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Weber's class number problem in the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$, II

par Takashi FUKUDA et Keiichi KOMATSU

Résumé. Soit h_n le nombres de classes du n-ième étage de la $\mathbb{Z}_2$-extension cyclotomique de $\mathbb{Q}$. Weber a prouvé que h_n $(n \geq 1)$ est impair et Horie a prouvé que h_n $(n \geq 1)$ n'est divisible par aucun nombre premier ℓ satisfaisant $\ell \equiv 3, 5 \pmod 8$. Dans un article précédent, les auteurs ont montré h_n $(n \geq 1)$ n'est divisible par aucun nombre premier ℓ inférieur à 10^7. Dans le présent article, en étudiant plus précisément les propriétés d'une unité particulière, nous montrons que h_n $(n \geq 1)$ n'est divisible par aucun nombre premier ℓ inférieur à $1.2 \cdot 10^8$. Notre argument conduit aussi à la conclusion que h_n $(n \geq 1)$ n'est divisible par aucun nombre premier ℓ satisfaisant $\ell \not\equiv \pm 1 \pmod{16}$.

Abstract. Let h_n denote the class number of n-th layer of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$. Weber proved that h_n $(n \geq 1)$ is odd and Horie proved that h_n $(n \geq 1)$ is not divisible by a prime number ℓ satisfying $\ell \equiv 3, 5 \pmod 8$. In a previous paper, the authors showed that h_n $(n \geq 1)$ is not divisible by a prime number ℓ less than 10^7. In this paper, by investigating properties of a special unit more precisely, we show that h_n $(n \geq 1)$ is not divisible by a prime number ℓ less than $1.2 \cdot 10^8$. Our argument also leads to the conclusion that h_n $(n \geq 1)$ is not divisible by a prime number ℓ satisfying $\ell \not\equiv \pm 1 \pmod{16}$.

1. Introduction

Let $\zeta_n = \exp(2\pi\sqrt{-1}/2^n)$ and $\mathbb{Q}_n = \mathbb{Q}(\zeta_{n+2} + \zeta_{n+2}^{-1})$. Then $\mathbb{Q}_n$, which is n-th layer of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$, is a cyclic extension of $\mathbb{Q}$ with degree 2^n. Weber [14] studied the class number h_n of $\mathbb{Q}_n$ and proved that h_n is odd for all $n \geq 1$. Weber also showed $h_1 = h_2 = h_3 = 1$. We note that h_{n-1} divides h_n because h_{n-1} is odd and $[\mathbb{Q}_n : \mathbb{Q}_{n-1}] = 2$.

Weber conjectured $h_4 > 1$. But Cohn [2], Bauer [1] and Masley [10] showed $h_4 = 1$. Furthermore Linden [11] showed $h_5 = 1$. It is also shown $h_6 = 1$ if GRH (Generalized Riemann Hypothesis) is valid. This phenomenon indicates a possibility that $h_n = 1$ for all $n \geq 1$. But the technique using root discriminant, which enables Masley and Linden to show $h_4 = 1$

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and $h_5 = 1$ respectively, is no longer applicable for h_n $(n \geq 7)$. We need a entirety new technique to calculate h_n or to show $h_n = 1$ for $n \geq 7$.

The calculation of the whole class number h_n is very difficult even if we use a modern computer. So we are led to study the odd part of h_n. In this aspect, there are preceding works of Washington [12] and [13]. He proved that the ℓ-part of h_n is bounded as n tends to ∞ for a fixed prime number ℓ. Precisely speaking, using the theory of $\mathbb{Z}_p$-extensions, he developed a method which enables us to obtain an explicit bound on n for which the growth of e_n stops, where $h_n = \ell^{e_n} q$ with q not divisible by ℓ.

There is also an approach of Horie [5], [6], [7], [8] which tries to attack h_n from another point of view. He proved that if ℓ satisfies a certain congruence relation and exceeds a certain bound, which is explicitly described, then ℓ does not divide h_n for all $n \geq 1$, namely the ℓ-part of h_n is trivial for all $n \geq 1$. The following is a part of Horie's results.

Proposition 1.1 (Horie, cf. Proposition 3 in [8])**.** *Let ℓ be a prime number such that $\ell \equiv 3, 5 \pmod 8$. Then ℓ does not divide h_n for all $n \geq 1$.*

Horie also obtained the following results which treat higher congruence.

Proposition 1.2 (Horie, cf. Theorem 1 in [5] and Theorem 1 in [7])**.** *Let ℓ be a prime number.*

(1) *If $\ell \equiv 9 \pmod{16}$ and $\ell > 34797970939$, then ℓ does not divide h_n for all $n \geq 1$.*
(2) *If $\ell \equiv -9 \pmod{16}$ and $\ell > 210036365154018$, then ℓ does not divide h_n for all $n \geq 1$.*

Although Horie's results were very striking and very effective, there were many small prime numbers ℓ for which we did not know whether ℓ divides h_n. For example, it was not known whether $\ell \mid h_n$ $(n \geq 6)$ for $\ell = 7, 17, 23, 31, 41, \ldots.$

The main purpose of this paper is to prove the following two theorems. The first,which is proved by investigating the properties of a special unit introduced by Horie, is considered an explicit version of Theorem 3 in [12] and is a refinement of Theorems 1.2 and 5.1 in [3], which were proved by relating the plus part of the class number with the non-divisibility of Bernoulli numbers. For a real number x, we denote by $[x]$ the largest integer not exceeding x.

Theorem 1.1. *Let ℓ be an odd prime number and 2^c the exact power of 2 dividing $\ell - 1$ or $\ell^2 - 1$ according as $\ell \equiv 1 \pmod 4$ or not. Put*

$$m_\ell = 2c - 3 + [\log_2 \ell]$$

and recall h_n denotes the class number of $\mathbb{Q}_n$. Then ℓ does not divide h_n / h_{m_ℓ} for any integer $n \geq m_\ell$.

Typical values of m_ℓ are as follows:

ℓ	7	17	31	257	8191	65537	524287	7340033	39845887
m_ℓ	7	9	14	21	38	45	56	59	66

Theorem 1.1 has a computational application. An algorithm verifying that ℓ does not divide h_n for given ℓ and n was established in [3] and the value of m_ℓ is small enough for this algorithm. So we are able to derive the following corollary which will supersede Corollary 1.3 in [3]. We implemented the algorithms in [3] on a computer with Xeon 2.0 GHz processor and 32 GB memory using TC. The calculating time was three months.

Corollary 1.1. *Let ℓ be a prime number less than $1.2 \cdot 10^8$. Then ℓ does not divide h_n for all $n \geq 1$.*

The second is considered a precise version of Proposition 1.2, which is a direct consequence of Corollary 1.1 and Lemma 2.3 in §2.

Theorem 1.2. *Notations being as in Theorem 1.1, if $\ell \equiv \pm 9 \pmod{16}$, then ℓ does not divide h_n for all $n \geq 1$.*

Remark. After we wrote this manuscript, we were aware of the preprint of K. Horie and M. Horie [9], in which they showed that a prime number ℓ does not divide h_n for all $n \geq 1$ if ℓ satisfies $\ell \equiv 9 \pmod{16}$ and $\ell > 7150001069$ or if $\ell \equiv -9 \pmod{16}$ and $\ell > 17324899980$.

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2. Proofs

We prove our theorems by using Horie's method in [8]. Notations being as in Theorem 1.1, let $\zeta_n = \exp(2\pi\sqrt{-1}/2^n)$ and put

$$\eta_n = \frac{\zeta_{n+2} - 1}{\sqrt{-1}(\zeta_{n+2} + 1)}$$

Then η_n is a unit and contained in $\mathbb{Q}_n$ because $\mathbb{Q}_n$ is the maximal real subfield of $\mathbb{Q}(\zeta_{n+2})$. This special unit, which played important role in Horie's work, takes an active part also in our proofs. First we note

$$(2.1) \quad N_{\mathbb{Q}_n/\mathbb{Q}_{n-1}}(\eta_n) = \frac{\zeta_{n+2} - 1}{\sqrt{-1}(\zeta_{n+2} + 1)} \, \frac{-\zeta_{n+2} - 1}{\sqrt{-1}(-\zeta_{n+2} + 1)} = -1.$$

An element α in $\mathbb{Z}[\zeta_n]$ is uniquely expressed in the form

$$\alpha = \sum_{j=0}^{2^{n-1}-1} a_j \zeta_n^j \quad (a_j \in \mathbb{Z}).$$

For each such α and each $\sigma \in G(\mathbb{Q}(\zeta_{n+2})/\mathbb{Q}(\zeta_2))$, we define the element α_σ in the group ring $\mathbb{Z}[G(\mathbb{Q}(\zeta_{n+2})/\mathbb{Q}(\zeta_2))]$ by

$$\alpha_\sigma = \sum_{j=0}^{2^{n-1}-1} a_j \sigma^j \,.$$

The following Horie's results are essential in this paper. Following the referee's advice that self-contained paper is convenient for readers, we give proofs here. The idea is due to the referee.

Proposition 2.1 (Horie, cf. Lemma 2 in [5])**.** *Let ℓ be an odd prime number, σ a generator of the Galois group $G(\mathbb{Q}(\zeta_{n+2})/\mathbb{Q}(\zeta_2))$ and F an extension in $\mathbb{Q}(\zeta_n)$ of the decomposition field of ℓ with respect to for $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. Then ℓ divide h_n/h_{n-1} if and only if there exists a prime ideal $\mathfrak{L}$ of F dividing ℓ such that $\eta_n^{\alpha_\sigma}$ is an ℓ-th power in $\mathbb{Q}_n$ for any element α of the ideal $\ell\mathfrak{L}^{-1}$ of F.*

Proof. We prove "only if part" which is sufficient for our purpose. We take an integer s with $\zeta_{n+2}^\sigma = \zeta_{n+2}^s$ and put

$$\rho = \sigma^{2^{n-1}}, \quad \xi = \frac{\zeta_{n+3} - \zeta_{n+3}^{-1}}{\zeta_{n+3}^s - \zeta_{n+3}^{-s}} \,.$$

Let E_n be the unit group of $\mathbb{Q}_n$ and C_n the cyclotomic unit group of $\mathbb{Q}_n$, which is generated by $\{\, \xi^{\sigma^i} \mid i = 1, 2, \dots, 2^n \,\}$. Then $\mathbb{Z}[\zeta_n]$ acts on $E_n^{1-\rho}$ by $(\varepsilon^{1-\rho})^\alpha = (\varepsilon^{1-\rho})^{\alpha_\sigma}$ for $\varepsilon \in E_n$ and $\alpha \in \mathbb{Z}[\zeta_n]$ and we have

$$\mathbb{Z}_\ell \otimes (E_n^{1-\rho}/C_n^{1-\rho}) \cong \prod_j \mathbb{Z}[\zeta_n]/\mathfrak{L}_j^{k_j} \,,$$

where $\mathfrak{L}_j$ runs through the prime ideals of $\mathbb{Q}(\zeta_n)$ lying above ℓ and k_j is a non-negative integer. Moreover the order of $E_n^{1-\rho}/C_n^{1-\rho}$ is h_n/h_{n-1} by analytic class number formula.

Now we assume that ℓ divides h_n/h_{n-1}. Then there exists a prime ideal $\mathfrak{L}_j$ of $\mathbb{Q}(\zeta_n)$ lying above ℓ with $k_j > 0$. Hence we have $(\xi^{1-\rho})^{\alpha_\sigma}$ is an ℓ-th power in $\mathbb{Q}_n$ for $\alpha \in (\ell)\mathfrak{L}_j^{-1}$. Since $(\eta_n^{1+\rho})^2 = 1$ by (2.1), we have

$$\eta_n^4 = \eta_n^{2-2\rho} = \eta_n^{2(1-\sigma)(1+\sigma+\cdots+\sigma^{2^{n-1}-1})} \,.$$

This shows

$$\eta_n^4 = (\xi^{1-\rho})^{2(1+\sigma+\cdots+\sigma^{2^{n-1}-1})}$$

by $\eta_n^{1-\sigma} = \xi^{1-\rho}$, which means $\eta_n^{\alpha_\sigma}$ is an ℓ-th power in $\mathbb{Q}_n$. $\square$

Proposition 2.2 (Horie, cf. Lemma 5 in [4]). *Let ℓ be an odd prime number and φ the Frobenius automorphism of ℓ in $\mathbb{Q}(\zeta_{n+2})/\mathbb{Q}$. If an element β in $\mathbb{Z}[\zeta_{n+2}]$ is an ℓ-th power in $\mathbb{Z}[\zeta_{n+2}]$, then $\beta^\varphi - \beta^\ell \in \ell^2\mathbb{Z}[\zeta_{n+2}]$.*

Proof. Put $\beta = x^\ell$ and $x^\varphi = x^\ell + \ell u$ with $x, u \in \mathbb{Z}[\zeta_{n+2}]$. Then

$$\beta^\varphi = (x^\varphi)^\ell = (x^\ell + \ell u)^\ell = (\beta + \ell u)^\ell \equiv \beta^\ell \pmod{\ell^2}.$$

□

Let ℓ and φ be as in Proposition 2.2, $\zeta = \zeta_{n+2}$, σ a generator of $G(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta_2))$ and put $\eta = \eta_n = (\zeta - 1)/(\sqrt{-1}(\zeta + 1))$. We choose $\mathbb{Q}(\zeta_c)$ as F. We assume $n \geq c$ and ℓ divides h_n/h_{n-1}. Then, by Proposition 2.1, there exists a prime ideal $\mathfrak{L}$ in $\mathbb{Q}(\zeta_c)$ dividing ℓ such that η^{α_σ} is an ℓ-th power of a unit in $\mathbb{Q}_n$ for any element α of the ideal $\ell\mathfrak{L}^{-1}$ of $\mathbb{Q}(\zeta_c)$. let

$$\alpha = \sum_{i=0}^{2^{c-1}-1} a_i (\zeta_n^{2^{n-c}})^i$$

be an element in $\ell\mathfrak{L}^{-1}$ with $a_i \in \mathbb{Z}$. we put $\tau = \sigma^{2^{n-c}}$. Then $\alpha_\sigma = \sum_{i=0}^{2^{c-1}-1} a_i\tau^i$ and $(\zeta^{\tau^i - 1})^{2^c} = 1$. Now, we start computations similar to Lemma 13 in [8]. Noting that

$$\begin{aligned}(\beta+\gamma)^{a\ell} &= \Big(\beta^\ell + \gamma^\ell + \sum_{k=1}^{\ell-1}\binom{\ell}{k}\beta^{\ell-k}\gamma^k\Big)^a \\ &\equiv (\beta^\ell+\gamma^\ell)^a + a(\beta^\ell+\gamma^\ell)^{a-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\beta^{\ell-k}\gamma^k \pmod{\ell^2}\end{aligned}$$

for $\beta, \gamma \in \mathbb{Z}[\zeta]$ with $\beta + \gamma$ prime to ℓ and for $a \in \mathbb{Z}$, it follows that

$$\begin{aligned}(\zeta^{\tau^i} - 1)^{a_i\ell} \equiv{}& (\zeta^{\ell\tau^i} - 1)^{a_i} \\ &+ a_i(\zeta^{\ell\tau^i} - 1)^{a_i - 1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\zeta^{\tau^i(\ell-k)}(-1)^k \pmod{\ell^2}, \\ (\zeta^{\tau^i} + 1)^{-a_i\ell} \equiv{}& (\zeta^{\ell\tau^i} + 1)^{-a_i} \\ &- a_i(\zeta^{\ell\tau^i} + 1)^{-a_i - 1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\zeta^{\tau^i(\ell-k)} \pmod{\ell^2}.\end{aligned}$$

From these congruence relations and a consequence

$$\begin{aligned}\frac{(\eta^{\alpha_\sigma})^\ell - (\eta^{\alpha_\sigma})^\varphi}{\sqrt{-1}^{\,-\ell\alpha^\sigma}} &= \prod_{i=0}^{2^{c-1}-1}\frac{(\zeta^{\tau^i} - 1)^{a_i\ell}}{(\zeta^{\tau^i} + 1)^{a_i\ell}} - \prod_{i=0}^{2^{c-1}-1}\Big(\frac{\zeta^{\ell\tau^i} - 1}{\zeta^{\ell\tau^i} + 1}\Big)^{a_i} \\ &\equiv 0 \pmod{\ell^2}\end{aligned}$$

of Propositions 2.1 and 2.2, we have

$$\sum_{i=0}^{2^{c-1}-1}\Big(\frac{a_i}{\zeta^{\ell\tau^i}-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}(-1)^k\zeta^{\tau^i(\ell-k)} - \frac{a_i}{\zeta^{\ell\tau^i}+1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\zeta^{\tau^i(\ell-k)}\Big) \equiv 0 \pmod{\ell^2}$$

because $\zeta^{\ell(\tau^i-1)} \pm \zeta^{-\ell}$ are prime to ℓ. Since

$$\binom{\ell}{k} \equiv \frac{\ell(-1)^{k-1}}{k} \pmod{\ell^2} \quad (1 \le k \le \ell-1)$$

and since

$$\prod_{i=0}^{2^{c-1}-1}(\zeta^{\ell\tau^i}-1)(\zeta^{\ell\tau^i}+1) = \prod_{i=0}^{2^{c-1}-1}(\zeta^{2\ell\tau^i}-1) = 1-\zeta^{2^c\ell},$$

we have

$$\begin{aligned}
&(1-\zeta^{2^c\ell})\sum_{i=0}^{2^{c-1}-1}\Big(\frac{a_i}{\zeta^{\ell\tau^i}-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}(-1)^k\zeta^{\tau^i(\ell-k)} - \frac{a_i}{\zeta^{\ell\tau^i}+1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\zeta^{\tau^i(\ell-k)}\Big)\\
&\equiv \ell\sum_{i=0}^{2^{c-1}-1}a_i\Big(\sum_{j=0}^{2^c-1}-\zeta^{\ell\tau^i(2^c-1-j)}\sum_{k=1}^{\ell-1}\frac{(-1)^{2k-1}}{k}\zeta^{\tau^i(\ell-k)}\\
&\qquad -\sum_{j=0}^{2^c-1}(-1)^{2^c-1-j}\zeta^{\ell\tau^i(2^c-1-j)}\sum_{k=1}^{\ell-1}\frac{(-1)^{k-1}}{k}\zeta^{\tau^i(\ell-k)}\Big)\\
&\equiv 0 \pmod{\ell^2}.
\end{aligned}$$

Hence we have

$$\sum_{i=0}^{2^{c-1}-1}a_i\sum_{j=0}^{2^c-1}\sum_{k=1}^{\ell-1}\Big(\frac{1}{k}+\frac{(-1)^{j+k+1}}{k}\Big)\zeta^{-\tau^i(\ell j+k)} \equiv 0 \pmod{\ell}$$

by $\zeta^{2^c(\tau^i-1)}=1$. Considering the complex conjugate of the left hand side of the above congruence relation, we have the following:

Lemma 2.1. *Let α be in Proposition 2.1 and*

$$\alpha = \sum_{i=0}^{2^{c-1}-1}a_i\Big(\zeta_n^{2^{n-c}}\Big)^i \tag{2.2}$$

with $a_i \in \mathbb{Z}$*. If* ℓ *divides* h_n/h_{n-1}*, then*

$$\sum_{i=0}^{2^{c-1}-1} a_i \sum_{j=0}^{2^c-1} \sum_{k=1}^{\ell-1} \frac{1+(-1)^{j+k+1}}{k} \zeta^{\tau^i(\ell j+k)} \equiv 0 \pmod{\ell}.$$

We put

$$S = \{\, b_0 2^{n-c+2} + b_1 2^{n-c+3} + \cdots + b_{c-1} 2^{n+1} \mid b_j = 0, 1 \text{ for } 0 \le j \le c-1 \,\}$$

and define the subset S' of S by

$$S' = \bigcup_{i=0}^{2^{c-1}-1} \{\, r \in S \mid \zeta^{\tau^i - 1} = \zeta^r \,\}.$$

Lemma 2.2. *Let* j *and* k *be rational integers with* $0 \le j \le 2^c - 1$*,* $1 \le k \le \ell - 1$ *and* $r \in S'$*. Let* ℓ *be an odd prime number with* $\ell < 2^{n-2c+3}$*. If* $(r+1)(\ell j+k) \equiv 2^{c-1}\ell - 1 \pmod{2^{n+1}}$*, then we have* $j = 2^{c-1}-1$*,* $k = \ell - 1$ *and* $r = 0$*.*

Proof. We have $-2^{n-c+2} < (2^{c-1} - j)\ell - k - 1 < 2^{n-c+2}$ because of $0 \le j \le 2^c - 1$, $1 \le k \le \ell - 1$ and $\ell < 2^{n-2c+3}$. Since $(2^{c-1} - j)\ell - k - 1 \equiv 0 \pmod{2^{n-c+2}}$, we have $(2^{c-1}-j)\ell - k - 1 = 0$. Since $2 \le k+1 = (2^{c-1}-j)\ell \le \ell$, we have $k = \ell - 1$ and $j = 2^{c-1} - 1$, which implies $r \equiv 0 \pmod{2^{n+1}}$. Hence $r = 0$ or $r = 2^{n+1}$. Since $r \in S'$, we have $r = 0$. □

Proof of Theorem 1.1. The assertion of the theorem is trivial when $n = m_\ell$. So we assume that ℓ divides h_n/h_{n-1} for some n greater than m_ℓ. Then ℓ satisfies $\ell < 2^{n-2c+3}$ and Lemma 2.1 yields

$$\sum_{i=0}^{2^{c-1}-1} a_i \sum_{j=0}^{2^c-1} \sum_{k=1}^{\ell-1} \frac{1+(-1)^{j+k+1}}{k} \zeta^{\tau^i(\ell j+k)} \equiv 0 \pmod{\ell},$$

where a_i is the rational integer defined by (2.2). We choose an element α in $\ell\mathfrak{L}^{-1}$ so that $\alpha \notin \ell\mathbb{Z}[\zeta_c]$. Since we may assume $a_0 \not\equiv 0 \pmod{\ell}$, we see that $a_i \dfrac{-1+(-1)^{j+k}}{k} \not\equiv 0 \pmod{\ell}$ for $i = 0$, $j = 2^{c-1} - 1$ and $k = \ell - 1$. This contradicts Lemma 2.2 because $\{\, \zeta^i \mid 0 \le i \le 2^{n+1} - 1 \,\}$ is an integral basis of $\mathbb{Q}(\zeta)$. □

We follow the arguments in [5] to prove Theorem 1.2. For an algebraic number α, let

$$||\, \alpha \,|| = \max_{\rho} |\, \alpha^\rho \,|,$$

where ρ runs through all isomorphism of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$. Then

$$||\, \beta\beta' \,|| \le ||\, \beta \,|| \cdot ||\, \beta' \,||, \quad ||\, \beta^m \,|| = ||\, \beta \,||^m$$

for any algebraic numbers β, β' and any positive rational integer m. The following is the key lemma in our proof.

Lemma 2.3. *Assume that an odd prime number ℓ divides h_n/h_{n-1}.*

(1) *If* $\ell \equiv 9 \pmod{16}$, *then we have* $2^{n-3} < \ell < 32(n+1)^4$.

(2) *If* $\ell \equiv -9 \pmod{16}$, *then we have* $2^{n-5} < \ell < 98(n+1)^4$.

Proof. It is known that $h_5 = 1$ by [11]. So we may assume $n \geq 6$. Recall that σ is a generator of $G(\mathbb{Q}(\zeta_{n+2})/\mathbb{Q}(\zeta_2))$. (1) The decomposition field of ℓ with respect to $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is $\mathbb{Q}(\zeta_3)$. Proposition 2.1 guarantees the existence of a prime ideal $\mathfrak{L}$ of $\mathbb{Q}(\zeta_3)$ dividing ℓ such that η^{α_σ} is an ℓ-th power in $\mathbb{Q}_n$ for each element α of $\mathbb{Q}(\zeta_3)$ with $\ell\mathfrak{L}^{-1} = (\alpha)$. We write $\alpha = a_0 + a_1\zeta_3 + a_2\zeta_3^2 + a_3\zeta_3^3$ with $a_i \in \mathbb{Z}$ and denote by $\overline{\alpha}$ the complex conjugate of α. Then we have

$$\alpha\overline{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2 + \sqrt{2}(a_0a_1 + a_1a_2 + a_2a_3 - a_3a_0).$$

We put $a = (a_0^2+a_1^2+a_2^2+a_3^2)/\ell^{3/2}$ and $b = (a_0a_1+a_1a_2+a_2a_3-a_3a_0)/\ell^{3/2}$. Since $N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\alpha) = \ell^3$, we have $a^2 - 2b^2 = 1$. Hence there exists a real number x with $a + b\sqrt{2} = (\sqrt{2}+1)^x$ and $a - b\sqrt{2} = (\sqrt{2}-1)^x$. Since $(\alpha) = (\alpha(1+\sqrt{2})^m)$ for $m \in \mathbb{Z}$, we may assume $-1 \leq x < 1$. Hence we have $0 \leq a \leq \sqrt{2}$, which implies $a_0^2 + a_1^2 + a_2^2 + a_3^2 \leq \sqrt{2}\ell^{3/2}$. This shows $|a_0| + |a_1| + |a_2| + |a_3| \leq 2^{5/4}\ell^{3/4}$. Noting that $\eta^{\alpha_\sigma} \neq \pm 1$ (cf. [5, p. 384]), we have

$$\begin{aligned} 2^\ell < \|\, \eta^{\alpha_\sigma} \,\| &= \|\, \eta^{a_0 + a_1\sigma^{2^{n-3}} + a_2\sigma^{2\cdot 2^{n-3}} + a_3\sigma^{3\cdot 2^{n-3}}} \,\| \\ &\leq \|\, \eta \,\|^{|a_0|+|a_1|+|a_2|+|a_3|} \\ &\leq \|\, \eta \,\|^{2^{5/4}\ell^{3/4}} < 2^{2^{5/4}(n+1)\ell^{3/4}} \end{aligned} \tag{2.3}$$

by the formula (2.1) and [5, Lemmas 3 and 4]. On the other hand, we have

$$n \leq m_\ell = 3 + [\,\log_2 \ell\,] < 3 + \log_2 \ell \tag{2.4}$$

by Theorem 1.1. Combining (2.3) and (2.4), we derive the desired inequality.

(2) In this case, the decomposition field F of ℓ with respect to $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is $\mathbb{Q}(\sqrt{-1}\sqrt{2-\sqrt{2}})$, which is contained in $\mathbb{Q}(\zeta_4)$. Proposition 2.1 again guarantees the existence of a prime ideal $\mathfrak{L}$ of F dividing ℓ such that η^{α_σ} is an ℓ-th power in $\mathbb{Q}_n$ for each element α of F with $\ell\mathfrak{L}^{-1} = (\alpha)$. We write $\alpha = a_0 + a_1\zeta_4 + \cdots + a_7\zeta_4^7$ with $a_i \in \mathbb{Z}$. For the Frobenius automorphism φ_ℓ of ℓ with respect to $\mathbb{Q}(\zeta_4)/\mathbb{Q}$, we have $\alpha^{\varphi_\ell} = \alpha$, which implies $a_4 = 0$, $a_5 = a_3$, $a_6 = -a_2$ and $a_7 = a_1$. Hence we have

$$\alpha = a_0 + a_1(\zeta_4 + \zeta_4^7) + a_2(\zeta_4^2 - \zeta_4^6) + a_3(\zeta_4^3 + \zeta_4^5).$$

This shows

$$\alpha\overline{\alpha} = a_0^2 + 2a_1^2 + 2a_2^2 + 2a_3^2 + \sqrt{2}(2a_0a_2 - a_1^2 + 2a_1a_3 + a_3^2).$$

We put $a = (a_0^2+2a_1^2+2a_2^2+2a_3^2)/\ell^{3/2}$ and $b = (2a_0a_2-a_1^2+2a_1a_3+a_3^2)/\ell^{3/2}$. Since $N_{F/\mathbb{Q}}(\alpha) = \ell^3$, we have $a^2 - 2b^2 = 1$. Hence there exists a real number

x with $a + b\sqrt{2} = (\sqrt{2}+1)^x$ and $a - b\sqrt{2} = (\sqrt{2}-1)^x$. In a way similar to that in the case $\ell \equiv 9 \pmod{16}$, we have $a_0^2 + 2(a_1^2 + a_2^2 + a_3^2) \le \sqrt{2}\ell^{3/2}$, which shows $|a_0| + 2(\,|a_1| + |a_2| + |a_3|\,) \le 2^{1/4}\sqrt{7}\ell^{3/4}$. Hence we have

$$
\begin{aligned}
(2.5) \quad 2^\ell &< ||\,\eta^{\alpha_\sigma}\,|| \\
&= ||\,\eta^{a_0 + a_1(\sigma^{2^{n-4}} + \sigma^{7\cdot 2^{n-4}}) + a_2(\sigma^{2\cdot 2^{n-4}} - \sigma^{6\cdot 2^{n-4}}) + a_3(\sigma^{3\cdot 2^{n-4}} + \sigma^{5\cdot 2^{n-4}})}\,|| \\
&\le ||\,\eta\,||^{|a_0| + 2(\,|a_1| + |a_2| + |a_3|\,)} \\
&\le ||\,\eta\,||^{2^{1/4}\sqrt{7}\ell^{3/4}} < \Big(\frac{2^{n+2}}{\pi}\Big)^{2^{1/4}\sqrt{7}\ell^{3/4}} < 2^{2^{1/4}\sqrt{7}(n+1)\ell^{3/4}}.
\end{aligned}
$$

In this case, Theorem 1.1 implies

$$
(2.6) \qquad n \le m_\ell = 5 + [\,\log_2 \ell\,] < 5 + \log_2 \ell
$$

and we combine (2.5) and (2.6) to derive the conclusion. □

Proof of Theorem 1.2. Assume that ℓ divides h_n/h_{n-1} for some $n \ge 1$. Then Lemma 2.3 implies $\ell < 32 \cdot 28^4 = 19668992$ if $\ell \equiv 9 \pmod{16}$ or $\ell < 98 \cdot 32^4 = 102760448$ if $\ell \equiv -9 \pmod{16}$. However this contradicts Corollary 1.1. Hence the proof is completed.

□

Remark. We are also able to prove Theorem 1.2 by combining Proposition 1.2 and Theorem 1.1. Namely, it suffices to verify that ℓ does not divide h_{m_ℓ} for all ℓ not exceeding a certain explicit bound. This bound on ℓ is 34797970939 in the case $\ell \equiv 9 \pmod{16}$ and 210036365154018 in the case $\ell \equiv -9 \pmod{16}$. The calculating time is estimated about one month or one thousand years.

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