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The circle method and pairs of quadratic forms

par Henryk IWANIEC et Ritabrata MUNSHI

Résumé. Nous donnons une majoration non triviale du nombre de solutions entières, de taille donnée, d'un système de deux formes quadratiques en cinq variables.

Abstract. We give non-trivial upper bounds for the number of integral solutions, of given size, of a system of two quadratic form equations in five variables.

1. Introduction

The purpose of the paper is to obtain a non-trivial upper bound for the number of integer solutions of height B of the system of quadratic equations

$$\psi_1(x_1, \ldots, x_5) = 0, \quad \psi_2(x_1, \ldots, x_5) = 0. \tag{1.1}$$

Here ψ_1 and ψ_2 are quadratic forms over $\mathbb{Z}$ with five variables, such that the above pair of equations define a quartic del Pezzo surface. Accordingly we define the counting function

$$N^*(B) = \#\{\mathbf{x} \in \mathbb{Z}^5 : \psi_1(\mathbf{x}) = \psi_2(\mathbf{x}) = 0, \ |x_i| \leq B\}. \tag{1.2}$$

It seems that one can easily obtain the upper bound $N^*(B) \ll B^{2+\varepsilon}$. This we will refer to as the trivial bound. If the surface $V \subset \mathbb{P}^4$ defined by the equations (1.1) contains a line defined over $\mathbb{Q}$, then one cannot hope to get a better upper bound. Indeed any of the $\mathbb{Q}$-lines will contain B^2 many points of height B. In such cases it is logical to modify the counting function (1.2) so that only those points which lie outside the lines are taken into account.

To make this precise, let U be the open subset of V obtained by deleting the lines. (Note that there are at most 16 lines on V.) Then we define the counting function

$$N(B) = \#\{\mathbf{x} \in \mathbb{Z}^5 : \mathbf{x} \text{ primitive}, \ \mathbf{x} \in U(\mathbb{Q}), |x_i| \leq B\}, \tag{1.3}$$

where by $\mathbf{x}$ primitive we mean that the gcd $(x_1, \ldots, x_5) = 1$ and the first non-zero coordinate is positive. We have the following conjecture.

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Conjecture.

$$N(B) \sim cB(\log B)^{t-1}, \tag{1.4}$$

where c and t are certain constants which depend only on ψ_1 and ψ_2.

This conjecture is a special case of a more general conjecture formulated by Batyrev, Franke, Manin, Tschinkel and others for Fano varieties. Here we are dealing with the case of del Pezzo surfaces (Fano varieties of dimension 2) of degree 4. More details about this conjecture can be found in [2].

Some progress has been made towards Conjecture 1 when the surface V contains isolated singularities (e.g. see [1]). However the conjecture is far from being proved for any non-singular del Pezzo surface of degree 4. The best known result in the smooth case is due to Salberger, who proves the upper bound $N(B) \ll B^{1+\varepsilon}$, in the special case where the surface V contains a conic defined over $\mathbb{Q}$. (The details of this result have not appeared in prints.) Manin and Tschinkel have proved that $N(B) \ll B^{5/4+\varepsilon}$ when all the 16 lines on the surface are $\mathbb{Q}$-rational.

In general it is difficult to obtain any non-trivial bound for $N(B)$. Our aim in this paper is to use circle method (combined with the square detector of Heath-Brown) to establish a bound of the form

$$N(B) \ll B^{2-\delta}, \tag{1.5}$$

with some absolute constant $\delta > 0$. For the sake of simplicity we will only focus on diagonal forms. In this case we can reduce the problem to a slightly different counting problem. To this end let

$$\phi_1(x_1, \dots, x_4) = a_1x_1^2 + \cdots + a_4x_4^2,$$
$$\phi_2(x_1, \dots, x_4) = b_1x_1^2 + \cdots + b_4x_4^2,$$

and consider the counting function

$$M(B) = \#\{\mathbf{x} \in \mathbb{Z}^4 : \phi_1(\mathbf{x}) = 0, \phi_2(\mathbf{x}) = \square, |x_i| \asymp B\}, \tag{1.6}$$

which counts the number of vectors $\mathbf{x} \in \mathbb{Z}^4$ of size B, such that $\phi_1(\mathbf{x}) = 0$ and $\phi_2(\mathbf{x})$ is a non-zero square. For convenience we assume that the discriminant $\alpha = a_1a_2a_3a_4$ is not a square, and that all the 2×2 minors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

are non-zero. Let $\mathcal{M}$ denote the product of all the 2×2 minors. Now we state our main result.

Theorem 1.1. *Suppose the quadratic forms ϕ_1 and ϕ_2 are such that α is not a square and $\mathcal{M} \neq 0$. Then we have*

$$M(B) \ll B^{9/5+\varepsilon}, \tag{1.7}$$

where the implied constant depends on the quadratic forms ϕ_i and on ε.

Suppose the forms ψ_i in (1.1) are compatible in the sense that there is an unimodular transformation that reduces the system (1.1) to that of the form considered in the above theorem. Then it follows that $N(B) \ll M(B)$. So as a corollary of Theorem 1.1 we obtain the following:

Corollary 1.1. *Suppose the forms ψ_1, ψ_2 in (1.1) are compatible in the above sense. Then we have*

$$N(B) \ll B^{9/5+\varepsilon}, \tag{1.8}$$

where the implied constant depends on the quadratic forms ψ_i and on ε.

Remark. Our method will yield non-trivial upper bound for the general quadratic forms ψ_i, provided they share a common eigenvector.

2. A square detector

We construct a device to detect the condition $n = \square$, i.e. n is a non-zero square. To this end let

$$\theta(n) = \begin{cases} 1 & \text{if } n = \square, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We will use the following construction of Heath-Brown which he calls a 'square-sieve'. Let $\mathcal{P}$ be a set of P primes of size $P \log P$. Then for integers n with $|n| < \exp(P)$, we have

$$\theta(n) \ll \frac{1}{P^2}\Big|\sum_{p\in\mathcal{P}} \chi_p(n)\Big|^2, \tag{2.2}$$

where χ_p is the quadratic residue character modulo p.

Let $W : \mathbb{R}^4 \to \mathbb{R}$ be a non-negative smooth function supported on $[-1,1]^4$. Consider

$$M^*(B) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^4 \\ \phi_1(\mathbf{x})=0}} W(B^{-1}\mathbf{x})\theta(\phi_2(\mathbf{x})). \tag{2.3}$$

This is a smooth version of the counting function (1.6). Clearly to prove Theorem 1.1, it is enough to show that

$$M^*(B) \ll B^{9/5+\varepsilon}. \tag{2.4}$$

Using (2.2) we get that

$$M^*(B) \ll \sum_{\substack{\mathbf{x}\in\mathbb{Z}^4\\ \phi_1(\mathbf{x})=0}} W(B^{-1}\mathbf{x})\frac{1}{P^2}\Big|\sum_{p\in\mathcal{P}}\chi_p(\phi_2(\mathbf{x}))\Big|^2$$
$$\ll \frac{1}{P^2}\sum_{p_1,p_2\in\mathcal{P}}\sum_{\substack{\mathbf{x}\in\mathbb{Z}^4\\ \phi_1(\mathbf{x})=0}} W(B^{-1}\mathbf{x})\chi_{p_1p_2}(\phi_2(\mathbf{x})).$$

To compute the contribution of the diagonal terms ($p_1 = p_2$) we need an upper bound for the number of integer solutions of $\phi_1(\mathbf{x}) = 0$ of height B. Precise asymptotic for this counting function can be obtained using circle method, however for our purpose the following crude bound, which can be established quite easily, suffices

$$\sum_{\substack{\mathbf{x}\in\mathbb{Z}^4\\ \phi_1(\mathbf{x})=0}} W(B^{-1}\mathbf{x}) \ll B^{2+\varepsilon}. \tag{2.5}$$

So it follows that

$$M^*(B) \ll \frac{B^{2+\varepsilon}}{P} + \frac{1}{P^2}\sum_{p_1\neq p_2\in\mathcal{P}} T_{p_1p_2}(B), \tag{2.6}$$

where

$$T_q(B) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^4\\ \phi_1(\mathbf{x})=0}} W(B^{-1}\mathbf{x})\chi_q(\phi_2(\mathbf{x})). \tag{2.7}$$

A trivial upper bound of $T_q(B)$ is given by (2.5). But we seek to utilize the cancellation coming from the sign change of $\chi_q(\phi_2(\mathbf{x}))$ to get a non-trivial bound. This is an interesting problem on its own right and we will devote the rest of the article in establishing the following proposition.

Proposition 2.1. *For square-free q, we have*

$$T_q(B) \ll (q^{-1}B^2 + B^{3/2} + q^2B)(qB)^{\varepsilon} \tag{2.8}$$

where the implied constant depends on the forms ϕ_i and ε.

Assuming this result we can complete the proof of our main theorem. Indeed replacing $T_{p_1p_2}(B)$ in (2.6) by the upper bound we get

$$M^*(B) \ll \Big\{\frac{B^2}{P} + B^{3/2} + P^4B\Big\}(PB)^{\varepsilon}.$$

Then by choosing $P = B^{1/5}$ we get (2.4) and hence the main theorem.

3. The δ-symbol

We will prove Proposition 2.1 using a version of circle method introduced in [3]. The starting point is a smooth approximation of the following arithmetic function - the δ-symbol:

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

Let $w(t)$ be an even function on $\mathbb{R}$ with $w(0) = 0$ and compactly supported and such that

$$\sum_{k=1}^{\infty} w(k) = 1.$$

Put

$$\delta_k(n) = w(k) - w(n/k).$$

Then we have

$$\delta(n) = \sum_{k|n} \delta_k(n).$$

Using exponential sum to pick the condition $k|n$, we get

$$\delta(n) = \sum_k k^{-1} \sum_{h (\mathrm{mod}\ k)} e_k(hn)\delta_k(n).$$

Here we have introduced the notation $e_c(a)$ for $\exp(2\pi i a/c)$. Putting

$$\Delta_c(n) = \sum_r r^{-1} \delta_{cr}(n)$$

and $r = (h,k)$, $a = h/r$, $c = k/r$ we get the following expression for the δ-symbol.

Lemma 3.1.

$$\delta(n) = \sum_c c^{-1} \sideset{}{^*}\sum_{a (\mathrm{mod\ c})} e_c(an)\Delta_c(n). \tag{3.2}$$

In practice, to detect the equation $n = 0$ for a sequence of integers in the range $|n| < N/2$, we apply the above identity with a smooth test function $w(t)$ supported on $K/2 < |t| < K$, with derivatives satisfying $w^{(j)}(t) \ll K^{-j-1}$. It is logical to choose $K = N^{1/2}$. Then $\delta_k(n)$ vanishes unless $1 \le k < K$, and accordingly $\Delta_c(n)$ vanishes unless $1 \le c < K$. Also it follows that $\Delta_c(n) \ll K^{-1}$.

Heath-Brown [4] has successfully employed this version of the circle method to get asymptotic formula for $T_1(B)$. Of course the new feature in the sum $T_q(B)$ that we consider is the twist $\chi_q(\phi_2(\mathbf{x}))$, which is forced by the diophantine problem we are investigating. Naturally the basic parts of our analysis come very close to that in [4]. Our next lemma gives an expression for $T_q(B)$ which is free of the condition $\phi_1(\mathbf{x}) = 0$.

Lemma 3.2. *We have*

$$T_q(B) = \sum_{\mathbf{u}\in\mathbb{Z}^4} \sum_{c=1}^{\infty} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{u}) I_{q,c}(\mathbf{u}), \tag{3.3}$$

where

$$S_{q,c}(\mathbf{u}) = \sideset{}{^*}\sum_{a(\text{mod c})} \sum_{\mathbf{k}(\text{mod [q,c]})} \chi_q(\phi_2(\mathbf{k}))e_c(a\phi_1(\mathbf{k}))e_{[q,c]}(\mathbf{u}.\mathbf{k}) \tag{3.4}$$

and

$$I_{q,c}(\mathbf{u}) = \int_{\mathbb{R}^4} W(B^{-1}(\mathbf{y}))\Delta_c(\phi_1(\mathbf{y}))e_{[q,c]}(-\mathbf{u}.\mathbf{y})d\mathbf{y}. \tag{3.5}$$

Proof. We use Lemma 3.1 to pick the condition $\phi_1(\mathbf{x}) = 0$ in the definition of $T_q(B)$, and get

$$T_q(B) = \sum_c c^{-1} \sideset{}{^*}\sum_{a(\text{mod } c)} \sum_{\mathbf{x}\in\mathbb{Z}^4} W(B^{-1}\mathbf{x})\chi_q(\phi_2(\mathbf{x}))e_c(a\phi_1(\mathbf{x}))\Delta_c(\phi_1(\mathbf{x})). \tag{3.6}$$

Splitting the sum over $\mathbf{x}$ into respective residue classes modulo the least common multiple $[q, c]$, we get that the inner sum is given by

$$\sum_{\mathbf{k}(\text{mod } [q,c])} \chi_q(\phi_2(\mathbf{k}))e_c(a\phi_1(\mathbf{k})) \sum_{\mathbf{v}\in\mathbb{Z}^4} f(\mathbf{v}),$$

where $f(\mathbf{v}) = W(B^{-1}(\mathbf{k} + [q, c]\mathbf{v}))\Delta_c(\phi_1(\mathbf{k} + [q, c]\mathbf{v}))$. By Poisson summation formula we have

$$\sum_{\mathbf{v}\in\mathbb{Z}^4} f(\mathbf{v}) = \sum_{\mathbf{u}\in\mathbb{Z}^4} \hat{f}(\mathbf{u}),$$

where

$$\begin{aligned}\hat{f}(\mathbf{u}) &= \int_{\mathbb{R}^4} f(\mathbf{y})e(-\mathbf{u}.\mathbf{y})d\mathbf{y}\\ &= \frac{e_{[q,c]}(\mathbf{u}.\mathbf{k})}{[q,c]^4} \int_{\mathbb{R}^4} W(B^{-1}(\mathbf{y}))\Delta_c(\phi_1(\mathbf{y}))e_{[q,c]}(-\mathbf{u}.\mathbf{y})d\mathbf{y}.\end{aligned}$$

Substituing this in (3.6) and rearranging the sums over c, a and $\mathbf{u}$, the lemma follows. □

4. The mixed character sum $S_{q,c}(\mathbf{u})$

We begin this section by proving a general multiplicativity property of the mixed character sum

$$S_{q,c}(\mathbf{u}) = \sideset{}{^*}\sum_{a(\text{mod } c)} \sum_{\mathbf{k}(\text{mod } [q,c])} \chi_q(\phi_2(\mathbf{k}))e_c(a\phi_1(\mathbf{k}))e_{[q,c]}(\mathbf{u}.\mathbf{k}).$$

Lemma 4.1. *For $q = q_1q_2$, $c = c_1c_2$ with $(q_1c_1, q_2c_2) = 1$, we have*

$$S_{q,c}(\mathbf{u}) = S_{q_1,c_1}(\mathbf{u})S_{q_2,c_2}(\mathbf{u}).$$

Proof. Since $(q_1c_1, q_2c_2) = 1$ we have $[q, c] = [q_1, c_1][q_2, c_2]$. For convenience let $l_i = [q_i, c_i]$. Set

$$\mathbf{k} = \mathbf{m}l_1\bar{l}_1 + \mathbf{n}l_2\bar{l}_2, \quad \text{and} \quad a = a'l_1\bar{l}_1 + a''l_2\bar{l}_2,$$

where $\mathbf{m}$, a' run modulo l_2, and $\mathbf{n}$, a'' run modulo l_1. Also $l_1\bar{l}_1 \equiv 1(\text{mod } l_2)$ and $l_2\bar{l}_2 \equiv 1(\text{mod } l_1)$. Then we get

$$\begin{aligned}\chi_q(\phi_2(\mathbf{k})) &= \chi_{q_1}(\phi_2(\mathbf{n}))\chi_{q_2}(\phi_2(\mathbf{m})),\\ e_c(a\phi_1(\mathbf{k})) &= e_{c_1}(a''l_2^*\phi_1(\mathbf{n}))e_{c_2}(a'l_1^*\phi_1(\mathbf{m})), \quad \text{and}\\ e_{[q,c]}(\mathbf{u}.\mathbf{k}) &= e_{l_1}(\bar{l}_2\mathbf{u}.\mathbf{n})e_{l_2}(\bar{l}_1\mathbf{u}.\mathbf{m}),\end{aligned}$$

where $l_i^* = \bar{l}_i(l_i/c_i)$. Then substitute $l_1\mathbf{m}$ and $l_2\mathbf{n}$ in place of $\mathbf{m}$ and $\mathbf{n}$ respectively. Accordingly, substitute a' and a'' in place of $a'l_1^*l_1^2$ and $a''l_2^*l_2^2$ respectively. Then by rearranging the sum the lemma follows. $\square$

Next we evaluate $S_{q,1}(\mathbf{u})$ for prime modulus. This will be sufficient for our purpose, as the modulus q comes from the off-diagonal terms of the square-sieve and hence it will be a product of two distinct primes $q = p_1p_2$. First we note the following basic result about quadratic character sum which will be used several times in this section.

To this end let $\phi(\mathbf{x}) = \sum_{i=1}^n a_ix_i^2$ be an n-ary diagonal quadratic form. Let $\alpha = \prod a_i$, and define the associated quadratic form

$$\tilde{\phi}(\mathbf{x}) = \sum_{i=1}^n \frac{\alpha x_i^2}{a_i}. \tag{4.1}$$

Also for any odd prime p, let $\varepsilon(p) = 1$ if $p \equiv 1(\text{mod } 4)$, and $\varepsilon(p) = i$ if $p \equiv 3(\text{mod } 4)$.

Lemma 4.2. *Let p be a prime with $p \nmid 2\prod a_i$. Then we have*

$$\sum_{k(\text{mod p}^\text{r})} e_{p^r}(\phi(\mathbf{k}) + \mathbf{u}.\mathbf{k}) = \begin{cases} p^{nr/2}e_{p^r}(-\overline{4\alpha}\tilde{\phi}(\mathbf{u})) & \textit{if } r \textit{ is even;}\\ p^{nr/2}\chi_p(\alpha)\varepsilon(p)^n e_{p^r}(-\overline{4\alpha}\tilde{\phi}(\mathbf{u})) & \textit{if } r \textit{ is odd.}\end{cases}$$

Proof. Let S denote the sum appearing on the left-hand side of the formula. Then

$$S = \prod_{i=1}^n \Big\{ \sum_{k(\text{mod } p^r)} e_{p^r}(a_ik^2 + u_ik)\Big\}.$$

Since $p \nmid 2a_i$, we can evaluate the quadratic sum within the braces by completing the square. This yields

$$\sum_{k(\bmod p^r)} e_{p^r}(a_i k^2 + u_i k) = e_{p^r}(-\overline{4a_i} u_i^2) \sum_{k(\bmod p^r)} e_{p^r}(a_i k^2).$$

The last sum is the well known quadratic Gauss sum, which is given by

$$\sum_{k(\bmod p^r)} e_{p^r}(a_i k^2) = \begin{cases} p^{r/2} & \text{if } r \text{ is even;} \\ \chi_p(a_i)\varepsilon(p)p^{r/2} & \text{if } r \text{ is odd.} \end{cases}$$

Substituting this in the above expression of S, the lemma follows. □

We will apply the above lemma to evaluate $S_{p,1}$. Recall that $\phi_2(\mathbf{x}) = b_1x_1^2 + \cdots + b_4x_4^2$. Let $\beta = b_1b_2b_3b_4$ be the determinant of the associated matrix.

Lemma 4.3. *For any prime $p \nmid 2\beta$, we have*

$$S_{p,1}(\mathbf{u}) = \chi_p(-\tilde{\phi}_2(\mathbf{u}))p^2.$$

Proof. Let $\tau(p)$ denote the Gauss sum associated with the quadratic residue character χ_p. Then we have the inversion formula

$$\chi_p(x) = \frac{1}{\tau(p)} \sideset{}{^*}\sum_{m(\bmod p)} \chi_p(m) e_p(mx).$$

Using this formula we get

$$S_{p,1}(\mathbf{u}) = \frac{1}{\tau(p)} \sideset{}{^*}\sum_{m(\bmod p)} \chi_p(m) \sum_{\mathbf{k}(\bmod p)} e_p(m\phi_2(\mathbf{k}) + \mathbf{u}.\mathbf{k}).$$

The inner sum can be evaluated using Lemma 4.2, and it follows that

$$S_{p,1}(\mathbf{u}) = \frac{p^2}{\tau(p)} \sideset{}{^*}\sum_{m(\bmod p)} \chi_p(m\beta) e_p(-\overline{4m\beta}\tilde{\phi}_2(\mathbf{u}))$$

If $p|\tilde{\phi}_2(\mathbf{u})$ then we get $S_{p,1}(\mathbf{u}) = 0$. Otherwise changing m to $-\overline{4\beta m}\tilde{\phi}_2(\mathbf{u})$, we get

$$\begin{aligned} S_{p,1}(\mathbf{u}) &= \chi_p(-\tilde{\phi}_2(\mathbf{u})) \frac{p^2}{\tau(p)} \sideset{}{^*}\sum_{m(\bmod p)} \chi_p(m) e_p(m) \\ &= \chi_p(-\tilde{\phi}_2(\mathbf{u}))p^2. \end{aligned}$$

□

The exponential sum $S_{1,c}(\mathbf{u})$ is intrinsically related to the problem of evaluating asymptotically the number of solutions of the equation $\phi_1(\mathbf{x}) = 0$ via the circle method. As such this sum has been studied in the literature, for example [4]. However, for our purpose we need a high level of uniformity, and for this we will need finer information about this sum. We begin by linking $S_{1,c}(\mathbf{u})$ with the well known Ramanujan sum (rather than with the Kloosterman sum), which is defined as follows:

$$h_m(n) = \sum_{\substack{k(\text{mod } m)\\(k,m)=1}} e_m(kn).$$

Recall that $\phi_1(\mathbf{x}) = a_1x_1^2 + \cdots + a_4x_4^2$. Let $\alpha = a_1a_2a_3a_4$, which we again assume to be non-zero, and as before we define the associated form $\tilde{\phi}_1(\mathbf{x})$.

Lemma 4.4. *For any prime $p \nmid 2\alpha$, we have*

$$S_{1,p^r}(\mathbf{u}) = \chi_p(\alpha)^r h_{p^r}(\tilde{\phi}_1(\mathbf{u}))p^{2r}.$$

Proof. We apply Lemma 4.2 to get

$$\begin{aligned} S_{1,p^r}(\mathbf{u}) &= \sum_{a(\text{mod } p^r)}^{*} \sum_{\mathbf{k}(\text{mod } p^r)} e_{p^r}(a\phi_1(\mathbf{k}) + \mathbf{u}.\mathbf{k}) \\ &= p^{2r}\chi_p(\alpha)^r \sum_{a(\text{mod } p^r)}^{*} e_{p^r}(-\overline{4a\alpha}\tilde{\phi}_1(\mathbf{u})). \end{aligned}$$

In the last sum we substitute k in place of $-\overline{4a\alpha}$, and the result follows. □

From the above lemma and using the standard properties of the Ramanujan sum we deduce the following corollaries.

Corollary 4.1. *For any prime $p \nmid 2\alpha$, we have*

$$S_{1,p}(\mathbf{u}) = \begin{cases} \chi_p(\alpha)p^2(p-1) & \text{if } p|\tilde{\phi}_1(\mathbf{u}); \\ -\chi_p(\alpha)p^2 & \text{if } p \nmid \tilde{\phi}_1(\mathbf{u}). \end{cases}$$

Using the multiplicativity of $S_{1,c}(\mathbf{u})$ and the well known bound for the Ramanujan sum, we get the following upper bound.

Corollary 4.2. *We have*

$$S_{1,c}(\mathbf{u}) \ll c^2(c, \tilde{\phi}_1(\mathbf{u})),$$

where the implied constant depends only on the form ϕ_1.

Remark. In the above bound the factor c^2 is achieved due to explicit evaluation of the complete character sums in this section. A weaker factor, say c^3, which may be obtained using much simpler estimates is not sufficient for our purpose.

Finally we turn our attention on the mixed character sum $S_{p,p^r}(\mathbf{u})$ for a prime p. The following lemma records a sharp upper bound for this sum. To state the result in a neat form, we define the quadratic form $\phi_3(\mathbf{x}) = \sum_i a_i^2 b_i x_i^2$. (Recall that a_i's are the coefficients of ϕ_1 and b_i's are those of ϕ_2.)

Lemma 4.5. *For any prime $p \nmid 2\alpha$, and $r > 1$, we have*

$$S_{p,p^r}(\mathbf{u}) = \chi_p(\alpha)^r \chi_p(\phi_3(\mathbf{u})) p^{2r} h_{p^r}(\tilde{\phi}_1(\mathbf{u})).$$

Proof. Using the inversion formula, we get

$$\begin{aligned} S_{p,p^r}(\mathbf{u}) &= \sideset{}{^*}\sum_{a(p^r)} \sum_{\mathbf{k} (\mathrm{mod}\ p^r)} \chi_p(\phi_2(\mathbf{k})) e_{p^r}(a\phi_1(\mathbf{k}) + \mathbf{u}.\mathbf{k}) \\ &= \frac{1}{\tau(p)} \sum_{n(p)} \sideset{}{^*}\sum_{a(p^r)} \chi_p(n) \sum_{\mathbf{k} (\mathrm{mod}\ p^r)} e_{p^r}(\psi_n(\mathbf{k}) + \mathbf{u}.\mathbf{k}), \end{aligned}$$

where $\psi_n(\mathbf{k}) = a\phi_1(\mathbf{k}) + np^{r-1}\phi_2(\mathbf{k})$. Let us write, temporarily, $c_i = aa_i + nb_ip^{r-1}$ the coefficients of the diagonal form ψ_n. Now since $p \nmid 2\alpha$, and $r > 1$ it follows that $p \nmid c_i$. Then writing $\overline{aa_i}$ for the multiplicative inverse of aa_i modulo p^r, we observe that

$$(aa_i + nb_ip^{r-1})(\overline{aa_i} - (\overline{aa_i})^2 nb_ip^{r-1}) \equiv 1 (\mathrm{mod}\ p^r),$$

and it follows from Lemma 4.2 that

$$\sum_{\mathbf{k} (\mathrm{mod}\ p^r)} e_{p^r}(\psi_n(\mathbf{k}) + \mathbf{u}.\mathbf{k}) = \chi_p(\alpha)^r p^{2r} e_{p^r}(-\overline{4a\alpha}\tilde{\phi}_1(\mathbf{u})) e_p(\bar{4}n \sum_{i=1}^{4} (\overline{aa_i})^2 b_i u_i^2).$$

Substituting this in the above expression of S_{p,p^r}, we get

$$\begin{aligned} S_{p,p^r}(\mathbf{u}) &= \frac{p^{2r}\chi_p(\alpha)^r}{\tau(p)} \sum_{n(p)} \sideset{}{^*}\sum_{a(p^r)} \chi_p(n) e_{p^r}(-\overline{4a\alpha}\tilde{\phi}_1(\mathbf{u})) e_p(\bar{4}n \sum_{i=1}^{4} (\overline{aa_i})^2 b_i u_i^2) \\ &= \frac{p^{2r}\chi_p(\alpha)^r}{\tau(p)} h_{p^r}(\tilde{\phi}_1(\mathbf{u})) \sum_{n(p)} \chi_p(n) e_p(n \sum_{i=1}^{4} \overline{a_i}^2 b_i u_i^2). \end{aligned}$$

The result follows. □

For the sum $S_{p,p}$ we will be satisfied with the following upper bound. We also impose, for the sake of simplicity, the condition that all the 2×2 minors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

are non-zero. Let $\mathcal{M}$ denote the product of all these minors.

Lemma 4.6. *Suppose that the quadratic forms ϕ_1 and ϕ_2 satisfy the above condition. Then for any prime $p \nmid 2\alpha\beta\mathcal{M}$, we have*

$$S_{p,p}(\mathbf{u}) \ll p^{5/2}.$$

Proof. The inversion formula gives

$$\begin{aligned} S_{p,p}(\mathbf{u}) &= \sum_{a(p)}^{*} \sum_{\mathbf{k}(\mathrm{mod}\ p)} \chi_p(\phi_2(\mathbf{k})) e_p(a\phi_1(\mathbf{k}) + \mathbf{u}.\mathbf{k}) \\ &= \frac{1}{\tau(p)} \sum_{n(p)} \sum_{a(p)}^{*} \chi_p(na) \sum_{\mathbf{k}(\mathrm{mod}\ p)} e_p(a\psi_n(\mathbf{k}) + \mathbf{u}.\mathbf{k}), \end{aligned}$$

where $\psi_n(\mathbf{k}) = \phi_1(\mathbf{k}) + n\phi_2(\mathbf{k})$. Let $c_i = (a_i + nb_i)$ be the coefficients of the diagonal form ψ_n, and let $\gamma = c_1c_2c_3c_4$. For n such that $p \nmid (a_i + nb_i)$, it follows from Lemma 4.2 that

$$\sum_{\mathbf{k}(\mathrm{mod}\ p)} e_p(a\psi_n(\mathbf{k}) + \mathbf{u}.\mathbf{k}) = \chi_p(\gamma) p^2 e_p(-\overline{4a\gamma}\tilde{\psi}_n(\mathbf{u})).$$

On the other hand if n is such that $p|(a_i+nb_i)$, then the above sum vanishes unless $p|u_i$. In this case we observe that our hypothesis implies that $p \nmid \gamma/c_i$. So we can apply Lemma 4.2 to evaluate the above sum, and then executing the sum over a, we get that each $n \equiv -a_i\bar{b}_i(\mathrm{mod}\ p)$ contributes at most $p^{5/2}$ to $S_{p,p}$. Hence we get

$$\begin{aligned} S_{p,p} &= \frac{p^2}{\tau(p)} \sum_{\substack{n(p) \\ n \neq -a_i\bar{b}_i(p)}} \sum_{a(p)}^{*} \chi_p(na\gamma) e_p(-\overline{4a\gamma}\tilde{\psi}_n(\mathbf{u})) + O(p^{5/2}) \\ &= p^2 \sum_{\substack{n(p) \\ n \neq -a_i\bar{b}_i(p)}} \chi_p(-n\tilde{\psi}_n(\mathbf{u})) + O(p^{5/2}). \end{aligned}$$

The result follows by applying the Weil bound. ☐

The following upper bound is a consequence of Lemmas 4.5, 4.6, and the Corollary 4.2.

Corollary 4.3. *For any prime $p \nmid 2\alpha\beta\mathcal{M}$ we have*

$$S_{p,p^r}(\mathbf{u}) \ll p^{5r/2}(p^r, \tilde{\phi}_1(\mathbf{u})).$$

Also for $q|c$ we have $S_{q,c} \ll c^3$.

5. Cancellation in the sum of $S_{q,c}(\mathbf{u})$

To prove Proposition 2.1 we have to estimate

$$\sum_{c=1}^{\infty} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{u}) I_{q,c}(\mathbf{u}).$$

The estimates that we have obtained in the previous section for the individual terms $S_{q,c}(\mathbf{u})$ will be sufficient for our purpose, save the case when $\tilde{\phi}_1(\mathbf{u})$ vanishes. In this particular situation we have to take advantage of the cancellation coming from the sum over the modulus c, à la Kloosterman.

We begin this section with the bound which does not exploit the cancellation in the c-sum.

Lemma 5.1. *Let* $(q, 2\alpha\beta\mathcal{M}) = 1$*. Then for any* $\mathbf{u} \in \mathbb{Z}^4$ *with* $\tilde{\phi}_1(\mathbf{u}) \neq 0$*, we have*

$$\sum_{\substack{c \le X \\ q|c}} |S_{q,c}(\mathbf{u})| \ll q^{-1/2} X^3 |\tilde{\phi}_1(\mathbf{u})|^{\varepsilon}, \tag{5.1}$$

where the implied constant depends only on the forms ϕ_i *and on* ε*.*

Proof. As $q|c$ using Corollaries 4.2 and 4.3, we get

$$|S_{q,c}(\mathbf{u})| \ll q^{-1/2} c^2 (c, q\tilde{\phi}_1(\mathbf{u})).$$

Hence

$$\sum_{\substack{c \le X \\ q|c}} |S_{q,c}(\mathbf{u})| \ll \sum_{\substack{c \le X \\ q|c}} q^{-1/2} c^2 (c, q\tilde{\phi}_1(\mathbf{u}))$$
$$\ll q^{-1/2} \sum_{d | q\tilde{\phi}_1(\mathbf{u})} d^3 \sum_{c \le X/d} c^2.$$

Executing the sums and applying the trivial bound for the number of divisors of $\tilde{\phi}_1(\mathbf{u})$, the lemma follows. ☐

In case $\tilde{\phi}_1(\mathbf{u}) = 0$, using Corollary 4.2, we get

$$\sum_{c \le X} |S_{1,c}(\mathbf{u})| \ll X^4,$$

which is too weak to yield anything useful. Our next lemma gives a non-trivial averaging over the modulus c.

Lemma 5.2. *Suppose* $q = q_1 q_2$ *is square-free,* $(q, 2\alpha\beta\mathcal{M}) = 1$ *and* $\alpha = a_1 a_2 a_3 a_4$ *is not a square. Then for* $\mathbf{u} \in \mathbb{Z}^4$ *with* $\tilde{\phi}_1(\mathbf{u}) = 0$*, we have*

$$\sum_{\substack{c \le X \\ (c,q) = q_1}} S_{q_1,c}(\mathbf{u}) \ll q_1^{-1} X^{7/2} (Bq)^{\varepsilon}, \tag{5.2}$$

where the implied constant depends only on the forms ϕ_i*, and on* ε*.*

Proof. The multiplicativity of the mixed exponential sum gives the following decomposition of the generating Dirichlet series

$$\sum_{(c,q) = q_1} S_{q_1,c}(\mathbf{u}) c^{-s} = \prod_{p|q_1} \Big\{ S_{p,p}(\mathbf{u}) p^{-s} + \dots \Big\} L_q(s; \mathbf{u}),$$

where

$$L_q(s; \mathbf{u}) = \sum_{(c,q)=1} S_{1,c}(\mathbf{u}) c^{-s} = \prod_{p \nmid q} \Big\{ \sum_n S_{1,p^n}(\mathbf{u}) p^{-ns} \Big\}.$$

It follows from Lemma 4.2 that $L_q(s;\mathbf{u})$ converges absolutely for $\sigma = \Re(s) > 4$, and from Lemmas 4.5, 4.6 it follows that the product over $p|q_1$ is analytic for $\sigma > 7/2$ and is bounded by $O(1/q_1)$. Now for $p \nmid 2\alpha$, we can use Lemmas 4.2 and 4.4 to get

$$\sum_n S_{1,p^n}(\mathbf{u})p^{-ns} = 1 + p^2(p-1)\chi_p(\alpha)p^{-s} + O(p^{-1-\delta}),$$

for $\sigma \geq (7+\delta)/2$. So in this half plane we can write

$$L_q(s;\mathbf{u}) = L(s-3,\chi)l(s;\mathbf{u})\prod_{p|q}\Big\{\sum_n S_{1,p^n}(\mathbf{u})p^{-ns}\Big\},$$

where χ is the Dirichlet character such that $\chi(p) = \chi_p(\alpha)$, and $l(s;\mathbf{u})$ is a Dirichlet series which is absolutely convergent for $\sigma > 7/2$. Hence the above expression gives the analytic continuation of $L_q(s;\mathbf{u})$ to the half plane $\sigma > 7/2$. Also in this half plane

$$l(s;\mathbf{u})\prod_{p|q}\Big\{\sum_n S_{1,p^n}(\mathbf{u})p^{-ns}\Big\} \ll_\alpha d(q),$$

where $d(q)$ denotes the number of divisors of q. The result now follows by contour integration (Perron's formula), and observing that $L(s-3,\chi)$ does not have a pole at $s=4$ as α is not a square. □

6. Bounds for the integral $I_{q,c}(\mathbf{u})$

In this section we investigate the integral

$$I_{q,c}(\mathbf{u}) = \int_{\mathbb{R}^4} W(B^{-1}(\mathbf{y}))\Delta_c(\phi_1(\mathbf{y}))e_{[q,c]}(-\mathbf{u}.\mathbf{y})d\mathbf{y},$$

which appears in Lemma 3.2. Recall that the function Δ_c comes from the smooth approximation for the δ-symbol. The smooth function $w(t)$ involved in the formula (Lemma 3.1) is taken to be an even function which vanishes outside $B/2 < |t| < B$. Also it is such that $w^{(j)}(t) \ll B^{-j-1}$. So it follows that

$$\Delta_c^{(j)}(t) \ll (cB)^{-j}B^{-1}.$$

Using this bound and by integration-by-parts we obtain the following bound for the integral.

Lemma 6.1. *For $\mathbf{u} \neq \mathbf{0}$, we have*

$$I_{q,c}(\mathbf{u}) \ll cB^3\Big(\frac{[q,c]}{c|\mathbf{u}|}\Big)^N, \tag{6.1}$$

where the implied constant depends on N and on the form ϕ_1.

As a consequence we get that $\mathbf{u}$ with $|\mathbf{u}| > c^{-1}[q,c]B^{\varepsilon}$ will make a negligible contribution in our analysis of $T_q(B)$. For $\mathbf{u}$ with $0 < |\mathbf{u}| \leq c^{-1}[q,c]B^{\varepsilon}$ we need a more refined bound. The following result comes from a closer study of the behaviour of the function Δ_c, and is essentially due to Heath-Brown [4]. The proof does not seem to simplify significantly in the special case where ϕ_1 is a diagonal quadratic form. Hence we opt to skip the proof entirely. The interested reader may find the details in sections 4 - 8 of [4]. Note that in the notation used in [4] we have $I_{q,c}(\mathbf{u}) = cB^2 I_r^*(\mathbf{v})$ with $r = cB^{-1}$ and $\mathbf{v} = c[q,c]^{-1}\mathbf{u}$. Let $q = q_1q_2$ be a decomposition of q as a product of two coprime integers, and suppose $(q,c) = q_1$. Let $c = q_1t$, then $I_{q,q_1t}(\mathbf{u}) = q_1tB^2I_r^*(\mathbf{v})$ with $r = q_1tB^{-1}$ and $\mathbf{v} = q_2^{-1}\mathbf{u}$. Hence $I_{q,q_1t}(\mathbf{u})$ is a differentiable function in t.

Lemma 6.2. *For $0 < |\mathbf{u}| \leq qB^{\varepsilon}$ and $q \leq B$, we have*

$$I_{q,q_1t}(\mathbf{u}) \ll q_1^2q_2t^2|\mathbf{u}|^{-1}B^{1+\varepsilon}, \quad \textit{and}$$
$$\frac{d}{dt}I_{q,q_1t}(\mathbf{u}) \ll q_1^2q_2t|\mathbf{u}|^{-1}B^{1+\varepsilon},$$

where the implied constants depend on the form ϕ_1 and on ε.

Remark. Here and after we are using the well accepted convention that ε is an arbitrary positive number, not the same in each occurance and the implied constants depend at least on ε among other parameters.

Finally for the zero frequency $\mathbf{u} = \mathbf{0}$ we have the following result.

Lemma 6.3. *We have*

$$I_{q,c}(\mathbf{0}) \ll cB^2, \tag{6.2}$$

where the implied constant depends only on ϕ_1.

7. Proof of Proposition 2.1

Using the bound obtained in Lemma 6.1, we get

$$T_q(B) = \sum_{c=1}^{\infty} \sum_{|\mathbf{u}| \leq c^{-1}[c,q]B^{\varepsilon}} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{u}) I_{q,c}(\mathbf{u}) + O(1).$$

Let $q = q_1q_2$ be a decomposition of q as a product of two coprime integers. Then for a $\mathbf{u}$ with $\tilde{\phi}_1(\mathbf{u}) \neq 0$, consider the sum

$$J(\mathbf{u}) = \sum_{\substack{c=1 \\ (c,q)=q_1}}^{\infty} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{u}) I_{q,c}(\mathbf{u}).$$

Recall that, by our choice, the function Δ_c vanishes unless $1 \leq c < B$. Hence $I_{q,c}(\mathbf{u})$ vanishes unless $1 \leq c < B$. Using the multiplicativity of the mixed exponential sum we get

$$J(\mathbf{u}) = q_2^{-4} S_{q_2,1}(\mathbf{u}) \sum_{\substack{c<B \\ (c,q)=q_1}} c^{-5} S_{q_1,c}(\mathbf{u}) I_{q,c}(\mathbf{u}).$$

An application of Lemmas 4.3, 5.1 and 6.2, gives

$$q_2^{-4} S_{q_2,1}(\mathbf{u}) \sum_{\substack{C<c<2C \\ (c,q)=q_1}} c^{-5} S_{q_1,c}(\mathbf{u}) I_{q,c}(\mathbf{u}) \ll \frac{B^{1+\varepsilon}}{q_1^{1/2} q_2 |\mathbf{u}|}.$$

Hence it follows that

$$J(\mathbf{u}) \ll \frac{B^{1+\varepsilon}}{q_1^{1/2} q_2 |\mathbf{u}|}.$$

We collect the contributions of those $\mathbf{u}$ for which $\tilde{\phi}_1(\mathbf{u}) \neq 0$, and define

$$J^* = \sum_{\substack{|\mathbf{u}|\leq q_2 B^\varepsilon \\ \tilde{\phi}_1(\mathbf{u})\neq 0}} J(\mathbf{u}).$$

Our next lemma furnishes a satisfactory bound for the above sum.

Lemma 7.1. *We have*

$$J^* \ll q_1^{-1/2} q_2^2 B (qB)^\varepsilon,$$

where the implied constant depends on the forms ϕ_i and on ε.

Proof. We have already shown above that

$$J^* \ll \sum_{\substack{|\mathbf{u}|\leq q_2 B^\varepsilon \\ \tilde{\phi}_1(\mathbf{u})\neq 0}} \frac{B^{1+\varepsilon}}{q_1^{1/2} q_2 |\mathbf{u}|}.$$

Now

$$\sum_{0<|\mathbf{u}|\leq U} |\mathbf{u}|^{-1} \ll \sum_{1\leq n\leq U^2} \frac{1}{\sqrt{n}} \sum_{|\mathbf{u}|^2=n} 1 \ll U^{3+\varepsilon}.$$

The lemma follows. □

Next we consider $J(\mathbf{u})$ for $\mathbf{u} \neq 0$ with $\tilde{\phi}_1(\mathbf{u}) = 0$. In this case we use Lemmas 5.2 and 6.2, together with partial summation to establish

$$\sum_{\substack{C<c\leq 2C \\ (c,q)=q_1}} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{u}) I_{q,c}(\mathbf{u}) = \frac{S_{q_2,1}(\mathbf{u}) q_1^{-1}}{q^4} \sum_{\substack{\frac{C}{q_1}<n\leq 2\frac{C}{q_1} \\ (n,q_2)=1}} \frac{I_{q,q_1 n}(\mathbf{u})}{n^5} S_{q_1,q_1 n}(\mathbf{u})$$

$$\ll \frac{q_2^2 q_1^{-1}}{q^4} \frac{q_1^4 q_2 B C^{1/2}}{|\mathbf{u}|} B^\varepsilon = \frac{B C^{1/2}}{q|\mathbf{u}|} B^\varepsilon.$$

This yields that for $\mathbf{u} \neq 0$ with $\tilde{\phi}_1(\mathbf{u}) = 0$, we have

$$J(\mathbf{u}) \ll (q|\mathbf{u}|)^{-1} B^{3/2+\varepsilon}.$$

Let

$$J^{**} = \sum_{\substack{0<|\mathbf{u}|\le q_2 B^\varepsilon \\ \tilde{\phi}_1(\mathbf{u})=0}} J(\mathbf{u}).$$

Lemma 7.2. *We have*

$$J^{**} \ll q_1^{-1} B^{3/2} (qB)^\varepsilon,$$

where the implied constant depends on the forms ϕ_i and on ε.

Proof. From the above analysis it follows that

$$J^{**} \ll \frac{B^{3/2+\varepsilon}}{q} \sum_{\substack{0<|\mathbf{u}|\le q_2 B^\varepsilon \\ \tilde{\phi}_1(\mathbf{u})=0}} |\mathbf{u}|^{-1}.$$

We can evaluate the last sum by breaking it up into dyadic blocks

$$\sum_{\substack{\mathbf{u} \\ U_i<u_i<2U_i \\ \tilde{\phi}_1(\mathbf{u})=0}} |\mathbf{u}|^{-1} \ll (\max U_i)^{-1} \sum_{\substack{\mathbf{u} \\ U_i<u_i<2U_i \\ \tilde{\phi}_1(\mathbf{u})=0}} 1 \ll (\max U_i)^{1+\varepsilon}.$$

The lemma follows. □

It remains to consider the contribution of the zero frequency

$$J(\mathbf{0}) = \sum_{\substack{c=1 \\ (c,q)=q_1}}^{\infty} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{0}) I_{q,c}(\mathbf{0}).$$

Using multiplicativity we get

$$S_{q,c}(\mathbf{0}) = S_{q_2,1}(\mathbf{0}) S_{q_1,c}(\mathbf{0}),$$

which, by Lemma 4.3, vanishes unless $q_2 = 1$, i.e. $q|c$. So $J(\mathbf{0}) = 0$ unless $q|c$, in which case we have the following lemma.

Lemma 7.3. *We have*

$$J(\mathbf{0}) \ll q^{-1} B^{2+\varepsilon},$$

where the implied constant depends on the forms ϕ_i and on ε.

Proof. We have

$$J(\mathbf{0}) = \sum_{\substack{c<B \\ q|c}} \frac{c^{-1}}{[q,c]^4} S_{q,c}(\mathbf{0}) I_{q,c}(\mathbf{0}) = \sum_{\substack{c<B \\ q|c}} c^{-5} S_{q,c}(\mathbf{0}) I_{q,c}(\mathbf{0}).$$

Then using Corollary 4.3 and Lemma 6.3, we get that the above expression is bounded by

$$B^{2+\varepsilon} \sum_{\substack{c<B \\ q|c}} c^{-1} \ll \frac{B^{2+\varepsilon}}{q}.$$

□

Now Proposition 2.1 follows from Lemmas 7.1, 7.2 and 7.3.

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