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On the slopes of the U_5 operator acting on overconvergent modular forms

par L. J. P KILFORD

Résumé. Nous démontrons que les pentes de l'opérateur U_5 agissant sur 5-adique formes modulaires surconvergentes de poids k avec caractère de Dirichlet primitif χ de conducteur 25 sont

$$\left\{\frac{1}{4}\cdot\left\lfloor\frac{8i}{5}\right\rfloor : i \in \mathbf{N}\right\} \text{ ou } \left\{\frac{1}{4}\cdot\left\lfloor\frac{8i+4}{5}\right\rfloor : i \in \mathbf{N}\right\}.$$

Nous prouvons aussi que l'espace de forms parabolique de poids k et caractère χ a une base des formes propres pour les opérateurs de Hecke T_p et U_5 définie sur $\mathbf{Q}_5(\sqrt[4]{5}, \sqrt{3})$.

Abstract. We show that the slopes of the U_5 operator acting on 5-adic overconvergent modular forms of weight k with primitive Dirichlet character χ of conductor 25 are given by either

$$\left\{\frac{1}{4}\cdot\left\lfloor\frac{8i}{5}\right\rfloor : i \in \mathbf{N}\right\} \text{ or } \left\{\frac{1}{4}\cdot\left\lfloor\frac{8i+4}{5}\right\rfloor : i \in \mathbf{N}\right\},$$

depending on k and χ.

We also prove that the space of classical cusp forms of weight k and character χ has a basis of eigenforms for the Hecke operators T_p and U_5 which is defined over $\mathbf{Q}_5(\sqrt[4]{5}, \sqrt{3})$.

1. Introduction

We first define the slope of a (normalized) cuspidal eigenform.

Definition 1. *Let f be a normalized cuspidal modular eigenform with Fourier expansion at ∞ given by $\sum_{n=1}^{\infty} a_n q^n$. The* slope *of f is defined to be the 5-valuation of a_5 viewed as an element of $\mathbf{C}_5$; we normalize the 5-valuation of 5 to be 1.*

As a consequence of the main result of this paper, we will prove the following theorem about the slopes of classical modular forms. Let us set up some notation.

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The cyclotomic polynomial $\Phi_{20}(x)$ factors over $\mathbf{Q}_5$ into two factors, f_1 and f_2, such that

$$f_1 \equiv x^4 + 2x^3 + 4x^2 + 3x + 1 \mod 5, \tag{1}$$

$$f_2 \equiv x^4 + 3x^3 + 4x^2 + 2x + 1 \mod 5. \tag{2}$$

Let χ be an odd primitive Dirichlet character of conductor 25 and let τ be an odd primitive Dirichlet character of conductor 5.

Let k be a positive integer. We fix an embedding of the field of definition of χ into $\mathbf{Q}_5(\sqrt[4]{5}, \sqrt{3})$. Then we have the following theorem which tells us what the slopes of the eigenforms are:

Theorem 2. *Using the preceding notation, the slopes of the U_5 operator acting on $S_k(\Gamma_0(25), \chi\tau^{k-1})$ are*

$$\left\{\frac{1}{4} \cdot \lfloor \frac{8i}{5} \rfloor : i \in \mathbf{N}\right\} \text{ if } \chi(6) \text{ is a root of } f_1,$$

$$\left\{\frac{1}{4} \cdot \lfloor \frac{8i+4}{5} \rfloor : i \in \mathbf{N}\right\} \text{ if } \chi(6) \text{ is a root of } f_2.$$

Because all of the slopes are distinct, we can also prove a result about the field of definition of a basis of eigenforms for this space.

Corollary 3. *Let k be a positive integer and let χ be a primitive Dirichlet character of conductor 25. Then if f is a normalized modular eigenform of weight k and character χ, its Fourier expansion is defined over $\mathbf{Q}_5(\sqrt[4]{5}, \sqrt{3})$.*

This gives some evidence towards Questions 4.3 and 4.4 of Buzzard [2]; these concern conjectural bounds on the degree of the field of definition of certain modular forms over $\mathbf{Q}_p$.

2. Previous work, and new directions

This paper uses methods introduced by Emerton in his PhD thesis [9], which deals with the action of the U_2 operator. It also uses methods developed by Smithline in his thesis [16] and applied by him in [17], which were then also used in the author's paper [11] and in the paper of the author with Buzzard [4].

In [17], the following theorem is proved about 3-adic modular forms:

Theorem 4 (Smithline [17], Theorem 4.3)**.** *We order the slopes of U_3 by size, beginning with the smallest.*

The sum of the first x nonzero slopes of the U_3 operator acting on 3-adic overconvergent modular forms of weight 0 is at least $3x(x-1)/2 + 2x$, and is exactly that if x is of the form $(3^j - 1)/2$ for some j.

His thesis also shows that the sum of the first x slopes of the U_5 operator acting on 5-adic overconvergent modular forms of weight 0 is at least x^2.

In Buzzard-Kilford [4], the following theorem was proved about the 2-adic slopes of U_2 acting on certain spaces of modular forms.

Theorem 5 (Buzzard-Kilford [4], Theorem B)**.** *Let k be an integer and let χ be a character of conductor 2^n such that $\chi(-1) = (-1)^k$.*

If $|5^k \cdot \chi(5) - 1|_2 > 1/8$, then the slopes of the overconvergent cuspidal modular forms of weight k and character χ are $\{t, 2t, 3t, \ldots\}$, where $t = v(5^k \cdot \chi(5) - 1)$, and each slope occurs with multiplicity 1.

This paper will prove a 5-adic analogue of these results for the specific level $\Gamma_0(25)$. However, the slopes are no longer in one arithmetic progression, as they are in the previously studied cases. Instead, there are five arithmetic progressions which interlace together; these all have a common difference between terms (which is 2). (One could, of course, view the arithmetic progression $1, 2, 3, \ldots$ as being made up of the two arithmetic progressions $1, 3, 5, \ldots$ and $2, 4, 6 \ldots$ but this point of view is only reasonable after one has considered the action of U_5 on forms of level 25, where the slopes form several arithmetic progressions).

In Buzzard-Calegari [3], the following theorem is proved, using similar techniques to those in [16], [11] and [4]:

Theorem 6. *The slopes of the U_2 operator acting on 2-adic overconvergent modular forms of weight 0 are*

$$\left\{1 + 2v_2\left(\frac{(3n)!}{n!}\right) : n \in \mathbf{N}\right\},$$

where v_2 is the normalized 2-adic valuation.

We see that the slopes at weight 0 do not fall into the arithmetic progressions found in [11] and [4]; we note that it appears that the behaviour of the slopes at the boundary of weight-space, which we consider in this paper, is rather different to the behaviour of the slopes at the centre of weight-space considered in [3].

In addition to the fact that the slopes do not lie in *one* arithmetic progression, there are some other interesting new features which appear when we consider 5-adic overconvergent modular forms that do not appear for 2-adic or 3-adic overconvergent modular forms.

Firstly, there is a computational issue. In previous work (such as [4] or [11]), the computations in MAGMA [1] could be carried out either over the rational numbers or over the field $\mathbf{Q}_p$. However, for $p = 5$ the calculation must be carried out over $\mathbf{Q}_5(\sqrt[4]{5}, \sqrt{3})$.

Secondly, and more importantly, there are now two different possibilities for the slopes, which depend on which character is chosen. These two possibilities correspond to the two factors of the cyclotomic polynomial $\Phi_{20}(x)$

over $\mathbf{Q}_5$. This is a departure from the situation in [4] and [11], where the slopes are independent of choice of character.

Thirdly, part of the complexity in the calculations in [11] was the fact that (in the notation of that paper) the modular function $U_2(z^{2i+1})$ was identically zero. This meant that the first "matrix of the U_2 operator" that was defined had identically zero determinant, which meant that some algebra had to be done to get a matrix to which Theorem 13 could be applied to. In the current note, this does not happen because the matrix M of the U_5 operator does not have any identically zero columns.

Finally, the strategy of Section 4 was chosen because the modular functions involved were unusually simple, thus making the calculations more tractable (the corresponding functions for (say) weight 2 were much less pleasant).

3. Defining 5-adic overconvergent modular forms

We now present the definition of the 5-adic overconvergent modular forms, first by defining overconvergent modular forms of weight 0, and then by defining forms with weight and character in terms of the weight 0 forms and a suitable Eisenstein series.

This section follows Section 3 of [11] in its layout and direction; more details on the specific steps can be found there.

Following Katz [10], section 2.1, we recall that, for C an elliptic curve over an $\mathbf{F}_5$-algebra R, there is a mod 5 modular form $A(C)$ called the *Hasse invariant*, which has q-expansion over $\mathbf{F}_5$ equal to 1.

We consider the Eisenstein series of weight 4 and tame level 1 defined over $\mathbf{Z}$, with q-expansion

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^3 \right) \cdot q^n.$$

We see that E_4 is a lifting of $A(C)$ to characteristic 0, as the reduction of E_4 to characteristic 5 has the same q-expansion as $A(C)$, and therefore E_4 mod 5 and $A(C)$ are both modular forms of level 1 and weight 4 defined over $\mathbf{F}_5$, with the same q-expansion. Note also that if C is an elliptic curve defined over $\mathbf{Z}_5$ then the valuation $v_5(E_4(C))$ can be shown to be well-defined.

It is interesting to note that one can use the same Eisenstein series, E_4, in this part of the definition for 2-adic, 3-adic and 5-adic overconvergent modular forms (as a lifting of the 4^{th}, 2^{nd} and 1^{st} power of the Hasse invariant, respectively).

We now let m be a positive integer, and we recall that the modular curve $X_0(p^m)$ is defined to be the moduli space that parametrizes pairs (C, P), where C is an elliptic curve and P is a subgroup of C of order p^m.

Using arguments exactly similar to those in [11], we define the affinoid subdomain $Z_0(5^m)$ of $X_0(5^m)$ to be the connected component containing the cusp ∞ of the set of points $t = (C, P)$ in $X_0(5^m)$ which have $v_5(E_4(t)) = 0$; we note that $v_5(E_4(t)) = 0$ means either that the point t corresponds to an ordinary elliptic curve or that t is a cusp.

We now define strict affinoid neighbourhoods of $Z_0(5^m)$.

Definition 7 (Coleman [7], Section B2). *We think of $X_0(5^m)$ as a rigid space over $\mathbf{Q}_5$, and we let $t \in X_0(5^m)(\overline{\mathbf{Q}}_5)$ be a point, corresponding either to an elliptic curve defined over a finite extension of $\mathbf{Q}_5$, or to a cusp. Let w be a rational number, such that $0 < w < 5^{2-m}/6$.*

We define $Z_0(5^m)(w)$ to be the connected component of the affinoid

$$\{t \in X_0(5^m) : \ v_5(E_4(t)) \leq w\}$$

which contains the cusp ∞.

Given this definition, we can now define 5-adic overconvergent modular forms.

Definition 8 (Coleman, [6], page 397). *Let w be a rational number, such that $0 < w < 5^{2-m}/6$. Let $\mathcal{O}$ be the structure sheaf of $Z_0(5^m)(w)$. We call sections of $\mathcal{O}$ on $Z_0(5^m)(w)$* w-overconvergent 5-adic modular forms of weight 0 and level $\Gamma_0(5^m)$. *If a section f of $\mathcal{O}$ is a w-overconvergent modular form, then we say that f is an* overconvergent 5-adic modular form.

Let K be a complete subfield of $\mathbf{C}_5$, and define $Z_0(5^m)(w)_{/K}$ to be the affinoid over K induced from $Z_0(5^m)(w)$ by base change from $\mathbf{Q}_5$. The space

$$M_0(5^m, w; K) := \mathcal{O}(Z_0(5^m)(w)_{/K})$$

of w-overconvergent modular forms of weight 0 and level $\Gamma_0(5^m)$ is a K-Banach space.

We now let χ be a primitive Dirichlet character of conductor 5^m and let k be an integer such that $\chi(-1) = (-1)^k$. Let $E^*_{k,\chi}$ be the normalized Eisenstein series of weight k and character χ with nonzero constant term.

The space of w-overconvergent 5-adic modular forms of weight k and character χ is given by

$$\mathcal{M}_{k,\chi}(5^m, w; K) := E^*_{k,\chi} \cdot \mathcal{M}_0(5^m, w; K).$$

This is a Banach space over K.

There are Hecke operators U_5 and T_p (where $p \nmid 5$) acting on the space of modular forms $\mathcal{M}_{k,\chi}(5^m, w; K)$; these are defined on the q-expansions of the overconvergent modular forms in exactly the same way as they are defined on the q-expansions of classical modular forms. One defines T_n for n a natural number in the usual way.

Using results of Coleman, we have the following theorem about the independence of the characteristic power series of U_5 acting on $\mathcal{M}_{k,\chi}(5^m, w; K)$:

Theorem 9 (Coleman [7], Theorem B3.2). *Let w be a real number such that $0 < w < \min(5^{2-m}/6, 1/6)$, let k be an integer and let χ be a character such that $\chi(-1) = (-1)^k$.*

The characteristic polynomial of U_5 acting on w-overconvergent 5-adic modular forms of weight k and character χ is independent of the choice of w.

We will now rewrite the definition of $Z_0(25)(w)$ in terms of a carefully chosen modular function of level 25, in order to prove the following theorem:

Theorem 10. *Let $w_0 = 1/12$. The space of w_0-overconvergent modular forms of weight 0 and level 25, with coefficients in $\mathbf{Q}_5(\sqrt[4]{5})$, is a Tate algebra in one variable over $\mathbf{Q}_5(\sqrt[4]{5})$.*

Proof. We have given a valuation on the points t of the rigid space $X_0(5^m)$, based on the lifting of the Hasse invariant by the Eisenstein series E_4. We recall that the modular j-invariant is defined to be $j := E_4^3/\Delta$. Therefore, we see that, if the elliptic curve corresponding to t has good reduction, then $\Delta(t)$ has valuation 0, and therefore that

$$v_5(t) = v_5(E_4(t)) = \frac{1}{3} \cdot v_5((E_4)(t)^3) = \frac{1}{3} \cdot v_5(j(t)).$$

We now recall that the modular curve $X_0(25)$ has genus 0. This means that there is a modular function t_{25} which is a uniformizer on $X_0(25)$:

$$t_{25} := \frac{\eta(q)}{\eta(q^{25})},$$

where η is the Dedekind η-function. We could write t_{25} as a rational function in j directly, but as the resulting rational function is very complicated, we will instead also work with the uniformizer t_5 of $X_0(5)$, defined as

$$t_5 := \left(\frac{\eta(q)}{\eta(q^5)}\right)^6$$

By explicit calculation, one can verify the following identities of modular functions:

$$j = \frac{(t_5^2 + 250t_5 + 3125)^3}{t_5^5} \text{ and } t_5 = \frac{t_{25}^5}{t_{25}^4 + 5t_{25}^3 + 15t_{25}^2 + 25t_{25} + 25}. \tag{3}$$

We note also that

$$j(\infty) = t_5(\infty) = t_{25}(\infty) = \infty;$$

this follows because the q-expansion of all of these functions begins $q^{-1}+\cdots$.

Let D be an element of $\mathbf{C}_5$ such that $0 < v_5(D) < 1/2$. We see that, if $0 \leq v_5(j) \leq D$, then $v_5(j) = v_5(t_5)$ (the valuation of $t_5^2 + 250t_5 + 3125$ is given by the valuation of t_5^2) and that $v_5(t_5) = v_5(t_{25})$ (because the valuation of $t_{25}^4 + 5t_{25}^3 + 15t_{25}^2 + 25t_{25} + 25$ is given by the valuation of t_{25}^4). Therefore, for this range of D we have that $v_5(j) = v_5(t_{25})$. Then, because $t_5(\infty) = t_{25}(\infty) = \infty$, this means that the connected components of $Z_0(5)$ and $Z_0(25)$ which contain the cusp ∞ of sufficiently small radius are of the form $v_5(t_5) < D_1$ and $v_5(t_{25}) < D_2$, for some rational numbers D_1 and D_2 with valuations between 0 and 1/2.

By considering the Newton polygons of the numerators and denominators of the rational functions in (3), we see that if $v_5(t_{25}) < 1/2$, then $v_5(t_{25}) = v_5(t_5) = v_5(j)$. This means that we have shown that

$$Z_0(25)(w) = \{x \in X_0(25) :\ v_5(t_{25}(x)) \leq 3w\}\,, \text{ for } 0 < w < 1/6.$$

Now, we choose $w = 1/12$, and therefore we obtain

$$Z_0(25)(1/12) = \{x \in X_0(25) :\ v_5(t_{25}(x)) \leq 1/4\}\,.$$

Let us define $W := \sqrt[4]{5}/t_{25}$. We can rewrite the definition of $Z_0(25)(1/12)$ again in terms of W to get

$$Z_0(25)(1/12) = \{x \in X_0(25) :\ v_5(W(x)) \geq 0\}\,.$$

Finally, we recall that the rigid functions on the closed disc over $\mathbf{Q}_5$ with centre 0 and radius 1 are defined to be power series of the form

$$\sum_{n\in\mathbf{N}} a_n z^n \ :\ a_n \in \mathbf{Q}_5,\ a_n \to 0.$$

Therefore, the space of 1/12-overconvergent modular forms of level $\Gamma_0(25)$ and weight 0 is

$$\mathbf{Q}_5(\sqrt[4]{5})\langle W\rangle,$$

which is what we wanted to show. □

We have written down this space of overconvergent modular forms as an explicit Banach space. This means that we can write down its *Banach basis*: the set $\{W, W^2, W^3, \ldots\}$ forms a Banach basis for the overconvergent modular forms of weight 0 and level $\Gamma_0(25)$. This Banach basis is composed of weight 0 modular functions — we want to be able to consider the action of the U_5 operator on overconvergent modular forms with non-zero weight k and character χ (here, as elsewhere in this note, χ has conductor 25 and $\chi(-1) = (-1)^k$). Using an observation from the work of Coleman [7], we will be able to move between weight 0 and weight k and character χ via multiplication by a suitable quotient of modular forms.

Let F be an overconvergent modular form of weight k and character χ which has nonzero constant term, and let z be an overconvergent modular function of weight 0. In particular, we note that F may have negative weight. From the discussion in Coleman [7, page 450] we see that the pullback $\widetilde{U}_5$ of the U_5 operator acting on overconvergent modular forms of weight k and character χ to weight 0 is $1/F \cdot U_5(z \cdot F)$.

Now by equation 3.3 of [8] we have that

$$U_5(z \cdot V(F)) = U_5(z) \cdot F. \tag{4}$$

We therefore consider the modular form $V(F)$, which has nonzero constant term because F does, and substitute $V(F)$ into the formula for the pullback of U_5 to weight zero, to obtain

$$\frac{1}{V(F)} \cdot U_5(z \cdot V(F)) = \frac{F}{V(F)} \cdot U_5(z). \tag{5}$$

We can also use (4) to see that the pullback of U_5 acting on overconvergent modular forms of weight k and character χ is

$$1/F \cdot U_5(z \cdot F) = U_5\left(z \cdot \frac{F}{V(F)}\right). \tag{6}$$

We now use equations (5) and (6) to define the twisted U_5 operator $\widetilde{U}$.

Definition 11 (The twisted U_5 operator)**.** *Let k be an integer and let χ be an odd character of conductor 25. Let τ be an odd character of conductor 5, and let W be the overconvergent modular form $\sqrt[4]{5}/t_{25}$ of level $\Gamma_0(25)$ and weight 0 that we defined above.*

We define $\widetilde{U}$ to be the "weight k" twisted U_5 operator *acting on the basis $\{W^i \cdot E^*_{1,\chi}/V(E^*_{1,\chi})\}$ by the formula:*

$$\widetilde{U}(W^i) := U_5\left(W^i \cdot \frac{E_{1,\chi}}{V(E_{1,\chi})}\right) \cdot \left(\frac{E_{1,\tau}}{V(E_{1,\tau})}\right)^{k-1}. \tag{7}$$

We note here that the operator $\widetilde{U}$ acts on overconvergent modular forms of weight 0; we consider it because it has the same characteristic power series as the standard U_5 operator acting on overconvergent modular forms with nonzero weight and character.

We can now consider the action of the operator $\widetilde{U}$ on these spaces of overconvergent modular forms.

Definition 12 (The matrix of the twisted U_5 operator)**.** *Let k be an integer and let χ be an odd character of conductor 25. Let τ be a character of conductor 5 such that $\chi(-1) = (-1)^k$. We let W be the modular function given in Definition 11.*

Let $M = (m_{i,j})$ be the infinite compact matrix of the twisted U_5 operator acting on overconvergent modular forms of weight $1+k$ and character $\chi \cdot \tau^k$, where $m_{i,j}$ is defined to be the coefficient of W^i in the W-expansion of the operator defined in equation (7)*.*

Here we will make the observation that the entries of our matrix M depend on χ and τ.

We know that U_5 is a compact operator, so we can show that the trace, determinant and characteristic power series of M are all well-defined. The following result follows directly from the general result given as Proposition 7 of [15], on the characteristic power series of compact operators (Serre uses the older term "completely continuous" for what we are calling "compact").

Theorem 13. (1) *Let M_n be an $n \times n$ matrix defined over a finite extension of $\mathbf{Q}_p$. Let $det(1 - tM_n) = \sum_{i=0}^{n} c_i t^i$. Let M_m be the matrix formed by the first m rows and columns of M_n.*

Let $s(i)$ be the formula for the i^{th} slope; in our specific case, this will mean that either

$$s(i) = \frac{1}{4} \cdot \left\lfloor \frac{8i}{5} \right\rfloor \quad or \; s(i) = \frac{1}{4} \cdot \left\lfloor \frac{8i+4}{5} \right\rfloor .$$

Assume that there exists a constant $r \in \mathbf{Q}^{\times}$ such that

(a) *For all positive integers m such that $1 \leq m \leq n$, the valuation of $det(M_m)$ is $r \cdot \sum_{i=1}^{m} s(i)$.*

(b) *The valuation of elements in column j is at least $r \cdot s(i)$.*

Then we have that, for all positive integers m such that $1 \leq m \leq n$, $v_2(c_m) = r \cdot \sum_{i=1}^{m} s(i)$.

(2) *Let M_∞ be a compact infinite matrix (that is, the matrix of a compact operator). If M_m is a series of finite matrices which tend to M_∞, then the finite characteristic power series $det(1 - tM_m)$ converge coefficientwise to $det(1 - tM_\infty)$, as $m \to \infty$.*

We now quote a result of Coleman that tells us that overconvergent modular forms of small slope are in fact classical modular forms:

Theorem 14 (Coleman [6], Theorem 1.1)**.** *Let k be a non-negative integer and let p be a prime. Every p-adic overconvergent modular eigenform of weight k with slope strictly less than $k - 1$ is a classical modular form.*

We now state the main theorem of this paper, which tells us exactly what the slopes of the U_5 operator acting on slopes of modular forms of level 25 are.

Theorem 15. *We recall and use the notation of Theorem 2.*

Let χ be an odd primitive Dirichlet character of conductor 25 and let τ be an odd primitive Dirichlet character of conductor 5.

Let k be a positive integer. We fix an embedding of the field of definition of χ into $\mathbf{Q}_5(\sqrt[4]{5})$, and recall the notation of f_1 and f_2 from Theorem 2.

The slopes of overconvergent modular forms of weight k and character $\chi\tau^{k-1}$ are given by

$$\left\{\frac{1}{4}\cdot\lfloor\frac{8i}{5}\rfloor : i\in\mathbf{N}\right\} \text{ if } \chi(6) \text{ is a root of } f_1,$$
$$\left\{\frac{1}{4}\cdot\lfloor\frac{8i+4}{5}\rfloor : i\in\mathbf{N}\right\} \text{ if } \chi(6) \text{ is a root of } f_2.$$

We can prove Theorem 2, assuming Theorem 15, by recalling the following theorem from Cohen-Oesterlé:

Theorem 16 (Cohen-Oesterlé [5], Théorème 1)**.** *Let χ be a primitive Dirichlet character of conductor 25 and let k be a positive integer greater than 1. The following formula holds:*

$$d(k,\chi) := \dim S_k(\Gamma_0(25),\chi) = \frac{5k-7}{2} + \varepsilon\cdot(\chi(8)+\chi(17)),$$

where ε is 0 for odd k, $-1/4$ if $k \equiv 2 \mod 4$, and $1/4$ if $k\equiv 0 \mod 4$.

Proof of Theorem 2. The classical theorem will follow, because when we substitute $d(k,\chi)$ into the formula $s(i)$ for the i^{th} slope, we see that the maximum value of $s(d(k,\chi))$ is $k-1$. We now apply either Theorem 14 or a standard argument shows that slopes greater than $k-1$ cannot be classical (see [4], the proof of the Corollary to Theorem B, which references the proof of Theorem 4.6.17(1) of [13]), so therefore as there are at most $k-1$ slopes which are smaller than or equal to $k-1$, we see that all of these small slopes are the slopes of classical eigenforms. □

Finally, we show that there is a basis of eigenforms which is defined over $K := \mathbf{Q}_5(\sqrt[4]{5},\sqrt{3})$, to prove Corollary 3.

Proof of Corollary 3. Let χ be a primitive Dirichlet character of conductor 25 and let k be a positive integer such that $\chi(-1) = (-1)^k$.

We recall a useful fact stated on page 21 of [14], that if $f(q) = \sum_{n=1}^{\infty} a_n q^n$ is the Fourier expansion of a nonzero normalized classical modular cuspidal eigenform of weight k and character χ, and σ is an element of $\mathrm{Gal}(\overline{K}/K)$, then

$$\sigma(f)(q) := \sum_{n=1}^{\infty} \sigma(a_n) q^n$$

is also the Fourier expansion of a normalized classical modular cuspidal eigenform of weight k and character χ; we see that because χ takes values in K, it is invariant under the action of σ.

Now, we see that the 5-valuation of a_5 is the same as the 5-valuation of $\sigma(a_5)$, because the characteristic polynomial of a_5 is invariant under the action of σ. Therefore, we see that $\sigma(f)$ is an eigenform of the same weight and character as f with the same slope as f, because there is only one eigenform of any given slope for any given χ and k. This means that $\sigma(f) = f$ for every choice of σ, so therefore $\sigma(a_n) = a_n$ for every σ, which means that $a_n \in K$ for every n, as required. □

4. The technical part; proof of Theorem 15

As the actual proof of Theorem 15 is somewhat technical, we will first outline a plan to show how the proof works.

Plan for the proof of Theorem 15. In this section, we will show that we can apply Theorem 13, which will prove Theorem 15. First we fix an arbitrary positive integer n, an integer k and a primitive Dirichlet character χ of conductor 25 such that $\chi(-1) = -1$.

We will begin with the matrix M_n; the matrix formed by the first n rows and n columns of M, the matrix of the twisted U_5 operator acting on forms of weight 1 and character χ defined in Definition 11. The proof will then proceed in the following way:

(1) *Define the matrix $D(\beta(i))$ to be the diagonal matrix with $\beta(j)$ in the j^{th} row and the j^{th} column. We define the matrix $O_n := D(\pi^{-2j}) \cdot M_n \cdot D(\pi^{2j})$.*

(2) *We then show that the valuation of elements in the jth column of O_n are $s(j)$; this verifies condition (b) of Theorem 13, with $r = s(j)$.*

(3) *We finally show that O_n has determinant of valuation $\sum_{i=1}^{n} s(i)$, by considering the matrix $P_n := D(\pi^{-4s(j)}) \cdot O_n$. We will see that P_n has determinant of valuation 0, which implies that the valuation of the determinant of O_n is the valuation of the determinant of $D(\pi^{4s(j)})$, which is $\sum_{i=1}^{n} s(i)$. This will verify condition (a) of Theorem 13, with determinant of valuation $\sum_{i=1}^{n} s(i)$.*

(4) *Finally, we will show that, after multiplication by the multiplier (as defined in Definition 12)*

$$\left(\frac{E_{1,\tau}}{V(E_{1,\tau})} \right)^k ,$$

the matrix of the twisted U_5 *operator acting on forms of weight* $1+k$ *and character* $\chi \cdot \tau^{k-1}$ *still satisfies properties* (a) *and* (b) *of Theorem 13.*

At each step of this plan, we must show that the characteristic polynomial of the new matrix defined is the same as that of M_n. In the last step, we will show that P_n has unit determinant by reducing it modulo a prime ideal above 5 and showing that this reduction has determinant 1. This means that we must prove that P_n has coefficients which are integers in $\mathbf{Q}_5(\sqrt[4]{5})$.

Proof of Theorem 15. In this section, we will use the modular function T instead of W; we define T as follows:

$$T := \frac{1}{t_{25}}.$$

This will make it easier to perform the calculations.

We will adapt a method used by Smithline in [16] and [17] to find the matrix M_n of the "weight 1" operator $\widetilde{U}$, which will use the crucial fact that this operator is what Smithline calls "a compact operator with rational generation". This means that $\widetilde{U}(T^i)$ satisfies a recurrence relation in terms of other $\widetilde{U}(T^j)$, for $j = i-5, \ldots, i-1$.

Firstly, we can show that $\widetilde{U}(T^i)$ is a *polynomial* in T of degree $5i$, for $1 \leq i \leq 5$. We prove this by explicit computation; we can see that these are polynomials by studying the zeroes and poles of the $\widetilde{U}(T^i)$, or by an explicit computation of sufficiently many Fourier coefficients of both sides. This directly generalizes the work of Smithline on the weight 0 operator U_p.

Secondly, we will show that a recurrence relation exists, and then find an easy to calculate form in terms of the $\widetilde{U}(T^j)$ and T. This will allow us to compute the determinant of the matrix of $\widetilde{U}$ acting on the basis $\{T^i\}$ and therefore to compute the slopes of the characteristic power series of $\widetilde{U}$.

We will just give the valuations of the coefficients of these, as the actual coefficients are elements of $\mathbf{Q}_5(\sqrt[4]{5})$ and thus take up a lot of space. If we have chosen χ such that $\chi(6)$ is a root of (1), then these are the valuations of the coefficients of T in the T-expansions of the $\widetilde{U}(T^i)$ for $1 \leq i \leq 5$:

$$\widetilde{U}(T): \quad \left[\tfrac{1}{4}, \tfrac{5}{4}, 2, 3, 4\right]$$

$$\widetilde{U}(T^2): \quad \left[\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{7}{4}, \tfrac{5}{2}, 4, \tfrac{17}{4}, \tfrac{21}{4}, 6, 7, 8\right]$$

$$\widetilde{U}(T^3): \quad \left[0, \tfrac{3}{4}, 1, 2, 3, \tfrac{17}{4}, \tfrac{19}{4}, \tfrac{23}{4}, \tfrac{13}{2}, 8, \tfrac{33}{4}, \tfrac{37}{4}, 10, 11, 12\right]$$

$$\widetilde{U}(T^4): \quad \left[0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 3, \tfrac{13}{4}, \tfrac{17}{4}, 5, 6, 7, \tfrac{17}{2}, \tfrac{35}{4}, \tfrac{39}{4}, \tfrac{21}{2}, 12, \tfrac{49}{4}, \tfrac{53}{4}, 14, 15, 16\right]$$

$$\widetilde{U}(T^5): \quad \Big[0, 1, 1, 2, 2, \tfrac{7}{2}, \tfrac{15}{4}, \tfrac{19}{4}, \tfrac{11}{2}, 7, \tfrac{29}{4}, \tfrac{33}{4}, \tfrac{37}{4}, \tfrac{41}{4}, 11, \tfrac{49}{4}, \tfrac{51}{4}, \tfrac{55}{4}, \tfrac{29}{2}, 16, \tfrac{65}{4}, \tfrac{69}{4}, 18, 19, 20\Big].$$

On the other hand, if we have chosen $\chi(6)$ to be a root of (2), then these are the valuations of the coefficients of T in the T-expansions of the $\widetilde{U}(T^i)$:

$$\begin{aligned}
\widetilde{U}(T):&\quad \left[\tfrac{1}{2},\tfrac{3}{2},\tfrac{9}{4},\tfrac{13}{4},4\right]\\
\widetilde{U}(T^2):&\quad \left[\tfrac{1}{2},1,2,\tfrac{11}{4},4,\tfrac{9}{2},\tfrac{11}{2},\tfrac{25}{4},\tfrac{29}{4},8\right]\\
\widetilde{U}(T^3):&\quad \left[\tfrac{1}{4},1,\tfrac{5}{4},\tfrac{9}{4},3,5,5,6,\tfrac{27}{4},8,\tfrac{17}{2},\tfrac{19}{2},\tfrac{41}{4},\tfrac{45}{4},12\right]\\
\widetilde{U}(T^4):&\quad \left[\tfrac{1}{4},\tfrac{3}{4},\tfrac{5}{4},\tfrac{7}{4},3,\tfrac{7}{2},\tfrac{9}{2},\tfrac{21}{4},\tfrac{25}{4},7,\tfrac{17}{2},9,10,\tfrac{43}{4},12,\tfrac{25}{2},\tfrac{27}{2},\tfrac{57}{4},\tfrac{61}{4},16\right]\\
\widetilde{U}(T^5):&\quad [0,1,1,2,2,\tfrac{7}{2},4,5,\tfrac{23}{4},7,\tfrac{15}{2},\tfrac{17}{2},\tfrac{19}{2},\tfrac{21}{2},11,\\
&\quad \tfrac{25}{2},13,14,\tfrac{59}{4},16,\tfrac{33}{2},\tfrac{35}{2},\tfrac{73}{4},\tfrac{77}{4},20]
\end{aligned}$$

(We note here that these are the valuations of the elements of the T^i-coefficients of the $\widetilde{U}(T^j)$ viewed as elements of $\mathbf{Q}_5(\sqrt[4]{5},\sqrt{3})$).

The T-expansion of $U(T)$ is not dependent on the choice of valuations; it is

$$U(T) = 5T + 25T^2 + 75T^3 + 125T^4 + 125T^5,$$

which is $[1,2,2,3,3]$ in the valuation scheme we have used for the $\widetilde{U}(T^j)$ above.

There is an explicit recurrence relation which the $\widetilde{U}(T^i)$ satisfy; we have that

$$\begin{aligned}
\widetilde{U}(T^{i+5}) = \frac{1}{5}\cdot U_5(T)\cdot (25\widetilde{U}(T^{i+4}) + 25\widetilde{U}(T^{i+3})\\
+ 15\widetilde{U}(T^{i+2}) + 5\widetilde{U}(T^{i+1}) + \widetilde{U}(T^i)),\ \text{for } i\geq 0.
\end{aligned}\tag{8}$$

This, together with the explicit values for $\widetilde{U}(T^i)$ given above, will allow us to determine all of the $\widetilde{U}(T^i)$. We see by induction on i that all of the $\widetilde{U}(T^i)$ are polynomials in T of degree $5i$; this, together with the fact that the relation involves powers of 5, also allow us to show that the matrix of the operator $\widetilde{U}$ is *compact*.

We also note that, because 5 divides the coefficients of $U_5(T)$, that $\widetilde{U}(T^i)$ is an *integral* polynomial in T.

We prove this recurrence relation by first showing that the same recurrence relation as in (8) holds for $U_5(T^i)$. Following the lucid account given in [12], Section 2, we recall that

$$U_5(T^j)(z) = \frac{1}{5}\cdot \sum_{t\in\mathbf{Z}/5\mathbf{Z}} T^j\left(\frac{z+t}{5}\right).$$

If we define e_k to be the k^{th} symmetric polynomial in the $\{T(\frac{z+t}{5}) :\ t\in \mathbf{Z}/5\mathbf{Z}\}$, then we see that we have the following recurrence relation:

$$U_5(T^{i+5}) - e_1U_5(T^{i+4}) + \cdots - e_5U_5(T^i) = 0,\ \text{for any } i\in\mathbf{N}.$$

Now, we can show that $e_1 = 25U_5(T)$, $e_2 = -25U_5(T)$, $e_3 = 15U_5(T)$, $e_4 = -5U_5(T)$ and $e_5 = U_5(T)$ by explicit computations of the symmetric polynomials in the $T(\frac{z+t}{5})$. For instance, we see that $e_1 = 25U_5(T)$ because that is how $U_5(T)$ is defined. For the others, we can compute these by changing the variable z to $5z$ and then using MAGMA to compute the e_i to a sufficient precision to show that they are the correct multiples of $U_5(T)$.

It is well-known that every modular function of weight 0 for $\Gamma_0(25)$ must be a rational function of T, which is a uniformizer of the genus 0 curve $X_0(25)$. Because the e_k have no poles on $X_0(p)$ except possibly at the cusp 0, they must actually be polynomials in T, and we can compute these polynomials effectively.

We now derive the recurrence relation for $\widetilde{U}(T^j)$ from that for $U_5(T^j)$. By definition, we have that

$$\begin{aligned}\widetilde{U}(T^j) &= U_5(T^j \cdot E_{1,\chi}/V_{1,\chi})(z) \\ &= \frac{1}{5}\sum_{t\in\mathbf{Z}/5\mathbf{Z}} T^j\left(\frac{z+t}{5}\right)\cdot(E_{1,\chi}/V_{1,\chi})\left(\frac{z+t}{5}\right).\end{aligned} \tag{9}$$

Now, the functions $T^i((z+t)/5)$, for $t \in \{0,1,2,3,4\}$, are the general solutions to the recurrence relation, because they are the roots of the polynomial associated to the recurrence. Therefore, if F is fixed, both $U(T^i)$ and $U(T^i \cdot F)$ satisfy the same recurrence relation, so in particular we see that the $\widetilde{U}(T^i)$ *do* satisfy a recurrence relation.

We note in passing that similar calculations to those in this section can be performed for modular functions on $X_0(4)$ and $X_0(9)$, and similar recurrence relations derived, involving a common multiple of a suitable equivalent of $U_5(T)$ as in (8).

Using this recurrence relation, we could derive a generating function for $\widetilde{U}$, in the style of [3] or [17], but this would be hard to display and not very illuminating. We will instead use the recurrence relation in the next Section to show that the valuations of the coefficients of the i^{th} column of the matrix O_n are at least $s(i)$, that the valuation of the $(i,i)^{\text{th}}$ entry of O_n is exactly $s(i)$, and that the valuation of elements below the diagonal in the i^{th} column is strictly greater than $s(i)$. This will show that the determinant of O_n has valuation $\sum_{j=1}^n s(j)$, which will be enough to prove our theorem.

4.1. Finding the valuation of elements of O_n. In this section we will define $X := \pi^2 T$ and $\overline{U}(X^i) := \pi^{2i}\cdot\widetilde{U}(X^i)$ for clarity of notation.

We check that the X-expansion of $\overline{U}(X^r)$ has integral coefficients, for $r = 1,\ldots,5$, that the valuation of the coefficient of X^r in $\overline{U}(X^r)$ is $s(i)$, and that the valuation of the coefficient of X^j in $\overline{U}(X^r)$ is greater than $s(i)$ for $j > i$. We call these statements I_r for short.

We will now prove by induction that I_r holds for every positive integer r. Firstly, we note that the base cases are satisfied; we can check that the valuations of the T-coefficients of the $\widetilde{U}(T^r)$ given above, when translated into the language of Xs, satisfy I_r for $r = 1, \ldots, 5$.

We suppose that I_s holds for every positive integer $s < r$. We now use the recurrence relation given in (8) to show that

$$\begin{aligned}\overline{U}(X^r) = (X + \pi^2 X^2 + 3X^3 + \pi^2 X^4 + X^5) \cdot (&25 \cdot \overline{U}(X^{r-1}) \\ &+ 25\pi^2 \overline{U}(X^{r-2}) + 15\pi^4 \overline{U}(X^{r-3}) + 5\pi^6 \overline{U}(X^{r-4}) \\ &+ \pi^8 \cdot \overline{U}(X^{r-5})).\end{aligned}$$

We see that the elements of strictly lowest valuation in this sum are those from the term $U := \pi^8 \cdot \overline{U}(X^{r-5}) \cdot (X + \pi^2 X^2 + 3X^3 + \pi^2 X^4 + X^5)$, so the valuations of $\overline{U}(X^r)$ are determined by this product. We also note that the valuation of the coefficient of X^r in U is exactly $s(r-5)+2$, that the valuations of the coefficients of X^s for $s > r$ in U are greater than $s(r-5)+2$, and that the valuation of every coefficient of $\overline{U}(X^r)$ is at least $s(r-5)+2$ (by using I_{r-5}). Therefore we have proved I_r by induction, because $s(r) = s(r-5)+2$.

We now postmultiply the matrix O_n by $D(\pi^{-4s(i)})$ and define this product to be P_n. The effect on the $\overline{U}(X^i)$ is to multiply them by $\pi^{-4s(i)}$. Because the elements of P_n are given by the coefficients of these X-expansions, we see that P_n has integral coefficients, that the coefficients on the diagonal are units, and that the coefficients below the diagonal have strictly positive valuation. We can therefore reduce P_n modulo π and take its determinant; we see that P_n must have unit determinant by considering the main diagonal.

This means that the determinant of O_n and also the determinant of M_n both have valuation $\sum_{i=1}^{n} s(i)$. This means that M_n satisfies condition (a) of Theorem 13 and therefore that we can apply this theorem to the matrix M to show that the slopes of the U_5 operator are given by $s(i)$.

4.2. Generalizing all this to other weights. We note that this part of the proof has shown that the matrices M_n of the twisted U_5 operator acting on overconvergent modular forms of weight 1 have determinants with valuation $\sum_{i=1}^{m} s(i)$, and that the valuations of elements in the j^{th} column are at least $s(i)$. We also note that we have expressed O_n in the form $O_n = P_n \cdot D$, where D is a diagonal matrix and P is a matrix which is upper triangular and invertible modulo π.

We will now prove that the matrix of the twisted U_5 operator acting on weights of the form $1 + k$, where k is an integer, also satisfies these two properties; this will be enough to prove Theorem 15.

We note that the following identity of modular functions holds:

$$\frac{E_{1,\tau}}{V(E_{1,\tau})} = 1 - \frac{5(T + (2+2I)T^2)}{1 + (2+I)T + (2+I)T^2}, \tag{10}$$

where τ is an odd primitive Dirichlet character of conductor 5, and I is a suitably chosen square root of -1.

From this, we see that the following Lemma holds:

Lemma 17. *Let $\mathcal{O}$ be the ring of integers of $\mathbf{Q}_5(\pi)$. Then*

$$\frac{E_{1,\tau}}{V(E_{1,\tau})} \in 1 + X\mathcal{O}[[X]]. \tag{11}$$

Proof of Lemma 17. We recall that $X = \pi^2 T$, and check that when we substitute this into (10) that the coefficients of the X^j remain integral. □

We now change weights from weight 1 to weight $1+k$; this means that we change the basis on which the $\widetilde{U}_5$ operator acts by multiplication by the power series $(E_{1,\tau}/V(E_{1,\tau}))^k$. We see that this change of basis has the property that X^i is sent to a power series in X whose exponents are at least i, and that the coefficient of X^i in this new power series is 1. Also, by (11), we see that these power series have integral coefficients. We call the matrix which gives this change of basis S.

From this, we see that this matrix S has the following properties: it has coefficients in the ring of integers $\mathcal{O}$, the diagonal entries of S are all 1, and it is lower triangular modulo π.

The change of basis by S has the effect of replacing O_n by

$$P_n \cdot D \cdot S = P_n \cdot (D \cdot S \cdot D^{-1}) \cdot D.$$

We will now show that $P_n \cdot D \cdot S \cdot D^{-1}$ is invertible and upper triangular modulo π. The action of conjugation by D on S is to divide elements above the diagonal by $\pi^{4s(i)}$ and to multiply elements below the diagonal by $\pi^{4s(i)}$; given the list of properties of S above, we see that this means that $D \cdot S \cdot D^{-1}$ must be the identity modulo π, which means that if we premultiply by P_n then the product will be upper triangular and invertible modulo π.

This means that we are in the same situation as before; the determinant of the $n \times n$ truncation of the matrix of the operator $\widetilde{U}_5$ acting on overconvergent modular forms of weight $1+k$ has valuation $\sum_{i=1}^{n} s(i)$. This means that the matrix satisfies condition (a) of Theorem 13 and therefore that we can apply this theorem to show that the slopes of the U_5 operator are given by $s(i)$, which is what we wanted to prove. □

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