ$. Indeed, for a pair (x_1, x_2) to lie in $\mathcal{L}_S^*$, the first two equations imply $s_1(x_1), s_2(x_2) \in \mathfrak{p}_E$ and $s_1(x_1) \equiv s_2(x_2) \pmod{\mathfrak{p}_E^{n+1}}$, and the third implies $(x_1 - s_1(x_1), x_2 - s_2(x_2)) \in \mathbf{V}^{n+1}$.

Claim (3) follows from the inclusion $\mathbf{V}^n\mathbf{V}^{n+1} \subset \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1}$, which is easily checked.

For claims (4) and (5), let $\mathcal{R}_S = \mathcal{L}_S/\mathcal{L}_S^\circ$. Fix a uniformizer π_E of E.

In the case that E/F is ramified, we have $\mathbf{V}^n = \mathbf{V}^{n+1}$, so there is an isomorphism

$$\mathcal{R}_S \cong \frac{\Delta(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n}{\Delta(\mathfrak{p}_E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1}}.$$

The "numerator" of the right-hand side is the ring of pairs $(x, x+\pi_E^n y) \in \mathcal{O}_E \times \mathcal{O}_E$ with $x, y \in \mathcal{O}_E$. Define a map

$$\begin{array}{rcl} \mathcal{R}_S & \to & k \times k \\ (x, x+\pi_E^n y) & \mapsto & (\overline{x}, \overline{y}), \end{array}$$

where if $z \in \mathcal{O}_E$ we have put $\overline{z} = z \pmod{\mathfrak{p}_E}$. It is easily checked that this map is an isomorphism of (additive) groups; the multiplication law induced on $k \times k$ is $(x_1, y_1)(x_2, y_2) = (x_1 x_2, x_1 y_2 + x_2 y_1)$, which is to say that $\mathcal{R}_S \cong k[X]/(X^2)$.

Now suppose E/F is unramified. In this case $V = \mathbf{V}^n/\mathbf{V}^{n+1}$ is a vector space over k_E of dimension 1. We have $\mathbf{V}^n\mathbf{V}^n \subset \mathfrak{p}_E^n \times \mathfrak{p}_E^n$. On the other hand the image of $\mathfrak{p}_E^n \times \mathfrak{p}_E^n$ in $\mathcal{R}_S$ may be identified with k_E via $(x_1, x_2) \mapsto \pi_E^{-n}(x_1 - x_2)$. For $v, w \in V$, let $v \cdot w$ be the image of $vw \in \mathfrak{p}_E^n \times \mathfrak{p}_E^n$ under this latter map. Then $(v, w) \mapsto v \cdot w$ is a pairing $V \times V \to k_E$ which is k_E-linear in the first variable and satisfies $w \cdot v = (v \cdot w)^q$. This pairing is nondegenerate by part (1) of Lemma 4.3.2: One of the factors of $\mathbf{V}^n\mathbf{V}^n$ is always $\mathfrak{p}_E^n$. Choose an isomorphism $\phi\colon V \to k_E$ of k_E vector spaces in such a way that $v \cdot w = \phi(v)\phi(w)^q$.

We are now ready to describe the ring $\mathcal{R}_S$: let R be the k-algebra of matrices

$$[\alpha, \beta, \gamma] = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^q & \beta^q \\ & & \alpha \end{pmatrix},$$

where $\alpha, \beta, \gamma \in k_E$. Any element of $\mathcal{L}_S$ is of the form $(x, x+\pi_E^n y) + v$, where $x, y \in \mathcal{O}_E$ and $v \in \mathbf{V}^n$. Define a map

$$\begin{array}{rcl} \mathcal{L}_S & \to & R \\ (x, x+\pi_E^n y) + v & \mapsto & [\overline{x}, \overline{y}, \phi(v)]; \end{array}$$

it is easy to see that this map descends to a ring isomorphism $\mathcal{R}_S \to R$. Therefore $\mathcal{R}_S$ is a noncommutative ring of order q^6 whose isomorphism class is independent of n.

For claim (6), we begin with the fact that any element b of $\mathcal{L}_S^\times$ is of the form $(x + \pi^n y, x) + v$, with $x \in \mathcal{O}_E^\times$, $y \in \mathcal{O}_E$, and $v \in \mathbf{V}^n = \mathbf{V}_1^n \times \mathbf{V}_2^n$. If such an element has B-component 1 we must have $x = 1$ and $v = (v_1, 0)$, which is to say that $b = (1 + \pi^n y, 1) + (v_1, 0) \in (1 + \mathfrak{p}_E^n + V^n) \times \{1\}$ is an element of H_S. The argument for $B^\times$ is similar. □

In the sequel, we will construct a representation ρ_S of the unit group $\mathcal{L}_S^\times$ inflated from a representation of the finite group $(\mathcal{L}_S/\mathcal{L}_S^\circ)^\times$. Then when ρ_S

is extended to $\Delta(E^\times)(F^\times \times F^\times)\mathcal{L}^\times$ and induced up to $\mathrm{GL}_2(F) \times B^\times$, the result will realize the Jacquet-Langlands correspondence for representations of $\mathrm{GL}_2(F)$ containing the stratum S. For completeness' sake, we also want to construct the correspondence for supercuspidal representations of level 0. To this end we define the linking order of level 0 by

$$\mathcal{L}_0 = M_2(\mathcal{O}_F) \times \mathcal{O}_B$$

and its double-sided ideal by

$$\mathcal{L}_0^\circ = \mathfrak{p}_F M_2(\mathcal{O}_F) \times \mathfrak{P}_B.$$

Let E be the unique unramified quadratic extension of F and choose embeddings $E \hookrightarrow M_2(F)$, $E \hookrightarrow B$ so that $M_2(\mathcal{O}_F) \cap E = \mathcal{O}_B \cap E = \mathcal{O}_E$. Let $s_1\colon M_2(\mathcal{O}_F) \to E$ and $s_2\colon B \to E$ be the projections as in the previous section, let ν be an additive character of E vanishing on $\mathfrak{p}_E$ but not on $\mathcal{O}_E$, and let $\nu_0\colon \mathbf{A} \to \mathbf{C}^\times$ be the character $\nu_0(x_1, y_1) = \nu(s_1(x_1) - s_2(y_1))$. Then Prop. 4.3.4 has the following analogue in level zero:

Proposition 4.3.5. *The linking order $\mathcal{L}_0$ has the following properties:*

(1) $\mathcal{L}_0^\times$ *is normalized by* $\Delta(E^\times)$.
(2) *With respect to ν_0, the annihilator of $\mathcal{L}_0$ is $\mathcal{L}_0^\circ$.*
(3) $\mathcal{L}_0/\mathcal{L}_0^\circ \cong M_2(k_F) \times k_E$.
(4) $\mathcal{L}_0^\times \cap \mathrm{GL}_2(F) = \mathrm{GL}_2(\mathcal{O}_F)$, *and* $\mathcal{L}_0^\times \cap B^\times = \mathcal{O}_B^\times$.

5. Representations of $\mathcal{L}_S^\times$ and the Fourier transform.

Keep the notations from the previous section: Let $S = (\mathfrak{A}_1, n_1, \alpha_1)$ be a simple stratum in $\mathrm{GL}_2(F)$, let $S' = (\mathfrak{A}_2, n_2, \alpha_2)$ be its corresponding simple stratum in $B^\times$, let $n = n_1$, let $\mathcal{L}_S$ be the associated linking order, let $\mathcal{R}_S$ be its quotient ring by the ideal $\mathcal{L}_S^\circ$, and let ν_S be the associated additive character on $\mathbf{A} = M_2(F) \times B$. Let $\mathbf{G} = \mathrm{GL}_2(F) \times B^\times$. For $g = (g_1, g_2) \in \mathbf{G}$, write

$$\|g\| = |\det g_1|_F \, |\mathrm{N}\, g_2|_F \, .$$

We let μ_S be the unique Haar measure on the additive group $\mathbf{A}$ which is self-dual with respect to ν_S, and let $\mathcal{F}_S$ be the Fourier transform with respect to ψ_S:

$$\mathcal{F}_S f(y) = \int_{\mathbf{A}} f(x)\nu_S(xy)d\mu_S(x).$$

There are translation operators $L, R\colon \mathbf{G} \to \mathrm{Aut}\, C_c^\infty(\mathbf{G})$, defined by $L_g f(y) = f(g^{-1}y)$ and $R_h f(y) = f(yh)$; we have the rules

$$L_g \mathcal{F}_S = \|g\|^2 \mathcal{F}_S R_g, \quad R_h \mathcal{F}_S = \|h\|^{-2} \mathcal{F}_S L_h. \tag{5.0.1}$$

Let $\mathcal{R}_S$ be the k_E-algebra $\mathcal{L}_S/\mathcal{L}_S^\circ$ as in the proof of Prop. 4.3.4.

Proposition 5.0.1. *The measure of $\mathcal{L}_S^\circ$ with respect to μ_S is $\#\mathcal{R}_S^{-1/2}$.*

Proof. Let $\chi_{\mathcal{L}_S}$ be the characteristic function of $\mathcal{L}_S$. Then

$$\mathcal{F}_S\chi_{\mathcal{L}_S}(y) = \int_{\mathcal{L}_S} \nu_S(xy)\, d\mu_S(x)$$

is supported on $\mathcal{L}_S^\perp = \mathcal{L}_S^\circ$ and equals $\mu_S(\mathcal{L}_S)$ there; *i.e.* $\mathcal{F}_S\chi_{\mathcal{L}_S} = \mu_S(\mathcal{L}_S)\chi_{\mathcal{L}_S^\circ}$. Similarly $\mathcal{F}_S^2\chi_{\mathcal{L}_S} = \mu_S(\mathcal{L}_S)\mu_S(\mathcal{L}_S^\circ)\chi_{\mathcal{L}_S}$. On the other hand, since μ_S is self-dual, we must have $\mathcal{F}_S^2\chi_{\mathcal{L}_S} = \chi_{\mathcal{L}_S}$, implying $\mu_S(\mathcal{L}_S)\mu_S(\mathcal{L}_S^\circ) = 1$. Since $\mu_S(\mathcal{L}_S) = \#\mathcal{R}_S\mu_S(\mathcal{L}_S^\circ)$, the result follows. □

Let $\mathcal{C}(\mathcal{R}_S)$ be the space of complex-valued functions on $\mathcal{R}_S$. Note that the character ν_S vanishes on $\mathcal{L}_S^\circ$ and therefore induces a well-defined additive character of $\mathcal{R}_S$. We identify $\mathcal{C}(\mathcal{R}_S)$ with a subspace of $C_c^\infty(\mathbf{A})$.

Prop. 5.0.1 together with the key property that $\mathcal{L}_S$ and $\mathcal{L}_S^\circ$ are dual lattices imply the following:

Proposition 5.0.2. *The Fourier transform $f \mapsto \mathcal{F}_S f$ preserves the space $\mathcal{C}(\mathcal{R}_S)$. For $f \in \mathcal{C}(\mathcal{R}_S)$, we have*

$$\mathcal{F}_S f(y) = \#\mathcal{R}_S^{-1/2} \sum_{x\in\mathcal{R}_S} f(x)\nu_S(xy). \tag{5.0.2}$$

Recall that the data of S and S' determine characters ψ_{α_1} and ψ_{α_2} of the subgroups $U_{\mathfrak{A}_1}^{n_1}$ and $U_{\mathfrak{A}_2}^{n_2}$ of $\mathfrak{A}_1^\times$ and $\mathfrak{A}_2^\times$, respectively. The product group $U_{\mathfrak{A}_1}^{n_1}\times U_{\mathfrak{A}_2}^{n_2} = 1+\mathfrak{p}_E^n\mathfrak{A}_1\times\mathfrak{p}_E^n\mathfrak{A}_2$ is a subgroup of $\mathcal{L}_S^\times$, and the product character $\psi_S = \psi_{\alpha_1}\times\psi_{\alpha_2}^{-1}$ vanishes on $\left(U_{\mathfrak{A}_1}^{n_1}\times U_{\mathfrak{A}_2}^{n_2}\right)\cap(1+\mathcal{L}_S^\circ) = U_{\mathfrak{A}_1}^{n+1}\times U_{\mathfrak{A}_2}^{n+1}$. Therefore if we let $\mathbf{U}_S$ be the image of $U_{\mathfrak{A}_1}^{n_1}\times U_{\mathfrak{A}_2}^{n_2}$ in $\mathcal{R}_S$, then ψ_S induces a well-defined nontrivial character of $\mathbf{U}_S$.

We are now ready to construct the special representation ρ_S. Its relevant properties are as follows:

Theorem 5.0.3. *There exists an irreducible representation ρ_S of $\mathcal{R}_S^\times$ satisfying the conditions:*

1. *ρ_S vanishes on $k^\times \subset \mathcal{R}_S^\times$.*
2. *$\rho_S|_{\mathbf{U}_S}$ is a sum of copies of ψ_S.*
3. *If $f \in \mathcal{C}(\rho_S)$ is a matrix coefficient, then $\mathcal{F}_S f$ is supported on $\mathcal{R}_S^\times$ and satisfies $\mathcal{F}_S f(y) = \pm f(y^{-1})$, all $y \in \mathcal{R}_S^\times$. The sign is 1 if E/F is ramified and -1 otherwise.*

Remark 5.0.4. These three properties correspond to the three desired properties of the representation Π_S listed at the beginning of Section 4.

Proof. First, consider the case where $E = F(\alpha)$ is a ramified extension of F. Then by Prop. 4.3.4 we have an isomorphism $\mathcal{R}_S \cong k[X]/(X^2)$ with respect to which ν_S is a nontrivial additive character which vanishes on $k \subset \mathcal{R}_S$. The subgroup $\mathbf{U}_S \subset \mathcal{R}_S^\times$ corresponds to $\{1 + aX \mid a \in k\}$. There is obviously a unique character ρ_S of $\mathcal{R}_S^\times$ lifting ψ_S and vanishing on $k^\times$. It takes the form

$$\rho_S(a+bX) = \Psi(a^{-1}b),$$

where $\Psi\colon k \to \mathbf{C}^\times$ is a nontrivial character determined by ψ_S. That ρ_S satisfies claim (3) is a simple calculation in the commutative ring $\mathcal{R}_S$.

The case of $e=1$ is far more subtle. The required representation ρ_S is related to the construction of the Weil representation of a symplectic group over a finite field. We present a self-contained version of the construction in the following section. □

5.1. Fourier transforms on the Heisenberg group. In this section, k is the finite field with q elements and k_2/k is a quadratic field extension. As in the proof of Prop. 4.3.4, let R be the k-algebra of matrices of the form

$$[\alpha,\beta,\gamma] = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^q & \beta^q \\ & & \alpha \end{pmatrix},$$

where $\alpha,\beta,\gamma \in k_2$. Let $U \subset R^\times$ be the subgroup of matrices of the form $[1,0,\gamma]$, and let $U^1 \subset U$ be the subgroup consisting of those $[1,0,\gamma]$ for which $\operatorname{Tr}_{k_2/k}\gamma = 0$. Note that the center of $R^\times$ is $k^\times U$.

Let ℓ be a prime not dividing q, and let $\nu_k\colon k \to \overline{\mathbf{Q}}_\ell^\times$ be a nontrivial additive character. Define an additive character ν_R of R by $\nu_R([\alpha,\beta,\gamma]) = \nu_k(\operatorname{Tr}_{k_2/k}\gamma)$. Let $\mathcal{F}$ be the Fourier transform with respect to ν_R.

Theorem 5.1.1. *For each character ψ of U which is nontrivial on U^1, there exists a representation ρ_ψ of $R^\times$ satisfying the properties:*

1. *ρ_ψ is trivial on $k^\times$.*
2. *$\rho_\psi|_U$ is a multiple of ψ.*
3. *For a matrix coefficient $f \in \mathcal{C}(\rho_\psi)$, the Fourier transform $\mathcal{F}f$ is supported on $R^\times$ and satisfies $\mathcal{F}f(y) = -f(y^{-1})$ for $y \in R^\times$.*

The proof will occupy the rest of the section. To construct ρ_ψ, we will build a nonsingular projective curve $X/\overline{k}$ admitting an action of $R^\times$, and find ρ_ψ in the ℓ-adic cohomology of X.

First, we recognize a relationship between $R^\times$ and the unitary group GU_3. Let Φ be the matrix

$$\Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

and let $\mathrm{GU}_3(k)$ be the subgroup of matrices $M \in \mathrm{GL}_3(k_2)$ satisfying $M^*\Phi M = \lambda(M)\Phi$ for a scalar $\lambda(M)$. (Here M^* is the conjugate transpose of M.) Then a large part of the Borel subgroup of $\mathrm{GU}_3(k)$ is contained in

$R^\times$. Indeed, if $M \in R^\times$, we can measure the defect of M from lying in $\mathrm{GU}_3(k)$ by a homomorphism $\delta\colon R^\times \to k$ defined by

$$\Phi^{-1}M^*\Phi M = \lambda(M)\begin{pmatrix}1 & & \delta(M)\\ & 1 & \\ & & 1\end{pmatrix}. \tag{5.1.1}$$

Explicitly, $\delta([\alpha,\beta,\gamma]) = \alpha\gamma^q + \alpha^q\gamma - \beta^{q+1}$. Let $R^1 = \ker\delta$; then $R^1 \subset \mathrm{GU}_3(\mathbf{F}_q)$.

The algebraic group GU_3 acts on the projective plane $\mathbf{P}^2_k$ in the usual manner; the group $\mathrm{GU}_3(k)$ preserves the equation $y^{q+1} = x^qz + xz^q$ in projective coordinates. This equation defines a nonsingular projective curve X^1 of genus $q(q-1)/2$ with an action of $\mathrm{GU}_3(k)$. Let $X = R^\times \times_{R^1} X^1$; this is a smooth projective curve with an action of $R^\times$. Let ℓ be a prime distinct from the characteristic of k, and let $\rho\colon R^\times \to H^1(X,\overline{\mathbf{Q}}_\ell)$ be the representation of $R^\times$ on the first cohomology of X. The degree of ρ is $q^2(q-1)$. Note that ρ is trivial on $k^\times \subset R^\times$.

Since U lies in the center of R^1, we have a decomposition $\rho = \oplus_\psi \rho_\psi$ of ρ into its irreducible ψ-isotypic components, where ψ runs over characters of U which are nontrivial on U^1; each has dimension q. We claim that ρ_ψ is irreducible. By the discrete Stone-von Neumann theorem there is a unique irreducible representation ς of the p-Sylow subgroup $H \subset R^\times$ which lies over ψ, and furthermore $\deg\varsigma = q$. Since the restriction of ρ_ψ to H lies over ψ and has degree q, it must agree with ς. Therefore ρ_ψ is irreducible.

Let $T \subset R^\times$ be the subgroup of diagonal matrices, so that $T \cong k_2^*$. The Lefshetz fixed-point theorem can easily be used to compute the restriction of ρ_ψ to T:

Proposition 5.1.2. *The restriction of ρ_ψ to T is exactly the direct sum of those characters χ of T which are nontrivial on $T/k^\times$.*

For a matrix coefficient $f \in \mathcal{C}(\rho_\psi)$, we consider the Fourier transform $\mathcal{F}f$. We claim that the Fourier transform $\mathcal{F}f$ is supported on $R^\times$. Indeed, if $y \in R$ is not invertible then $uy = y$ for all $u \in U$. It follows from this that $\mathcal{F}f(uy) = \mathcal{F}f(uy) = \psi(u)^{-1}\mathcal{F}f(y)$ for all $u \in U$; since ψ is nontrivial we see that $\mathcal{F}f(y) = 0$.

Next we claim that for $y \in R^\times$ we have

$$\mathcal{F}f(y) = -f(y^{-1}). \tag{5.1.2}$$

Formally, we have $\mathcal{F}f(y) = f(y^{-1})\mathcal{F}(1)$, so in fact is suffices to show that

$$\mathcal{F}f(1) = -f(1). \tag{5.1.3}$$

It is enough to prove Eq. 5.1.3 in the case that f equals the character of ρ_χ. This is because the character of ρ_ψ generates $\mathcal{C}(\rho_\psi)$ as an $(R^\times \times R^\times)$-module, and because the property in Eq. 5.1.2 is invariant when we replace f by any of its $(R^\times \times R^\times)$-translates. Therefore let $f = \operatorname{Tr}\rho_\psi$ be the character of ρ_ψ.

We have

$$\mathcal{F}f(1) = \frac{1}{q^3}\sum_{x\in R^\times} \operatorname{Tr}\rho_\psi(x)\nu_R(x).$$

We observe that the term $\operatorname{Tr}\rho_\psi(x)\nu_R(x)$ only depends on the conjugacy class of x in $R^\times$. We first dispense with those terms in the above sum for which x has eigenvalues in $k^\times$. The sum over these terms vanishes, because for such an x we have $\operatorname{Tr}\rho_\psi(xu)\nu_R(xu) = \psi(u)\operatorname{Tr}\rho_\psi(x)\nu_R(x)$ for all $u \in U^1$. All that remains are the elements $x = [\alpha, \beta, \gamma]$ with $\alpha \in k_2^\times\backslash k^\times$, and each of these are conjugate to a unique element of the form tu, with $t \in T\backslash k^\times$ and $u \in U$. Each such conjugacy class has cardinality q^2, and the value of $\operatorname{Tr}\rho_\psi(tu)$ on such a class is $-\psi(u)$. Therefore

$$\mathcal{F}f(1) = -\frac{1}{q}\sum_{t\in T\backslash k^\times}\sum_{u\in U}\psi(u)\nu_r(tu).$$

This reduces to $-q = -f(1)$ by a simple calculation, thus completing the proof of Theorem 5.1.1.

Remark 5.1.3. The curve X is isomorphic (over $\overline{k}$) to the Fermat curve $x^{q+1} + y^{q+1} + z^{q+1} = 0$. It appears in the construction of the so-called unipotent representation of $\mathrm{GU}_3(k)$; see [Lus78].

There is also a connection to the theory of the discrete Weil representation. We have $R^\times = T \rtimes H$, where H is the p-Sylow subgroup of $R^\times$. Furthermore, $U \cap H = U^1$ is the center of H. Write ψ^1 for the (nontrivial) restriction of ψ to U^1. The group $H/\ker\psi^1$ is a discrete Heisenberg group. By the Stone-von Neumann theorem, there is a unique irreducible representation V_ψ of H lying over ψ.

The group T embeds as a nonsplit torus in $\mathrm{SL}_2(k)$, and the conjugation action of T on $H/\ker\psi^1$ extends to an action of $\mathrm{SL}_2(k)$ in a manner which fixes each element of U^1. The uniqueness property of V_ψ means that if $\alpha \in \mathrm{SL}_2(k)$ and ${}^\alpha V_\psi$ is the conjugate representation $g \mapsto V_\psi(\alpha(g))$, then there is an isomorphism $W(\alpha)\colon {}^\alpha V_\psi \cong V_\psi$ which is well-defined up to a scalar. The operators $W(\alpha)$ give an *a priori* projective representation of $\mathrm{SL}_2(k)$ on the underlying space of V_ψ which in fact lifts to a proper representation W, the Weil representation. See for instance [Gér77]. The operators $W(\alpha)$ together with the representation V_ψ give a q-dimensional representation of $\mathrm{SL}_2(k) \rtimes H$; restricting this to $T \rtimes H/\ker\psi^1 = R^\times/\ker\psi^1$ gives the representation ρ_ψ we have constructed in Theorem 5.1.1.

When W is restricted to a nonsplit torus of $\mathrm{SL}_2(k)$, each nontrivial character appears at most once, see Theorem 3 of [GH08]; this implies the property of ρ_ψ given in Prop. 5.1.2. The equation of part (3) of Thm. 5.1.1 may be established once one has a formula for the character of the "Heisenberg-Weil representation" of $\mathrm{SL}_2(k) \rtimes H$; for this, see Theorem 2.2.1 of [GH07]. We have chosen to provide a cohomological proof, however, because of its relative simplicity and because we believe the curve X appears as a connected component of the stable reduction of the Lubin-Tate curve, *cf.* the introduction.

The case of $e = 1$ in Theorem 5.0.3 follows from Theorem 5.1.1 once we observe the following:

(1) There exists an isomorphism $\mathcal{R}_S \to R$.
(2) Under this isomorphism, ν_S is identified with an additive character of the form ν_R described above.
(3) The subgroup $\mathbf{U}_S \in \mathcal{R}_S^\times$ is identified with $U \subset R^\times$.
(4) Choose an isomorphism $\iota\colon \mathbf{C} \to \overline{\mathbf{Q}}_\ell$, then the complex character ψ_S of $\mathbf{U}_S$ is identified with an ℓ-adic character ψ of U.
(5) The condition that $S = (M_2(\mathcal{O}_F), n, \alpha)$ be a simple stratum implies that the reduction of $\pi_F^n\alpha$ has irreducible characteristic polynomial, which in turn implies that ψ is nontrivial on U^1.
(6) The ℓ-adic representation ρ_ψ constructed in Theorem 5.1.1 with respect to the data of ν_R and ψ may be transported via ι^{-1} to a complex representation of $\mathcal{R}_S^\times$ which satisfies the requirements of Theorem 5.0.3.

5.2. The case of level 0. The linking order of level 0 is $\mathcal{L}_0 = M_2(\mathcal{O}_F) \times \mathcal{O}_B$, and its quotient ring $\mathcal{R}_0$ is $M_2(k) \times k_E$. The additive character ν_0 is of the form

$$\nu_0(x, y) = \nu(\mathrm{Tr}_{M_2(k)/k}\, x - \mathrm{Tr}_{k_E/k}\, y),$$

where ν is a nontrivial additive character of k, and $\mathcal{F}_0$ is the Fourier transform with respect to this character. Let θ be a character of $k_E^\times$. Assume that θ is *regular*, meaning that it does not factor through the norm map $k_E^\times \to k^\times$. It is well-known that there is an irreducible cuspidal representation η_θ of $\mathrm{GL}_2(k_F)$ corresponding to θ. The character of this representation takes the value $-(\theta(\alpha) + \theta(\alpha^q))$ on an element $g \in \mathrm{GL}_2(k_F)$ with distinct eigenvalues $\alpha, \alpha^q \in k_E$ not lying in k_F.

Let ρ_θ be the character $\eta_\theta \otimes \theta^{-1}$ of $\mathcal{R}_0^\times = \mathrm{GL}_2(k_F) \times k_E^\times$. The following proposition concerns the Fourier transforms of matrix coefficients of ρ_θ.

Proposition 5.2.1. *For $f \in \mathcal{C}(\rho_\theta)$ we have that $\mathcal{F}_0 f$ is supported on $\mathcal{R}_0^\times$ and satisfies $\mathcal{F}_0 f(y) = -f(y^{-1})$ for $y \in \mathcal{R}_0^\times$.*

Proof. We reduce this to two calculations relative to the rings $M_2(k)$ and k_E, respectively. Let $R_1 = M_2(k)$, $R_2 = k_E$, and for $i = 1, 2$ let ν_i be the additive character of R_i defined by $\nu_i(x) = \nu_0(\mathrm{Tr}_{R_i/k}\, x)$, so that $\nu_S(x, y) = \nu_1(x)\nu_2(-y)$.

Write $\tau_{\theta,\nu}$ for the Gauss sum $\sum_{\alpha\in k_E^\times}\theta(\alpha)\nu(\mathrm{Tr}_{k_E/k_F}\alpha)$. We claim that for all $f\in\mathcal{C}(\eta_\theta)$ we have that $\mathcal{F}_1 f$ is supported on $R_1^\times=\mathrm{GL}_2(k)$ and satisfies

$$\mathcal{F}_1 f(y)=-\tau_{\theta,\nu}f(y^{-1}).$$

This is a straightforward calculation. It is a special case of a calculation of epsilon factors of irreducible representations of GL_n which appears in [Kon63]; these can always be expressed as a product of Gauss sums. See also [Mac73], Chap. IV.

The corresponding analysis for $R_2=k_E$ is simpler: define a Fourier transform $\mathcal{F}_2$ on $\mathcal{C}(R_2)$ by $\mathcal{F}f(y)=q^{-1}\sum_{x\in k_E^\times}f(x)\nu_2(-xy)$. Then the Fourier transform of the character θ^{-1} is supported on $k_E^\times$ and equals $q^{-1}\tau_{\theta^{-1},\nu^{-1}}\theta$.

We may now complete the proof of the proposition. For a decomposable element $f=f_1\otimes f_2$ of $\mathcal{C}(R_1\times R_2)$, we have $\mathcal{F}_0 f=\mathcal{F}_1 f_1\otimes\mathcal{F}_2 f_2$. If this same f is a matrix coefficient for $\rho_\theta=\eta_\theta\otimes\theta^{-1}$ then we must have $\mathcal{F}_0 f=-q^{-2}\tau_{\theta,\nu}\tau_{\theta^{-1},\nu^{-1}}f(y^{-1})$. We now use the classical identity of Gauss sums $\tau_{\theta,\nu}\tau_{\theta^{-1},\nu^{-1}}=\#k_E=q^2$, and the proof is complete. □

6. Construction of the Jacquet-Langlands Correspondence

The construction of the family of rings $\mathcal{L}_S$ together with the representations ρ_S of $\mathcal{L}_S^\times$ will now be used to construct certain representations Π_S of $GL_2(F)\times B^\times$. We will then use Cor. 3.0.2 to show that the family Π_S realizes the Jacquet-Langlands Correspondence. This will involve showing that the matrix coefficients of Π_S satisfy the functional equation in Eq. 3.0.2 for sufficiently many χ. The heart of that calculation has already been completed in Theorem 5.0.3.

Recall that $\mathbf{G}=\mathrm{GL}_2(F)\times B^\times$; this group has center $Z(\mathbf{G})=F^\times\times F^\times$. Let $S=(\mathfrak{A},n,\alpha)$ be a simple stratum in $M_2(F)$, and let $S'=(\mathfrak{A}',n',\alpha')$ be its corresponding simple stratum in B. From these data we have constructed a linking order $\mathcal{L}_S$ and an irreducible representation ρ_S of $\mathcal{L}_S^\times$. Let $\ell=n/e$, so that every supercuspidal representation of GL_2 containing S has level ℓ, and likewise for $B^\times$. The intersection of $Z(\mathbf{G})$ with $\mathcal{L}_S^\times$ is

$$Z(\mathbf{G})\cap\mathcal{L}_S^\times=\left\{(z_1,z_2)\in\mathcal{O}_F^\times\times\mathcal{O}_F^\times\mid v_F(z_1-z_2)\geq\ell\right\}.$$

Here v_F is the valuation on F. By Theorem 5.0.3, ρ_S vanishes on the diagonally embedded subgroup $\Delta(F^\times)\cap\mathcal{L}_S^\times$. Choose a character ω of $Z(\mathbf{G})$ which vanishes on $\Delta(F^\times)$ and agrees with the central character of ρ_S on $Z(\mathbf{G})\cap\mathcal{L}_S^\times$. We identify ω with a character of $F^\times$ via its restriction to $F^\times\times\{1\}$.

We now extend ρ_S to a representation on a larger group which contains $Z(\mathbf{G})$ and which intertwines ρ_S. Define a group $\mathcal{K}_S$ by

$$\mathcal{K}_S=Z(\mathbf{G})\Delta(E^\times)\mathcal{L}_S^\times.$$

(Recall that $\Delta(E^\times)$ normalizes $\mathcal{L}_S^\times$, so this is indeed a group.) There is a unique extension of ρ_S to a representation $\rho_{S,\omega}$ of $\mathcal{K}_S$ which satisfies the conditions:

(1) $\rho_{S,\omega}|_{Z(\mathbf{G})} = \omega$,
(2) For $\beta \in E^\times$, $\rho_{S,\omega}(\Delta(\beta)) = (-1)^{v_E(\beta)}$ if E/F is ramified,
(3) For $\beta \in E^\times$, $\rho_{S,\omega}(\Delta(\beta)) = 1$ if E/F is unramified.

The group $\mathcal{K}_S$ is open and compact modulo its center. We may now define the representation $\Pi_{S,\omega}$ of $\mathbf{G}$ as the induction of $\rho_{S,\omega}$ with compact support:

$$\Pi_{S,\omega} = \operatorname{Ind}_{\mathcal{K}_S}^{\mathbf{G}} \rho_{S,\omega}.$$

We wish to confirm that $\Pi_{S,\omega}$ satisfies the desired properties (1)-(3) listed at the beginning of Section 4. It is already apparent that (1) $\Pi_{S,\omega}$ vanishes on $\Delta(F^\times)$. For property (2) we have the following:

Theorem 6.0.1. $\Pi_{S,\omega}$ *is the direct sum of representations of* $\mathbf{G}$ *of the form* $\pi \otimes \check{\pi}'$, *where* π *(resp.,* π'*) is a minimal supercuspidal irreducible representation of* $\mathrm{GL}_2(F)$ *(resp.,* $B^\times$*) having central character* ω *and containing the stratum* S *(resp.,* S'*). Every representation of either group having the above properties is contained in* $\Pi_{S,\omega}$.

Proof. Note that $\mathcal{K}_S \subset J_S \times J_{S'}$ is a subgroup of finite index. Let

$$M = \operatorname{Ind}_{\mathcal{K}_S}^{J_S \times J_{S'}} \rho_{S,\omega}.$$

Then M is a direct sum of irreducible representations of $J_S \times J_{S'}$ of the form $\Lambda \otimes \check{\Lambda}'$. By Theorem 5.0.3, such a $\Lambda \otimes \check{\Lambda}'$ lies over the character $\psi_S = \psi_\alpha \otimes \psi_{\alpha'}^{-1}$ of $U_S \times U_{S'}$. Therefore we have $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ and $\Lambda' \in C(\psi_{\alpha'}, \mathfrak{A}')$. By Theorem 2.2.5, $\pi = \operatorname{Ind}_{J_S}^{\mathrm{GL}_2(F)} \Lambda$ is an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ containing S. Since $\rho_{S,\omega}$ has central character ω, the same is true of π. The reasoning is similar for $\pi' = \operatorname{Ind}_{J'_S}^{B^\times} \Lambda'$.

Now assume π is an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ containing S with central character ω. We claim that π is contained in $\Pi_{S,\omega}|_{\mathrm{GL}_2(F)}$. Since $\Pi_{S,\omega}$ is induced from the representation $\rho_{S,\omega}$ of $\mathcal{K}_S$, the restriction of $\Pi_{S,\omega}$ to $\mathrm{GL}_2(F)$ contains $\operatorname{Ind}_{\mathcal{K}_S \cap \mathrm{GL}_2(F)}^{\mathrm{GL}_2(F)} \rho_{S,\omega}$. Therefore to show that π is contained in $\Pi_{S,\omega}|_{\mathrm{GL}_2(F)}$ it suffices to prove that $\pi|_{\mathcal{K}_S \cap \mathrm{GL}_2(F)}$ meets $\rho_{S,\omega}|_{\mathcal{K}_S \cap \mathrm{GL}_2(F)}$. By Prop. 4.3.4 we have

$$\mathcal{K}_S \cap \mathrm{GL}_2(F) = F^\times H_S.$$

The central characters of π and $\rho_{S,\omega}$ agree on $F^\times$ by hypothesis. Therefore it suffices to show that $\pi|_{H_S}$ meets $\rho_S|_{H_S}$. By Theorem 2.2.5, π contains a representation $\Lambda \in C(\mathfrak{A}, \psi_\alpha)$. This means that the restriction of π to H_S contains $\Lambda|_{H_S}$, which must agree with $\rho_S|_{H_S}$ by Theorem 4.2.1. The case of a representation of $B^\times$ is similar. □

The third required property of $\Pi_{S,\omega}$, concerning the zeta functions attached to matrix coefficients of this representation, shall follow from Prop. 5.0.3. We will start by translating Prop. 5.0.3 into a statement concerning the Fourier transforms of matrix coefficients of $\Pi_{S,\omega}$.

For a function f on $\mathbf{G}$, and a real number s, let f_s be the function

$$f_s(g) = f(g) \left\| g_1 \right\|^{s-2} \left\| g_2 \right\|^{-s}.$$

If $f \in \mathcal{C}(\Pi_{S,\omega})$, we wish to consider Fourier transforms of the functions f_s. The functions f_s are supported on $\mathcal{K}_S$, which is not compact, so their Fourier transforms do not *a priori* converge. Nonetheless we may formally define the Fourier transform $\widehat{f_s}$ by integrating $f_s(x)\psi_{\mathbf{A}}(xy)$ over each of the (compact) cosets of $\mathcal{L}_S^\times$ in $\mathbf{G}$. Since f is a linear combination of $\mathbf{G} \times \mathbf{G}$-translates of vectors in $\mathcal{C}(\rho_S)$, which are in turn supported on $\mathcal{L}_S^\times$, we see that the integral vanishes on all but finitely many of the cosets. We now evaluate $\widehat{f_s}$.

Proposition 6.0.2. *For a matrix coefficient $f \in \mathcal{C}(\Pi_{S,\omega})$, we have*

$$\widehat{f_s} = -\check{f}_{2-s}. \tag{6.0.1}$$

Proof. We will first prove the corresponding statement relative to the Fourier transform $\mathcal{F}_S$:

$$\mathcal{F}_S f_s = \pm \check{f}_{2-s}, \tag{6.0.2}$$

where the sign is 1 if E/F is ramified and -1 otherwise. It will suffice to prove Eq. 6.0.2 for matrix coefficients $f \in \mathcal{C}(\rho_S)$ supported on the group $\mathcal{L}_S^\times$. Indeed, glancing at the rules in Eq. 5.0.1 shows that the validity of Eq. 6.0.2 is unchanged upon replacing f by $L_g R_h f$ for elements $g, h \in \mathbf{G}$, and these translates span $\mathcal{C}(\Pi_{S,\omega})$ as f runs through $\mathcal{C}(\rho_S)$. But for $f \in \mathcal{C}(\rho_S)$, Eq. 6.0.2 follows from Theorem 5.0.3, because $f_s = f$.

To derive Eq. 6.0.1 from Eq. 6.0.2 we must compare the Fourier transforms $\hat{f}$ and $\mathcal{F}_S f$. The first transform is taken relative to the additive character $\psi_{\mathbf{A}}$, while the second is taken relative to the character ν_S. The characters are related by $\nu_S(x) = \psi_{\mathbf{A}}(\Delta(\beta)^{-1}x)$ for an element $\beta \in E^\times$ of valuation n; formally we have $\hat{f} = \left\| \Delta(\beta) \right\|^{-1} R_\beta \mathcal{F}_S f$. Applying this to the function f_s, we see that

$$\begin{aligned} \widehat{f_s} &= \left\| \Delta(\beta) \right\|^{-1} R_\beta \mathcal{F}_S f_s \\ &= \pm \left\| \Delta(\beta) \right\|^{-1} R_\beta (\check{f})_{2-s} \\ &= \pm (R_\beta \check{f})_{2-s}, \end{aligned}$$

where the sign is positive if and only if E/F is ramified. If E/F is ramified, then $\beta \in E^\times$ has odd valuation, and $R_\beta \check{f} = -\check{f}$ because $\rho_{S,\omega}$ takes the value -1 on such elements. If E/F is unramified, then $\rho_{S,\omega}(\Delta(\beta)) = 1$, and therefore $R_\beta \check{f} = \check{f}$. The proposition follows. $\square$

We are ready to prove the appropriate functional equation for the zeta functions attached to $\Pi_{S,\omega}$. Recall that for an admissible representation Π of $\mathbf{G}$, and for $\Phi \in \mathbf{C}_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi_{S,\omega})$, we defined the zeta function

$$\begin{aligned}\zeta(\Phi, f, s) &= \int_{\mathbf{G}} \Phi(g) f(g) \left\|g_1\right\|^s \left\|g_2\right\|^{2-s}\ d\mu^\times(g)\\ &= \int_{\mathbf{G}} \Phi(g) f_s(g)\ d\mu(g)\end{aligned}$$

where μ is a Haar measure on $\mathbf{A}$.

Theorem 6.0.3. *For all $\Phi \in \mathcal{C}_c^\infty(\mathbf{A})$ all $f \in \mathcal{C}(\Pi_\omega)$, and all characters χ of $F^\times$ of conductor not exceeding ℓ, we have*

$$\zeta(\Phi, \chi f, s) = -\zeta(\hat{\Phi}, \chi^{-1}\check{f}, 2-s).$$

Proof. It suffices to prove the claim for $\chi = 1$. Indeed, if $f \in \mathcal{C}(\Pi_{S,\omega})$, then χf lies in $\mathcal{C}(\Pi_{S',\chi^2\omega})$ for a different simple stratum $S' = (\mathcal{A}_1, n_1, \alpha_1')$. (Explicitly: let $\beta \in \mathfrak{p}_E^{-n}$ be such that $(\chi \circ N_{E/F})(1+x) = \psi_F(\mathrm{Tr}_{E/F}\, \beta x)$ for all $x \in \mathfrak{p}_E^n$; then $\alpha_1' = \alpha_1 + \beta$.)

Assume therefore that $\chi = 1$. We will take the measure $d\mu$ to equal $d\mu_\psi$, the measure dual to the character $\psi_{\mathbf{A}}$. Since $\hat{\hat{\Phi}}(x) = \Phi(-x)$ we have that $\zeta(\hat{\hat{\Phi}}, f, s) = \zeta(\Phi, f, s)$ by a change of variable $g \mapsto -g$ in the integral. Now we apply Prop. 6.0.1:

$$\begin{aligned}\zeta(\Phi, f, s) &= \zeta(\hat{\hat{\Phi}}, f, s)\\ &= \int_{\mathbf{G}} \hat{\hat{\Phi}}(g) f_s(g)\ d\mu_\psi(g)\\ &= \int_{\mathbf{G}} \hat{\Phi}(g) \hat{f}_s(g)\ d\mu_\psi(g)\\ &= -\int_{\mathbf{G}} \hat{\Phi}(g) \check{f}_{2-s}(g)\ d\mu_\psi(g)\\ &= -\zeta(\hat{\Phi}, \check{f}, 2-s).\end{aligned}$$

□

6.1. The construction in level zero. The preceding constructions carry over easily to the case of level zero. Let E be an unramified quadratic extension of F. Letting θ denote a regular character of $k_E^\times$, we constructed in Section 5.2 a representation ρ_χ of the unit group of the linking order $\mathcal{L}_0$. Choose a central character ω of $F^\times \times F^\times$ which agrees with the central character of ρ_θ on $(F^\times \times F^\times) \cap \mathcal{L}_0^\times = \mathcal{O}_F^\times \times \mathcal{O}_F^\times$. Extend ρ_θ to a representation $\rho_{\theta,\omega}$ of $\mathcal{K}_0 = (F^\times \times F^\times)\mathcal{L}_0$ agreeing with ω on the center. Finally, let $\Pi_{\theta,\omega}$ be the induced representation of $\rho_{\theta,\omega}$ from $\mathcal{K}_0$ up to $\mathrm{GL}_2(F) \times B^\times$.

Then Thm. 6.0.1 has the following analogue:

Theorem 6.1.1. *Let π be a minimal irreducible admissible representation of $\mathrm{GL}_2(F)$ (resp., $B^\times$) with central character ω (resp., ω^{-1}). The following are equivalent:*

(1) *π has level zero, and the restriction of π to $\mathrm{GL}_2(\mathcal{O}_F)$ (resp., $\mathcal{O}_B^\times$) contains a representation inflated from the representation η_θ of $\mathrm{GL}_2(k)$ (resp., the character θ of $k_E^\times$.)*

(2) *π is contained in $\Pi_{\theta,\omega}|_{\mathrm{GL}_2(F)}$ (resp., $\check{\pi}$ is contained in $\Pi_{\theta,\omega}|_{B^\times}$).*

Similarly, Prop. 6.0.3 has this analogue:

Theorem 6.1.2. *For $\Phi \in C_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi_{\omega,\theta})$, we have*

$$\zeta(\Phi, \chi f, s) = -\zeta(\hat{\Phi}, \chi^{-1}\check{f}, 2-s)$$

for all characters χ of $F^\times$ which are trivial on $1 + \mathfrak{p}_F$.

The proofs of Thm. 6.1.1 and Prop. 6.1.2 run exactly the same as those of Thm. 6.0.1 and Prop. 6.0.3.

6.2. Conclusion of the construction. Our construction of the Jacquet-Langlands correspondence is nearly complete.

Theorem 6.2.1. *For every irreducible representation π' of $B^\times$ of dimension greater than one, there is a supercuspidal representation π of $\mathrm{GL}_2(F)$ for which π and π' correspond. Every supercuspidal representation of $\mathrm{GL}_2(F)$ arises this way.*

Proof. By Theorem 2.2.2 we may twist π' to assume either that $\check{\pi}'$ contains a simple stratum S', or else that it is level zero. In the first case, let $S = (\mathfrak{A}, n, \alpha)$ be the corresponding stratum in $M_2(F)$. Applying Theorem 6.0.1, $\check{\pi}'$ is contained in $\Pi_{S,\omega}|_{B^\times}$, where ω is the central character of π'. Suppose π is a representation of $\mathrm{GL}_2(F)$ appearing in $\mathrm{Hom}_{B^\times}(\check{\pi}, \Pi_{S,\omega})$. Then $\pi \otimes \check{\pi}'$ appears in $\Pi_{S,\omega}$.

Applying Theorem 6.0.1 again, we find that π contains S. Combining Cor. 3.0.2 with Prop. 6.0.3 shows that π' and π correspond.

The logic is the same if π' has level zero: In this case $\check{\pi}'$ contains a character of $\mathcal{O}_B^\times$ inflated from a character θ of a quadratic extension of k, so that $\check{\pi}'$ is contained in $\Pi_{\theta,\omega}|_{B^\times}$. Proceeding as above, we find a representation π of $\mathrm{GL}_2(F)$ corresponding to π'.

If π is a given supercuspidal representation of $\mathrm{GL}_2(F)$, the argument above may be reversed to find a representation π' of $B^\times$ which corresponds to it. This concludes the proof. □

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