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Artin formalism for Selberg zeta functions of co-finite Kleinian groups

par Eliot BRENNER et Florin SPINU

Résumé. Soit Γ un sous-groupe discret de $\mathrm{SL}(2,\mathbb{C})$ tel que le quotient $\Gamma\backslash\mathbb{H}^3$ ait un volume fini. On associe à une représentation unitaire de dimension finie χ de Γ la fonction zêta de Selberg $Z(s;\Gamma;\chi)$. Dans cet article, on prouve le formalisme d'Artin pour cette fonction zêta de Selberg. Plus précisément, si $\tilde{\Gamma}$ est une extension de Γ d'indice fini dans $\mathrm{SL}(2,\mathbb{C})$, et si $\pi = \mathrm{Ind}_{\Gamma}^{\tilde{\Gamma}}\chi$ est la représentation induite, alors $Z(s;\Gamma;\chi) = Z(s;\tilde{\Gamma};\pi)$. Dans la deuxième partie de l'article, on prouve par une méthode directe l'identité analogue pour la fonction de dispersion. Plus précisément, $\phi(s;\Gamma;\chi) = \phi(s;\tilde{\Gamma};\pi)$ pour une certaine normalisation de la série d'Eisenstein.

Abstract. Let $\Gamma\backslash\mathbb{H}^3$ be a finite-volume quotient of the upper-half space, where $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ is a discrete subgroup. To a finite dimensional unitary representation χ of Γ one associates the Selberg zeta function $Z(s;\Gamma;\chi)$. In this paper we prove the Artin formalism for the Selberg zeta function. Namely, if $\tilde{\Gamma}$ is a finite index group extension of Γ in $\mathrm{SL}(2,\mathbb{C})$, and $\pi = \mathrm{Ind}_{\Gamma}^{\tilde{\Gamma}}\chi$ is the induced representation, then $Z(s;\Gamma;\chi) = Z(s;\tilde{\Gamma};\pi)$. In the second part of the paper we prove by a direct method the analogous identity for the scattering function, namely $\phi(s;\Gamma;\chi) = \phi(s;\tilde{\Gamma};\pi)$, for an appropriate normalization of the Eisenstein series.

1. Introduction

The Artin formalism for the Selberg zeta function associated to finite area surfaces of constant negative curvature has been proved by Venkov and al. in [9] and [11]. Two arguments are presented in these papers: a direct term-by-term comparison of the infinite products defining the Selberg zeta; a term-by-term comparison of the spectral expansions (via trace formula) of the the logarithmic derivative of the Selberg zeta functions. In generalizing the first approach to the 3-dimensional case, the obstacle consists in the more complicated structure of centralizers of hyperbolic elements in discrete subgroups of $\mathrm{SL}(2,\mathbb{C})$ than of $\mathrm{SL}(2,\mathbb{R})$. The difficulty in extending the

Mots clefs. Artin Formalism, Selberg Zeta function, Kleinian groups, Fuchsian groups hyperbolic 3-manifolds, scattering matrix, Eisenstein series.

spectral method to quotients of $\mathbb{H}^3$ has to do with the continuous spectrum contribution to the trace formula. For discussions of the Artin formalism in the context of the trace formula, see [1], and also, for the case of quotients of compact manifolds, [7].

The proof of the identity of Selberg zeta functions presented in this paper is very similar to the one in [5], which studied independently the special case when Γ is a normal subgroup of $\tilde{\Gamma}$. The proof differs from previous approaches in that it avoids entirely the trace formula. Instead of comparing traces of automorphic kernels, we compare the (partial) automorphic kernels themselves. This method can be extended easily to quotients of real hyperbolic spaces of arbitrary dimension.

In the second part of the paper we prove the Artin formalism for the scattering function by a direct method (independent of the Selberg zeta function), and compare that to Venkov's original formula in the 2-dimensional case.

2. Definitions and notations

Let $\mathbb{H}^3 := \mathbb{R}^2 \times (0, +\infty)$ be the upper half-space. For an element $\omega = (x_1, x_2, y) \in \mathbb{H}^3$ we will denote $y(\omega) = y$ and $z(\omega) = x_1 + ix_2$. $\mathbb{H}^3$ carries $\mathrm{SL}(2, \mathbb{C})$-invariant metric $ds^2 = y^{-2}(dx_1^2 + dx_2^2 + dy^2)$ and hyperbolic volume element $d\omega = y^{-3}\, dx_1 dx_2 dy$. The (positive) Laplace operator associated to this metric is

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial y}. \tag{2.1}$$

The **basic point-pair invariant** is given at [3, Ch.1] by

$$\delta(\omega, \omega') = \cosh\ d_{\mathbb{H}^3}(\omega, \omega') = \frac{\|\omega - \omega'\|^2}{2y(\omega)y(\omega')} + 1. \tag{2.2}$$

Assume $\Gamma \leq \mathrm{SL}(2, \mathbb{C})$ is a discrete subgroup[1] such that the quotient $\Gamma \backslash \mathbb{H}^3$ has finite hyperbolic volume. Let χ be a finite dimensional unitary representation of Γ. Then χ determines a vector bundle over $\Gamma \backslash \mathbb{H}^3$, and we denote by $L^2(\Gamma \backslash \mathbb{H}^3, \chi)$ the Hilbert space of L^2 sections.

Associated to χ, the Selberg zeta function $Z(s; \Gamma; \chi)$ is a function of the complex parameter s usually given by an infinite product analogous to the Euler factorization of the Artin L-functions ([8, 3, 4, 10]). Its logarithmic derivative has a simpler expression and we will work with that instead.

First, we need to introduce some notations. (We follow [3, Ch. 5] and [4].) We denote the set of hyperbolic[2], parabolic, and elliptic elements of Γ,

[1] We assume from now on that all discrete subgroups of $\mathrm{SL}(2, \mathbb{C})$ contain $\pm I_2$.

[2] We will not need the distinction between hyperbolic and loxodromic elements in this paper.

respectively, by

$$\Gamma_{\text{hyp}} := \{P \in \Gamma : \text{tr}\,(P) \notin [-2,2]\}$$
$$\Gamma_{\text{par}} := \{P \in \Gamma : \text{tr}\,(P) = \pm 2, P \neq \pm I_2\}$$
$$\Gamma_{\text{ell}} := \{P \in \Gamma : \text{tr}\,(P) \in (-2,2)\}$$

Note that we have a disjoint union

$$\Gamma = \{\pm I_2\} \cup \Gamma_{\text{hyp}} \cup \Gamma_{\text{par}} \cup \Gamma_{\text{ell}}$$

For $P \in \Gamma_{\text{hyp}}$, $a(P)$ denotes the eigenvalue with $|a(P)| > 1$ and $N(P) = |a(P)|^2$ is the norm of P. Let $C_\Gamma(P)$ denote the centralizer of P in Γ. This is a finitely generated abelian group of rank one. Let

$$C_\Gamma(P) = C_\Gamma(P)^{\text{free}} \times C_\Gamma(P)^{\text{tor}}$$

be its decomposition into the free and torsion part. A generator P_0 of $C_\Gamma(P)^{\text{free}}$ is called the primitive hyperbolic element of P in Γ. Set $m(P) := |C_\Gamma(P)^{\text{tor}}|$.

One can define uniquely the Selberg zeta function on $\text{Re}\,\text{s} > 1$ by way of the logarithmic derivative

$$(2.3)\qquad d\log Z(s;\Gamma;\chi) = \sum_{[P]_\Gamma} \frac{\text{tr}\,\chi(P)\log(NP_0)}{m(P)|a(P)-a(P)^{-1}|^2}\, NP^{-s}, \quad \text{Re}\,(\text{s}) > 1$$

subject to the condition $\lim_{\text{Re}\,\text{s}\to+\infty} Z_\Gamma(s,\chi) = 1$. The above sum is over Γ-conjugacy classes $[P]_\Gamma$ of hyperbolic elements, with P a representative of such a class. We use the notation $\text{tr}\,A$ for the trace of an operator A.

Remark. Let $\lambda_j = 1 + t_j^2$, $j \geq 0$, the eigenvalues of (the self-adjoint extension of) Δ on $L^2(\Gamma\backslash\mathbb{H}^3,\chi)$. With this definition, $Z(s;\Gamma;\chi)$ has zeros at $s = \pm it_j$, hence on the line $\text{Re}\,\text{s} = 0$.

Definition. Let $z(s;\Gamma;\chi)$ be the right-hand side of the identity (2.3). This is also the notation used in [6].

It is known that $z(s;\Gamma;\chi)$ has a meromorphic continuation to $s \in \mathbb{C}$ and its polar divisor is well understood, being related to the spectrum of Δ on $L^2(\Gamma\backslash\mathbb{H}^3,\chi)$. The meromorphic continuation of $Z(s;\Gamma;\chi)$ itself is more problematic, in that it depends on the specific shape of the polar divisor of $z(s;\Gamma;\chi)$. Various instances are known when $Z(s;\Gamma;\chi)$ has meromorphic continuation to $s \in \mathbb{C}$: [6] in the general rank one case, when Γ has no elliptic elements (and $\chi = 1$); [4] when Γ has no cuspidal elliptic elements (and arbitrary χ). Although, in most cases, [4] established the meromorphic continuation of an integer power of $Z(s;\Gamma;\chi)$, the question is, in general, still open (Conjecture in [4, Section 6]).

3. Main Results

3.1. The Artin formalism of the Selberg Zeta function. Let $\tilde{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$ be a second Kleinian group containing Γ with finite index: $\Gamma \subset \tilde{\Gamma}$ and $n = [\tilde{\Gamma} : \Gamma]$. Let χ be a finite dimensional, unitary representation of Γ, and set $\pi = \mathrm{Ind}_{\Gamma}^{\tilde{\Gamma}} \chi$. In this paper we prove the following identity of meromorphic functions

Theorem 1. *For $s \in \mathbb{C}$,*

$$z(s;\Gamma;\chi) = z(s;\tilde{\Gamma};\pi).$$

As a corollary, we have

$$Z(s;\Gamma;\chi) = Z(s;\tilde{\Gamma};\pi), \quad \mathrm{Re}\, s > 1, \tag{3.1}$$

and clearly this identity extends to the entire domain where the two Selberg zeta functions admit meromorphic continuation.

3.2. The Artin formalism of the scattering function. Let $\phi(s;\Gamma;\chi)$ denote the scattering function associated to the continuous spectrum of Δ on $L^2(\Gamma\backslash\mathbb{H}^3,\chi)$. (This is the determinant of the scattering matrix in the functional equation of the Eisenstein series.) It was observed earlier [11] that the identity (3.1) is linked, via the trace formula, with a relationship between $\phi(s;\Gamma;\chi)$ and $\phi(s;\tilde{\Gamma};\pi)$. The second part of this paper is concerned with a proof of this relationship that can be developed separately from (3.1), by a direct matching of the Eisenstein series on the two spaces. Namely, for a specific normalization of the Eisenstein series, we have the following

Theorem 2. *For $s \in \mathbb{C}$,*

$$\phi(s;\Gamma;\chi) = \phi(s;\tilde{\Gamma};\pi).$$

4. Preliminaries

4.1. The map $\mathfrak{L}$. Let $\mathcal{C}(\Gamma\backslash\mathbb{H}^3)$ be the linear space of Γ-invariant, continuous functions on $\mathbb{H}^3$, and $\mathcal{BC}(\Gamma\backslash\mathbb{H}^3)$ the subspace of bounded, continuous functions. We use the analogous notation for $\tilde{\Gamma}$ instead of Γ.

For $\{\alpha_i : 1 \le i \le n\}$ a set of coset representatives of $\Gamma\backslash\tilde{\Gamma}$ one can define the map

$$\mathfrak{L} : \mathcal{C}(\Gamma\backslash\mathbb{H}^3) \to \mathcal{C}(\tilde{\Gamma}\backslash\mathbb{H}^3), \quad \mathfrak{L}f(\omega) = \sum_{i=1}^{n} f(\alpha_i\,\omega). \tag{4.1}$$

It can be easily checked that the map $\mathfrak{L}$ is independent of the particular choice of α_i. The following lemma is straightforward.

Lemma 4.1. *Let $f \in \mathcal{BC}(\Gamma\backslash\mathbb{H}^3)$. Then $\mathfrak{L}f \in \mathcal{BC}(\tilde{\Gamma}\backslash\mathbb{H}^3)$ and*

$$\int_{\Gamma\backslash\mathbb{H}^3} f(\omega)d\omega = \int_{\tilde{\Gamma}\backslash\mathbb{H}^3} \mathfrak{L}f(\omega)d\omega \tag{4.2}$$

4.2. Orbital functions. Assume $\Phi \in \mathcal{C}((1,+\infty))$ is a continuous function. This defines a point-pair invariant by

$$k(\omega,\omega') \in \mathcal{C}(\mathbb{H}^3 \times \mathbb{H}^3), \quad k(\omega,\omega') := \Phi\left(\delta(\omega,\omega')\right). \tag{4.3}$$

Convergence issues set aside for the moment, by summing over all the Γ-translates of k one obtains the automorphic kernel

$$K_\chi(\omega,\omega') = \sum_{\gamma\in\Gamma} k(\omega,\gamma\omega')\chi(\gamma) \tag{4.4}$$

which is the kernel of an operator K_χ on $L^2(\Gamma\backslash\mathbb{H}^3,\chi)$ (see [4] for details). Instead of working with the automorphic kernel, we will add translates over proper subsets of Γ to obtain Γ-automorphic functions, rather than kernels, as follows.

Suppose that $\Omega \subset \Gamma$ is a proper subset invariant under conjugation. For our purposes we may assume that Ω acts without fixed points on $\mathbb{H}^3$. That means that Ω does not contain elliptic elements or the identity matrix.

Under the assumption that the function Φ has enough decay at infinity, to each such a subset Ω we can associate the Γ-invariant function

$$F_\chi^\Omega(\omega) := \sum_{g\in\Omega} k(\omega,g\omega)\operatorname{tr}\chi(g). \tag{4.5}$$

We will call such an object an orbital function, as it is obtained by summing over disjoint unions of conjugacy orbits of Γ. Note that, formally, the orbital function associated to $\Omega = \Gamma$ is the diagonal restriction of the automorphic kernel.

The next lemma is needed to establish the convergence of the series defining F_χ^Ω. To state the lemma we first introduce some notation.

Definition. Let $h = h(\Gamma)$ be the number of inequivalent cusps of Γ, and $\mathfrak{a}_1,\dots,\mathfrak{a}_h$ a complete set of Γ-inequivalent cusps. For each j, $1 \le j \le h$, we fix $\sigma_j \in \mathrm{SL}(2,\mathbb{C})$ such that $\sigma_j \cdot \infty = \mathfrak{a}_j$. The Γ-invariant height function is defined as

$$y_\Gamma : \mathbb{H}^3 \to (0,+\infty), \quad y_\Gamma(\omega) := \max_{1\le j\le h}\max_{\gamma\in\Gamma} y(\sigma_j^{-1}\gamma\omega). \tag{4.6}$$

The properties that make the height useful are:

i) $\inf\limits_{\omega\in\mathbb{H}^3} y_\Gamma(\omega) > 0$.

ii) For a sequence $\{\omega_n\} \subset \mathbb{H}^3$, $\{y_\Gamma(\omega_n)\}$ is unbounded if and only if $\{\omega_n\}$ has a subsequence that converges to a cusp of Γ.

Definition. In general, if $\mathfrak{a}$ is an arbitrary cusp, we will use the notation $\Gamma_\mathfrak{a}$ for the stabilizer of $\mathfrak{a}$ in Γ, and $\Gamma'_\mathfrak{a} := \Gamma_\mathfrak{a} \cap \Gamma_{\mathrm{par}}$ for the subset of parabolic elements in the stabilizer.

Lemma 4.2. *Assume that $\sigma > 0$ is a positive real number, and $\omega \in \mathbb{H}^3$. Then, with $\delta(\omega, \omega')$ defined at (2.2), we have:*

(a) $\sum_{\gamma \in \Gamma_{par}} \delta(\gamma\omega, \omega)^{-\sigma} \ll_\sigma [y_\Gamma(\omega)]^{2\sigma}$, *for $\sigma > 1$;*

(b) $\sum_{\gamma \in \Gamma_{hyp}} \delta(\gamma\omega, \omega)^{-\sigma} \ll_\sigma [y_\Gamma(\omega)]^{2(2-\sigma)}$, *for $\sigma > 2$.*

Both estimates are uniform in $\omega \in \mathbb{H}^3$, with the implicit constant depending on σ (and Γ) only.

Proof. We assume for simplicity that Γ has only one cusp, namely at ∞.

Let Γ_∞ be the stabilizer of ∞ in Γ. Then

$$\Gamma'_\infty := \Gamma_\infty \cap \Gamma_{\text{par}} = \pm \left\{ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} : 0 \neq \lambda \in \Lambda_\infty \right\}, \tag{4.7}$$

where Λ_∞ is a lattice in $\mathbb{C}$ (see [3, Thm. 2.1.8]).

By Shimizu's lemma ([3, Prop. 2.3.7]), the parabolic elements of Γ fix cusps, hence we have the disjoint union

$$\Gamma_{\text{par}} = \bigcup_{g \in \Gamma/\Gamma_\infty} \Gamma'_{g\infty} = \bigcup_{g \in \Gamma/\Gamma_\infty} g\Gamma'_\infty g^{-1} = \bigcup_{g \in \Gamma_\infty \backslash \Gamma} g^{-1}\Gamma'_\infty g \tag{4.8}$$

Since $\delta\left(g^{-1}\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} g\,\omega, \omega\right) = \delta(\lambda + g\,\omega, g\,\omega)$ we have, for $\sigma > 1$,

$$\begin{aligned}\sum_{\gamma \in \Gamma_{\text{par}}} \delta(\gamma\omega, \omega)^{-\sigma} &= \sum_{g \in \Gamma_\infty \backslash \Gamma} \sum_{0 \neq \lambda \in \Lambda_\infty} \delta(\lambda + g\omega, g\omega)^{-\sigma} \\ &= \sum_{g \in \Gamma_\infty \backslash \Gamma} \sum_{0 \neq \lambda \in \Lambda_\infty} \left[\frac{|\lambda|^2}{2y^2(g\omega)} + 1\right]^{-\sigma} \ll \sum_{g \in \Gamma_\infty \backslash \Gamma} [y(g\omega)]^{2\sigma}.\end{aligned}$$

In the range $\sigma > 1$ the latter series is, by a standard estimate on the Eisenstein series, $O\left(y^{2\sigma}(\omega)\right)$ as $y(\omega) \to +\infty$. This finishes the proof of part a) of the Lemma.

The proof of part b) follows the argument in [3, p. 157]. Once again, we suppose for simplicity that Γ has only one cusp. Since $\Gamma_{\text{hyp}} \subset \Gamma \setminus \Gamma_\infty$ we have, for $\sigma > 2$,

$$\sum_{\gamma \in \Gamma_{\text{hyp}}} \delta(\gamma\omega, \omega)^{-\sigma} \leq \sum_{\gamma \in \Gamma - \Gamma_\infty} \delta(\gamma\omega, \omega)^{-\sigma} = \sum_{t \in \Gamma_\infty} \sum_{g \in \Gamma_\infty \backslash (\Gamma - \Gamma_\infty)} \delta(tg\omega, \omega)^{-\sigma}.$$

This is

$$
\begin{aligned}
&\ll \sum_{\lambda\in\Lambda_\infty}\ \sum_{g\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)} \delta(g\omega+\lambda,\omega)^{-\sigma}\\
&= \sum_{g\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)}\ \sum_{\lambda\in\Lambda_\infty} \left[\frac{|\lambda+z(g\omega)-z(\omega)|^2+y^2(\omega)+y^2(g\omega)}{2y(\omega)y(g\omega)}\right]^{-\sigma}\\
&\ll \sum_{g\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)} y^\sigma(\omega)\ y^\sigma(g\omega) \sum_{\lambda\in\Lambda_\infty} \frac{1}{[\ |\lambda+z(g\omega)-z(\omega)|^2+y^2(\omega)]^\sigma}\\
&\ll \sum_{g\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)} y^\sigma(\omega)y^\sigma(g\omega)\cdot[y^{-2\sigma}(\omega)+y^{2-2\sigma}(\omega)],
\end{aligned}
$$

with the implied constant depending on σ only. As $y(\omega)\to+\infty$, this is

$$
\ll y^{2-\sigma}(\omega) \sum_{g\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)} y^\sigma(g\omega)
$$

By the same standard estimate on the Eisenstein series, when $\sigma>2$,

$$
\sum_{g\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)} y^\sigma(g\omega) = O(y^{2-\sigma}(\omega)), \quad \text{as } y(\omega)\to+\infty,
$$

which finishes the proof of part b). □

Note. In the general case (several cusps), one has to decompose the fundamental domain into a finite union of cuspidal sectors (as in [10, Thm. 1.2.4]) and analyze each cuspidal sector separately, but no new ideas are needed as far as the estimates are concerned.

Corollary 4.1. *We fix the following assumptions:*

- $\Omega\subset\Gamma$ *is a conjugation-invariant subset, not containing elliptic elements or* $\pm I_2$*. That is,* $\Omega\subset\Gamma_{hyp}\cup\Gamma_{par}$*;*
- $\Phi\in\mathcal{C}((1,\infty))$ *is a continuous function with decay* $\Phi(x)=O(x^{-\alpha})$ *as* $x\to+\infty$*.*

Then:

(a) *If* $\alpha>1$*, the series defining* F_χ^Ω *converges absolutely and uniformly on compact subsets of* $\mathbb{H}^3$*. Hence* $F_\chi^\Omega\in\mathcal{C}(\Gamma\backslash\mathbb{H}^3)$*;*

(b) *If* $\alpha>2$ *and* $\Omega\subset\Gamma_{hyp}$*, then* $F_\chi^\Omega\in\mathcal{BC}(\Gamma\backslash\mathbb{H}^3)$*.*

Remark. The point here is that there is no requirement on the behavior of Φ near $x=1$.

Proof. Fix a compact subset $K\subset\mathbb{H}^3$. Since Ω does not fix any points in $\mathbb{H}^3$,

$$
\inf_{\gamma\in\Omega,\ \omega\in K} \delta(\gamma\omega,\omega)>1
$$

Therefore, there exists a constant $C = C(K)$ such that

$$\Phi(\delta(\gamma\omega,\omega)) \le C\delta(\gamma\omega,\omega)^{-\alpha}, \quad \forall\gamma \in \Omega,\ \omega \in K.$$

Using part a) of the Lemma we see now that the series defining $F_\chi^\Omega(\omega)$ converges uniformly on compact subsets of $\mathbb{H}^3$.

In the case $\Omega \subset \Gamma_{\mathrm{hyp}}$, one can choose the constant C uniformly in $\omega \in \mathbb{H}^3$, hence for $\alpha > 2$ we have

$$F_\chi^\Omega(\omega) \ll \sum_{\gamma\in\Gamma_{\mathrm{hyp}}} \delta(\gamma\omega,\omega)^{-\alpha} \ll [y_\Gamma(\omega)]^{2(2-\alpha)} = O(1),$$

uniformly in $\omega \in \mathbb{H}^3$. □

4.3. Assume now that $\tilde\Omega \subset \tilde\Gamma$ is a conjugacy-invariant subset of $\tilde\Gamma$, and $\Omega = \tilde\Omega \cap \Gamma$. With $\pi = \mathrm{Ind}_\Gamma^{\tilde\Gamma}$ we have the following.

Proposition 4.1. *Provided the series defining F_χ^Ω and $F_\pi^{\tilde\Omega}$ are uniformly convergent on compact sets, we have*

$$\mathfrak{L}F_\chi^\Omega = F_\pi^{\tilde\Omega}. \tag{4.9}$$

Proof. We follow the notation and the formula for the trace of the induced representation from [11, p.483]. Let

$$\tilde\chi(g) = \begin{cases} \chi(g), & g \in \Gamma \\ 0, & g \notin \Gamma. \end{cases}$$

The trace of π is then given by

$$\mathrm{tr}\,\pi(g) = \sum_{i=1}^n \mathrm{tr}\,\tilde\chi(\alpha_i g \alpha_i^{-1}), \quad g \in \tilde\Gamma. \tag{4.10}$$

In the following computation we will be using two facts: $\alpha_i\tilde\Omega\alpha_i^{-1} = \tilde\Omega$; for $g \in \tilde\Omega$, $\mathrm{tr}\,\tilde\chi(g) = 0$ unless $g \in \tilde\Omega \cap \Gamma = \Omega$.

$$\begin{aligned}
F_\pi^{\tilde\Omega}(\omega) &= \sum_{i=1}^n \sum_{g\in\tilde\Omega} k(\omega, g\omega)\cdot \mathrm{tr}\,\tilde\chi(\alpha_i g\alpha_i^{-1}) \\
&= \sum_{i=1}^n \sum_{g\in\tilde\Omega} k(\omega, \alpha_i^{-1} g\alpha_i\omega)\cdot \mathrm{tr}\,\tilde\chi(g) = \sum_{i=1}^n \sum_{g\in\Omega} k(\omega, \alpha_i^{-1} g\alpha_i\omega)\cdot \mathrm{tr}\,\chi(g) \\
&= \sum_{i=1}^n \sum_{g\in\Omega} k(\alpha_i\omega, g\alpha_i\omega)\cdot \mathrm{tr}\,\chi(g) = \sum_{i=1}^n F_\chi^\Omega(\alpha_i\omega) \\
&= \mathfrak{L}F_\chi^\Omega(\omega)
\end{aligned}$$

□

By taking $f = F_\chi^\Omega$ in Lemma 4.1 we obtain

Corollary 4.2. *Assume that the series defining F_χ^Ω is uniformly bounded on $\mathbb{H}^3$. Then $F_\chi^\Omega \in \mathcal{B}C(\Gamma\backslash\mathbb{H}^3)$, $F_\pi^{\tilde\Omega} \in \mathcal{B}C(\tilde\Gamma\backslash\mathbb{H}^3)$, and*

$$\int_{\Gamma\backslash\mathbb{H}^3} F_\chi^\Omega(\omega)d\omega = \int_{\tilde\Gamma\backslash\mathbb{H}^3} F_\pi^{\tilde\Omega}(\omega)d\omega.$$

In the next two sections, we will show how the appropriate choice of Ω and Φ translates the identity of this Corollary into the result stated at Theorem 3.1.

5. Proof of Theorem 1.

5.1. Hyperbolic Orbital Functions. In this section we restrict to the case $\Omega = \Gamma_{\rm hyp}$, and denote by $F_\chi^{\rm hyp} := F_\chi^{\Gamma_{\rm hyp}}$ the corresponding orbital function. By Proposition 4.1, $F_\chi^{\rm hyp} \in \mathcal{B}C(\Gamma\backslash\mathbb{H}^3)$ whenever $\Phi(x) = O(x^{-2-\epsilon})$, for some $\epsilon > 0$. The computation of its integral is standard in the theory of Selberg trace formula (see [3, Ch.5]):

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}^3} F_\chi^{\rm hyp}(\omega)d\omega &= \sum_{[P]_\Gamma} \operatorname{tr}\chi(P) \int_{\Gamma_P\backslash\mathbb{H}^3} k(\omega, P\omega)\, d\omega \\ &= \sum_{[P]_\Gamma} \frac{\operatorname{tr}\chi(P)\ \log(NP_0)}{m(P)|a(P)-a(P)^{-1}|^2}\, g(\log NP).\end{aligned} \tag{5.1}$$

Here the sum is over hyperbolic Γ-conjugacy classes $[P]_\Gamma$, Γ_P is the stabilizer of P in Γ, and $g(t)$ is obtained from $\Phi(x)$ via the Selberg transform

$$g(t) = 2\pi \int_{\cosh t}^{\infty} \Phi(x)\, dx. \tag{5.2}$$

Remark. Under more restrictive conditions on the test function $\Phi(x)$, the right-hand side of the identity (5.1) represents precisely the contribution of the hyperbolic conjugacy classes to the geometric side of the trace formula.

5.2. Green function kernel. For $s \in \mathbb{C}$, with $\mathrm{Re}\,(\mathrm{s}) > 1$, we consider, as in [3, p.185], the test function

$$\Phi_s(x) = \frac{2^{-s}s}{\pi} \cdot \frac{(x+\sqrt{x^2-1})^{-s}}{\sqrt{x^2-1}}. \tag{5.3}$$

This is a smooth function on $(1,\infty)$, and clearly $\Phi_s(x) = O\left(x^{-\mathrm{Re}\,(\mathrm{s})-1}\right)$ as $x \to +\infty$. The corresponding transform given by (5.2) is

$$g_s(t) = e^{-s|t|}. \tag{5.4}$$

We fix now the parameter $s \in \mathbb{C}$, with $\mathrm{Re}\, s > 1$. Let $F_\chi^{\rm hyp}(\omega; s)$ be the corresponding orbital function on $\Gamma\backslash\mathbb{H}^3$ associated to $\chi, \Gamma_{\rm hyp}, \Phi_s$. Hence

$F_\chi^{\rm hyp}(\omega; s) \in \mathcal{BC}(\Gamma\backslash\mathbb{H}^3)$ and the formula (5.1) yields

$$\int_{\Gamma\backslash\mathbb{H}^3} F_\chi^{\rm hyp}(\omega; s) d\omega = z(s; \Gamma; \chi), \quad \mathrm{Re\,(s)} > 1. \tag{5.5}$$

where $z(s; \Gamma; \chi)$ has been defined at (2.3).

5.3. Conclusion. With the choice of test function $\Phi = \Phi_s$, $\mathrm{Re\,s} > 1$, and the choice of conjugacy-invariant subsets $\Omega = \Gamma_{\rm par} \subset \Gamma$ and $\tilde{\Omega} = \tilde{\Gamma}_{\rm par} \subset \tilde{\Gamma}$, the result of Cor. 4.2,

$$\int_{\Gamma\backslash\mathbb{H}^3} F_\chi^{\rm hyp}(\omega; s) d\omega = \int_{\tilde{\Gamma}\backslash\mathbb{H}^3} F_\pi^{\rm hyp}(\omega; s) d\omega,$$

combined with the identity (5.5), yields

$$z(s; \Gamma; \chi) = z(s; \tilde{\Gamma}; \pi), \quad \mathrm{Re\,s} > 1. \tag{5.6}$$

6. Proof of Theorem 2.

It was observed in [11] that the identity stated at Theorem 2 can be obtained as a by-product of the Artin formalism for the Selberg zeta functions combined with the Selberg trace formula.

We present here a direct proof which is independent of the identity of Selberg zeta functions. For the simplicity of exposition we restrict ourselves, throughout this section only, to the case when

$$\tilde{\Gamma} \textit{ has only one cusp (at } \infty\textit{) and } \dim\chi = 1, \tag{6.1}$$

but we remark that our argument works in complete generality (see Remark 6.4 below).

For $1 \le i \le h = h(\Gamma)$, let $\Gamma_i := \Gamma_{\mathfrak{a}_i}$ the stabilizer of the cusp $\mathfrak{a}_i$ in Γ, and $\tilde{\Gamma}_i = \tilde{\Gamma}_{\mathfrak{a}_i}$ the stabilizer of $\mathfrak{a}_i$ in $\tilde{\Gamma}$. We set

$$n_i = [\tilde{\Gamma}_i : \Gamma_i], \quad 1 \le i \le h \tag{6.2}$$

Since all the cusps of $\tilde{\Gamma}$ are equivalent to ∞, we will choose the scaling matrices (introduced in Definition 4.2) such that $\sigma_i \in \tilde{\Gamma}$.

6.1. Eisenstein series associated to χ. The degree of singularity of χ (cf. [11], [4]) is the integer $1 \le \kappa \le h$ with the following property: the restriction of χ to Γ_j is trivial for $1 \le j \le \kappa$, and non-trivial for $\kappa + 1 \le j \le h$.

The Eisenstein series on $\Gamma\backslash\mathbb{H}^3$ associated to χ are defined, for $\mathrm{Re\,s} > 2$, by the absolutely convergent series

$$E_j(\omega, s; \chi) := \sum_{g \in \Gamma_j\backslash\Gamma} y^s(\sigma_j^{-1} g \cdot \omega)\chi(g^{-1}) \in \mathcal{A}(\Gamma, \chi), \quad 1 \le j \le \kappa. \tag{6.3}$$

Here $\mathcal{A}(\Gamma, \chi)$ is the linear space of χ-automorphic forms of polynomial growth on $\Gamma\backslash\mathbb{H}^3$ (see [3, Chap. 3] and [4], for a more precise definition).

It is known that the Eisenstein series have meromorphic continuation to $s \in \mathbb{C}$ ([2, 3, 4]).

Given a choice of coset representatives $\{\alpha_i : 1 \le i \le n\}$ of $\Gamma\backslash\tilde{\Gamma}$, it was shown in [11] that the representation $\pi = \mathrm{Ind}_{\Gamma}^{\tilde{\Gamma}} \chi$ can be realized on $V := \mathbb{C}^n$ (column vectors) via left multiplication by the $n \times n$ matrices

$$\pi(g) = \left[\tilde{\chi}\left(\alpha_i g \alpha_j^{-1}\right)\right]_{1 \le i,j \le n}, \quad g \in \tilde{\Gamma}. \tag{6.4}$$

6.2. Coset decomposition. We will use a specific choice of representatives described by the following proposition.

Proposition 6.1. *For $1 \le i \le h$, let $\{\beta_{it} : 1 \le t \le n_i\}$ be a set of coset representatives of $\Gamma_i\backslash\tilde{\Gamma}_i$. Then $\tilde{\Gamma} = \bigcup_{i=1}^{h}\bigcup_{t=1}^{n_i} \Gamma_i \beta_{it}\sigma_i$ is a disjoint union. That is, we can take $\{\alpha_\nu : 1 \le \nu \le n\} = \{\beta_{it}\sigma_i : 1 \le i \le h, 1 \le t \le n_i\}$. In particular,*

$$\sum_{i=1}^{h} n_i = n.$$

Proof. Let $g' \in \tilde{\Gamma}$. There exists i, $1 \le i \le h$, and $g \in \Gamma$ such that $g'\infty = g\mathfrak{a}_i$. Since $\infty = \sigma_i^{-1}\mathfrak{a}_i \Rightarrow g'\sigma_i^{-1} \cdot \mathfrak{a}_i = g\mathfrak{a}_i \Rightarrow g^{-1}g'\sigma_i^{-1} \in \tilde{\Gamma}_i$. Consequently $g' \in \Gamma\tilde{\Gamma}_i\sigma_i$. But $\tilde{\Gamma}_i = \bigcup_t \Gamma_i\beta_{it}$, hence

$$g' \in \bigcup_{t=1}^{n_i} \Gamma\Gamma_i\beta_{it}\sigma_i \subset \bigcup_{t=1}^{n_i} \Gamma\beta_{it}\sigma_i \subset \bigcup_{i=1}^{h}\bigcup_{t=1}^{n_i} \Gamma\beta_{it}\sigma_i. \tag{6.5}$$

It is then straightforward to check that the cosets $\Gamma\beta_{it}\sigma_i$ are actually disjoint for different pairs of (i, t). □

From now on we will identify the sets $\{\alpha_\nu\}$ and $\{\beta_{it}\sigma_i\}$ in the following order:

$$\alpha_\nu = \beta_{it}\sigma_i, \quad \text{if} \quad \nu = n_1 + \cdots + n_{i-1} + t \tag{6.6}$$

6.3. Eisenstein series associated to $\boldsymbol{\pi}$. The singular space of the induced representation π is

$$V_\infty := \{v \in V : \pi(g)v = v, \quad \forall g \in \Gamma_\infty\}. \tag{6.7}$$

It is known that it has dimension κ and orthonormal basis

$$\mathbf{e}_j = \frac{1}{\sqrt{n_j}}[0, \ldots 0, 1 \ldots, 1, 0, \ldots, 0]^t, \quad 1 \le j \le \kappa, \tag{6.8}$$

where the 1's occur in the j^{th} block (of length n_j), according to the identification (6.6). The Eisenstein series on $\tilde{\Gamma}\backslash\mathbb{H}^3$ associated to π are given, for $\mathrm{Re}\, s > 2$, by

$$E_j(\omega, s; \pi) = \sum_{g \in \tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} y^s(g \cdot \omega)\pi(g^{-1})\mathbf{e}_j \in \mathcal{A}(\tilde{\Gamma}, \pi), \quad 1 \le j \le \kappa. \tag{6.9}$$

6.4. The map $\mathcal{T}$. In [11] the following map is defined:

$$\mathcal{T} : \mathcal{A}(\Gamma, \chi) \to \mathcal{A}(\tilde{\Gamma}, \pi), \quad \mathcal{T}f(\omega) = [f(\alpha_1\omega), \dots, f(\alpha_n\omega)]^t. \tag{6.10}$$

The following proposition represents the main result of this section.

Proposition 6.2. *For $1 \le j \le \kappa$,*

$$\mathcal{T}E_j(\omega, s; \chi) = n_j^{1/2}\, E_j(\omega, s; \pi).$$

Proof. Let $\nu = n_1 + \cdots + n_{i-1} + t$, that is, $\alpha_\nu = \beta_{it}\sigma_i$. The ν^{th} component of $E_j(\omega, s; \pi)$, as a vector in $\mathbb{C}^n$, is

$$\frac{1}{\sqrt{n_j}} \sum_{g \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} y^s(g \cdot \omega) \sum_{k=1}^{n_j} \tilde{\chi}\left(\beta_{it}\sigma_i g^{-1}(\beta_{jk}\sigma_j)^{-1}\right).$$

With the change of variable $g \mapsto \sigma_j g$ and using the fact that $\sigma_j \tilde{\Gamma}_\infty \sigma_j^{-1} = \tilde{\Gamma}_j$, the above sum becomes

$$\frac{1}{\sqrt{n_j}} \sum_{g \in \tilde{\Gamma}_j \backslash \tilde{\Gamma}} y^s(\sigma_j^{-1} g \cdot \omega) \sum_{k=1}^{n_j} \tilde{\chi}\left(\beta_{it}\sigma_i g^{-1}\beta_{jk}^{-1}\right).$$

With a further change of variable $g \mapsto \beta_{jk} g$, this equals:

$$\begin{aligned}
&\frac{1}{\sqrt{n_j}} \sum_{g \in \tilde{\Gamma}_j \backslash \tilde{\Gamma}} y^s(\sigma_j^{-1}\beta_{jk}^{-1} g \cdot \omega) \sum_{k=1}^{n_j} \tilde{\chi}\left(\beta_{it}\sigma_i g^{-1}\right) \\
&= \frac{1}{n_j^{3/2}} \sum_{g \in \Gamma_j \backslash \tilde{\Gamma}} \sum_{k=1}^{n_j} \cdots \qquad [\text{summing over } \Gamma_j \backslash \tilde{\Gamma} \text{ instead of } \tilde{\Gamma}_j \backslash \tilde{\Gamma}] \\
&= \frac{1}{n_j^{3/2}} \sum_{g \in \Gamma_j \backslash \Gamma} \sum_{\theta \in \Gamma \backslash \tilde{\Gamma}} \sum_{k=1}^{n_j} y^s(\sigma_j^{-1}\beta_{jk}^{-1} g\theta \cdot \omega)\, \tilde{\chi}\left(\beta_{it}\sigma_i\theta^{-1}\right) \chi(g^{-1})
\end{aligned}$$

Now, $\tilde{\chi}(\beta_{it}\sigma_i\theta^{-1}) = 0$ unless $\beta_{it}\sigma_i\theta^{-1} \in \Gamma$. This means that $\Gamma\beta_{it}\sigma_i = \Gamma\theta$, which forces θ in the sum $\sum_{\theta \in \Gamma \backslash \tilde{\Gamma}}$ to equal $\beta_{it}\sigma_i$. Hence the ν^{th} component of $E_j(\omega, s; \pi)$ equals

$$\frac{1}{n_j^{3/2}} \sum_{g \in \Gamma_j \backslash \Gamma} \sum_{k=1}^{n_j} y^s(\sigma_j^{-1}\beta_{jk}^{-1} g\beta_{it}\sigma_i \cdot \omega)\, \chi(g^{-1}).$$

Note that

$$y(\sigma_j^{-1}\beta_{jk}^{-1} \cdot P) = y(\sigma_j^{-1}\beta_{jk}^{-1}\sigma_j \cdot \sigma_j^{-1} P) = y(\sigma_j^{-1} P), \quad P \in \mathbb{H}^3,$$

since $\sigma_j^{-1}\beta_{jk}^{-1}\sigma_j \in \tilde{\Gamma}_\infty$. Therefore the terms in the k-sum are all equal, and the ν^{th} component of $E_j(\omega, s; \pi)$ equals

$$\begin{aligned}
&\frac{1}{n_j^{3/2}} \sum_{g\in\Gamma_j\backslash\Gamma} \sum_{k=1}^{n_j} y^s(\sigma_j^{-1} g\beta_{it}\sigma_i\cdot\omega)\,\chi(g^{-1}) \\
&= \frac{1}{\sqrt{n_j}} \sum_{g\in\Gamma_j\backslash\Gamma} y^s(\sigma_j^{-1} g\beta_{it}\sigma_i\cdot\omega)\,\chi(g^{-1}) \\
&= \frac{1}{\sqrt{n_j}} E_j(\beta_{it}\sigma_i\cdot\omega, s;\chi) = \frac{1}{\sqrt{n_j}} E_j(\alpha_\nu\omega, s;\chi) \qquad [\text{recall that } \alpha_\nu = \eta_{it}\sigma_i].
\end{aligned}$$

This shows that $E_j(\omega, s; \pi) = \frac{1}{\sqrt{n_j}} \mathfrak{T} E_j(\omega, s; \chi)$. □

Remark. The general version of Proposition 6.2 (for an arbitrary number of cusps of $\tilde{\Gamma}$ and χ of arbitrary dimension) was stated and proved in [1, Prop. 14]. The proof of Theorem 2, which is an immediate corollary of Prop. 6.2 (as shown in the next section), thus carries over to the general case.

6.5. The Artin formalism of the scattering function. Let

$$\mathcal{E}(\omega, s;\chi) = [E_1(\omega, s;\chi), \ldots, E_\kappa(\omega, s;\chi)]^t$$

be the column vector that encodes all the Eisenstein series associated to χ. The scattering matrix $\mathfrak{S}_\Gamma(s;\Gamma;\chi) = [\mathfrak{S}_{ij}(s;\chi)]_{1\le i,j\le\kappa}$ is determined by the functional equation (we refer to [4] for more details)

$$\mathcal{E}(\omega, s;\chi) = \mathfrak{S}(s;\Gamma;\chi)\mathcal{E}(\omega, 2-s;\chi).$$

That is,

$$E_i(\omega, s;\chi) = \sum_{j=1}^{\kappa} \mathfrak{S}_{ij}(s;\chi) E_j(\omega, 2-s;\chi), \quad 1\le i\le\kappa. \tag{6.11}$$

The scattering function is

$$\phi(s;\Gamma;\chi) := \det\mathfrak{S}(s;\Gamma;\chi). \tag{6.12}$$

Similarly, the scattering matrix of π is determined by

$$E_i(\omega, s;\pi) = \sum_{j=1}^{\kappa} \mathfrak{S}_{ij}(s;\pi) E_j(\omega, 2-s;\pi), \quad 1\le i\le\kappa. \tag{6.13}$$

By applying the $\mathfrak{T}$ operator on both sides of the equation (6.11) we obtain, in view of Proposition 6.2,

$$n_i^{1/2} E_i(\omega, s;\pi) = \sum_{j=1}^{\kappa} \mathfrak{S}_{ij}(s;\chi) n_j^{1/2} E_j(\omega, s;\pi).$$

Comaparison to (6.13) gives the relation between the two scattering matrices:
(6.14)

$$\mathfrak{S}_{ij}(s;\pi) = n_i^{-1/2}\mathfrak{S}_{ij}(s;\chi)n_j^{1/2}, \quad \text{hence} \quad \mathfrak{S}(s;\tilde\Gamma;\pi) = D^{-1}\mathfrak{S}(s;\Gamma;\chi)D,$$

where D is the $\kappa\times\kappa$ diagonal matrix with $n_i^{1/2}$ on the diagonal. This implies that the two scattering matrices have the same determinant, which is the statement of Theorem 2.

6.6. Comparison with Thm. 3.2 in [11]**.** The aim of this section is to reconcile Theorem 2 with the slightly different version of Artin formalism proved in [11] for the 2-dimensional case.

In the remaining section $\Gamma \subset \tilde\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ are cofinite Fuchsian groups. The result (and proof) of Theorem 2 carries over to this case almost word by word: if χ is a finite dimensional unitary representation and $\pi = \mathrm{Ind}_\Gamma^{\tilde\Gamma}\chi$, then $\phi(s;\Gamma;\chi) = \phi(s;\tilde\Gamma;\pi)$. However, Theorem 3.2 in [11] states the following (we will use the upper index VZ for the analogous concepts introduced in VZ):

$$\phi^{VZ}(s;\Gamma;\chi)\Omega(\chi)^{1-2s} = \phi^{VZ}(s;\tilde\Gamma;\pi)\Omega(\pi)^{1-2s}, \tag{6.15}$$

(with the constants $\Omega(\pi)$ and $\Omega(\chi)$ to be defined shortly). The goal of this section is a direct proof that the two formulas are equivalent.

We will keep the assumption that $\tilde\Gamma$ has only one cusp (at ∞) and $\dim\chi = 1$, for simplicity of exposition. We keep the notation $\mathfrak{a}_1,\dots,\mathfrak{a}_h$ for the cusps of Γ.

In the 2-dimensional case, stabilizers of cusps in Fuchsian groups are cyclic (mod $\pm\,\mathrm{I}_2$). Let $\tilde S_\infty$ be a generator of $\tilde\Gamma_\infty$. It is easy to see that $S_i := \sigma_i\tilde S_\infty^{n_i}\sigma_i^{-1}$ is a generator of $\Gamma_i := \Gamma_{\mathfrak{a}_i}$, for $1 \le i \le h$.

The constants $\Omega(\chi)$ and $\Omega(\pi)$ are defined as follows. First,

$$\Omega(\chi) := \prod_{j=1}^{\kappa} |1-\chi(S_j)|. \tag{6.16}$$

Let now $\det'(A)$ denote the product of the non-zero eigenvalues of a matrix A. Then

$$\Omega(\pi) := \left|\det{}'(I-\pi(\tilde S_\infty))\right|, \tag{6.17}$$

with I the identity matrix.

The difference between the two versions of the Artin formalism for the scattering function stems from different choices of of the scaling matrices of cusps (Def. 4.2). This in turn leads to a different normalization of the Eisenstein series (associated to χ), and thus to an extra scalar factor in the scattering function. In [11] the scaling matrices of cusps (see Def. 4.2) are chosen such that the lattices associated to stabilizers of cusps have (relative) co-volume one. That is,

$$\sigma_j^{VZ} = \sigma_j \cdot \begin{bmatrix} n_j^{1/2} & \\ & n_j^{-1/2} \end{bmatrix},$$

which leads to

$$E_j^{VZ}(\omega, s; \chi) = n_j^{-s} E_j(\omega, s; \chi), \quad 1 \le j \le \kappa.$$

Therefore, with D the $\kappa \times \kappa$ diagonal matrix with n_j on the diagonal, we have

$$\begin{aligned} \mathcal{E}^{VZ}(\omega, s; \chi) &= D^{-s}\mathcal{E}(\omega, s; \chi) \\ &= D^{-s}\mathfrak{S}(s; \Gamma; \chi)\mathcal{E}(\omega, 1-s; \chi) \\ &\qquad \text{[functional equation in the 2-dimensional case]} \\ &= D^{-s}\mathfrak{S}(s; \Gamma; \chi)D^{1-s}\mathcal{E}^{VZ}(\omega, s; \chi), \end{aligned}$$

which gives

$$\mathfrak{S}^{VZ}(s; \Gamma; \chi) = D^{-s}\mathfrak{S}(s; \Gamma; \chi)D^{1-s}. \tag{6.18}$$

The scattering determinants then satisfy the relation

$$\phi_\Gamma^{VZ}(s; \Gamma; \chi) = \phi(s; \Gamma; \chi) \cdot \prod_{j=1}^{\kappa} n_j^{1-2s} \tag{6.19}$$

On the other hand, $E_j^{VZ}(\omega, s; \pi) = E_j(\omega, s; \pi)$, hence $\phi^{VZ}(s; \tilde{\Gamma}; \pi) = \phi(s; \tilde{\Gamma}; \pi)$.

Therefore, to show the equivalence of the two versions of Artin formalism for the scattering function, we will prove the following formula.

Proposition 6.3. *With the above notations,*

$$\frac{\Omega(\pi)}{\Omega(\chi)} = \prod_{j=1}^{\kappa} n_j.$$

Proof. As remarked before, in the 2-dimensional case stabilizers of cusps are cyclic, hence we have the following coset representatives for $\Gamma_i \backslash \tilde{\Gamma}_i$:

$$\beta_{ia} = \sigma_i \tilde{S}_\infty^a \sigma_i^{-1}, \quad 1 \le a \le n_i, \quad 1 \le i \le h. \tag{6.20}$$

Recall that

$$\pi(\tilde{S}_\infty) = [\tilde{\chi}(\alpha_i \tilde{S}_\infty \alpha_j^{-1})]_{1 \le i,j \le n}$$

where $\{\alpha_\nu : 1 \le \nu \le n\} = \{\beta_{ia}\sigma_i : 1 \le i \le h, 1 \le a \le n_i\}$. It is easy to verify that $\beta_{ia}\sigma_i\tilde{S}_\infty(\beta_{jb}\sigma_j)^{-1} \notin \Gamma$, unless $i = j$. Since $\beta_{ia}\sigma_i\tilde{S}_\infty(\beta_{ib}\sigma_i^{-1})^{-1} =$

$\sigma_i\tilde{S}_\infty^{a+1-b}\sigma_i^{-1}$, we find that $\pi(\tilde{S}_\infty)$ is a block-diagonal matrix

$$\pi(\tilde{S}_\infty) = \bigoplus_{i=1}^{h} [\, \tilde{\chi}(\sigma_i\, \tilde{S}_\infty^{a+1-b}\, \sigma_i^{-1})\,]_{1\leq a,b\leq n_i} \tag{6.21}$$

Moreover, $\sigma_i\tilde{S}_\infty^{a+b-1}\sigma_i^{-1} \in \Gamma$ if and only if $a+b-1 \equiv 0 \,(\mathrm{mod}\, n_i)$. Hence

$$\pi(\tilde{S}_\infty) = \bigoplus_{j=1}^{h} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ \chi(S_j) & 0 & \dots & 0 & 0 \end{pmatrix} \tag{6.22}$$

A straightforward computations gives the characteristic polynomial of $\pi(\tilde{S}_\infty)$:

$$P(\lambda) = \prod_{j=1}^{h} (\lambda^{n_j} - \chi(S_j)) \tag{6.23}$$

Therefore the eigenvalue $\lambda = 1$ comes only from the first κ blocks, where $\chi(S_j) = 1$. The product of non-zero eigenvalues of $I - \pi(\tilde{S}_\infty)$ is

$$\begin{aligned} \det{}'(I - \pi(\tilde{S}_\infty)) &= \prod_{1\leq j\leq \kappa} \left[\frac{\lambda^{n_j} - 1}{\lambda - 1}\Big|_{\lambda=1} \right] \cdot \prod_{i=\kappa+1}^{h} (1 - \chi(S_i)) \\ &= \prod_{j=1}^{\kappa} n_j \cdot \prod_{i=\kappa+1}^{h} (1 - \chi(S_j)) \end{aligned}$$

Taking the absolute value on both sides of this identity finishes the proof of Proposition 6.3. □

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