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Variants of the Brocard-Ramanujan equation

par Omar KIHEL et Florian LUCA

Résumé. Dans cet article, nous étudions quelques variations sur l'équation diophantienne de Brocard-Ramanujan.

Abstract. In this paper, we discuss variations on the Brocard-Ramanujan Diophantine equation.

1. Introduction

Brocard (see [4, 5]), and independently Ramanujan (see [15, 16]), posed the problem of finding all integral solutions to the diophantine equation

$$n! + 1 = x^2. \tag{1}$$

Although it is unlikely that equation (1) has any solution with $n > 7$, the fact that it has only finitely many solutions has only been conditionally proved by Overholt (see [13]). He showed that the weak form of Szpiro's conjecture implies that equation (1) has only finitely many solutions. The weak form of Szpiro's conjecture is a special case of the ABC conjecture and asserts that there exists a constant s such that if A, B, and C are positive integers satisfying $A + B = C$ with $\gcd(A, B) = 1$, then

$$C \ \leq \ N(ABC)^s,$$

where $N(k)$ is the product of all primes dividing k taken without repetition. Berend and Osgood [1] showed that if $P \in \mathbb{Z}[X]$ is a polynomial of degree ≥ 2, then the density of the set of positive integers n for which there exists an integer x satisfying the more general diophantine equation

$$n! = P(x) \tag{2}$$

is zero. Erdős and Obláth [7] and Pollack and Shapiro [14] showed that if $P(x) = x^d \pm 1$ and $d \geq 3$, then equation (2) has no solution with $n > 1$. Generalizing Overholt's result, the second author [11] showed that the full ABC conjecture implies that equation (2) has only finitely many solutions. A wealth of information about this equation can be found in the recent paper [2].

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In this paper, we discuss some variations on the above diophantine equations. We look at the following diophantine equations:

$$x^p \pm y^p = \prod_{\substack{k \times n \\ k=1}}^{n} k, \tag{3}$$

and

$$P(x) = \prod_{\substack{k \times n \\ k=1}}^{n} k, \tag{4}$$

where $P \in \mathbb{Q}[x]$ is a polynomial of degree ≥ 2. Here and in what follows, we write $k \times n$ to mean that k does not divide n.

2. New results

In what follows, we use the Vinogradov symbols $\gg$, $\ll$ and $\asymp$ as well as the Landau symbols O and o with their regular meanings. Recall that $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent and mean that $|A| \leq c|B|$ holds with some positive constant c.

Theorem 1. *The Diophantine equation*

$$x^p \pm y^p = \prod_{\substack{k \times n \\ k=1}}^{n} k$$

admits only finitely many integer solutions (x, y, p, n) *with* $p \geq 3$ *a prime number and* $\gcd(x, y) = 1$.

Proof. There is no loss of generality to consider only the '+' sign and to assume that $|x| > |y|$. Since the right hand side is positive, we get that x is positive. Note that $\gcd(x, y) = 1$ implies that no prime $q \leq n$ coprime to n divides either x or y. Now either $x \leq n$, or $x \geq n + 1$. In the first case,

$$(n-1)!^{1/2} \leq x^p + |y|^p \leq 2n^p,$$

therefore by Stirling's formula

$$\frac{(n-1)}{2}(1 + o(1)) \log n \leq p \log n + \log 2.$$

For large n, the above inequality implies that $p \geq n/3$. Note however that $x^p + y^p = (x + y)(x^p + y^p)/(x + y)$, and, by Fermat's Little Theorem, it follows easily that $(x^p + y^p)/(x + y) = \delta m$, where $\delta \in \{1, p\}$, and every prime factor of m is $1 \pmod p$. Since every prime factor of m is $\leq n \leq 3p$, it follows that

$$\frac{x^p + y^p}{x + y} \leq p(p + 1)(2p + 1).$$

However,

$$\frac{x^p + y^p}{x+y} \geq x^{p-2} \geq 2^{p-2}.$$

Indeed, the above inequality holds for positive y because

$$x^p + y^p > x^p \geq 2x^{p-1} = (2x)x^{p-2} \geq (x+y)x^{p-2},$$

and for negative y because

$$\frac{x^p + y^p}{x+y} = x^{p-1} + x^{p-2}(-y) + \cdots + (-y)^{p-1} > x^{p-1} > x^{p-2}.$$

We thus get the inequality

$$2^{p-2} \leq p(p+1)(2p+1),$$

which shows that p is bounded, and since $p \geq n/3$, we get that n is bounded as well in this case.

We now assume that $x \geq n+1$. If $y > 0$, then $n^n > n! > x^p \geq (n+1)^p$, and if $y < 0$, then

$$n^n > (x+y)\left(x^{p-1} + x^{p-2}(-y) + \cdots + (-y)^{p-1}\right) > x^{p-1} \geq (n+1)^{p-1}.$$

In both cases, we get $p \leq n$. We write again

$$x^p + y^p = (x+y)\left(\frac{x^p + y^p}{x+y}\right),$$

and we use the fact that $(x^p + y^p)/(x+y) = \delta m$, where $\delta \in \{1, p\}$, and every prime factor of m is $1 \pmod{p}$. If $y > 0$, we get

$$m = \frac{x^p + y^p}{\delta(x+y)} > \frac{x^p}{2xp} = \frac{x^{p-1}}{2p},$$

while if $y < 0$, then

$$m = \frac{1}{\delta}\left(\frac{x^p + y^p}{x+y}\right) = \frac{1}{\delta}\left(x^{p-1} + x^{p-2}(-y) + \cdots + (-y)^{p-1}\right) > \frac{x^{p-1}}{2p}.$$

Thus, we always have

$$m \geq \frac{x^{p-1}}{2p} > \frac{(2x^p)^{(p-1)/p}}{4p} > \frac{\left(\prod_{\substack{k=1 \\ k \times n}}^{n} k\right)^{(p-1)/p}}{4p},$$

where we used the fact that $\prod_{\substack{k=1 \\ k \times n}}^{n} k = x^p + y^p < 2x^p$. Let M be the largest divisor of $\prod_{\substack{k=1 \\ k \times n}}^{n} k$ build up only from primes of the form $q \equiv 1 \pmod{p}$. Then $m \mid M$, and so

$$\log M > \frac{p-1}{p} \log\left(\prod_{\substack{k=1 \\ k \times n}}^{n} k\right) - \log(4p) > \frac{p-1}{2p}(n-1)\log\left(\frac{n-1}{e}\right) - \log(4p).$$

In the above inequality, we used Stirling's formula as well as the fact that $\prod_{\substack{k\times n\\k=1}}^{n} k > (n-1)!^{1/2}$. It is clear that for all $q \le n$, the order of q in $n!$ is

$$\left\lfloor\frac{n}{q}\right\rfloor + \left\lfloor\frac{n}{q^2}\right\rfloor + \cdots < n\sum_{i\ge 1}\frac{1}{q^i} = \frac{n}{q-1}.$$

Thus,

$$\log M < n \sum_{\substack{q\le n\\ q\equiv 1 \pmod p}} \frac{\log q}{q-1}.$$

Comparing the above inequalities, we get

$$\frac{p-1}{2p}\log(n-1) - \frac{p-1}{p} - \frac{\log 4p}{n-1} < \frac{n}{n-1}\sum_{\substack{q\le n\\ q\equiv 1 \pmod p}} \frac{\log q}{q-1},$$

which together with the fact that $p \le n$ leads to

$$\frac{p-1}{2p}\log(n-1) \le \frac{n}{n-1}\sum_{\substack{q\le n\\ q\equiv 1 \pmod p}} \frac{\log q}{q-1} + O(1).$$

Writing $q = 1 + pt$ for some $t \le n/p$ and using the trivial inequality

$$\sum_{\substack{q\le n\\ q\equiv 1 \pmod p}} \frac{\log q}{q-1} \le \frac{\log n}{p}\sum_{t\le n/p}\frac{1}{t} \ll \frac{\log^2 n}{p},$$

we get

$$\frac{1}{3}\log(n-1) \le \frac{p-1}{2p}\log(n-1) \ll \frac{\log^2 n}{p} + O(1),$$

therefore $p \ll \log n$. Using the Montgomery-Vaughan Theorem concerning primes in arithmetic progressions (see [12]) as well as partial summation, we deduce that

$$\sum_{\substack{q\le n\\ q\equiv 1 \pmod p}} \frac{\log q}{q-1} \ll \frac{\log n}{p},$$

and therefore get

$$\frac{1}{3}\log(n-1) \le \frac{p-1}{2p}\log(n-1) \ll \frac{\log n}{p} + O(1),$$

which leads to $p \ll 1$. Since now p may be assumed fixed, we may apply Dirichlet's theorem on primes in arithmetical progressions, to get that

$$\sum_{\substack{q\le n\\ q\equiv 1 \pmod p}} \frac{\log q}{q-1} = \frac{\log(n-1)}{p-1} + O(1),$$

and now we are led to

$$\frac{p-1}{2p}\log(n-1) \leq \frac{\log(n-1)}{p-1} + O(1),$$

which tells us that

$$\left(\frac{p-1}{2p} - \frac{1}{p-1}\right) = O\left(\frac{1}{\log(n-1)}\right),$$

which admits only finitely many solutions (p, n) with $p \geq 5$ because

$$\frac{p-1}{2p} - \frac{1}{p-1} = \frac{p^2-4p+1}{2p(p-1)} \geq \frac{6}{40}$$

for $p \geq 5$. Finally, since $n^n > x^{p-1}$ and n and p are bounded, we get that x is also bounded. The statement with $p = 3$ follows from the same arguments by strengthening the inequality

$$\prod_{\substack{k=1 \\ k \times n}}^{n} k > (n-1)!^{1/2},$$

to say

$$\prod_{\substack{k=1 \\ k \times n}}^{n} k > n!^{2/3}$$

for n sufficiently large. To see that this last inequality holds, note that

$$\prod_{\substack{k=1 \\ k \times n}}^{n} k \geq \frac{n!}{n^{\tau(n)}},$$

where $\tau(n)$ is the number of divisors of n. Thus, it suffices to show that the inequality

$$n^{\tau(n)} < n!^{1/3}$$

holds for large n, and this inequality is implied by

$$\tau(n)\log n < \frac{n}{3}\log(n/e).$$

Since $\log(n/e) > (\log n)/2$ if n is large, it follows that it is enough that the inequality

$$\tau(n) < \frac{n}{6}$$

holds for large n, and this last inequality is obvious. □

Before stating and proving Theorem 2, we restate the *ABC* conjecture mentioned already in the introduction. The *ABC* conjecture asserts that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ depending only on ε, such that if A, B and C are three nonzero coprime integers satisfying $A + B = C$, then

$$\max(|A|, |B|, |C|) < C(\varepsilon)N(ABC)^{1+\varepsilon}.$$

Theorem 2. *Let $P \in \mathbb{Q}[x]$ be a polynomial of degree ≥ 2. Then the ABC conjecture implies that the equation*

$$P(x) = \prod_{\substack{k \times n \\ k=1}}^{n} k$$

has only finitely many solutions (x, n), where x is a rational number and n is a positive integer.

Proof. We write the equation as

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_{d+1} \prod_{\substack{k \times n \\ k=1}}^{n} k,$$

where $a_0, \ldots, a_{d+1}$ are integers with $a_0a_{d+1} \neq 0$. Here, one may choose a_{d+1} to be the least common denominator of all the coefficients of $P(x)$. Multiplying both sides of the above equation by a_0^{d-1} and letting $y = a_0x$, it follows that we arrive at the equation

$$y^d + b_1y^{d-1} + \cdots + b_d = b_{d+1} \prod_{\substack{k \times n \\ k=1}}^{n} k,$$

where $b_i = a_ia_0^{i-1}$ for $i = 1, \ldots, d$, and $b_{d+1} = a_{d+1}a_0^{d-1}$. Note now that y is an integer because for every fixed positive integer n the above equation shows that the rational number y is a root of a monic polynomial with integer coefficients; hence, an algebraic integer, and therefore a rational integer. With the substitution $z = y + b_1/d$, we may rewrite the above equation as

$$z^d + c_2z^{d-2} + \cdots + c_d = c_{d+1} \prod_{\substack{k \times n \\ k=1}}^{n} k,$$

where c_i are rational numbers whose denominator divides d^d. Finally, we multiply the above equation by d^d and use the substitution $t = dz$ to arrive at

$$t^d + e_2t^{d-2} + \cdots + e_d = e_{d+1} \prod_{\substack{k \times n \\ k=1}}^{n} k,$$

where t and e_i are integers for $i = 2, \ldots, d + 1$. Let $j \leq d$ be the largest index in $\{2, \ldots, d\}$ such that $e_j \neq 0$. If this index does not exist, then the above equation is

$$t^d = e_{d+1} \prod_{\substack{k \times n \\ k=1}}^{n} k.$$

By the Prime Number Theorem, for large n, the interval $(n/2, n)$ contains $\approx n/(2\log n)$ prime numbers p and none of those divides n. Since $d > 1$ and t is an integer, it follows that every such prime number divides e_{d+1}. In particular, $n/2 < e_{d+1}$, which shows that n is bounded.

Assume now that j exists and rewrite the equation as

$$t^j + (e_2 t^{j-2} + \cdots + e_j) = \frac{e_{d+1}}{t^{d-j}} \prod_{\substack{k \times n \\ k=1}}^{n} k. \tag{5}$$

Since for large t we have that $|t^d + e_2 t^{d-2} + \cdots + e_j| \asymp |t|^d$, it follows that

$$|t|^d \gg |t^d + e_2 t^{d-2} + \cdots + e_d| \gg \prod_{\substack{k \times n \\ k=1}}^{n} k \gg (n-1)!^{1/2},$$

therefore, by taking logarithms and invoking Stirling's formula, we get

$$|t| \geq \exp\left(\frac{1}{2d}(1 + o(1))n \log n + O(1)\right). \tag{6}$$

We now set $A = t^j$, $B = (e_2 t^{j-2} + \cdots + e_j)$, $C = \frac{e_{d+1}}{t^{d-j}} \prod_{\substack{k \times n \\ k=1}}^{n} k$, and we apply the ABC conjecture to equation (5). We note that our A, B, C are not necessarily coprime, but their greatest common divisor is $O(1)$. Indeed, let $D_1 = \gcd(t, e_j)$. Clearly, $D_1 \leq |e_j|$, and

$$\gcd(A, B) = \gcd(t^j, e_2 t^{j-2} + \cdots + e_j) \mid \Big(\gcd(t, e_2 t^{j-2} + \cdots + e_j)\Big)^j \mid D_1^d.$$

Thus, we may apply the ABC-conjecture and get that

$$|t|^j \ll \left(|t||B| \prod_{p \leq n} p\right)^{1+\varepsilon} \ll |t|^{(j-1)(1+\varepsilon)} \cdot 4^{(1+\varepsilon)n}.$$

Choosing $\varepsilon = 1/j$, we get that $(j-1)(1+\varepsilon) = j - 1/j$, and that the inequality

$$|t|^{1/j} \ll 4^{(1+\varepsilon)n} < 4^{2n}$$

holds. This last inequality leads to

$$|t| \leq \exp\left(2jn \log 4 + O(1)\right) \leq \exp(2dn \log 4 + O(1)). \tag{7}$$

Comparing (6) and (7), we get

$$2dn \log 4 - \frac{1}{2d}(1 + o(1))n \log n = O(1),$$

which certainly implies that n is bounded. □

Dabrowski (see [6]), showed that if $P(x) = x^2 - A$, where A is an integer which is not a perfect square, then equation (2) has only finitely many solutions. We consider the diophantine equation

$$x^2 - A = \prod_{\substack{k=1 \\ k \times n}}^{n} k, \tag{8}$$

and prove the following result.

Theorem 3. *If the integer A is not a perfect square, and n and x are positive integers satisfying equation* (8)*, then either*

$$n \le p \;\text{ or }\; n = 2p,$$

where p is the smallest prime such that $(\frac{A}{p}) = -1$. Here, $(\frac{\bullet}{p})$ stands for the Legendre symbol. In particular, equation (8) *has only finitely many positive integer solutions.*

Before proving Theorem 3, we need the following Lemma.

Lemma 4. *Every prime $p \le n$ divides $\prod_{\substack{k=1 \\ k \times n}}^{n} k$, except when $n = p,\ 2p$, cases in which p is the only prime $\le n$ which does not divide $\prod_{\substack{k=1 \\ k \times n}}^{n} k$.*

Proof. Suppose that $p \le n$ and that $n \ne p,\ 2p$. If $p \in (n/2, n)$, then p does not divide n, therefore it divides $\prod_{\substack{k=1 \\ k \times n}}^{n} k$. We now assume that $p < \frac{n}{2}$. Hence, there exists a positive integer i such that

$$\frac{n}{2} \le 2^i p < n.$$

(i) If $\frac{n}{2} < 2^i p < n$, then $2^i p$ does not divide n, and so it divides $\prod_{\substack{k=1 \\ k \times n}}^{n} k$.

(ii) If $\frac{n}{2} = 2^i p$, then $3 \cdot 2^{i-1} p < n$, and does not divide n, therefore it divides $\prod_{\substack{k=1 \\ k \times n}}^{n} k$.

□

Proof of Theorem 3. Since A is not a perfect square, there exists a prime p such that $(\frac{A}{p}) = -1$. Then p does not divide $\prod_{\substack{k=1 \\ k \times n}}^{n} k$. Lemma 4 now shows that either $n \le p$ or $n = 2p$. □

In the general case in which A is any integer in equation (8), we have the conditional result given in the following theorem.

Theorem 5. *If the weak form of Hall's conjecture is true, then equation* (8) *has only finitely many solutions.*

The weak form of Hall's conjecture is a special case of the ABC-conjecture and asserts that for every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ depending only on $\varepsilon > 0$, such that if x, y, and k are nonzero integers satisfying $x^2 = y^3 + k$, then

$$\max\left(|x^2|,\ |y^3|\right) \quad \leq \quad C(\varepsilon)|k|^{6+\varepsilon}.$$

Proof of Theorem 5. We assume that $n \geq 3$. Let d and y be the two integers with d cubefree such that $\prod_{\substack{k \times n \\ k=1}}^{n} k = dy^3$. Then, from Chebyshev's bound we obtain that

$$d \leq \left(\prod_{\substack{p < n \\ p \text{ prime}}} p \right)^2 < 4^{2(n-1)}.$$

Equation (8) gives

$$dy^3 + A = x^2,$$

so

$$Y^3 + d^2 A = X^2,$$

where $X = dx$ and $Y = dy$. Taking $\varepsilon = 1$ in the weak form of Hall's conjecture we get that

$$d^2 \prod_{\substack{k \times n \\ k=1}}^{n} k = d^3 y^3 = Y^3 \leq C(1)|d^2 A|^7.$$

Since

$$(n-1)!^{1/2} \leq \prod_{\substack{k \times n \\ k=1}}^{n} k,$$

it follows, from Stirling's formula, that

$$\left(4(n-1)^{(n-1)} e^{-(n-1)}\right)^{\frac{1}{2}} \leq (n-1)!^{\frac{1}{2}} \leq \prod_{\substack{k \times n \\ k=1}}^{n} k.$$

Hence,

$$\left(4(n-1)^{(n-1)} e^{-(n-1)}\right)^{\frac{1}{2}} \leq \prod_{\substack{k \times n \\ k=1}}^{n} k \ll |d^2|^6 |A|^7 \ll 4^{24(n-1)} |A|^7.$$

Thus,

$$\left(\frac{n-1}{4^{48}}\right)^{\frac{n-1}{2}} \ll |A|^7.$$

This proves that n is bounded. □

We considered equation (8) with $A=1$, namely

$$\prod_{\substack{k\times n\\ k=1}}^{n} k+1 \;=\; y^2, \tag{9}$$

and did some computations. Except from the obvious solutions $n=4$ and $n=5$, we didn't find any other solution for equation (9) up to $n=10^5$.

Finally, we look at yet another variant of the Brocard-Ramanujan diophantine equation, namely

$$1+\prod_{\substack{k\le n\\ \gcd(k,n)=1}} k \;=\; y^2. \tag{10}$$

Theorem 6. *Suppose that there exist integers $n>4$ and y satisfying equation* (10)*. Then either n is equal to p^α or $2p^\alpha$ for some prime p and positive integer α, or all odd primes dividing n are* $\pm 1 \pmod 8$.

Proof. We know from Gauss generalization to Wilson's Theorem (see [17]) that

$$\prod_{\substack{k\le n\\ \gcd(k,n)=1}} k \;=\; \begin{cases} -1 \pmod n, & \text{if } n=4,\ p^\alpha,\ 2p^\alpha,\\ 1 \pmod n, & \text{otherwise.}\end{cases}$$

Thus, if $n\neq 4,\ p^\alpha$, or $2p^\alpha$, then $\prod_{\substack{k\le n\\ \gcd(k,n)=1}} k+1\equiv 2 \pmod n$. This implies that $y^2\equiv 2 \pmod n$. In particular, $y^2\equiv 2 \pmod q$ holds for all odd prime factors q of n. Hence, $\left(\frac{2}{q}\right)=1$, leading to the conclusion that $q\equiv \pm 1 \pmod 8$. □

We remark that results of Landau (see pages 668–669 in [10]), together with the Prime Number Theorem, imply that if x is any positive real number, then the number of positive integers $n\le x$ such that $n=p^\alpha,\ 2p^\alpha$, or n is free of prime factors $\equiv \pm 3 \pmod 8$ is $\ll x/\sqrt{\log x}$. In particular, the set of n for which equation (10) can have a positive integer solution y is of asymptotic density zero, which is an analogue of the result of Berend and Osgood from [1] for the particular polynomial $P(X)=X^2-1$ and our variant of the Brocard-Ramanujan equation.

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