JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Benoît RITTAUD

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Tome 22, n° 2 (2010), p. 449-474.

<http://jtnb.cedram.org/item?id=JTNB_2010__22_2_449_0>

Journal de Théorie des Nombres
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On subsequences of convergents to a quadratic irrational given by some numerical schemes

par Benoît RITTAUD

Résumé. Un irrationnel quadratique α étant donné, nous nous intéressons à la manière dont une fonction f convenablement choisie produit des sous-suites de réduites de α. Nous étudions trois schémas numériques : les méthodes type sécante et certaines généralisations formelles, qui conduisent à des sous-suites à récurrence linéaire ; la méthode de la fausse position, qui conduit à des sous-suites arithmétiques de réduites et donne quelques intéressants développement en série ; la méthode de Newton, pour laquelle nous complétons un résultat d'Edward Burger [1] sur l'existence de fonctions f qui fournissent des sous-suites arithmétiques de réduites.

Abstract. Given a quadratic irrational α, we are interested in how some numerical schemes applied to a convenient function f provide subsequences of convergents to α. We investigate three numerical schemes: secant-like methods and formal generalizations, which lead to linear recurring subsequences; the false position method, which leads to arithmetical subsequences of convergents and gives some interesting series expansions; Newton's method, for which we complete a result of Edward Burger [1] about the existence of some functions f which provide arithmetical subsequences of convergents.

1. Introduction

Given a real function f regular enough with root α, there exists several ways to approximate this root. The most classical ones are, in increasing order of efficiency, the false position method, the secant method and Newton's method.

The false position method, whose order of convergence is linear, consists in choosing two initial approximations of α, x_0 and x_1, such that $f(x_0)$ and $f(x_1)$ are of opposite signs, and in defining each new x_n in the following way: i_{n-1} being the smallest integer such that $f(x_{n-1})$ and $f(x_{n-1-i_{n-1}})$ are of opposite signs, x_n is the x-coordinate of the intersection with the x-axis of the straight line defined by the points $(x_{n-1}, f(x_{n-1}))$ and

Manuscrit reçu le 28 octobre 2008.

$(x_{n-1-i_{n-1}}, f(x_{n-1-i_{n-1}}))$. (When f is convex or concave in a neighborhood of α, then the sequence $(n-1-i_n)_n$ is ultimately constant.)

The secant method consists in defining the sequence $(x_n)_n$ of successive approximations to α by the following induction: x_0 and x_1 being two given approximations of α, each x_n is defined as the intersection with the x-axis of the straight line defined by the points $(x_{n-1}, f(x_{n-1}))$ and $(x_{n-2}, f(x_{n-2}))$. Its order of convergence is equal to $\varphi := (1+\sqrt{5})/2$, that is, the difference $|x_n - \alpha|$ is asymptotically upper-bounded by $|x_{n-1} - \alpha|^\varphi$ up to a multiplicative constant. If we replace, in the definition of x_n, the points $(x_{n-1}, f(x_{n-1}))$ and $(x_{n-2}, f(x_{n-2}))$ by the points $(x_{n-s}, f(x_{n-s}))$ and $(x_{n-t}, f(x_{n-t}))$ (where s and t are fixed integers), we obtain what we call here "secant-like methods".

Newton's method for approximating α consists in choosing a first approximation x_0, close enough to α, and in defining each new approximation x_n by the intersection with the x-axis of the tangent at $(x_{n-1}, f(x_{n-1}))$ of the curve $y = f(x)$. The convergence of Newton's method is quadratic, that is, its order of convergence is equal to 2.

When f is a polynomial with integral coefficients, all these methods allow to find rational approximations to one of its roots, so it is natural to ask whether these approximations are linked in any way with rational approximations given by continued fraction expansion.

If c is a positive integer which is not a perfect square, Joseph-Alfred Serret proved [6] that, if $f(x) = x^2 - c$ and x_0 is the integer part of $\sqrt{c}$, then, for every $n \geq 0$, Newton's method gives that, for all n, $x_n = p_{2^n-1}/q_{2^n-1}$, where p_i/q_i is the i-th convergent to $\sqrt{c}$. In other words, applying Newton's formula where f stands for the minimal polynomial of some quadratic surd α and x_0 the integral part of α gives for x_n the (2^n-1)-th convergent to α. This can be seen as an echo to the fact that the sequence $(x_n)_n$ converges quadratically to α.

In section 2, we show that a similar phenomenon occurs for the secant method (Theorems 2.1 and 2.2): for a large class of quadratic irrational numbers, starting with suitable x_0 and x_1, we get a subsequence of convergents to α of the form p_{F_n}/q_{F_n} where the F_ns verify the "quasi-Fibonacci relation" $F_n = F_{n-1} + F_{n-2} + z_n$ where $(z_n)_n$ is a bounded sequence. Again, this property is an echo of the fact that the order of convergence of the secant method is equal to φ, the only solution of $x^2 = x + 1$ such that $x > 1$. We show that this phenomenon holds also for secant-like methods; thus, we get other subsequences of convergents to α given by linear recurring sequences (Theorem 2.3).

When f is a quadratic polynomial, the false position method can be seen as a limit case of one of these generalizations of the secant method. We show in section 3 that, again for a large class of quadratic irrationals, this method gives arithmetical subsequences of convergent to α (Theorem 3.1). In passing, these arithmetical subsequences leads to some nice formulas which express α by series of Egyptian fractions.

As for Newton's methods, we extend here (section 4) some previous results given by various authors. In 1999, Georg Rieger [4] exhibited a function f for which Newton's formula converges to a quadratic irrational number and gives more convergents of it than the use of its minimal polynomial; his example was limited to $\varphi-1 = (\sqrt{5}-1)/2$, for which he obtained a sequence x_n which describes all even-indexed convergents. In 2001, Takao Komatsu [3] extended this result to every real number with a continued fraction expansion of the form $[0, \overline{a, b}]$ (a, b positive integers). Eventually, Edward Burger [1] gave, for every quadratic number α of the form $[0, \overline{a_1, \ldots, a_L}]$ where L is even, an explicit function f_α from which the sequence $(x_n)_n$ obtained by Newton's formula starting from $x_0 = 0$ gives all convergents to α of the form p_{nL}/q_{nL}.

Here, for any quadratic irrational α, we show an explicit way to construct a function f_α and an initial value x_0 for which Newton's formula gives exactly the convergents of the sequence $(p_{nL+k}/q_{nL+k})_n$, where k is any integer and L the length of any period of the partial quotients in the continued fraction expansion of α (Theorem 4.2). Contrary to the previously mentioned works, we are not limited anymore to the case L even, even if the parity of L plays a role in the study (explained in Proposition 4.1). We mention that, under reasonable hypotheses, f_α is essentially unique, and we show that there is no "reasonable" function f for which Newton's method gives the whole subsequence of convergents to the quadratic irrational α, apart from the cases $L = 1$ and $L = 2$ (Proposition 4.2 and Corollary 4.1). It is nevertheless possible, by applying Newton's method "circularily" to different functions, to get this whole sequence (Theorem 4.3).

The last section of the present paper gives, in an informal way, some possible extensions of some of the previous results, mainly in the case of the so-called λ-*continued fractions*.

2. Secant-like methods and generalization

Let $t > s > 0$ be two integers. A natural generalization of the secant method consists in defining x_n with the straight line given by the points $(x_{n-s}, f(x_{n-s}))$ and $(x_{n-t}, f(x_{n-t}))$, the values x_0, ..., x_{t-1} being given. We call this variant the (s,t)-*secant method*; it corresponds to the iteration of the following induction formula:

$$x_n = \frac{f(x_{n-s})}{f(x_{n-s}) - f(x_{n-t})} x_{n-t} - \frac{f(x_{n-t})}{f(x_{n-s}) - f(x_{n-t})} x_{n-s}. \tag{2.1}$$

In all this section, $\alpha > 0$ is a quadratic irrational number with minimal polynomial given by $f(x) = ax^2 + bx + c$ (with $a > 0$ and a, b, $c \in \mathbb{Z}$). We write $\Delta := b^2 - 4ac$. We denote by $(p_n/q_n)_n$ the sequence of convergents to α. The conjugate root of α is denoted by $\overline{\alpha}$.

Definition. We define the function h_α on $\mathbb{Q}(\alpha)$ by:

$$h_\alpha(u + v\alpha) := a(u + v\alpha)(u + v\overline{\alpha}) \quad (u,\, v \in \mathbb{Q}).$$

An index i is said to be *suitable* if it has the following properties:

- $h_\alpha(p_i - q_i\alpha) = \pm 1$;
- $p_i/q_i > \alpha - |\sqrt{\Delta} - 2|/a$.

The main result of the present section is the following

Theorem 2.1. *Let $t > s > 0$ be two integers, and assume that $f(x) = ax^2 + bx + c$ is such that $b \in a\mathbb{Z}$ and $2a < \sqrt{\Delta}$. Suppose that there exists a pair of integers (u, v) such that $h_\alpha(u - v\alpha) = \pm 1$. Then, there exists suitable (explicit) indices $i_0, \ldots, i_{t-1}$ such that, defining x_j as p_{i_j}/q_{i_j} for all $j < t$, the iteration of the (s,t)-secant method leads to the sequence of general term $x_n = p_{\phi(n)}/q_{\phi(n)}$, where $\phi(n)$ is defined by*

$$\phi(n) = \begin{cases} i_n & \text{for } n < t; \\ \phi(n-s) + \phi(n-t) + z_n & \text{for } n \geq t, \end{cases} \tag{2.2}$$

where z_n is a bounded sequence.

In all the examples we tested, the sequence $(z_n)_n$ appeared to be constant and equal to 1; unfortunately, we do not know how to prove it in full generality. In subsection 2.3 we give a proof for some particular cases, which makes use of some ideas that could be helpful for the general case.

The present section is subdivised in three parts: the first one gives some complements about the necessity of the hypotheses of Theorem 2.1 and gives an additional theorem; the second one gives the proof of Theorem 2.1 and of this additional theorem; the third one makes use of an element of the proof to give some more generalizations of secant-like methods to get other subsequences of convergents.

2.1. Some examples, and an additional theorem. Theorem 2.1 applies for example to the usual secant method ($s = 1$, $t = 2$) and $\alpha = \sqrt{2}$, with $f(x) = x^2 - 2$, $x_0 = 1$, $x_1 = 3/2$. The first terms of the sequence $(x_n)_n$ are

$$\frac{1}{1} = \frac{p_0}{q_0}, \quad \frac{3}{2} = \frac{p_1}{q_1}, \quad \frac{7}{5} = \frac{p_2}{q_2}, \quad \frac{41}{29} = \frac{p_4}{q_4}, \quad \frac{577}{408} = \frac{p_7}{q_7}, \quad \frac{47321}{33461} = \frac{p_{12}}{q_{12}} \ldots$$

Denoting by $(F_n)_n$ the Fibonacci sequence defined by $F_0 = 1$, $F_1 = 2$ and, for any $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, we have that $x_n = p_{F_n - 1}/q_{F_n - 1}$ for all $n \geq 0$. Another example of number α for which we get the same formula is the positive root of $x^2 - kx - 1$ (where $k > 0$ is any integer), with $x_0 = k$ and $x_1 = k + 1/k$ (see section 2.3).

The assumption that $b \in a\mathbb{Z}$ cannot be removed: for example, defining α as the only positive root of $f(x) := 8x^2 + 5x - 23$ (we have $\alpha = (\sqrt{761} - 5)/16$) and applying the usual secant method ($s = 1$, $t = 2$), starting with the suitable pairs $(u_0, v_0) = (p_3, q_3) = (24, 17)$ and $(u_1, v_1) = (p_8, q_8) = (38398, 27201)$ gives that $x_2 = p_{12}/q_{12}$, but some calculation shows that the next x_ns are not anymore convergents to α.

As regards the hypothesis $2a < \sqrt{\Delta}$, here is a way to weaken it:

Theorem 2.2. *Remove the hypothesis $2a < \sqrt{\Delta}$ in the statement of Theorem 2.1. Let $(d(n))_n$ be the sequence of integers defined by*

$$d(n) = \begin{cases} 0 & \text{if } n < t; \\ 1 + d(n-s) + d(n-t) & \text{if } n \geq t. \end{cases}$$

For any n such that $d(n)$ is even, x_n is a convergent to α.

An example in which Theorem 2.2 applies but not Theorem 2.1 is given by the usual secant method ($s = 1, t = 2$) applied to $f(x) = 2x^2 - 1$ starting with $(u_0, v_0) = (1, 1)$ (so $x_0 = 1$) and $(u_1, v_1) = (2, 3)$ (so $x_1 = 2/3$). The successive approximations to $\alpha = 1/\sqrt{2}$ are then the ratios u_n/v_n given by the following table, the last line telling if u_n/v_n is a convergent to α.

n	0	1	2	3	4	5	6
u_n	1	2	7	58	1632	378 568	2 471 293 760
v_n	1	3	10	82	2308	535 376	3 494 937 152
$d(n)$	0	0	1	2	4	7	12
convergent?	yes	yes	no	yes	yes	no	yes

The indices of the convergents to α obtained under the assumptions of Theorem 2.2 can certainly be precized. To show this, let us consider the example of the $(1,3)$-secant method applied to $\alpha = \sqrt{2/3}$, which has $f(x) = 3x^2 - 2$ as minimal polynomial. A simple verification shows that $2a > \sqrt{\Delta}$ and that the pairs $(u_0, v_0) = (1, 1)$, $(u_1, v_1) = (9, 11)$ and $(u_2, v_2) = (881, 1079)$ are suitable. The iteration of the $(1,3)$-secant method then gives the following table, in which the second line indicates the value of the index i_n such that the equality $x_n = p_{i_n}/q_{i_n}$ holds.

n	0	1	2	3	4	5	6	7	8	9	10
index	1	3	7	(8)	11	(18)	(26)	37	55	81	(118)

Indices written between parentheses are not "true" indices: the equality $x_n = p_{i_n}/q_{i_n}$ does not hold for these one. The value of these "false indices" is given by the relation $i_n = i_{n-1} + i_{n-3}$, which is satisfied by the full sequence of indices (true and false) given in the table. As stated in Theorem 2.2, the set of n for which a "false index" occurs is the set of integers n such that $d(n)$ is odd.

Note also that, as a consequence of the fact that α is a quadratic surd, if i_n is a "false index", then $x_n = q_{i_n}/(3p_{i_n}/2)$.

As regards the definition of initial "suitable" indices, we do not know whether there is a simple way to weaken it. The particular case of the usual secant method applied to $\alpha = \sqrt{3}$ (of minimal polynomial $f(x) = x^2 - 3$) starting with $(u_0, v_0) = (1, 1)$ and $(u_1, v_1) = (2, 1)$ is an example in which $h_{\sqrt{3}}(u_0, v_0) = 2$ but for which, nevertheless, all the fractions u_n/v_n (to be reduced) are convergents to $\sqrt{3}$.

2.2. Proof of Theorem 2.1 and Theorem 2.2. Here, we prove simultaneously both Theorem 2.1 and Theorem 2.2.

We start with some general properties of the function h_α and of the (s, t)-secant method.

Proposition 2.1. *The function $h := h_\alpha$ satisfies the following properties:*
1) for any z and $z' \in \mathbb{Z}(\alpha)$, $ah(zz') = h(z)h(z')$;
2) for any u, $v \in \mathbb{Z}$, $h(u + v\alpha) = au^2 - buv + cv^2$;
3) for any $z \in \mathbb{Z}(\alpha)$ and any $m \in \mathbb{Z}$, we have $h(mz) = m^2 h(z)$.

Proof. Simple calculation. □

Lemma 2.1. *The discriminant $\Delta = b^2 - 4ac$ is lower-bounded by* 5*.*

Proof. We have $\Delta = b^2 \pmod 4$, and $b^2 = 0$ or $1 \pmod 4$. Since 0, 1 and 4 are perfect squares, we must have $\Delta \geq 5$. □

Proposition 2.2. *If there exists a pair of integers (u, v) such that $h(u - v\alpha) = \pm 1$, then there exists infinitely many suitable indices.*

Proof. Let ε be any unit of $\mathbb{Q}(\sqrt{\Delta})$ of the form $\mu + \nu\sqrt{\Delta}$ (μ, ν integers). A calculation shows that $\varepsilon(u - v\alpha) \in \mathbb{Z}(\alpha)$ and that $h(\varepsilon) = a$. Thus, $h(\varepsilon(u - v\alpha)) = h(\varepsilon)h(u - v\alpha)/a = \pm 1$, so we have proved that there are infinitely many pairs of integers (x, y) such that $h(x - y\alpha) = \pm 1$. Our aim is now to prove that infinitely many of these pairs are convergents to α, and that these convergents satisfy the inequality $p_i/q_i > \alpha - |\sqrt{\Delta} - 2|/a$. Note that, thanks to Lemma 2.1, $\sqrt{\Delta} - 2$ is positive.

Assume that y goes to infinity. The relation $f(x/y) = h(x - y\alpha)/y^2 = \pm 1/y^2$ shows that x/y remains bounded and that the accumulation points of x/y are included in the set $\{\alpha, \overline{\alpha}\}$. A calculation shows that, defining x/y by the sequence $\varepsilon^n(u - v\alpha)$, we have that x/y goes to α (resp. $\overline{\alpha}$) for $\mu\nu < 0$ (resp. $\mu\nu > 0$). Thus, we can find a sequence x/y converging to α, so the inequality $x/y > \alpha - (\sqrt{\Delta} - 2)/a$ is true for infinitely many pairs (x, y), and it only remains to show that these pairs are convergents to α.

An order 1 approximation gives that

$$\frac{1}{y^2} = \left| f\left(\frac{x}{y}\right) - f(\alpha) \right| \approx |f'(\alpha)| \cdot \left| \frac{x}{y} - \alpha \right|,$$

so, since $f'(\alpha) = \sqrt{\Delta} > 2$ thanks to Lemma 2.1:

$$\left| \frac{x}{y} - \alpha \right| < \frac{1}{2y^2},$$

hence, thanks to a classical result on approximation of an irrational number by its convergents (see for example [2], Theorem 184), x/y is a convergent to α, and the proposition is proved. □

Since it is assumed in the statement of Theorem 2.1 that there exists a pair (u, v) of integers such that $h(u-v\alpha) = \pm 1$, Proposition 2.2 allows us to choose t convergents to α which are suitable in the sense of our Definition. These suitable convergents define the x_ns for $n < t$. We have now to prove that the x_ns for $n \geq t$, defined by formula (2.1), have the desired property.

For any $n < t$, write x_n as u_n/v_n, where u_n and v_n are mutually prime integers. The integers u_n and v_n for $n \geq t$ are defined by induction, using formula (2.1) of the (s, t)-secant method:

$$\begin{aligned}
x_n &= \frac{\left(a\frac{u_{n-s}^2}{v_{n-s}^2} + b\frac{u_{n-s}}{v_{n-s}} + c\right) \cdot \frac{u_{n-t}}{v_{n-t}} - \left(a\frac{u_{n-t}^2}{v_{n-t}^2} + b\frac{u_{n-t}}{v_{n-t}} + c\right) \cdot \frac{u_{n-s}}{v_{n-s}}}{\left(a\frac{u_{n-s}^2}{v_{n-s}^2} + b\frac{u_{n-s}}{v_{n-s}} + c\right) - \left(a\frac{u_{n-t}^2}{v_{n-t}^2} + b\frac{u_{n-t}}{v_{n-t}} + c\right)} \\
&= \frac{(au_{n-t}u_{n-s}^2 v_{n-t} + cu_{n-t}v_{n-t}v_{n-s}^2) - (au_{n-t}^2 u_{n-s}v_{n-s} + cu_{n-s}v_{n-t}^2 v_{n-s})}{(au_{n-s}^2 v_{n-t}^2 + bu_{n-s}v_{n-t}^2 v_{n-s}) - (au_{n-t}^2 v_{n-s}^2 + bu_{n-t}v_{n-t}v_{n-s}^2)} \\
&= \frac{(au_{n-t}u_{n-s} - cv_{n-t}v_{n-s})(u_{n-s}v_{n-t} - u_{n-t}v_{n-s})}{(au_{n-s}v_{n-t} + au_{n-t}v_{n-s} + bv_{n-t}v_{n-s})(u_{n-s}v_{n-t} - u_{n-t}v_{n-s})} \\
&= \frac{au_{n-t}u_{n-s} - cv_{n-t}v_{n-s}}{au_{n-s}v_{n-t} + au_{n-t}v_{n-s} + bv_{n-t}v_{n-s}}.
\end{aligned}$$

This leads to the following definition of u_n and v_n for $n \geq t$:

$$\begin{cases} u_n = au_{n-t}u_{n-s} - cv_{n-t}v_{n-s} \\ v_n = au_{n-s}v_{n-t} + au_{n-t}v_{n-s} + bv_{n-t}v_{n-s} \end{cases} \tag{2.3}$$

so we can write $x_n = u_n/v_n$ for any n. Remind that, apart from the case $n < t$, the fraction u_n/v_n is non necessarily irreducible.

Lemma 2.2. *For any $n \geq t$, we have*

$$u_n - v_n\alpha = a(u_{n-s} - v_{n-s}\alpha)(u_{n-t} - v_{n-t}\alpha).$$

Proof. Simply expand the expression $a(u_{n-s}-v_{n-s}\alpha)(u_{n-t}-v_{n-t}\alpha)$, use the relation $a\alpha^2 = -b\alpha - c$ and compare the result to the definition of u_n and v_n given by relation (2.3) (of course, we also make use of the irrationality of α). □

Recall that, as defined in the statement of Theorem 2.2, for any n we have

$$d(n) = \begin{cases} 0 & \text{if } n < t; \\ 1 + d(n-s) + d(n-t) & \text{if } n \geq t. \end{cases}$$

Proposition 2.3. *For any n, we have $|h(u_n - v_n\alpha)| = a^{d(n)}$.*

Proof. The proposition is true for any $n < t$, since these x_ns were chosen to be convergents to α of suitable indices (so $|h_\alpha(u_n - v_n\alpha)| = 1 = a^0$ for these n). For $n \geq t$, we write

$$\begin{aligned} |h(u_n - v_n\alpha)| &= h(|a(u_{n-s} - v_{n-s}\alpha)(u_{n-t} - v_{n-t}\alpha)|) \\ &= a^2 h(|(u_{n-s} - v_{n-s}\alpha)(u_{n-t} - v_{n-t}\alpha)|) \\ &= a^2 h(|u_{n-s} - v_{n-s}\alpha|) \cdot h(|u_{n-t} - v_{n-t}\alpha)|)/a \\ &= a \cdot a^{d(n-s)} \cdot a^{d(n-t)}. \end{aligned}$$

The first equality comes from Lemma 2.2, the two following ones by Proposition 2.1 (point 3, then point 1), and the last one by the induction hypothesis. □

Corollary 2.1. *For any n, if p is a prime number which divides both u_n and v_n, then p divides a.*

Proof. Assume that p divides both u_n and v_n. Thus, for some integers u and v, Proposition 2.1 gives that $h(u_n - v_n\alpha) = p^2 h(u - v\alpha)$. Since $h(u - v\alpha)$ is integer-valued, the right side of this equality is a multiple of p^2. Since $h(u_n - v_n\alpha)$ is a power of a by Proposition 2.3, we get that p divides a. □

Proposition 2.4. *For any $n \geq 0$, let us denote by $y(n)$ (resp. $z(n)$) the biggest exponent $e \in \mathbb{N}$ such that u_n (resp. v_n) is a multiple of a^e. We have, for any n:*

$$y(n) \geq \frac{d(n) - 1}{2} \quad \textit{and} \quad z(n) \geq \frac{d(n)}{2}.$$

Proof. Since $d(n) = 0$ for any $n < t$, the property is true for any $n < t$. Let assume that it is true until some $n - 1$. Recall that we assume that b is a multiple of a: we write $b = b'a$, where b' is an integer. The general expression for u_n and v_n gives that, for some integers $\tilde{u}_{n-s}$, $\tilde{u}_{n-t}$, $\tilde{v}_{n-s}$ and $\tilde{v}_{n-t}$:

$$\begin{aligned} u_n &= a \cdot a^{y(n-t)} \tilde{u}_{n-t} \cdot a^{y(n-s)} \tilde{u}_{n-s} - c \cdot a^{z(n-t)} \tilde{v}_{n-t} \cdot a^{z(n-s)} \tilde{v}_{n-s} \\ v_n &= a \cdot a^{y(n-s)} \tilde{u}_{n-s} \cdot a^{z(n-t)} \tilde{v}_{n-t} + a \cdot a^{y(n-t)} \tilde{u}_{n-t} \cdot a^{z(n-s)} \tilde{v}_{n-s} \\ &\quad + b'a \cdot a^{z(n-t)} \tilde{v}_{n-t} \cdot a^{z(n-s)} \tilde{v}_{n-s}. \end{aligned}$$

The induction hypothesis and the definition of $d(n)$ then give

$$\begin{aligned} 1+y(n-t)+y(n-s) &\geq 1+\frac{d(n-t)-1}{2}+\frac{d(n-s)-1}{2} \\ &\geq \frac{d(n-t)+d(n-s)}{2} \\ &\geq \frac{d(n)-1}{2} \\ z(n-t)+z(n-s) &\geq \frac{d(n-t)}{2}+\frac{d(n-s)}{2} \\ &\geq \frac{d(n)-1}{2} \end{aligned}$$

so $y(n) \geq (d(n)-1)/2$. The same calculation for the exponents of a in the expression of v_n gives that $z(n) \geq d(n)/2$. □

The following lemma will conclude the proof of Theorem 2.2.

Lemma 2.3. *If $a/\sqrt{\Delta} < 1/2$ then $x_n = u_n/v_n$ is a convergent to α. Else, for any n such that $d(n)$ is even, $x_n = u_n/v_n$ is a convergent to α.*

Proof. We have, by Proposition 2.3:

$$\left(\frac{u_n}{v_n}-\alpha\right)\left(\frac{u_n}{v_n}-\overline{\alpha}\right)=\frac{h(u_n-v_n\alpha)/a}{v_n^2}=\frac{\pm a^{d(n)-1}}{v_n^2}.$$

For any n we denote by ε_n the value such that $u_n/v_n-\alpha=\varepsilon_n$. We have

$$a\left(\frac{u_n}{v_n}-\overline{\alpha}\right)=a\left(\alpha-\overline{\alpha}+\varepsilon_n\right)=\sqrt{\Delta}+a\varepsilon_n.$$

Thus, we can write

$$\frac{u_n}{v_n}-\alpha=\frac{1}{\sqrt{\Delta}+a\varepsilon_n}\frac{\pm a^{d(n)}}{v_n^2} \tag{2.4}$$

- Case 1: $d(n)$ is even

  Then, the lower bound of $y(n)$ and $z(n)$ given in Proposition 2.4 become the same, equal to $d(n)/2$. There exists two integers u'_n and v'_n such that $u_n = a^{d(n)/2}u'_n$ and $v_n = a^{d(n)/2}v'_n$, so equation (2.4) becomes

  $$\frac{u_n}{v_n}-\alpha=\frac{u'_n}{v'_n}-\alpha=\frac{1}{\sqrt{\Delta}+a\varepsilon_n}\frac{\pm 1}{{v'_n}^2}.$$

  We know that, for any $n < t$, the x_ns are convergents to α with suitable indices, so, by our Definition, we have that $\sqrt{\Delta}+a\varepsilon_n > 2$ for any $n < t$. Moreover, for any $n \geq t$, it is proved in the same way as for the usual secant method that the succession of values given by the (s,t)-secant method is such that $|x_n - \alpha| \leq \min(|x_{n-s} - \alpha|, |x_{n-t} - \alpha|)$, that is, $|\varepsilon_n| \leq \min(|\varepsilon_{n-s}|, |\varepsilon_{n-t}|)$. Thus, for any $n \geq t$, we also have $\sqrt{\Delta} + a\varepsilon_n > 2$, so we finally get that:

$$\left|\frac{u'_n}{v'_n} - \alpha\right| \leq \frac{1}{2{v'_n}^2},$$

an inequality that implies that u_n/v_n is a convergent to α, by the classical result already mentionned in the proof of Proposition 2.2. Thus, Theorem 2.2 is proved.

- Case 2: $d(n)$ is odd

  In this case, we have $y(n) \geq (d(n) - 1)/2$ and $z(n) \geq (d(n) + 1)/2$. As in the previous case, writing $u_n = a^{(d(n)-1)/2} u'_n$ and $v_n = a^{(d(n)-1)/2} v'_n$ gives, in equation (2.4):

$$\left|\frac{u'_n}{v'_n} - \alpha\right| = \frac{a}{\sqrt{\Delta} + a\varepsilon_n} \frac{1}{{v'_n}^2}.$$

  Recall the assumption, specific to Theorem 2.1, that $a/\sqrt{\Delta} < 1/2$. When we chose the t first x_ns by Proposition 2.2, we can ask for these convergents to be close enough to α so as to get that $a/(\sqrt{\Delta} + a\varepsilon_n) < 1/2$ for any $n < t$. Since the sequence $(|\varepsilon_n|)_n$ is decreasing, this choice implies that $a/(\sqrt{\Delta} + a\varepsilon_n) < 1/2$ for any n, thus, we finally get

$$\left|\frac{u'_n}{v'_n} - \alpha\right| < \frac{1}{2{v'_n}^2},$$

  an inequality that implies, as before, that u'_n/v'_n is a convergent to α.

□

Note that neither the fractions u_n/v_n nor the fractions u'_n/v'_n given in the previous proof are necessarily irreducible (even if they correspond to convergents to α). Nevertheless, we have the following property.

Lemma 2.4. *Let g_n be the greatest common divisor between the values u'_n and v'_n defined in the proof of Lemma 2.3. The sequence $(g_n)_n$ is bounded.*

Proof. For n even or odd, the proof of Lemma 2.3 shows that the inequality

$$\left|\frac{u'_n}{v'_n} - \alpha\right| \leq \frac{a}{\sqrt{\Delta} + a\varepsilon_n} \frac{1}{{v'_n}^2}$$

holds. Let $u'_n = g_n u''_n$ and $v'_n = g_n v''_n$. We thus have

$$\left|\frac{u''_n}{v''_n} - \alpha\right| \leq \frac{a}{g_n^2(\sqrt{\Delta} + a\varepsilon_n)} \frac{1}{{v''_n}^2}.$$

Since α is a quadratic number, it has bounded partial quotients, so the coefficient $a/(g_n^2(\sqrt{\Delta}+a\varepsilon_n))$ cannot become arbitrarily small. Therefore, g_n cannot become arbitrarily big, and the lemma is proved. □

Lemma 2.5. *Let L be the period length of the continued fraction expansion of α. There exists a quadratic number β such that its conjugate, $\overline{\beta}$, is equal to $\pm 1/\beta$ and such that, for any n big enough and any m between* 0 *and* $L-1$*:*

$$p_{nL+m} = \rho_m \beta^n + \overline{\rho}_m \overline{\beta}^n \quad \textit{and} \quad q_{nL+m} = (\rho_m/\alpha)\beta^n + (\overline{\rho}_m/\overline{\alpha})\overline{\beta}^n$$

for some ρ_m.

Proof. Classical formulae for numerators of convergents give that, for any n big enough:

$$\begin{aligned} p_{nL+1} &= a_1 p_{nL} + p_{nL-1} \\ p_{nL+2} &= a_2 p_{nL+1} + p_{nL} \\ &\vdots \\ p_{(n+1)L} &= a_L p_{(n+1)L-1} + p_{(n+1)L-2}, \end{aligned}$$

where the a_is are the partial quotients belonging to the periodic part of the continued fraction expansion of α. The smallest n_0 such that these equalities are true for any $n \geq n_0$ is the index of the first partial quotient to α from which the continued fraction expansion of α is periodic.

Linear combinations of these relations allow to express $p_{nL+1}, p_{nL+2}, \dots, p_{(n+1)L}$ as a linear combination of p_{nL} and p_{nL-1}. By induction, let us define the following vectors of $\mathbb{R}^L$: $\ell_1 := (0,\dots,0,1,a_1)$, $\ell_2 = (0,\dots,0,a_2,a_1a_2+1)$ and, for any $3 \leq i \leq L$, $\ell_i := a_i \ell_{i-1} + \ell_{i-2}$. Define, then, the $L \times L$ matrix M as the matrix with ℓ_i as i-th row. We write P_n for the vector whose coordinates are $p_{nL+1}, \dots, p_{(n+1)L}$. We thus have, for any $n \geq n_0$, that $P_n = MP_{n-1}$.

Due to the form of M, the characteristic polynomial of M is of the form $X^{L-2}Q(X)$, where Q is of degree 2. Moreover, $Q(0)$ is equal to the determinant of the 2×2-submatrix of M at the bottom and the right of M. Thanks to the definition of the ℓ_is, this determinant is equal (up to a possible change of sign) to the determinant of the 2×2-submatrix of M at the top and the right of M, that is: $Q(0) = \begin{vmatrix} 1 & a_1 \\ a_2 & a_1a_2+1 \end{vmatrix} = 1$. Hence, the roots of Q are of the form β and $\pm 1/\beta$. The diagonalization of M then gives the expected result for p_{nL+m}.

The same reasoning gives also that, for some ρ'_m, we have $q_{nL+m} = \rho'_m \beta^n + \overline{\rho'}_m \overline{\beta}^n$ for $n \geq n_0$ and $m < L$. Let us fix m. The ratio p_{nL+m}/q_{nL+m} goes to α as n goes to infinity and, taking $|\beta| > 1$, it also converges to ρ_m/ρ'_m. Hence, we have $\rho'_m = \rho_m/\alpha$, and we are done. □

We are now ready to end the proof of Theorem 2.1.

Denote by $\delta(n)$ the value $d(n)/2$ if $d(n)$ is even, $(d(n)-1)/2$ if $d(n)$ is odd. Thus, using the notation of the proof of Lemma 2.3, we have $u_n = a^{\delta(n)}u'_n$ and $v_n = a^{\delta(n)}v'_n$. In Lemma 2.4, we defined g_n as the greatest common divisor between u'_n and v'_n, and defined, in the proof, $u'_n = g_n u''_n$ and $v'_n = g_n v''_n$. Hence, u''_n/v''_n is irreducible. Since it is equal to a convergent to α (because u_n/v_n is, by Lemma 2.3), we can define $\phi(n)$ such that $u''_n/v''_n = p_{\phi(n)}/q_{\phi(n)}$. We thus get

$$u_n = a^{\delta(n)} g_n p_{\phi(n)},$$

$$v_n = a^{\delta(n)} g_n q_{\phi(n)}.$$

It remains to show that the sequence $(\phi(n))_n$ satisfies the relation (2.2) of the statement of Theorem 2.1.

Lemme 2.5 allows to write $p_{\phi(n)}$ and $q_{\phi(n)}$ as a linear combination of $\beta^{\lfloor n/L\rfloor L}$ and $(\pm\beta)^{-\lfloor n/L\rfloor L}$. In the induction definition of u_n, express u_n, u_{n-t}, u_{n-s}, v_{n-t} and v_{n-s} using all of this. Writing $\phi(n) = \mu L + \nu$ with $0 \leq \nu < L$, $\phi(n-t) = iL + j$ with $0 \leq j < L$ and $\phi(n-s) = i'L + j'$ with $0 \leq j' < L$, we get, by considering only the most rapidly increasing terms in n in both sides of the equality:

$$a^{\delta(n)} g_n \rho_\nu \beta^\mu = a^{\delta(n-t)+\delta(n-s)} g_{n-t} g_{n-s} (a - c\alpha^{-2}) \rho_j \rho_{j'} \beta^{i+i'}.$$

Thus

$$\rho_\nu \beta^\mu = \left((a - c\alpha^{-2}) \frac{g_{n-t} g_{n-s}}{g_n} a^{\delta(n-s)+\delta(n-t)-\delta(n)} \rho_j \rho_{j'} \right) \beta^{i+i'}.$$

Since $\delta(n) - \delta(n-t) - \delta(n-s)$ and $(g_n)_n$ are bounded, there exists a bounded value z such that $\beta^{i+i'+z} = \beta^\mu$. Thus, we get that $\phi(n) = \phi(n-s) + \phi(n-t) + z_n$ for some bounded z_n, and Theorem 2.1 is proved.

2.3. A proof that $z_n = 1$ in Theorem 2.1 for some particular cases. We prove here the following result.

Proposition 2.5. *Let k be a positive integer, let $\alpha > 0$ be such that $\alpha^2 - k\alpha - 1 = 0$. Theorem 2.1 applies to this α with $z_n = 1$ for all n.*

Proof. Let $(p_n/q_n)_n$ be the sequence of convergents to α. Since $\alpha = k + 1/\alpha$, we have $\alpha = [\overline{k}]$, so $q_n = p_{n+1}$ for any $n \geq 0$, $p_0 = k$, $p_1 = k^2 + 1$ and $p_n = kp_{n-1} + p_{n-2}$ for any $n \geq 2$. A calculation thus shows that, for any $n \geq 0$:

$$p_n = \rho\alpha^n + \overline{\rho\alpha}^n$$

with $\rho = \alpha^2/(\alpha - \overline{\alpha})$.

Since $a = 1$, Theorem 2.1 together with Corollary 2.1 give that $u_n = p_{\phi(n)}$ and $v_n = q_{\phi(n)}$ for any n. Thus, applying the induction formula for u_n and using the relations $\overline{\alpha} = -\alpha^{-1}$ and $q_n = p_{n-1}$ give

$$\begin{aligned} p_{\phi(n)} &= u_n \\ &= u_{n-t}u_{n-s} + v_{n-t}v_{n-s} \\ &= p_{\phi(n-t)}p_{\phi(n-s)} + q_{\phi(n-t)}q_{\phi(n-s)} \\ &= \rho^2\alpha^{\phi(n-t)+\phi(n-s)} + \rho^2\alpha^{\phi(n-t)+\phi(n-s)-2}) \\ &\quad + \overline{\rho^2\alpha^{\phi(n-t)+\phi(n-s)} + \rho^2\alpha^{\phi(n-t)+\phi(n-s)-2}}) \\ &= \rho\alpha^{\phi(n-t)+\phi(n-s)+1} \cdot (\rho(\alpha^{-1} + \alpha^{-3})) \\ &\quad + \overline{\rho\alpha^{\phi(n-t)+\phi(n-s)+1} \cdot (\rho(\alpha^{-1} + \alpha^{-3}))}. \end{aligned}$$

A simple calculation shows that $\rho(\alpha^{-1} + \alpha^{-3}) = 1$, so $\phi(n) = \phi(n-t) + \phi(n-s) + 1$, thus $z_n = 1$ for any n, and we are done. □

The previous proof can be generalized to many other particular cases. For example, we leave as an exercice to the reader to show that Theorem 2.1 applies with $z_n = 1$ for all n whenever $\alpha > 0$ is a root of $x^2 - kx - l = 0$ with $k \in l\mathbb{N}$. We did not found a way to extend the previous proof in the full generality of Theorem 2.1.

2.4. A generalization of secant-like methods. The (s, t)-secant method allows to find subsequences of convergents to some quadratic irrationals; these subsequences are asymptotically geometrically increasing with growth rate equal to the only $x > 1$ such that $x^t = x^{t-s} + 1$. (Note that this growth rate gives also the order of convergence of the (s, t)-secant method, as is it shown by a straightforward generalization of the classical corresponding result for the usual secant method, which is of order φ where $\varphi^2 = \varphi + 1$.) A natural question is then to ask for other variants of the secant method which give raise to subsequences of other growth rate. A way to do it is to make use of the crucial relation given in Lemma 2.2. Taking it as a definition of u_n and v_n instead of a consequence of properties of the (s, t)-secant method leads to the following

Definition. Let $k > 0$ be an integer and let $0 < s_0 \le s_1 \le s_2 \le \cdots \le s_k$ be integers. The quadratic irrational α being defined as in Theorem 2.1, we define the $(s_0, s_1, \ldots, s_k)$-*secant method* of approximation to α as the iteration of the formula

$$u_n - v_n\alpha := a^k(u_{n-s_0} - v_{n-s_0}\alpha)(u_{n-s_1} - v_{n-s_1}\alpha)\cdots(u_{n-s_k} - v_{n-s_k}\alpha).$$

We do not know whether this definition of the $(s_0, \ldots, s_k)$-secant method has an elementary geometric interpretation which extends in a natural way the geometric interpretation of the (s, t)-secant method.

For the sake of brievety, we give the result only in the case $a = 1$.

Theorem 2.3. *Let $k > 0$ be an integer, let $f(x) = x^2 + bx + c$ the minimal polynomial over $\mathbb{Z}$ of a quadratic irrational number $\alpha > 0$. Assume that there exists u and v such that $|h(u - v\alpha)| = 1$, where $h(x + y\alpha)$ is defined as $(x + y\alpha)(x - y\alpha) = x^2 + bxy + cy^2$. There exists (explicit) indices $i_0, \ldots, i_{\max(s_k)-1}$ such that, defining x_j as p_{i_j}/q_{i_j} for all $j < \max(s_k)$, the iteration of the $(s_0, \ldots, s_k)$-secant method leads to the sequence of general term $x_n = p_{\phi(n)}/q_{\phi(n)}$, where $\phi(n)$ is defined by*

$$\phi(n) = \begin{cases} i_n & \text{for } n < \max(s_k); \\ \sum_{j=0}^{k} \phi(n - s_j) + z_n & \text{for } n \geq \max(s_k), \end{cases}$$

where $(z_n)_n$ is a bounded sequence.

The proof of Theorem 2.3 goes in the same way as the proof of Theorem 2.1. Since $a = 1$, Lemma 2.1 remains true and Proposition 2.3 becomes $|h(u_n - v_n\alpha)| = 1$ for any n. Since $a = 1$, Lemma 2.1 gives that $a\sqrt{\Delta} < 1/2$, so Lemma 2.3 implies that $x_n = u_n/v_n$ is a convergent to α for any n. The end of the proof is the same.

3. The method of false position

Let α be a (irrational) root of the polynomial $f(x) := ax^2 + bx + c$, where a, b and c are integers without common divisor and $a > 0$. In particular, f is convex. We define $\Delta := b^2 - 4ac$.

If $f'(\alpha) > 0$ (resp. $f'(\alpha) < 0$), consider two first approximations to α, x_0 and x_1, close enough to α and such that $x_1 < \alpha < x_0$ (resp. $x_0 < \alpha < x_1$). Thus, the iteration of the method of false position gives the sequence $(x_n)_n$ of approximations of α obtained by the following induction formula for all $n \geq 1$:

$$x_{n+1} = x_0 - \frac{x_0 - x_n}{f(x_0) - f(x_n)} \cdot f(x_0) \tag{3.1}$$

In some way, it can be seen as a limit case of the secant-like methods presented in the beginning of the previous section, where $s = 1$ and $t = +\infty$.

Theorem 3.1. *Under the previous hypotheses on α and f, assume that there exists a pair of integers (u, v) such that $au^2 + buv + cv^2 = \pm 1$ and such that $bv \in a\mathbb{Z}$. Put $x_0 := u/v$, and let $x_1 = u_1/v_1$ be a convergent to α. Let $h_1 := au_1^2 + bu_1v_1 + cv_1^2$.*

- *Assume $v_1 \in a\mathbb{Z}$.*
  If $2|h_1| \leq |a|\sqrt{\Delta}$ (resp. $2|h_1| \leq \sqrt{\Delta}$), then, the sequence $(x_{2n})_{n\geq 1}$ (resp. $(x_{2n-1})_{n\geq 1}$) obtained by the iteration of the false position

method starting from x_0 and x_1 is an arithmetical subsequence of convergents to α.

- *Do not assume $v_1 \in a\mathbb{Z}$.*
  If $2|h_1| \le |a|^{-1}\sqrt{\Delta}$ (resp. $2|h_1| \le \sqrt{\Delta}$), then, the sequence $(x_{2n})_{n\ge1}$ (resp. $(x_{2n-1})_{n\ge1}$) obtained by the iteration of the false position method starting from x_0 and x_1 is an arithmetical subsequence of convergents to α.

In the particular case $a = 1$, if $2|h_1| \le \sqrt{\Delta}$, then the full sequence $(x_n)_{n\ge1}$ is an arithmetical subsequence of convergents to α.

In any case, the common difference of the arithmetical subsequence of convergents to α is equal to mL, where L is the length of the periodic part of the continued fraction expansion of α and $m = m(u, v)$ is an integer explicit in u and v.

In the next subsection, we discuss the hypotheses of Theorem 3.1 and give some relevant examples. In the second subsection we prove the theorem and, in the last one, we give some interesting series expansions of some quadratic irrational numbers deduced from the proof.

3.1. Some examples. The hypothesis $bv \in a\mathbb{Z}$ cannot be removed. We can illustrate this by the same example used in subsection 2.1: consider the root $\alpha = (\sqrt{761} - 5)/16$ of the polynomial $f(x) = 8x^2 + 5x - 23$ and take $u = 24$, $v = 17$, $u_1 = 38398$ and $v_1 = 27201$. We have $h = 1$, and $h_1 = -1$ (so the hypothesis $2|h_1| < |a|^{-1}\sqrt{\Delta}$ is satisfied), but a calculation shows that the x_ns for $n \ge 2$ are not convergents to α.

Nevertheless, there are possible generalizations of Theorem 3.1 which make use of weaker assumptions on bv (and/or v_1). For example, assume that a is of the form p^i, where p is a prime number and $i \ge 2$. If p divides bv, then our proof of Theorem 3.1 may leads to the result that some arithmetical subsequences of $(u_n)_n$ consist in arithmetical subsequences of convergents to α. More generally, the same kind of results probably holds under the assumption that any prime factor of a divides bv.

The hypothesis on $2|h_1|$ cannot be removed. Let us give two examples of that, the first in the case $v_1 \notin a\mathbb{Z}$, the second in the case $v_1 \in a\mathbb{Z}$.

Consider the polynomial $f(x) = 3x^2 - 2$, which has $\alpha = \sqrt{2/3}$ as a root. Take $u = 1$, $v = 1$ (so $h = 1$), $u_1 = 4$ and $v_1 = 5$ (so $h_1 = -2$ and $v_1 \notin a\mathbb{N}$). We have $\Delta = 24$, so $2|h_1| < \sqrt{\Delta}$ and $2|h_1| > |a^{-1}|\sqrt{\Delta}$. The first terms of the sequence $(x_n)_n$, starting from x_0, are, after simplifications:

$$1 \quad \frac{4}{5} \quad \frac{22}{27} \quad \frac{40}{49} \quad \frac{218}{267} \quad \frac{396}{485} \quad \frac{2158}{2643} \quad \frac{3920}{4801} \ldots,$$

and, 1 being excluded, only 4/5, 40/49, 396/485, 3920/4801, etc. are convergents to α, that is, the elements of the subsequence $(x_{2n+1})_n$.

Now, consider the polynomial $f(x) = 2x^2 - 1$, and start with $u = 1$, $v = 1$, $u_1 = 1$ and $v_1 = 2$. We then have $v_1 \in a\mathbb{Z}$, $\sqrt{\Delta} < 2|h_1| < |a|\sqrt{\Delta}$ (that is, $\sqrt{8} < 4 < 2\sqrt{8}$), and a calculation shows that the successive x_ns are, starting from x_0:

$$1 \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{7}{10} \quad \frac{12}{17} \quad \frac{41}{58} \dots,$$

that is, the x_{2n} are convergents to $\alpha = 1/\sqrt{2}$ whereas the x_{2n+1} are not, since we have, for any n:

$$x_{2n} = [1, \overbrace{2, \dots, 2}^{2n-1}] \quad \text{and} \quad x_{2n+1} = [1, \overbrace{2, \dots, 2}^{2n}, 1].$$

Apart from the case $a = 1$ (for which Theorem 3.1 asserts that it is impossible), we do not know whether we could find a case for which the convenient hypotheses of Theorem 3.1 are satisfied to get that both sequences $(x_{2n})_n$ and $(x_{2n+1})_n$ are arithmetical subsequences of convergents, but such that the full sequence $(x_n)_n$ is not.

3.2. Proof of Theorem 3.1. Without loss of generality, we assume $f'(\alpha) > 0$, the other case being similar.

We write $x_0 := u_0/v_0 := u/v$ (with $u = u_0$ and $v = v_0$), $x_1 := u_1/v_1$; for any $n \geq 1$, we define u_{n+1} and v_{n+1} by identifying the numerators and the denominators in the following equality (obtained by replacing x_n by u_n/v_n and x_0 by u/v in the expression of x_{n+1} given by relation (3.1)):

$$\frac{u_{n+1}}{v_{n+1}} = x_{n+1} = \frac{auu_n - cvv_n}{avu_n + (au + bv)v_n}.$$

Let $h_n := au_n^2 + bu_nv_n + cv_n^2$ for any $n \geq 0$. For some reason that will become clearer in the next subsection, we write h instead of h_0 (we thus have $h = 1$). Note that for any $n \geq 1$, we have $h_n < 0$.

An elementary calculation shows that $h_{n+1} = ahh_n$. We thus have

$$\left(\frac{u_n}{v_n} - \alpha\right)\left(\frac{u_n}{v_n} - \overline{\alpha}\right) = \frac{h_n}{av_n^2},$$

so, since $u_n/v_n - \overline{\alpha} < \alpha - \overline{\alpha} = \sqrt{\Delta}/a$ (where $\Delta = b^2 - 4ac$):

$$\left|\frac{u_n}{v_n} - \alpha\right| < \frac{|h_1|(ah)^{n-1}}{\sqrt{\Delta}}\frac{1}{v_n^2}. \tag{3.2}$$

In the following, we take $v_1 \in a\mathbb{Z}$. By the hypothesis $bv \in a\mathbb{Z}$, an easy induction then shows that, for any $n \geq 0$, $u_n \in a^{\lfloor n/2 \rfloor}\mathbb{N}$ and $v_n \in a^{\lceil n/2 \rceil}\mathbb{N}$.

- If n is even

  Let us write, $u_n = a^{n/2}u'_n$ and $v_n = a^{n/2}v'_n$. Inequality (3.2) becomes

$$\left|\frac{u_n}{v_n}-\alpha\right| = \left|\frac{u'_n}{v'_n}-\alpha\right| < \frac{|h_1|(ah)^{n-1}}{\sqrt{\Delta}}\frac{1}{a^n {v'_n}^2} = \frac{|h_1|h^{n-1}}{a\sqrt{\Delta}}\frac{1}{{v'_n}^2}.$$

Since $h=1$, the assumption $2|h_1| \leq a\sqrt{\Delta}$ gives that $|u'_n/v'_n - \alpha| < 1/(2{v'_n}^2)$ and so u'_n/v'_n is a convergent to α (same argument as in the proof of Theorem 2.1).

- If n is odd

  We write $u_n = a^{(n-1)/2}u'_n$ and $v_n = a^{(n-1)/2}v'_n$. Inequality (3.2) now becomes

$$\left|\frac{u_n}{v_n}-\alpha\right| = \left|\frac{u'_n}{v'_n}-\alpha\right| < \frac{|h_1|(ah)^{n-1}}{\sqrt{\Delta}}\frac{1}{a^{n-1} {v'_n}^2} = \frac{|h_1|h^{n-1}}{\sqrt{\Delta}}\frac{1}{{v'_n}^2}.$$

  Since $h=1$, the assumption $2|h_1| \leq \sqrt{\Delta}$ gives that $|u'_n/v'_n - \alpha| < 1/(2{v'_n}^2)$ and so u'_n/v'_n is a convergent to α.

The reader may check that, when $v_1 \notin a\mathbb{Z}$, we only have $u_n \in a^{\lfloor (n-1)/2 \rfloor}\mathbb{N}$ and $v_n \in a^{\lfloor n/2 \rfloor}\mathbb{N}$; it does not change the previous study in the case n odd, but when n is even, the convenient assumption to get that u'_n/v'_n is a convergent to α is now: $2|h_1| \leq a^{-1}\sqrt{\Delta}$.

Now that we know that $(x_{2n})_n$ or $(x_{2n+1})_n$, or even $(x_n)_n$ (depending on the assumed assumptions) is a sequence of convergents to α, it remains to show that this is also an arithmetical subsequence of convergents to α.

Define, for any $n \geq 0$:

$$X_n := \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad M := \begin{pmatrix} au & -cv \\ av & au+bv \end{pmatrix}.$$

We thus have, for any $n \geq 1$, $X_{n+1} = MX_n$.

Assume $v_1 \in a\mathbb{Z}$ (otherwise, the study is essentially the same) and define also $X'_n := \begin{pmatrix} u_{2n}/a^n \\ v_{2n}/a^n \end{pmatrix}$: thanks to a previous remark, the coordinates of X_{2n} are integers.

Observe that $X_{2(n+1)} = M^2 X_{2n}$ and that

$$M^2 = \begin{pmatrix} a(au^2-cv^2) & -acv(2u+w) \\ a^2v(2u+w) & a(-cv^2+a(u+w)^2) \end{pmatrix},$$

where w is the integer defined by $bv = aw$ (recall the hypothesis $bv \in a\mathbb{Z}$).

Dividing each entry of M by a, we put

$$M' := \begin{pmatrix} au^2-cv^2 & -cv(2u+w) \\ av(2u+w) & -cv^2+a(u+w)^2 \end{pmatrix},$$

so we get $X'_n = M'X'_{n-1}$ for any $n \geq 1$.

Note that, considered in the projective space, X_{2n} and X'_n are equal. Note also that, since $\det(M) = a$, we have $\det(M') = 1$, so $M' \in \mathrm{PSL}(2, \mathbb{Z})$. Moreover, written as a homography, M' admits α as a fixed point, so iterating it starting from u_0/v_0 leads to an arithmetical subsequence of convergents to α with common difference in $L\mathbb{N}$, where L is the length of the period of the continued fraction expansion of α. Note that the exact value of this common difference can be easily computed with the help of the standard decomposition of the homography M' into product of generators of the monoid $\mathrm{PSL}(2, \mathbb{N})$ which are $z \longmapsto z + 1$ and $z \longmapsto -1/z$.

The same argument applies when starting from $Y'_n := \begin{pmatrix} u_{2n+1}/a^n \\ v_{2n+1}/a^n \end{pmatrix}$, and leads to the same result.

In the case $a = 1$, we do not need to consider M^2 and M', since M already belongs to $\mathrm{PSL}(2, \mathbb{N})$. Thus, in this case, the previous reasoning applies to the full sequence $(x_n)_n$, and not only to each subsequences (x_{2n+1}) and (x_{2n}) separately.

Thus, Theorem 3.1 is proved.

3.3. Series expansions of some quadratic irrational numbers. We remove here the assumptions $bv \in a\mathbb{Z}$ and $2|h_1| \leq |a|\sqrt{\Delta}$. Again, without loss of generality, we assume $a > 0$.

A simple calculation shows that, for any $n \geq 2$, we have $x_n - x_{n-1} = u_n/v_n - u_{n-1}/v_{n-1} = -vh_{n-1}/(v_{n-1}v_n)$ for any n. Thus, we have $x_n - x_{n-1} = -vh_1(ah)^{n-2}/(v_{n-1}v_n)$. The sequence $(v_n)_n$ is a linear recurring sequence determined by the choice of the values $v_0 = v$, v_1 and $v_2 = avu_1 + (au + bv)v_1$, and the characteristic polynomial of the matrix M (see previous subsection), this polynomial being $x^2 - (2au + bv)x + ah$.

Thus, writing α as $x_1 + \sum_{n\geq 2}(x_n - x_{n-1})$ and since $h_1 < 0$, we finally get:

$$\alpha = \frac{u_1}{v_1} + v|h_1| \sum_{n\geq 2} \frac{(ah)^{n-2}}{v_{n-1}v_n},$$

$$\text{with } \begin{cases} v_0 = v \\ v_1 = v_1 \\ v_2 = avu_1 + (au + bv)v_1 \\ v_n = (2au + bv)v_{n-1} - ahv_{n-2} \text{ for } n \geq 3. \end{cases}$$

When $bv \in a\mathbb{Z}$, as we already noticed, v_n is a multiple of $a^{\lfloor n/2 \rfloor}$, so $v_{n-1}v_n$ is a multiple of a^{n-1}, and the general term of the series hence can be simplified. In particular, in the case $h = 1$, the previous expression leads to an expression of α made of Egyptian fractions (that is, fractions with numerator equal to 1). For example, for $\alpha = \sqrt{2}$ with $m \in \mathbb{N}^*$, taking $u = 3$, $v = 2$, $u_1 = 1$ and $v_1 = 1$ gives

$$\sqrt{2} = 1 + \frac{2}{5} + \frac{2}{145} + \frac{2}{4901} + \frac{2}{166465} + \cdots = 1 + 2\sum_{n\geq 2} \frac{1}{v'_n v'_{n-1}},$$

where $v'_0 = 2$, $v'_1 = 1$, $v'_2 = 5$, and $v'_n = 6v'_{n-1} - v'_{n-2}$ for $n \geq 3$.

Note that, in this case, the sequence $t_n = v'_n v'_{n-1}$ of denominators in the series can also be expressed in the following simpler form: $t_2 = 5$, $t_3 = 145$ and $t_n = 34t_{n-1} - t_{n-2} - 24$ for $n \geq 4$.

Other rules of the same kind can be given in other cases: for example, staying with $\sqrt{2}$ and starting with $u/v = 17/12$ and $u_1/v_1 = 1$ leads to the equality $\sqrt{2} = 1 + 12\sum_{n\geq 1} 1/t_n$ with $t_1 = 29$, $t_2 = 28565$ and $t_n = 1154t_{n-1} - t_{n-2} - 4896$ for $n \geq 3$. More generally, expressing v_n as a linear combination of the roots of the characteristic polynomial of M leads to an expression of $v_n v_{n-1}$ which can be used to find a induction formula for t_n.

Again for $\alpha = \sqrt{2}$, taking $u = 2$, $v = 1$, $u_1 = 1$ and $v_1 = 1$ gives

$$\sqrt{2} = 1 + \sum_{n\geq 2} \frac{2^{n-2}}{v_n v_{n-1}},$$

where $v_0 = v_1 = 1$, $v_2 = 3$ and $v_n = 4v_{n-1} - 2v_{n-2}$ for $n \geq 3$. Since v_n belongs to $2^{\lfloor (n-1)/2 \rfloor}\mathbb{N}$ for any $n \geq 0$, $v_n v_{n-1}$ is multiple of 2^{n-2}, and the expression can be simplified to get Egyptian fractions (even if h is not equal to 1). A study shows that we get

$$\sqrt{2} = \frac{1}{1} + \frac{1}{3} + \frac{1}{15} + \frac{1}{85} + \frac{1}{493} + \frac{1}{2871} + \cdots,$$

where the denominators t_n are defined by the rule: $t_1 = 1$, $t_2 = 3$ and $t_n = 6t_n - t_{n-1} - 2$ for $n \geq 3$.

A last example, where h and $|h_1|$ are both different from 1, is given by $\alpha = \sqrt{2}$, $u/v = 2/1$ and $u_1/v_1 = 4/3$, which leads to the sequence

$$\sqrt{2} = \frac{1}{1} + \frac{1}{3} + \frac{1}{15} + \frac{1}{85} + \frac{1}{493} + \cdots,$$

where the denominators are given by the rule $t_0 = 1$, $t_1 = 3$ and $t_n = 6t_{n-1} - t_{n-2} - 2$ for $n \geq 2$.

4. Newton's method

We recall that the analytic expression of Newton's method is given by the following induction formula for all $n > 0$:

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}.$$

Definition. For any quadratic irrational number α, we denote by $L(\alpha)$ the number of partial quotients belonging to the shortest period of its partial quotients, and by $K(\alpha)$ the number of elements of the (shortest) aperiodic part of its continued fraction expansion (so $K(\alpha)=0$ when this aperiodic part is empty).

We start with the existence result; it slightly extends the one given by Burger in [1], and the proof is quite similar:

Theorem 4.1. *Let $(p_n/q_n)_n$ be the sequence of convergents to a fixed quadratic irrational number α, L an element of $L(\alpha)\mathbb{N}$, k a non-negative integer and $n_0 := K(\alpha)+k$. There exists an explicit function f_α such that the sequence defined by*

$$x_0 = \frac{p_{n_0}}{q_{n_0}} \quad \textit{and} \quad x_{n+1} = x_n - \frac{f_\alpha(x_n)}{f'_\alpha(x_n)}$$

is the sequence $(p_{n_0+nL}/q_{n_0+nL})_n$.

Proof. Let us fix an α for which $K(\alpha)=0$, and let us choose for L any positive multiple of $L(\alpha)$. We have $\alpha=[a_0,\ldots,a_{L-2},a_{L-1},1/\alpha]=(a\alpha+b)/(c\alpha+d)$, where a, b, c and d are integers. (Classical facts about continued fractions asserts that we can choose $a=p_{L-1}$, $b=p_{L-2}$, $c=q_{L-1}$ and $d=q_{L-2}$, where p_n/q_n is the n-th convergent to α — see [2].) We define $u(x)$ by $u(x):=(ax+b)/(cx+d)$ for all $x\neq -d/c$. If $x=p_i/q_i$, then $u(x)=p_{L+i}/q_{L+i}$, so our goal is to find a function f_α such that, for any x close enough to α, the following equality holds:

$$x-\frac{f_\alpha(x)}{f'_\alpha(x)}=u(x).$$

This leads to a differential equation wich can be easily solved. Denoting by $\overline{\alpha}$ the conjugate of α, we get, on each interval $I^- :=]-\infty,\min(\alpha,\overline{\alpha})[$, $I :=]\min(\alpha,\overline{\alpha}),\max(\alpha,\overline{\alpha})[$ and $I^+ :=]\max(\alpha,\overline{\alpha}),+\infty[$ the following solution (up to a multiplicative constant, which can be fixed equal to 1):

$$f_\alpha(x)=|x-\alpha|^s\cdot|x-\overline{\alpha}|^t,$$

where s and t are such that $s+t=1$ and $\overline{\alpha}s+\alpha t=-d/c$.

We thus obtain that, when $K(\alpha)=0$, for any integer $k\geq 0$, Newton's algorithm applied to the initial value $x_0=p_k/q_k$ and the previous function f_α gives the sequence $(p_{nL+k}/q_{nL+k})_n$ of convergents to α.

Let us now consider the case $K(\alpha)\geq 1$. We write $K=K(\alpha)$, so α can be written as $\alpha=[\tilde{a}_0,\ldots,\tilde{a}_{K-1},\overline{a_0,\ldots,a_{L-1}}]$. Let us start with $k=0$, that is, $x_0=[\tilde{a}_0,\ldots,\tilde{a}_{K-1}]$. Our aim is to get, for every $n\geq 0$, the equality $x_n=[\tilde{a}_0,\ldots,\tilde{a}_{K-1}+y_n]$, where y_n is defined by $y_0=0$ and $y_{n+1}=[0,a_0,\ldots,a_{L-1}+y_n]$ for all $n\geq 0$. Let us denote by v the homography defined by $v(y)=[0,a_0,\ldots,a_{L-1}+y]$ and by $\tilde{v}$ the homography defined by $\tilde{v}(x)=[\tilde{a}_0,\ldots,\tilde{a}_{K-1}+x]$. Then, we have $x_{n+1}=\tilde{v}(y_{n+1})=\tilde{v}(v(y_n))=\tilde{v}(v(\tilde{v}^{-1}(x_n)))$. Now, we choose for u the function $\tilde{v}\circ v\circ\tilde{v}^{-1}$ and we apply to u the same method we previously used for the case $K=0$. The case $x_0=[\tilde{a}_0,\ldots,\tilde{a}_{K-1+k}]$ where $k>0$ is solved with the same technique. □

Let us explain now in which way the f_α found in the proof of Theorem 4.1 is essentially the unique solution of the problem. We consider a quadratic irrational number α for which $K(\alpha) = 0$. Let u be any continuous function for which $u(p_{nL}/q_{nL}) = p_{(n+1)L}/q_{(n+1)L}$ for all n. Solving the differential equation $x - f(x)/f'(x) = u(x)$ gives $|f| = c \cdot e^{\int 1/(x-u(x))}$ which is a solution to the problem.

To avoid theses too numerous and irrelevant solutions, it is then logical to ask for a holomorphic property since the isolated zeroes theorem implies, then, that there is (at most) one solution to the problem. The relation $f'/f = 1/(x - u(x))$ gives a natural corresponding property of f given in the following

Definition. A function f defined in a neighborhood of α is said to be *reasonable* if its logarithmic derivative F is continuous at α and if there exists an $\varepsilon > 0$ such that the restrictions of F to the intervals $]\alpha - \varepsilon, \alpha[$ and $]\alpha, \alpha + \varepsilon[$ are holomorphic.

These considerations allow us to understand the assumption made in [1], [3] and [4] that $L(\alpha)$ is even, since theses studies are restricted to the case f'/f holomorphic *in a neighborhood of* α; asking for f'_α/f_α to be holomorphic only in the intervals $]\alpha - \varepsilon, \alpha[$ and $]\alpha, \alpha + \varepsilon[$ allows us to avoid this restriction; for example, for $\alpha = (1 + \sqrt{5})/2$, starting with $x_0 = 1$ and using

$$f_\alpha(x) = \left| x - \frac{1+\sqrt{5}}{2} \right|^{\frac{5+\sqrt{5}}{10}} \cdot \left| x - \frac{1-\sqrt{5}}{2} \right|^{\frac{5-\sqrt{5}}{10}},$$

we get every convergents to the golden ratio.

All of this can be synthetized in the following

Theorem 4.2. *For any quadratic irrational number α, there exists a x_0 explicit in α and a (essentially unique) reasonable function f_α, also explicit in α, for which the sequence defined for every $n > 0$ as*

$$x_{n+1} = x_n - \frac{f_\alpha(x_n)}{f'_\alpha(x_n)}$$

corresponds to the sequence of convergents $(p_{n_0+nL}/q_{n_0+nL})_n$, where $L \in L(\alpha)\mathbb{N}$ and $n_0 \geq K(\alpha) - 1$.

It the same way, if we choose for x_0 any Farey approximant to α which does not come before p_{K-1}/q_{K-1}, then the sequence of the x_n is an arithmetical subsequence of the sequence of all successive Farey approximants to α. More precisely, if p_{K+k}/q_{K+k} is a convergent to α and if we chose for x_0 the i-th Farey approximant which comes after p_{K+k}/q_{K+k}, then x_n is the i-th Farey approximant which comes after p_{nL+K+k}/q_{nL+K+k}.

Let us remark also that, if we work with $v := u^{-1}$ in spite of u, we find another function, g_α, obtained by simply exchanging s and t in the expression of f. This new function "kills" the convergents, *i.e.* if we start from $x_0 = p_{n_0L+K+k}/q_{n_0L+K+k}$, then $x_n = p_{(n_0-n)L+K+k}/q_{(n_0-n)L+K+k}$ for all $n \leq n_0$. For $K(\alpha) = k = 0$, we then have $x_{n_0+1} = \infty$, and after that we get an arithmetical subsequence of the convergents to $\overline{\alpha}$. This remark allows us to understand that the choice $x_0 = 0$ made in [1], [3] and [4] is the real reason to the limitation, in those studies, to the case $a_0 = 0$ and L even. Indeed, in the case $\alpha = [0, \overline{a_1, \dots, a_{L-1}}]$, and in this case only, our study leads us to define $u(x) = [0, a_1, \dots, a_{L-1} + 1/x]$, and taking $x_0 = 0$ gives, in the case L even, $x_1 = p_L/q_L$. If L is odd, then $x_1 = \infty$, and $x_2 = p_L/q_L$, etc.

Here is another part of the explanation of the difference that can be made between the cases in which $L(\alpha)$ is even and odd.

Proposition 4.1. *Using the notations of Theorem 4.2, let us take $L = L(\alpha)$. If L is even, then f_α is of class C^1 in a neighborhood of α, and $f'_\alpha(\alpha) = 0$. Else, $\lim_{x\to\alpha}(|f'_\alpha(x)|) = +\infty$.*

Proof. Instead of making a quite long and tiresome calculation, let us give a qualitative argument. We write f instead of f_α and assume, without loss of generality, that f is positive on $]\alpha, \alpha + \varepsilon[$.

Classical facts about continued fractions assert that the even-indexed convergents to a real number x are smaller than x and the odd-indexed ones are bigger than x. So if L is even, then all the x_n are bigger to α (or smaller, this second case leading to the same study), so f is convex on $]\alpha, \alpha+\varepsilon[$, so $f'(x)$ is increasing on this interval. Thus, $f'(x)$ converges to an $l \in \mathbb{R} \cup \{-\infty\}$ as $x \in]\alpha, \alpha + \varepsilon[$ tends to α. Since $f > 0$ on $]\alpha, \alpha + \varepsilon[$, we have $l \geq 0$. If $f'(\alpha) \neq 0$, then Newton's method applied to f converges quadratically to α, and this contradicts the fact that the sequence obtained by Newton's method for this f is a arithmetical subsequence of the convergents to α, since this subsequence converges with order 1 (α having periodic partial quotients). So, we must have $l = 0$. The same argument starting from a $x_n < \alpha$ gives that the left derivative of f is also 0, and we are done in the case L even.

If L is odd, then $x_n < \alpha$ implies $x_{n+1} > \alpha$, so, since $f > 0$ on $]\alpha, \alpha + \varepsilon[$, f is concave on $]\alpha, \alpha+\varepsilon[$ (that is, $-f$ is convex), so we find also that $f'(x)$ tends to a limit $l \in \mathbb{R}^{+*} \cup \{+\infty\}$ when $x \in]\alpha, \alpha+\varepsilon[$ tends to α. By the same argument as before, the assumption $l \in \mathbb{R}^{+*}$ leads to a contradiction since Newton's method would converge quadratically. So we must have $l = +\infty$; the same argument works in the same way on $]\alpha - \varepsilon, \alpha[$, and the proof is complete. □

We may wonder if other arithmetical subsequences of convergents can be obtained by Newton's method. The following gives a partial answer.

Proposition 4.2. *Let α be a quadratic irrational number such that $L(\alpha) = 2$. There exists an $x_0 \in \mathbb{R}$ and a reasonable function f_α such that Newton's formula applied to f_α starting from x_0 gives* all *the convergents to α.*

Proof. It is enough to consider the case $\alpha = [\overline{a, b}]$, since the arguments given for Theorem 4.2 show how to extend the purely periodic case to the ultimately periodic one.

Let us define $u(x) := a + a/(bx)$. It is easily seen that, denoting by u^n the n-th iterate of u, we have $u^n(p_0/q_0) = p_n/q_n$, so the same study as in Theorem 4.2 leads to the construction of the desired function. □

An extension of this result to other values of L can be made in the following way:

Theorem 4.3. *Let α be a quadratic irrational number whose convergents are denoted by p_n/q_n. For any integer $k \geq K(\alpha)$, there exists (explicit) reasonable functions $f_0, \dots, f_{L-1}$ and an (explicit) initial value x_0 such that Newton's formula applied circularily to the f_i starting from x_0 gives the whole sequence of convergents $(p_{n+k}/q_{n+k})_{n\geq 0}$.*

Before giving the proof, let us indicate that, by applying Newton's method circularily, we mean that, for every integer m and any integer r such that $0 \leq r < L$, we define x_{mL+r+1} as

$$x_{mL+r+1} = x_{mL+r} - \frac{f_r(x_{mL+r})}{f_r'(x_{mL+r})}.$$

Proof. Again, we consider only the case $\alpha = [\overline{a_0, \dots, a_{L-1}}]$. We begin with the following lemma.

Lemma 4.1. *For any integer k, there exists a unique homography u_k with integral coefficients for which, for all integer $n \geq 0$, we have $u_k(p_{nL+k}/q_{nL+k}) = p_{nL+k+1}/q_{nL+k+1}$.*

Proof. By a continuity argument, such a homography satisfies $u(\alpha) = \alpha$. Let denote by u_α the homography such that $u_\alpha(x) = [a_0, \dots, a_{L-1}, x]$ for all x. We have also $u_\alpha(\alpha) = \alpha$, so u and u_α have α and $\overline{\alpha}$ as fixed points, so u and u_α commute.

The conditions $u(\alpha) = \alpha$ and $u(p_k/q_k) = p_{k+1}/q_{k+1}$ define a unique homography with integral coefficients, denoted by u_k. We then have

$$u_k\left(\frac{p_{nL+k}}{q_{nL+k}}\right) = u_k \circ u_\alpha^n\left(\frac{p_k}{q_k}\right) = u_\alpha^n \circ u_k\left(\frac{p_k}{q_k}\right) = u_\alpha^n\left(\frac{p_{k+1}}{q_{k+1}}\right) = \frac{p_{nL+k+1}}{q_{nL+k+1}},$$

so we are done for the lemma. □

To conclude the proof of Theorem 4.3, it is then enough to define f_k from u_k as in Theorem 4.2. □

The same kind of consideration allows to build a finite set of functions for which Newton's method applied circularily to them gives subsequences of convergents of the form $(p_{\phi(n)}/q_{\phi(n)})_n$, where $\phi(\mathbb{N})$ is a finite union of subsets of $\mathbb{N}$ of the form $N_i\mathbb{N}+k_i=\{N_in+k_i,\ n\in\mathbb{N}\}$, where the N_is and the k_is are integers.

Another consequence of the previous proof is the following result, which shows that, in some sense, the "circular" way given in the previous theorem is the only sensible way. (It is likely that this theorem could be extended: apart from some possible exceptions, it is probably impossible to get arithmetical subsequences of convergents with a common difference non multiple of $L(\alpha)$ — possibly $L(\alpha)/2$ for $L(\alpha)$ even.)

Corollary 4.1. *For any quadratic irrational α for which $L(\alpha)\geq 3$, there exists no reasonable function f such that Newton's formula applied to f and starting from any x_0 gives the full sequence of convergents to α.*

Proof. The proof of Theorem 4.3 gives the existence of a finite set of functions u_k holomorphic in a neighborhood of α $(i=1\dots n)$ for which the relationship $x-f(x)/f'(x)=u_k(x)$ should be true for a converging sequence of x. By the isolated zeroes theorem, all of the u_ks must be the same homography. By a conjugation, we assume that α has a purely periodic continued fraction expansion. Hence, the matrix form of a homography with integer entries and which admits α as a fixed point is necessarily of the form $\begin{pmatrix} p_{L-1}+\delta & p_{L-2} \\ q_{L-1} & q_{L-2}+\delta \end{pmatrix}$, where $L=L(\alpha)$ and $\delta\in\mathbb{R}$.

Since $u(\infty)=u(p_{-1}/q_{-1})=p_0/q_0=a_0$, we have $p_{L-1}+\delta=a_0q_{L-1}$, so $\delta=a_0q_{L-1}-p_{L-1}$.

Now, since $u(p_{L-2}/q_{L-2})=p_{L-1}/q_{L-1}$, we have:

$$\frac{p_{L-1}}{q_{L-1}}=\frac{(p_{L-1}+\delta)\cdot\frac{p_{L-2}}{q_{L-2}}+p_{L-2}}{q_{L-1}\cdot\frac{p_{L-2}}{q_{L-2}}+(q_{L-2}+\delta)}.$$

Simplifying this equality and using the equality $p_{L-1}q_{L-2}-p_{L-2}q_{L-1}=(-1)^L$ twice, we get that $\delta=-q_{L-2}$. Joining this equality with the first one concerning δ gives $-q_{L-2}=a_0q_{L-1}-p_{L-1}$, so $q_L=p_{L-1}$.

We know, then, that the matrix form of the homography u is $\begin{pmatrix} a_0q_{L-1} & p_{L-2} \\ q_{L-1} & 0 \end{pmatrix}$. We should have $u(a_0)=a_0+1/a_1$ but, with the expression of u, we get also $u(a_0)=a_0+p_{L-2}/(a_0q_{L-1})$, so $q_{L-1}=(a_1/a_0)p_{L-2}$. Replacing q_{L-1} by this latter expression and simplifying by p_{L-2}/a_0 the matrix form of u, we get $u = \begin{pmatrix} a_0 a_1 & a_0 \\ a_1 & 0 \end{pmatrix}$. We must then have

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = u\left(a_0 + \frac{1}{a_1}\right) = a_0 + \frac{1}{a_1 + \frac{1}{a_0}},$$

so $a_2 = a_0$. In the same way:

$$[a_0, a_1, a_2, a_3] = u([a_0, a_1, a_2]) = u([a_0, a_1, a_0]) = [a_0, a_1, a_0, a_1],$$

so $a_3 = a_1$. Assuming that $a_i = a_{i-2}$ for all $i \le n$, we get (for n even, but the same calculation could be made for n odd):

$$\begin{aligned}[a_0, \dots, a_{n+1}] &= u([a_0, \dots, a_n]) \\ &= u([a_0, a_1, \dots, a_0, a_1, a_0]) \\ &= [a_0, a_1, \dots, a_0, a_1, a_0, a_1]),\end{aligned}$$

so $a_{n+1} = a_0$. Thus, we have proved that $L = 1$ or $L = 2$. □

5. Some generalizations to other forms of continued fraction expansion

A positive real number λ being given, the λ-*continued fraction expansion* of a number x is an expression of the form

$$x = a_0\lambda + \cfrac{1}{a_1\lambda + \cfrac{1}{a_2\lambda + \cdots}} := [a_0, a_1, a_2, \dots]_\lambda,$$

where $(a_n)_n$ is a sequence in $\mathbb{Z}^*$ (apart from a_0, which may be equal to zero). A natural definition of the λ-convergents p_n/q_n to x is given by the formulae $p_n = a_n\lambda p_{n-1} + p_{n-2}$ and $q_n = a_n\lambda q_{n-1} + q_{n-2}$ with $p_0 = a_0\lambda$, $q_0 = 1$, $p_1 = a_0 a_1 \lambda^2 + 1$ and $q_1 = a_1\lambda$.

Since the a_ns are not assumed to be positive, there is no unicity of the λ-expansion of an x in general. If $\lambda < 2$, every real number x admits a λ-expansion; if $\lambda > 2$ the set of x that admit a λ-expansion is closed and of null measure. Note also that the most well-known cases of λ-continued fraction are *Rosen continued fractions* [5], which correspond to the case $\lambda = \lambda_k := 2\cos(\pi/k)$ with k integer, $k \ge 3$ (the first values of λ_k are $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$, $\lambda_5 = (1+\sqrt{5})/2$ and $\lambda_6 = \sqrt{3}$; λ_k is algebraic for all k, but for $k > 6$ λ_k is not quadratic anymore).

Some of the results given in the present paper easily extend to λ-continued fractions. Indeed, let us call λ-*quadratic* any number x which admits a periodic λ-expansion. It is easily seen that Theorems 4.1 and 4.2, and Propositions 4.1 and 4.2 remain true in the context of λ-continued fractions, since the fact that the partial quotients and the convergents are integers does not intervene anywhere in these results. (In particular, for $\lambda = 1$, it can be used with continued fractions with partial quotients in $\mathbb{Z}$). It could also be extended to even more general continued fractions, in which no assumption at all is made about the form of the partial quotients (the only constraint being their periodicity.)

As regards the secant-like methods, it is also highly probable that some extensions of our results to λ-continued fractions can be obtained. For example, some tests lead us to think that Theorem 2.1 (with $z_n = 1$ for all n) holds for any $\alpha > 0$ such that $\alpha^2 - m\lambda\alpha - 1 = 0$ where $m \in \mathbb{N}^*$ (that is: $\alpha = [\overline{m}]_\lambda$).

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Benoît RITTAUD
Laboratoire Arithmétique, Géométrie et Applications
Université Paris-13, Institut Galilée
99 avenue Jean-Baptiste Clément
F - 93 430 Villetaneuse.
E-mail: `rittaud@math.univ-paris13.fr`
URL: `http://www.math.univ-paris13.fr/~rittaud`