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Résumé. On fait voir comment K. Hensel aurait pû étendre le théorème de Wilson de $\mathbf{Z}$ à l'anneau des entiers $\mathfrak{o}$ d'un corps de nombres, pour trouver le produit de tous les éléments inversibles d'un quotient fini de $\mathfrak{o}$.

Abstract. We show how K. Hensel could have extended Wilson's theorem from $\mathbf{Z}$ to the ring of integers $\mathfrak{o}$ in a number field, to find the product of all invertible elements of a finite quotient of $\mathfrak{o}$.

1. Introduction

> *...puisque de tels hommes n'ont pas cru ce sujet indigne de leurs méditations...* [1].

More than two hundred years ago, Gauss generalised Wilson's theorem ($(p-1)! \equiv -1 \pmod p$ for a prime number p) to an arbitrary integer $A > 0$ in §78 of his *Disquisitiones* :

Theorem 1.1. ([1]) *Poductum ex omnibus numeris, numero* quocunque *dato A minoribus simulque ad ipsum primis, congruum est secundum A, vnitati vel negatiue vel positiue sumtae.*

(The product of all elements in $(\mathbf{Z}/A\mathbf{Z})^\times$ is $\bar{1}$ or $-\bar{1}$). He then specifies that the product in question is $-\bar{1}$ if A is 4, or p^m, or $2p^m$ for some odd prime p and integer $m > 0$; it equals $\bar{1}$ in the remaining cases.

According to Gauss ([1], §76) the elegant theorem according to which " upon augmenting the product of all numbers less than a given prime number by the unity, it becomes divisible by that prime number " was first stated by Waring in his *Meditationes* — which appeared in Cambridge in 1770 — and attributed to Wilson, but neither could prove it. Waring remarks that the proof must be all the more difficult as there is no *notation* which might express a prime number. *Nach unserer Meinung aber müssen derartige Wahrheiten vielmehr aus Begriffen (*notionibus*) denn aus Bezeichnungen (*notationibus*) geschöpft werden* [1]. The first proof was given by Lagrange (1771).

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Some hundred years later, Hensel [2] developed his local notions, which could have allowed him to extend the result from $\mathbf{Z}$ to rings of integers in number fields ; our aim here is to show how he could have done it.

Proposition 1.1. (“ Wilson’s theorem ”) *For an ideal* $\mathfrak{a} \subset \mathfrak{o}$ *in the ring of integers of a number field* K*, the product of all elements in* $(\mathfrak{o}/\mathfrak{a})^\times$ *is* $\bar{1}$*, except that it is*

1. $-\bar{1}$ *when* $\mathfrak{a}$ *has precisely one odd prime divisor, and* $v_\mathfrak{p}(\mathfrak{a}) < 2$ *for every even prime ideal* $\mathfrak{p}$*,*
2. $\bar{1} + \bar{\pi}$ *(resp.* $\bar{1} + \bar{\pi}^2$*) when all prime divisors of* $\mathfrak{a}$ *are even and for precisely one of them, say* $\mathfrak{p}$*,* $v_\mathfrak{p}(\mathfrak{a}) > 1$ *with moreover* $v_\mathfrak{p}(\mathfrak{a}) = 2$*,* $f_\mathfrak{p} = 1$ *(resp.* $v_\mathfrak{p}(\mathfrak{a}) = 3$*,* $f_\mathfrak{p} = 1$*,* $e_\mathfrak{p} > 1$*) ; here* π *is any element of* $\mathfrak{p}$ *not in* $\mathfrak{p}^2$*, and we have indentified* $(\mathfrak{o}/\mathfrak{p}^2)^\times$ *(resp.* $(\mathfrak{o}/\mathfrak{p}^3)^\times$*) with a subgroup of* $(\mathfrak{o}/\mathfrak{a})^\times$*.*

The notation and the terminology are unambiguous : a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ is even if $2 \in \mathfrak{p}$, odd if $2 \notin \mathfrak{p}$; $v_\mathfrak{p}(\mathfrak{a})$ is the exponent of $\mathfrak{p}$ in the prime decomposition of $\mathfrak{a}$; $f_\mathfrak{p}$ is the residual degree and $e_\mathfrak{p}$ the ramification index of $K_\mathfrak{p}|\mathbf{Q}_p$ (p being the rational prime which belongs to $\mathfrak{p}$).

(It may happen that $\bar{1}+\bar{\pi} = -\bar{1}$ in $(\mathfrak{o}/\mathfrak{p}^2)^\times$ (resp. $\bar{1}+\bar{\pi}^2 = -\bar{1}$ in $(\mathfrak{o}/\mathfrak{p}^3)^\times$) for some even prime $\mathfrak{p} \subset \mathfrak{o}$. Example : $\mathfrak{o} = \mathbf{Z}$ (resp. $\mathbf{Z}[\sqrt{2}]$) and $\mathfrak{p}$ the unique even prime of $\mathfrak{o}$. More banally, we have $-\bar{1} = \bar{1}$ in $(\mathfrak{o}/\mathfrak{p}^n)^\times$ when $\mathfrak{p}$ is an even prime and n is between 1 and $e_\mathfrak{p}$.)

2. d_2

The elementary observation behind the proof of Gauss’s th. 1.1, also used in our proof of prop. 1.1, is that the sum s of all the elements in a finite commutative group G is 0, unless G has precisely one order-2 element τ, in which case $s = \tau$. Anyone can supply a proof ; he can then skip this section, and take the condition “ $d_2(G) = 1$ ” as a shorthand for “ G has precisely one order-2 element ”.

Define $d_2(G) = \dim_{\mathbf{F}_2}({}_2G)$, where ${}_2G$ is the subgroup of G killed by 2. It is clear that G has $2^{d_2(G)} - 1$ order-2 elements.

Example. For a prime number p and a positive integer n, we have $d_2((\mathbf{Z}/p^n\mathbf{Z})^\times) =$

1. 1 if $p \neq 2$,
2. 0 if $p = 2$ and $n = 1$,
3. 1 if $p = 2$ and $n = 2$,
4. 2 if $p = 2$ and $n > 2$.

In this example, the unique order-2 element is $-\bar{1}$ whenever $d_2 = 1$.

Lemma 2.1. *The sum* s *of all elements in* G *is* 0 *unless* $d_2(G) = 1$*, in which case* s *is the unique order-*2 *element of* G*.*

The involution $\iota : g \mapsto -g$ fixes every element of the subgroup ${}_2G = \mathrm{Ker}(x \mapsto 2x)$. As the sum of elements in the remaining orbits of ι is 0, we are reduced to the case $G = {}_2G$ of a vector $\mathbf{F}_2$-space, and the proof is over by induction on the dimension $d_2(G)$ of ${}_2G$, starting with dimension 2.

Proof of Gauss's th. 1.1 : Let $A = \prod_p p^{m_p}$ be the prime decomposition of A. By the Chinese remainder theorem, $(\mathbf{Z}/A\mathbf{Z})^\times$ is the product over p of $(\mathbf{Z}/p^{m_p}\mathbf{Z})^\times$, so $d_2((\mathbf{Z}/A\mathbf{Z})^\times)$ is the sum over p of $d_2((\mathbf{Z}/p^{m_p}\mathbf{Z})^\times)$. In view of the foregoing Example, the only way for this sum to be 1 is for A to be 2^2, or p^{m_p}, or $2p^{m_p}$ for some odd prime p and integer $m_p > 0$.

3. Local units

Let's enter Hensel's world : let p be a prime number, $K \mid \mathbf{Q}_p$ a finite extension, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$. Let $n > 0$ be an integer. We would like to know when $d_2((\mathfrak{o}/\mathfrak{p}^n)^\times) = 1$, and, when such is the case, which one the unique order-2 element is.

Proposition 3.1. *Denoting by e the ramification index and by f the residual degree of $K \mid \mathbf{Q}_p$, we have $d_2((\mathfrak{o}/\mathfrak{p}^n)^\times) =$*

(1) 1 *if* $p \neq 2$,
(2) 0 *if* $p = 2$, $n = 1$,
(3) 1 *if* $p = 2$, $n = 2$, $f = 1$,
(4) 1 *if* $p = 2$, $n = 3$, $f = 1$, $e > 1$,
(5) > 1 *in all other cases.*

For any $\mathfrak{o}$-basis π of $\mathfrak{p}$, the unique order-2 element in the cases $d_2 = 1$ is

(1) $-\bar{1}$ *if* $p \neq 2$,
(2) $\bar{1} + \bar{\pi}$ *if* $p = 2$, $n = 2$, $f = 1$,
(3) $\bar{1} + \bar{\pi}^2$ *if* $p = 2$, $n = 3$, $f = 1$, $e > 1$.

Proof : For every $j > 0$, denote by U_j the kernel of $\mathfrak{o}^\times \to (\mathfrak{o}/\mathfrak{p}^j)^\times$. If $p \neq 2$, the group $(\mathfrak{o}/\mathfrak{p}^n)^\times$ is the direct product of the even-order cyclic group $(\mathfrak{o}/\mathfrak{p})^\times$ and the p-group U_1/U_n, so $d_2 = 1$.

Assume now that $p = 2$. When $n = 1$, the group $(\mathfrak{o}/\mathfrak{p})^\times$ is (cyclic) of odd order, so $d_2 = 0$. If $f > 1$, then the d_2 of U_1/U_2 is f and hence the d_2 of $(\mathfrak{o}/\mathfrak{p}^n)^\times$ is > 1 for every $n > 1$.

Assume further that $f = 1$. When $n = 2$, the d_2 of $(\mathfrak{o}/\mathfrak{p}^2)^\times = U_1/U_2$ is $f = 1$. If moreover $e = 1$, then the d_2 of U_1/U_n is 2 for $n > 2$ (see Example).

Assume finally that, in addition, $e > 1$. We see that U_1/U_3 is generated by $\bar{1} + \bar{\pi}$, since $(1 + \pi)^2 = 1 + \pi^2 + 2\pi$ is in U_2 but not in U_3. However, U_1/U_4 is not cyclic because its order is 8 whereas every element has order at most 4 : for every $a \in \mathfrak{o}$,

$$(\bar{1} + \bar{a}\bar{\pi})^4 = \bar{1} + \bar{4}\bar{\pi}\bar{a} + \bar{6}\bar{\pi}^2\bar{a}^2 + \bar{4}\bar{\pi}^3\bar{a}^3 + \bar{\pi}^4\bar{a}^4 = \bar{1}$$

in U_1/U_4. Hence U_1/U_n is not cyclic for $n > 3$ (cf. Narkiewicz, *Elem. and anal. theory of alg. numbers*, 1990, p. 275). This concludes the proof.

(For $p = 2$ and $n > 2e$, we have $d_2((\mathfrak{o}/\mathfrak{p}^n)^\times) = 1 + ef$; cf. Hasse, *Zahlentheorie*, Kap. 15.)

Corollary 3.1. *The only cases in which the group* $(\mathfrak{o}/\mathfrak{p}^n)^\times$ *has precisely one order-*2 *element* s *are :* $p \neq 2$ *;* $p = 2$, $n = 2$, $f = 1$ *;* $p = 2$, $n = 3$, $f = 1$, $e > 1$. *In these three cases,* $s = -\bar{1}$, $\bar{1} + \bar{\pi}$, $\bar{1} + \bar{\pi}^2$, *respectively. The group* $(\mathfrak{o}/\mathfrak{p}^n)^\times$ *has no order-*2 *element precisely when* $p = 2$, $n = 1$.

4. The proof

Let us return to the global situation of an ideal $\mathfrak{a} \subset \mathfrak{o}$ in the ring of integers of a number field $K \mid \mathbf{Q}$. The proof can now proceed as in the case $\mathfrak{o} = \mathbf{Z}$ (§2). Everything boils down to deciding if the d_2 of $(\mathfrak{o}/\mathfrak{a})^\times$ is 1 — we know that the product of all elements is 1 if $d_2 \neq 1$ (lemma 2.1). Writing $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{m_\mathfrak{p}}$ the prime decomposition of $\mathfrak{a}$, the Chinese remainder theorem tells us that $d_2((\mathfrak{o}/\mathfrak{a})^\times)$ is the sum, over the various primes $\mathfrak{p}$ of $\mathfrak{o}$, of $d_2((\mathfrak{o}/\mathfrak{p}^{m_\mathfrak{p}})^\times)$. This sum can be 1 only when one of the terms is 1, the others being 0.

For each $\mathfrak{p}$, the group $(\mathfrak{o}/\mathfrak{p}^{m_\mathfrak{p}})^\times$ is the same as $(\mathfrak{o}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^{m_\mathfrak{p}})^\times$, where $\mathfrak{o}_\mathfrak{p}$ is the completion of $\mathfrak{o}$ at $\mathfrak{p}$ and $\mathfrak{p}_\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{o}_\mathfrak{p}$. Running through the possibilities enumerarted in prop. 3.1 completes the proof of prop. 1.1.

Example. Let $\zeta \in \bar{\mathbf{Q}}^\times$ be an element of order 2^t $(t > 1)$; take $K = \mathbf{Q}(\zeta)$ and $\mathfrak{p}$ the unique even prime of its ring of integers $\mathbf{Z}[\zeta]$. We have $e_\mathfrak{p} = 2^{t-1}$ and $f_\mathfrak{p} = 1$; we may take $\pi = 1 - \zeta$. The product of all elements in $(\mathbf{Z}[\zeta]/\mathfrak{p}^n)^\times$ is respectively $\bar{1}$, $\bar{1} + \bar{\pi}$, $\bar{1} + \bar{\pi}^2$, $\bar{1}$ for $n = 1$, $n = 2$, $n = 3$ and $n > 3$.

5. Acknowledgements

We thank Herr Prof. Dr. Peter Roquette for suggesting the present definition $d_2(G) = \dim_{\mathbf{F}_2}({}_2G)$ instead of the original $d_2(G) = \dim_{\mathbf{F}_2}(G/2G)$. After this Note was completed, a search in the literature revealed M. Laššák, *Wilson's theorem in algebraic number fields*, Math. Slovaca, **50** (2000), no. 3, pp. 303–314. We solicited a copy from Prof. G. Grekos, and thank him for supplying one ; it contains substantially the same result as our prop. 1.1. Our proof is shorter, simpler, more direct, and more conceptual ; it is based on *notionibus* rather than *notationibus*, of which there is now-a-days a surfeit. In any case, our aim was to show how Hensel could have proved prop. 1.1.

References

[1] C. Gauss, *Disquisitiones arithmeticae.* Gerh. Fleischer, Lipsiae, 1801, xviii+668 pp.

[2] K. Hensel, *Die multiplikative Darstellung der algebraischen Zahlen für den Bereich eines beliebigen Primteilers.* J. f. d. reine und angewandte Math., **146** (1916), pp. 189–215.

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