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Density of rational points on cyclic covers of $\mathbb{P}^n$

par Ritabrata MUNSHI

Résumé. Nous obtenons une majoration de la densité des points rationnels sur les revêtements cycliques de $\mathbb{P}^n$. Quand $n \to \infty$ notre estimation tend vers la majoration conjecturale de Serre.

Abstract. We obtain upper bound for the density of rational points on the cyclic covers of $\mathbb{P}^n$. As $n \to \infty$ our estimate tends to the conjectural bound of Serre.

1. Introduction

Let $\mathbb{P}^n$ denote the n-dimensional projective space over the field of rational numbers. For a finite cover $f : X \to \mathbb{P}^n$ over $\mathbb{Q}$, one can define a counting function

$$N(f, B) = |\{P \in X(\mathbb{Q}) : H(f(P)) \leq B\}|,$$

where H is the standard multiplicative height function defined on $\mathbb{P}^n$. If the degree of f is at least two, Serre [10] proved that

$$N(f, B) \ll B^{n+\frac{1}{2}}(\log B)^{\gamma}$$

with $\gamma < 1$. Here the implied constant depends both on f and γ. Serre's proof is based on an analogous result proved by Cohen [3] in the affine case. The proof is a beautiful application of the large sieve inequalities. Serre [10] also conjectures that one should have

$$N(f, B) \ll B^n(\log B)^c,$$

for some c. Broberg [2] almost settled Serre's conjecture for covers of the projective line and the projective plane. More precisely, he proved:

(1) If $f : X \to \mathbb{P}^1$ is a cover of degree d, then for every $\varepsilon > 0$ we have

$$N(f, B) \ll_{\epsilon,f} B^{\frac{2}{d}+\varepsilon}.$$

(2) If $f : X \to \mathbb{P}^2$ is a cover of degree $d > 2$, then for every $\varepsilon > 0$ we have

$$N(f, B) \ll_{\epsilon,f} B^{2+\varepsilon}.$$

In the case $d = 2$, we have

$$N(f, B) \ll_{\epsilon,f} B^{\frac{9}{4}+\varepsilon}$$

for every $\varepsilon > 0$.

Broberg's method is based on that introduced by Heath-Brown in [7]. These methods, though elementary, have proved to be powerful in many situations. In fact the last result of Broberg for degree two covers of $\mathbb{P}^2$, is the best known result towards Manin's conjecture, which predicts the exponent to be $1+\varepsilon$, for del Pezzo surfaces of degree 2. Here we will give a slightly better estimate albeit at the cost of employing heavy machinery.

To this end let $F(x_0,\dots,x_n)$ be an irreducible homogenous polynomial of degree md, with $d>1$. Moreover suppose that the projective hypersurface defined by $F(\mathbf{x})=0$ is smooth. Then the equation

$$y^d = F(x_0,\dots,x_n)$$

defines a variety X in the weighted projective space $\mathbb{P}(m,1,\dots,1)$, where y is given weight m and each x_i is given weight 1. This variety can be viewed as a cyclic d-sheeted cover of the projective space $\mathbb{P}^n$, via the natural map

$$f : X \to \mathbb{P}^n;$$
$$(y,x_0,\dots,x_n) \mapsto (x_0,\dots,x_n).$$

This cover is ramified above the hypersurface $F(\mathbf{x})=0$. Using adjunction formula one easily verifies the canonical sheaf of X to be

$$\omega_X = O_X(-m+md-n-1).$$

Hence, if the parameters n, m and d are such that

$$md-m<n+1$$

then the variety X is a Fano variety, i.e. the anticanonical sheaf is ample. In this case we have a more precise conjecture due to Manin et al ([1], [5]), regarding the density of rational points.

In this paper, we employ the power sieve together with Poisson summation formula, and deep results of Deligne [4], and of Katz [9] about cancellation in mixed character sums over finite fields, to estimate the number of d-powers in the set

$$\{F(x_0,\dots,x_n) : -B<x_i<B\}$$

with some smooth weight W. This will give us our main result.

Theorem 1.1. *Let $f : X \to \mathbb{P}^n$ be a finite cyclic cover of $\mathbb{P}^n$ over $\mathbb{Q}$. Then we have*

$$N(f,B) \ll_f B^{n+\frac{1}{n+2}} (\log B)^{\frac{n+1}{n+2}}.$$

Remark. In the special case of cyclic covers our result is better than that of Serre [10]. Also it is exciting to note that as the dimension n gets larger our bound comes closer to the conjectured bound of Serre.

Remark. For $n = 2$ and $d = 2$ our result gives

$$N(f, B) \ll B^{\frac{9}{4}} (\log B)^{\frac{3}{4}},$$

which improves the result of Broberg by an ε. However, for $n = 2$, $d > 2$ our result is worse.

2. Power sieve

In this section we briefly recall the concept of d-power sieve (see [6] for $d = 2$). Let p be a prime such that $p \equiv 1 \pmod{d}$. Then for every non-zero element a in the finite field $\mathbb{F}_p$, we can define a d-th root of unity by $a^{\frac{p-1}{d}}$. Now since $\mathbb{F}_p^*$ is cyclic, we can define a non-canonical isomorphism

$$\theta_p : \mathbb{F}_p^* \to \mu_{p-1},$$

where μ_{p-1} denotes the group of $(p-1)$-th roots of unity in $\mathbb{C}^*$. Using this we can define a primitive Dirichlet character modulo p,

$$\chi_p(n) = \theta_p\Big(\bar{n}^{\frac{p-1}{d}}\Big)$$

for $(p, n) = 1$, and $\chi_p(n) = 0$ for $p|n$. The crucial property of this character is the following:

$$\chi_p(n) = 1 \quad \text{if } (p, n) = 1, \text{ and } n = m^d \text{ for some } m.$$

Now let $\mathcal{A} = (a(n))$ be a finite sequence of non-negative quantities. We are interested in estimating the sum over the d-powers

$$S(\mathcal{A}) = \sum_n a(n^d).$$

Let $\mathcal{P}$ be a set of P primes each having residue 1 modulo d. Suppose that $a(n) = 0$ if $|n| \geq e^P$, and consider the expression

$$S = \sum_n a(n) |\sum_{p \in \mathcal{P}} \chi_p(n)|^2$$

Each n is counted with a non-negative weight, and if $n = m^d$ and $0 < |n| < e^P$ then

$$\sum_{p \in \mathcal{P}} \chi_p(n) = \sum_{\substack{p \in \mathcal{P} \\ (p,n)=1}} 1 \gg P.$$

Hence $P^2 S(\mathcal{A}) \ll S$. Moreover

$$\begin{aligned} S &= \sum_{p,q \in \mathcal{P}} \sum_n a(n) \chi_p(n) \bar{\chi}_q(n) \\ &\leq P \sum_n a(n) + \sum_{p \neq q \in \mathcal{P}} |\sum_n a(n) \chi_p(n) \bar{\chi}_q(n)|. \end{aligned}$$

Hence we obtain the following:

Lemma 2.1. *Let $\mathcal{P}$ be a set of P primes each having residue* 1 *modulo d. Suppose that $a(n) = 0$ for $n = 0$ and for $|n| \geq e^P$. Then*

$$S(\mathcal{A}) \ll P^{-1} \sum_n a(n) + P^{-2} \sum_{p \neq q \in \mathcal{P}} |\sum_n a(n)\chi_p(n)\bar{\chi}_q(n)|.$$

3. Proof of Theorem 1

Let $W : \mathbb{R}^{n+1} \to \mathbb{R}$ be a non-negative smooth function supported in the dyadic box $[-B, B]^{n+1}$, and such that the partial derivatives satisfy the following bound:

$$|\frac{d^{i_0+\cdots+i_n} W(x_0, \ldots, x_n)}{dx_0^{i_0} \ldots dx_n^{i_n}}| \ll B^{-(i_0+\cdots+i_n)}.$$

Then via integration-by-parts we note the following bound for the Fourier transform

$$\hat{W}(\mathbf{u}) = \int_{\mathbb{R}^{n+1}} W(\mathbf{x}) e(-\langle \mathbf{x}, \mathbf{u} \rangle) d\mathbf{x} \ll B^{n+1} \prod_{i=0}^{n} (1 + |u_i| B)^{-2}$$

where $\mathbf{u} = (u_0, \ldots, u_n)$.

Then we define a sequence of non-negative numbers by

$$a(k) = \sum_{\mathbf{x} \in \mathbb{Z}^{n+1}, F(\mathbf{x}) = k} W(\mathbf{x}).$$

We wish to obtain an upper bound for

$$S(\mathcal{A}) = \sum_k a(k^d)$$

using the d-power sieve. Now clearly we have

$$\sum_k a(k) \ll B^{n+1}.$$

Hence to apply the lemma we need to estimate the sum

$$S(p, q) = \sum_k a(k)\chi_p(k)\bar{\chi}_q(k) = \sum_{\mathbf{x} \in \mathbb{Z}^{n+1}} W(\mathbf{x})\chi_p(F(\mathbf{x}))\bar{\chi}_q(F(\mathbf{x})),$$

where p and q are two distinct primes. Using Poisson summation formula we obtain

$$\begin{aligned} S(p, q) &= \sum_{\mathbf{a} \pmod{pq}} \chi_p(F(\mathbf{a}))\bar{\chi}_q(F(\mathbf{a})) \sum_{\mathbf{x} \equiv \mathbf{a}} W(\mathbf{x}) \\ &= (pq)^{-(n+1)} \sum_{\mathbf{u} \in \mathbb{Z}^{n+1}} g(\mathbf{u}; p, q) \hat{W}\Big(\frac{\mathbf{u}}{pq}\Big), \end{aligned}$$

where

$$g(\mathbf{u};p,q)=\sum_{\mathbf{a}\ (\mathrm{mod}\ pq)}\chi_p(F(\mathbf{a}))\bar{\chi}_q(F(\mathbf{a}))e\Big(\frac{\langle\mathbf{a},\mathbf{u}\rangle}{pq}\Big)$$

is a mixed character sum. Then we observe that the above sum satisfies the following multiplicative property

$$g(\mathbf{u};p,q)=g(\bar{q}\mathbf{u};\chi_p)g(\bar{p}\mathbf{u};\bar{\chi}_q),$$

where $p\bar{p}\equiv 1 \pmod q$, $q\bar{q}\equiv 1 \pmod p$ and

$$g(\mathbf{u};\chi_p)=\sum_{\mathbf{a}\ (\mathrm{mod}\ p)}\chi_p(F(\mathbf{a}))e\Big(\frac{\langle\mathbf{a},\mathbf{u}\rangle}{p}\Big).$$

So our job is reduced to estimating the mixed character sum over the finite field $\mathbb{F}_p$. Let V be the hypersurface defined by $F(\mathbf{x})=0$ over the field of rationals. Let V^* be its dual variety, which in this case is again a hypersurface. Assuming that the prime p is such that the reduction $V_p = V \pmod p$ is smooth, we have three situations:

(1) First suppose that the vector $\mathbf{u}\in\mathbb{Z}^{n+1}$ is such that $\mathbf{u}\equiv\mathbf{0}\pmod p$, then

$$g(\mathbf{u};\chi_p)=\sum_{\mathbf{a}\ (\mathrm{mod}\ p)}\chi_p(F(\mathbf{a}))\ll p^{\frac{n+1}{2}},$$

where the implied constant depends only on n and the degree of F. This follows from the multiplicative analogue of Deligne's bound established by Katz [8].

(2) If $\mathbf{u} \pmod p$ is non-zero and the associated hyperplane $\langle\mathbf{u},\mathbf{x}\rangle=0$ is not a tangent to the hypersurface V_p, then by Theorem 1.1 in [9] we have square-root cancellation i.e.

$$g(\mathbf{u};\chi_p)\ll p^{\frac{n+1}{2}}.$$

Here again the implied constant depends only on n and the degree of F. In particular, it does not depend on $\mathbf{u}$.

(3) If $\mathbf{u} \pmod p$ is non-zero and the associated hyperplane $\langle\mathbf{u},\mathbf{x}\rangle=0$ is a tangent to the hypersurface V_p, then using the primitivity of the character we get

$$g(\mathbf{u};\chi_p)=\frac{\chi_p(-1)\tau(\chi_p)}{p}\sum_{\mathbf{b}\ (\mathrm{mod}\ p)}\bar{\chi}_p(\mathbf{b})\sum_{\mathbf{a}\ (\mathrm{mod}\ p)}e\Big(\frac{\mathbf{b}F(\mathbf{a})+\langle\mathbf{a},\mathbf{u}\rangle}{p}\Big),$$

where $\tau(\chi)$ stands for the Gauss sum of the character χ. Then we apply Deligne's bound [4] to the inner sum to obtain

$$g(\mathbf{u};\chi_p)\ll\sqrt{p}p^{\frac{n+1}{2}}.$$

Again the implied constant depends only on n and the degree of F.

In the first two cases we say that the vector $\mathbf{u} \in \mathbb{Z}^{n+1}$ is 'good' modulo p. In the third case we say that $\mathbf{u}$ is 'bad' modulo p.

We first compute the contribution to $S(p,q)$ from those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are 'good' modulo both p and q. Using square-root cancellation we see that this is bounded by

$$\ll B^{n+1}\frac{(pq)^{\frac{n+1}{2}}}{(pq)^{n+1}} \sum_{\mathbf{u}\in\mathbb{Z}^{n+1}} \prod_{i=0}^{n}\Big(1+\frac{|u_i|B}{pq}\Big)^{-2} \ll (pq)^{\frac{n+1}{2}}.$$

In the last inequality we are assuming that $pq \geq B$, which follows from our choice of the set $\mathcal{P}$. Now the contribution of those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are 'bad' for the prime p but 'good' for the prime q is given by

$$(pq)^{-(n+1)} \sum_{\mathbf{v}\in V_p^*(\mathbb{F}_p)} \sum_{\lambda=1}^{p} \sideset{}{^*}\sum_{\mathbf{u}\equiv\lambda\mathbf{v} \pmod p} g(\mathbf{u};p,q)\hat{W}\Big(\frac{\mathbf{u}}{pq}\Big).$$

The $*$ above the sum indicates that the sum is restricted over those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are 'good' modulo q. The above expression is bounded by

$$\begin{aligned} &B^{n+1}\frac{p^{\frac{n+2}{2}}q^{\frac{n+1}{2}}}{(pq)^{n+1}} \sum_{\mathbf{v}\in V_p^*(\mathbb{F}_p)} \sum_{\lambda=1}^{p} \sideset{}{^*}\sum_{\mathbf{u}\equiv\lambda\mathbf{v} \pmod p} \prod_{i=0}^{n}\Big(1+\frac{|u_i|B}{pq}\Big)^{-2} \\ \ll &B^{n+1}\frac{p^{\frac{n+2}{2}}q^{\frac{n+1}{2}}}{(pq)^{n+1}} \sum_{\mathbf{v}\in V_p^*(\mathbb{F}_p)} \sum_{\lambda=1}^{p} \Big(\frac{q}{B}\Big)^{n+1} \ll p^{\frac{n}{2}}q^{\frac{n+1}{2}}. \end{aligned}$$

In the last inequality we have used the fact that $|V_p^*(\mathbb{F}_p)| \ll p^{n-1}$. Similarly one can show that the contribution of those vectors $\mathbf{u} \in \mathbb{Z}^{n+1}$ which are 'bad' for both p and q, is bounded by $(pq)^{\frac{n}{2}}$. Hence we conclude that

$$S(p,q) \ll (pq)^{\frac{n+1}{2}}.$$

Now for the set $\mathcal{P}$ we pick P consecutive primes in the progression $(1+dk)_k$ of size $\sim P\log P$. Then using the d-power sieve inequality, we get

$$S(\mathcal{A}) \ll \frac{B^{n+1}}{P} + (P\log P)^{n+1}.$$

Then choosing

$$P = \Big(\frac{B}{\log B}\Big)^{\frac{n+1}{n+2}}$$

the theorem follows.

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