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András BIRÓ

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Résumé. Ces dernières années, depuis l'article [B-D-S], nous avons étudié la possibilité de caratériser les sous-groupes dénombrables du tore $T = \mathbf{R}/\mathbf{Z}$ par des sous-ensembles de $\mathbf{Z}$. Nous considérons ici de nouveaux types de sous-groupes: soit $K \subseteq T$ un ensemble de Kronecker (un ensemble compact sur lequel toute fonction continue $f : K \to T$ peut être approchée uniformément par des caractéres de T) et G le groupe engendré par K. Nous prouvons (théorème 1) que G peut être caractérisé par un sous-ensemble de $\mathbf{Z}^2$ (au lieu d'un sous-ensemble de $\mathbf{Z}$). Si K est fini, le théorème 1 implique notre résultat antérieur de [B-S]. Nous montrons également (théorème 2) que si K est dénombrable alors G ne peut pas être caractérisé par un sous-ensemble de $\mathbf{Z}$ (ou une suite d'entiers) au sens de [B-D-S].

Abstract. In recent years, starting with the paper [B-D-S], we have investigated the possibility of characterizing countable subgroups of the torus $T = \mathbf{R}/\mathbf{Z}$ by subsets of $\mathbf{Z}$. Here we consider new types of subgroups: let $K \subseteq T$ be a Kronecker set (a compact set on which every continuous function $f : K \to T$ can be uniformly approximated by characters of T), and G the group generated by K. We prove (Theorem 1) that G can be characterized by a subset of $\mathbf{Z}^2$ (instead of a subset of $\mathbf{Z}$). If K is finite, Theorem 1 implies our earlier result in [B-S]. We also prove (Theorem 2) that if K is uncountable, then G cannot be characterized by a subset of $\mathbf{Z}$ (or an integer sequence) in the sense of [B-D-S].

1. Introduction

Let $T = \mathbf{R}/\mathbf{Z}$, where $\mathbf{R}$ denotes the additive group of the real numbers, $\mathbf{Z}$ is its subgroup consisting of the integers. If $x \in \mathbf{R}$, then $\|x\|$ denotes its distance to the nearest integer; this function is constant on cosets by $\mathbf{Z}$, so it is well-defined on T. A set $K \subseteq T$ is called a Kronecker set if it

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is nonempty, compact, and for every continuous function $f : K \to T$ and $\delta > 0$ there is an $n \in \mathbf{Z}$ such that

$$\max_{\alpha \in K} \|f(\alpha) - n\alpha\| < \delta.$$

If $K \subseteq T$ is a finite set, it is a Kronecker set if and only if its elements are independent over $\mathbf{Z}$ (this is essentially Kronecker's classical theorem on simultaneous diophantine approximation). There are many uncountable Kronecker sets, see e.g. [L-P], Ch. 1.

In [B-D-S] and in [B-S], we proved for a subgroup $G \subseteq T$ generated by a finite Kronecker set that G can be characterized by a subset of the integers in certain ways. In fact we dealt with any countable subgroup of T in [B-D-S], and the result of [B-S] was generalized also for any countable subgroup in [B]. For further generalizations and strengthenings of these results, see [Bi1], [Bi2], [D-M-T], [D-K], [B-S-W].

In the present paper, we prove such a characterization of a group generated by a general Kronecker set by a subset of $\mathbf{Z}^2$ (instead of a subset of $\mathbf{Z}$). We also show, on the contrary, that using a subset of $\mathbf{Z}$, the characterization is impossible, if K is uncountable. More precisely, we prove the following results.

Throughout the paper, let K be a fixed Kronecker set, G the subgroup of T generated by K, and let $\epsilon > 0$ be a fixed number. Write

$$l\,(x) = \frac{-1}{\log_2 x} \quad \text{for} \quad 0 < x < 1/2,$$

and extend it to every $x \geq 0$ by $l(0) = 0$, and $l(x) = 1$ for $x \geq 1/2$.

Theorem 1. *There is an infinite subset $A \subseteq \mathbf{Z}^2$ such that for every $\alpha \in G$ we have*

$$\sum_{\mathbf{n}=(n_1,n_2)\in A} l^{1+\epsilon}\left(\min\left(\|n_1\alpha\|, \|n_2\alpha\|\right)\right) < \infty, \tag{1.1}$$

and if $\beta \in T$ satisfies

$$\min\left(\|n_1\beta\|, \|n_2\beta\|\right) < \frac{1}{10} \tag{1.2}$$

for all but finitely many $\mathbf{n} = (n_1, n_2) \in A$, then $\beta \in G$. Moreover, A has the additional property that if $\alpha_1, \alpha_2, \ldots, \alpha_t \in G$ are finitely many given elements, then there is a function $f : A \to \mathbf{Z}$ such that $f(\mathbf{n}) = n_1$ or $f(\mathbf{n}) = n_2$ for every $\mathbf{n} = (n_1, n_2) \in A$, and for every $1 \leq i \leq t$ we have

$$\sum_{\mathbf{n}\in A} l^{1+\epsilon}\left(\|f(\mathbf{n})\alpha_i\|\right) < \infty. \tag{1.3}$$

If K is finite, the theorem of [B-S] follows at once from Theorem 1, since we can take all elements of K as $\alpha_1, \alpha_2, \ldots, \alpha_t$ (see also Lemma 2 (i) in

Section 3). Note that the statement of the Theorem in [B-S] contains a misprint: lim inf should be replaced by lim sup there.

Theorem 2. *If K is uncountable, and $A \subseteq \mathbf{Z}$ is an infinite subset, then*

$$G \neq \left\{ \beta \in T : \lim_{n \in A} \|n\beta\| = 0 \right\}.$$

This is in fact an easy corollary of a result of Aaronson and Nadkarni, but since the proof of that result is very sketchy in [A-N], we present its proof (see Section 4, Prop. 1.).

We give the proof of Theorem 1 in Section 2. We mention that the basic idea is the same as in [Bi2]. Some lemmas needed in the proof of Theorem 1 are presented in Section 3. We remark that Lemma 4 is very important in the proof, and it provides the main reason why we need an $\epsilon > 0$ in the theorem. The proof of Theorem 2 is given in Section 4. Section 5 contains a few comments and open questions.

2. Proof of Theorem 1

We will use Lemmas 2, 3 and 4, these lemmas are stated and proved in Section 3, so see that section if we refer to one of these lemmas.

If $x \in \mathbf{R}$, we also write x for the coset of x modulo $\mathbf{Z}$, so we consider x as an element of T. The fractional part function $\{x\}$ is well-defined on T. Let $T^{(2)}$ be the subgroup of T defined by

$$T^{(2)} = \left\{ \frac{a}{2^N} : \ N \geq 0, \ 1 \leq a \leq 2^N \right\}.$$

For $N \geq 0$ and $1 \leq a \leq 2^N$ let

$$K_{N,a} = \left\{ \alpha \in K : \ \frac{a-1}{2^N} < \{\alpha\} < \frac{a}{2^N} \right\}.$$

Since K is a Kronecker set, we can easily see that $K \cap T^{(2)} = \emptyset$, and so every $K_{N,a}$ is an open-closed subset of K, and

$$K = \bigcup_{a=1}^{2^N} K_{N,a}$$

(disjoint union). Let F be the set of functions $f : K \to T^{(2)}$ which are constant on each small set of one of these subdivisions, i.e.

$$F = \left\{ f : K \to T^{(2)} : |f(K_{N,a})| \leq 1 \quad \begin{array}{l} \text{for some } N \geq 0 \\ \text{and for every } 1 \leq a \leq 2^N \end{array} \right\},$$

where $|f(K_{N,a})|$ denotes the cardinality of the set $f(K_{N,a})$, and we write ≤ 1 because it may happen that some set $K_{N,a}$ is empty. Observe that F is countable. Every element of F is a continuous function on K, and

F is a group under pointwise addition. For a pair (N,a) with $N\geq 0$ and $1\leq a\leq 2^N$ let $F_{N,a}\leq F$ be the subgroup

$$F_{N,a}=\{f\in F: f(\alpha)=0 \text{ for } \alpha\in K\setminus K_{N,a},\ |f(K_{N,a})|\leq 1\}.$$

For any $N\geq 0$ let $g_N\in F$ be defined by

$$g_N(\alpha)=\frac{a}{2^N} \text{ for every } \alpha\in K_{N,a} \text{ and for every } 1\leq a\leq 2^N,$$

and let $f_{N,a,r}\in F_{N,a}$ be defined by ($N\geq 0$, $1\leq a\leq 2^N$, $r\geq 1$ are fixed):

$$f_{N,a,r}(\alpha)=\begin{cases}2^{-r}, & \text{if } \alpha\in K_{N,a}\\ 0, & \text{if } \alpha\in K\setminus K_{N,a}.\end{cases}$$

Clearly

$$\max_{\alpha\in K}\|g_N(\alpha)-\alpha\|\leq 2^{-N} \text{ for every } N\geq 0. \tag{2.1}$$

Remark that the functions g_N are not necessarily distinct, but if $N\geq 0$ is fixed, then

$$|\{\nu\geq 0:\ g_\nu=g_N\}|<\infty, \tag{2.2}$$

since otherwise (2.1), applied for the elements ν of this set, would give $g_N(\alpha)=\alpha$ for every $\alpha\in K$, which is impossible by $K\cap T^{(2)}=\emptyset$.

For every $f\in F$ take a number $C(f)>0$, and for every $N\geq 0$ a number $R(N)>0$, we assume the following inequalities:

$$\sum_{f\in F}C(f)^{-\epsilon}<\infty, \qquad \sum_{N=0}^{\infty}R(N)^{-\epsilon}<\infty, \tag{2.3}$$

and (it is possible by (2.2)):

$$C(g_N)>N \text{ for every } N\geq 0. \tag{2.4}$$

For every $f\in F$ and for every integer $j\geq 1$ we take an integer $m_j(f)$ such that

$$\max_{\alpha\in K}\|f(\alpha)-m_j(f)\alpha\|<2^{-j-2^jC(f)}, \tag{2.5}$$

which is possible, since K is a Kronecker set. Moreover, we can assume that if $j,j^\star\geq 1$, $f,f^\star\in F$, then

$$m_{j^\star}(f^\star)\neq m_j(f) \text{ if } (j,f)\neq(j^\star,f^\star). \tag{2.6}$$

Indeed, there are countably many pairs (j,f), and for a fixed pair (j,f) there are infinitely many possibilities for $m_j(f)$ in (2.5), so we can define recursively the integers $m_j(f)$ to satisfy (2.5) and (2.6).

Let $j(N,a,r)\geq 1$ be integers for every triple $(N,a,r)\in V$, where

$$V=\Big\{(N,a,r):\ N\geq 0,\ 1\leq a\leq 2^N,\ r>R(N)\Big\},$$

satisfying that if $(N^\star, a^\star, r^\star) \in V$ is another such triple, then

$$j(N,a,r) \neq j(N^\star, a^\star, r^\star), \text{ if } (N,a,r) \neq (N^\star, a^\star, r^\star). \tag{2.7}$$

We easily see from (2.6) and (2.7) that for (N,a,r),$(N^\star, a^\star, r^\star) \in V$ we have

$$m_{j(N,a.r)}\left(f_{N,a,r}\right) \neq m_{j(N^\star,a^\star,r^\star)}\left(f_{N^\star,a^\star,r^\star}\right), \text{ if } (N,a,r) \neq (N^\star, a^\star, r^\star). \tag{2.8}$$

Define

$$H_1 = \left\{ m_{j(N,a,r)}(f_{N,a,r}) : \ (N,a,r) \in V \right\}. \tag{2.9}$$

We claim that

$$\sum_{n \in H_1} l^{1+\epsilon}\left(\|n\alpha\|\right) < \infty \tag{2.10}$$

for every $\alpha \in K$. Indeed, let $\alpha \in K$ be fixed. We have

$$\left\| m_{j(N,a,r)}(f_{N,a,r})\alpha \right\| \leq \|f_{N,a,r}(\alpha)\| + 2^{-1-2^{j(N,a,r)}C(f_{N,a,r})} \tag{2.11}$$

by (2.5). Now, on the one hand,

$$\sum_{a=1}^{2^N} l^{1+\epsilon}\left(\|f_{N,a,r}(\alpha)\|\right) = l^{1+\epsilon}(2^{-r}), \qquad \sum_{N=0}^{\infty} \sum_{r > R(N)} l^{1+\epsilon}(2^{-r}) < \infty \tag{2.12}$$

by (2.3); on the other hand, using (2.7) and (2.3), we get

$$\sum_{(N,a,r) \in V} l^{1+\epsilon}\left(2^{-1-2^{j(N,a,r)}C(f_{N,a,r})}\right) \leq \sum_{f \in F} \sum_{j \geq 1} \left(C(f)2^j\right)^{-(1+\epsilon)} < \infty. \tag{2.13}$$

In view of Lemma 2 (i), (2.11)-(2.13), and the definition of H_1 in (2.9), we get (2.10).

If s is a nonnegative integer, the following set is a compact subset of T :

$$K_s = \left\{ \alpha = \sum_{i=1}^{t} k_i \alpha_i : \begin{array}{l} t \geq 1, \ \alpha_1, \alpha_2, \ldots, \alpha_t \in K, \\ k_1, k_2, \ldots, k_t \in \mathbf{Z}, \ \sum_{i=1}^{t} |k_i| \leq s \end{array} \right\}.$$

Lemma 1. *There is a subset H of the integers such that $H_1 \subseteq H$ and on the one hand we have*

$$\sum_{n \in H} l^{1+\epsilon}\left(\|n\alpha\|\right) < \infty \tag{2.14}$$

for every $\alpha \in K$; on the other hand, if $\beta \in T$ has the property that

$$\|n\beta\| < \frac{1}{10} \tag{2.15}$$

for all but finitely many $n \in H$, then there is a group homomorphism $\phi_\beta = \phi : F \to T$ which satisfies the following properties:

(i) for all but finitely many pairs (f,j) with $f \in F$, $j \geq 1$ we have

$$\|\phi(f) - m_j(f)\beta\| < 2^{-C(f)-j}; \tag{2.16}$$

(ii) for every (N, a) *pair with* $N \geq 0$, $1 \leq a \leq 2^N$, *if* $K_{N,a} \neq \emptyset$, *there is a unique integer* $k_{N,a}$ *for which*

$$\phi(f) = k_{N,a} f(\alpha) \tag{2.17}$$

for every $f \in F_{N,a}$, *where* $\alpha \in K_{N,a}$ *is arbitrary; if* $K_{N,a} = \emptyset$, *we put* $k_{N,a} = 0$, *and then for large* N *we have*

$$\max_{1 \leq a \leq 2^N} |k_{N,a}| \leq 2^{R(N)}; \tag{2.18}$$

(iii) if N *is large enough, then writing* $s = \sum_{a=1}^{2^N} |k_{N,a}|$, *there is an* $\alpha \in K_s$ *such that*

$$\|\alpha - \beta\| \leq \frac{1}{N} + s2^{-N}. \tag{2.19}$$

Proof. Define

$$H_2 = \left\{2^r \left(m_{j+1}(f) - m_j(f)\right) : \ f \in F, \ j \geq 1, \ 0 \leq r \leq j - 1 + C(f)\right\}.$$

Let us choose for every triple $f_1, f_2, f_3 \in F$ with $f_3 = f_1 + f_2$ an infinite subset J_{f_1,f_2,f_3} of the positive integers such that (the first summation below is over every such triple from F)

$$\Sigma := \sum_{f_3 = f_1 + f_2} \sum_{j \in J_{f_1,f_2,f_3}} \left(2^j \min\left(C(f_1), C(f_2), C(f_3)\right)\right)^{-\epsilon} < \infty. \tag{2.20}$$

Since $C(f) > 0$ for every $f \in F$, $\epsilon > 0$ and F is countable, this is obviously possible. Then define (we mean again that f_1, f_2, f_3 run over every such triple from F)

$$H_3 = \left\{2^r \left(m_j(f_1) + m_j(f_2) - m_j(f_3)\right) : \ \begin{array}{l} f_3 = f_1 + f_2, \ j \in J_{f_1,f_2,f_3}, \\ 0 \leq r \leq j - 2 \end{array} \right\},$$

$$H_4 = \left\{2^r \left(m_1(g_N) - 1\right) : \ N \geq 1, \ 0 \leq r \leq \log_2 N\right\}.$$

Let $H = \bigcup_{i=1}^4 H_i$. We first prove (2.14). If $f \in F$, $j \geq 1$ and $\alpha \in K$, then

$$\left\|\left(m_{j+1}(f) - m_j(f)\right)\alpha\right\| \leq 2^{-(j + C(f) - 1) - (2^j - 1)C(f)} \tag{2.21}$$

by (2.5), therefore, using also Lemma 2 (ii) and (2.3), we obtain

$$\sum_{n \in H_2} \max_{\alpha \in K} l^{1+\epsilon}\left(\|n\alpha\|\right) \leq m \sum_{f \in F} \sum_{j \geq 1} C(f)^{-\epsilon}(2^j - 1)^{-\epsilon} < \infty. \tag{2.22}$$

If $\alpha \in K$, $f_1, f_2, f_3 \in F$, $f_3 = f_1 + f_2$ and $j \in J_{f_1,f_2,f_3}$, then by (2.5) we get

$$\left\|\left(m_j(f_1) + m_j(f_2) - m_j(f_3)\right)\alpha\right\| \leq 2^{-(j-2)} 2^{-2^j \min(C(f_1), C(f_2), C(f_3))}, \tag{2.23}$$

and so by Lemma 2 (ii) and (2.20) we get

$$\sum_{n \in H_3} \max_{\alpha \in K} l^{1+\epsilon}\left(\|n\alpha\|\right) \leq m\Sigma < \infty. \tag{2.24}$$

If $N \geq 1$ and $\alpha \in K$, then

$$\|(m_1(g_N) - 1)\,\alpha\| \leq \|m_1(g_N)\alpha - g_N(\alpha)\| + \|g_N(\alpha) - \alpha\| \leq 2^{1-N} \quad (2.25)$$

by (2.1), (2.4) and (2.5), so by the definition of H_4, we obtain

$$\sum_{n\in H_4} \max_{\alpha\in K} l^{1+\epsilon}(\|n\alpha\|) \leq \sum_{N=1}^{\infty} (1+\log_2 N)\, l^{1+\epsilon}\left(2^{1-N-\log_2 N}\right) < \infty. \quad (2.26)$$

The relations (2.10), (2.22), (2.24) and (2.26) prove (2.14).

Now, assume that for a $\beta \in T$ we have an $n_0 > 0$ such that (2.15) is true if $n \in H$ and $|n| > n_0$. Since K is a Kronecker set, so $\|n\alpha\| > 0$ for $0 \neq n \in \mathbf{Z}$, $\alpha \in K$. Therefore, we see from (2.21) (and (2.3)) that

$$0 < |m_{j+1}(f) - m_j(f)| \leq n_0$$

can hold only for finitely many pairs $f \in F$, $j \geq 1$; we see from (2.23) that if $f_1, f_2, f_3 \in F$ are given with $f_3 = f_1 + f_2$, then

$$0 < |m_j(f_1) + m_j(f_2) - m_j(f_3)| \leq n_0$$

can hold only for finitely many $j \geq 1$; and from (2.25) that

$$0 < |m_1(g_N) - 1| \leq n_0$$

can hold only for finitely many N. Then, by Lemma 3, we obtain the following inequalities (using $H_2 \subseteq H$, $H_3 \subseteq H$, $H_4 \subseteq H$, respectively):

$$\|(m_{j+1}(f) - m_j(f))\,\beta\| < \frac{1/10}{2^{j-2+C(f)}} \quad (2.27)$$

for all but finitely many pairs $f \in F$, $j \geq 1$;

$$\|(m_j(f_1) + m_j(f_2) - m_j(f_3))\,\beta\| < \frac{1/10}{2^{j-2}} \quad (2.28)$$

for every triple $f_1, f_2, f_3 \in F$ with $f_3 = f_1 + f_2$ and for large enough $j \in J_{f_1,f_2,f_3}$;

$$\|(m_1(g_N) - 1)\,\beta\| < \frac{1/10}{N/2} \quad (2.29)$$

for large enough N.

Then from (2.27), for all but finitely many pairs $f \in F$, $j_1 \geq 1$ we have

$$\|(m_{j_2}(f) - m_{j_1}(f))\,\beta\| < \frac{2/5}{2^{C(f)}} \sum_{j=j_1}^{j_2-1} 2^{-j} \quad (2.30)$$

for every $j_2 > j_1$. This implies that $m_j(f)\beta$ is a Cauchy sequence for every $f \in F$, so

$$\phi(f) := \lim_{j\to\infty} m_j(f)\beta \quad (2.31)$$

exists, (2.16) is satisfied for all but finitely many pairs $f \in F$, $j \geq 1$ by (2.30), and since every J_{f_1,f_2,f_3} is an infinite set, $\phi : F \to T$ is a group homomorphism by (2.28) and (2.31). We also see that for large N, by (2.16), (2.4) and (2.29), we have

$$\|\phi(g_N)-\beta\| \le \frac{1}{N}. \tag{2.32}$$

If (N,a) is a fixed pair with $N\ge 0,\ 1\le a\le 2^N$ and $K_{N,a}\ne\emptyset$,, then

$$\|\phi(f_{N,a,r})\| \le \left\|\phi(f_{N,a,r}) - m_{j(N,a,r)}(f_{N,a,r})\beta\right\| + \left\|m_{j(N,a,r)}(f_{N,a,r})\beta\right\|,$$

and so

$$\limsup_{r\to\infty} \|\phi(f_{N,a,r})\| \le \frac{1}{10}$$

by (2.16), (2.7), using also the assumption on β, (2.8) and $H_1\subseteq H$. Then (2.17) follows from Lemma 4, because $F_{N,a}$ is obviously isomorphic to $T^{(2)}$. We now prove (2.18). Assume that N is large and

$$|k_{N,a}| > 2^{R(N)} \tag{2.33}$$

for some $1\le a\le 2^N$. Take an integer r such that

$$2|k_{N,a}| \le 2^r \le 4|k_{N,a}|. \tag{2.34}$$

Then $r>R(N)$, so $m_{j(N,a,r)}(f_{N,a,r})\in H_1\subseteq H$, and so for large N we have (see (2.8)) that

$$\left\|m_{j(N,a,r)}(f_{N,a,r})\beta\right\| < \frac{1}{10}. \tag{2.35}$$

But (2.34) and (2.17) imply

$$\|\phi(f_{N,a,r})\| \ge \frac{1}{4},$$

which contradicts (2.35) for large N by (2.16) and (2.7). Therefore (2.33) cannot be true for large N, so (2.18) is proved. To prove (2.19), if $N\ge 0$, $1\le a\le 2^N$ are arbitrary and $k_{N,a}\ne 0$, which implies $K_{N,a}\ne\emptyset$ by definition, we take an $\alpha_{N,a}\in K_{N,a}$, and then, by the definition of g_N and by the already proved properties of ϕ, we have

$$\|\phi(g_N) - \sum_{1\le a\le 2^N,\, k_{N,a}\ne 0} k_{N,a}\alpha_{N,a}\| \le 2^{-N}\sum_{a=1}^{2^N}|k_{N,a}|,$$

and together with (2.32), this proves (2.19). □

Proof of Theorem 1. For every $N\ge 0$ we take some integer $j(N)\ge 1$ such that the sequence $j(N)$ is strictly increasing and

$$\sum_{N=0}^{\infty} 2^{N-1}(R(N)+2)^2\, l^{1+\epsilon}\left(2^{-j(N)}\right) < \infty. \tag{2.36}$$

Let

$$U=\left\{(N,a): N\ge 0, 1\le a\le 2^{N-1}, K_{N,2a-1}\ne\emptyset,, K_{N,2a}\ne\emptyset\right\},$$

define $A^\star \subseteq \mathbf{Z}^2$ as

$$A^\star = \left\{ \left(m_{j(N)}\left(f_{N,2a-1,r_1}\right), m_{j(N)}\left(f_{N,2a,r_2}\right) \right) : \begin{array}{l} (N,a) \in U, \\ 1 \le r_1, r_2 \le R(N) + 2 \end{array} \right\},$$

and let $A = A^\star \cup \{(n,n) : n \in H\}$. Note that if $(N,a), (N^\star, a^\star) \in U$, and $1 \le r_1 \le R(N)+2$, $1 \le r_1^\star \le R(N^\star)+2$, then

$$m_{j(N)}\left(f_{N,2a-1,r_1}\right) \ne m_{j(N^\star)}\left(f_{N^\star,2a^\star-1,r_1^\star}\right), \text{ if } \ (N,a) \ne (N^\star, a^\star). \quad (2.37)$$

Indeed, this follows from the fact that j is strictly increasing (so one-to-one), using (2.6) and the definition of U.

Assume that $\beta \in T$ satisfies (1.2) for all but finitely many $\mathbf{n} = (n_1, n_2) \in A$. Then (2.15) is true for all but finitely many $n \in H$, we can apply Lemma 1. If N is large, and we assume that $k_{N,2a-1} \ne 0$ and $k_{N,2a} \ne 0$ for some $1 \le a \le 2^{N-1}$ (this implies $(N,a) \in U$ by the definitions), then by (2.18) we can take a pair $1 \le r_1, r_2 \le R(N)+2$ such that

$$2\left|k_{N,2a-1}\right| \le 2^{r_1} \le 4\left|k_{N,2a-1}\right|, \qquad 2\left|k_{N,2a}\right| \le 2^{r_2} \le 4\left|k_{N,2a}\right|.$$

Then by (2.17), we have

$$\left\|\phi\left(f_{N,2a-1,r_1}\right)\right\| \ge \frac{1}{4}, \qquad \left\|\phi\left(f_{N,2a,r_2}\right)\right\| \ge \frac{1}{4},$$

and, in view of (2.16), $j(N) \to \infty$, the definition of A, (2.37) and the property of β, this is a contradiction for large N. Therefore, if N is large, then $k_{N,2a-1}k_{N,2a} = 0$ for every $1 \le a \le 2^{N-1}$, and since clearly $k_{N,2a-1} + k_{N,2a} = k_{N-1,a}$, this easily implies that $\sum_{a=1}^{2^N} |k_{N,a}|$ is constant for large N. In view of (2.19) and the compactness of the sets K_s, this proves that $\beta \in G$.

Now, let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be given distinct elements of K. Then it is clear that if N is large enough ($N \ge N_0$), then for any $1 \le a \le 2^{N-1}$ we can take a $\delta(N,a) \in \{0,1\}$ such that

$$\alpha_1, \alpha_2, \ldots, \alpha_t \notin K_{N,2a-\delta(N,a)},$$

i.e.

$$f_{N,2a-\delta(N,a),r}(\alpha_i) = 0$$

for every $r \ge 1$, $1 \le i \le t$. Then, defining $\delta(N,a) \in \{0,1\}$ arbitrarily for $0 \le N < N_0$, $1 \le a \le 2^{N-1}$, by (2.5) and (2.36) we have

$$\sum_{N=0}^{\infty} \sum_{a=1}^{2^{N-1}} \sum_{1 \le r_1, r_2 \le R(N)+2} l^{1+\epsilon}\left(\left\|m_{j(N)}(f_{N,2a-\delta(N,a),r_{2-\delta(N,a)}})\alpha_i\right\|\right) < \infty$$

for $1 \le i \le t$. This, together with (2.14), means that defining f on $A^\star$ by

$$f\left(\left(m_{j(N)}\left(f_{N,2a-1,r_1}\right), m_{j(N)}\left(f_{N,2a,r_2}\right)\right)\right) = m_{j(N)}(f_{N,2a-\delta(N,a),r_{2-\delta(N,a)}}),$$

(the definition is correct by (2.37)), and extending f to A by $f((n,n)) = n$ for $n \in H$, we have (1.3) for every $1 \le i \le t$. We proved the existence of such an f for $\alpha_1, \alpha_2, \dots, \alpha_t \in K$, but since K generates G, such an f exists also for $\alpha_1, \alpha_2, \dots, \alpha_t \in G$, in view of Lemma 2 (i). Then (1.1) follows easily, so the theorem is proved. □

3. Some lemmas

Lemma 2. *(i) There is a constant $M > 0$ such that if $x, y \ge 0$, then*

$$l^{1+\epsilon}(x+y) \le M\left(l^{1+\epsilon}(x) + l^{1+\epsilon}(y)\right).$$

(ii) There is an $m > 0$ constant such that for any $a > 0$ we have

$$\sum_{r=0}^{\infty} l^{1+\epsilon}(2^{-r-a}) \le m a^{-\epsilon}.$$

Proof. For statement (i) we may obviously assume that $0 < x, y < 1/4$. Then

$$x + y \le 2\max(x,y) \le \sqrt{\max(x,y)},$$

and so

$$l^{1+\epsilon}(x+y) \le l^{1+\epsilon}\left(\sqrt{\max(x,y)}\right) = \left(-\log_2\left(\sqrt{\max(x,y)}\right)\right)^{-(1+\epsilon)}$$
$$= 2^{1+\epsilon} l^{1+\epsilon}\left(\max(x,y)\right),$$

which proves (i). Statement (ii) is trivial from the definitions. □

Lemma 3. *If $\omega \in T$, $k \ge 1$ is an integer, and*

$$\|\omega\|, \|2\omega\|, \|4\omega\|, \dots, \|2^k\omega\| \le \delta < \frac{1}{10},$$

then $\|\omega\| \le \frac{\delta}{2^k}$.

Proof. This is easy, and proved as Lemma 3 of [B-S]. □

Lemma 4. *If $\phi : T^{(2)} \to T$ is a group homomorphism and*

$$\limsup_{r\to\infty} \left\| \phi\left(\frac{1}{2^r}\right) \right\| < \frac{1}{4}, \tag{3.1}$$

then there is a unique integer k such that $\phi(\alpha) = k\alpha$ for every $\alpha \in T^{(2)}$.

Proof. The uniqueness is obvious, we prove the existence. It is well-known that the Pontriagin dual of the discrete group $T^{(2)}$ is the additive group $\mathbf{Z_2}$ of 2-adic integers. Hence there is a 0-1 sequence b_r $(r \ge 0)$ such that

$$\phi(\alpha) = \left(\sum_{r=0}^{\infty} b_r 2^r\right)\alpha \tag{3.2}$$

for every $\alpha \in T^{(2)}$, hence

$$\phi\left(\frac{1}{2^r}\right) = \frac{b_0}{2^r} + \frac{b_1}{2^{r-1}} + \ldots + \frac{b_{r-1}}{2} \tag{3.3}$$

for every $r \geq 1$. We see from (3.3) that if $b_{r-1} = 1$, $b_{r-2} = 0$, then

$$\frac{1}{2} \leq \left\{\phi\left(\frac{1}{2^r}\right)\right\} \leq \frac{3}{4},$$

which is impossible for large enough r, in view of (3.1). Consequently, the sequence b_r is constant for large enough r. If this constant is 0, i.e. $b_r = 0$ for $r \geq r_0$, then using (3.2), we get the lemma at once. If the constant is 1, so $b_r = 1$ for $r \geq r_0$, then, since

$$\sum_{r=0}^{\infty} 2^r = -1$$

in $\mathbf{Z_2}$, one obtains the lemma from (3.2) with

$$k = -1 - \Big((1-b_0) + 2(1-b_1) + \ldots + 2^{r_0-1}(1-b_{r_0-1})\Big).$$

□

4. Proof of Theorem 2

If G is a group and d is a metric on G, we say that (G,d) is a Polish group, if d is a complete metric, and G with this metric is a separable topological group.

The following proposition essentially appears on p. 541. of [A-N], but since they give only a brief indication of the proof, we think that it is worth to include a proof here.

Proposition 1. *Assume that K is an uncountable compact subset of T, and K is independent over $\mathbf{Z}$. Let $G \leq T$ be the subgroup generated by K. Let d be a metric defined on G such that (G,d) is a Polish group. Then the injection map*

$$i : (G,d) \to T, \qquad i(g) = g \text{ for every } g \in G$$

is not continuous (we take on T its usual topology, inherited from $\mathbf{R}$).

Proof. Let Q be a countable dense subgroup in (G,d) (such a subgroup clearly exists, since (G,d) is separable). Consider Q with the discrete topology (discrete metric). Then (Q,G) is a Polish (polonais) transformation group in the sense of [E], moreover, it clearly satisfies Condition C on p. 41. of [E]. Since Q is not locally closed in G by our conditions, conditon (5) of Theorem 2.6 of [E] is not satisfied. Hence (9) of that theorem is also false, therefore there is a Borel measure μ on G with $\mu(G) = 1$ such that

(i) each Q-invariant measurable subset of G has measure 0 or 1;

(ii) each point of G has measure 0.

Indeed, $\mu(G) = 1$ can be assumed, since μ is nontrivial and finite by [E], (i) follows since μ is ergodic in the sense of [E], and (ii) is true by (i), because μ is not concentrated in a Q-orbit.

The measure μ then has the following additional property, which is a strengthening of (ii):

(iii) if $F \subseteq G$ is a closed subset (in the d-topology) and $\mu(F) > 0$, then there is an $A \subseteq F$ with $0 < \mu(A) < \mu(F)$.

It follows by another application of Theorem 2.6 of [E]. Indeed, let $\{0\}$ be the trivial group, then $(\{0\}, F)$ is a polonais transformation group satisfying Conditon C on p.41. of [E], (5) of Theorem 2.6 is true, hence (8) of Theorem 2.6, using (ii), gives (iii).

Now, we are able to prove the proposition. Assume that $i : (G, d) \to T$ is continuous, and we will get a contradiction. For $t \geq 1$, $n_1, n_2, \dots, n_t \in \mathbf{Z}$ set

$$E(n_1, n_2, \dots, n_t) = \{n_1 x_1 + n_2 x_2 + \ldots + n_t x_t : \ x_1, x_2, \dots, x_t \in K\}.$$

Every $E(n_1, n_2, \dots, n_t)$ is a closed set in (G, d), since it is closed in T and i is continuous. Since

$$G = \bigcup_{t \geq 1} \bigcup_{n_1, n_2, \dots, n_t \in \mathbf{Z}} E(n_1, n_2, \dots, n_t),$$

hence $\mu\left(E(n_1, n_2, \dots, n_t)\right) > 0$ for some values of the parameters.

Let $g \in G$, $t \geq 1$, $n_1, n_2, \dots, n_t \in \mathbf{Z}$ be minimal with the property that

$$\mu\left(g + E(n_1, n_2, \dots, n_t)\right) > 0,$$

in the sense that

$$\mu\left(h + E(m_1, m_2, \dots, m_r)\right) = 0 \tag{4.1}$$

for every $h \in G$, $r \geq 1$, $m_1, m_2, \dots, m_r \in \mathbf{Z}$ with

$$|m_1| + |m_2| + \ldots + |m_r| + |r| < |n_1| + |n_2| + \ldots + |n_t| + |t|. \tag{4.2}$$

By (iii), writing $F = g + E(n_1, n_2, \dots, n_t)$, there is an $A \subseteq F$ with $0 < \mu(A) < \mu(F)$. Then $\mu\left(\bigcup_{q \in Q}(q + A)\right) > 0$, hence $\mu\left(\bigcup_{q \in Q}(q + A)\right) = 1$ by (i). We prove that

$$\mu\left(\left(\bigcup_{q \in Q}(q + A)\right) \bigcap (F \setminus A)\right) = 0.$$

This will give a contradiction, because $\mu\left(F \setminus A\right) > 0$. Since Q is countable, it is enough to prove that $\mu\left((q + A) \bigcap F\right) = 0$ for every $0 \neq q \in Q$, which follows, if we prove

$$\mu\left((q + F) \bigcap F\right) = 0 \tag{4.3}$$

for every $0 \neq q \in Q$.

Assume that $q + f_1 = f_2$, $f_1 = g + e_1$, $f_2 = g + e_2$, where $f_1, f_2 \in F$, $e_1, e_2 \in E(n_1, n_2, \ldots, n_t)$. For $i = 1, 2$ let

$$e_i = n_1 x_{i1} + n_2 x_{i2} + \ldots n_t x_{it}$$

with $x_{ij} \in K$ for $i = 1, 2$, $1 \le j \le t$. Let

$$q = \nu_1 x_{01} + \nu_2 x_{02} + \ldots \nu_s x_{0s}$$

with $s \ge 1$, and $\nu_l \in \mathbf{Z}$, $x_{0l} \in K$ for $1 \le l \le s$. Since $q + e_1 = e_2$, $q \ne 0$, and K is independent over $\mathbf{Z}$, there are integers $1 \le i \le 2$, $1 \le j \le t$ and $1 \le l \le s$ such that $x_{ij} = x_{0l}$. Therefore, if

$$E := \bigcup_{1 \le l \le s} \bigcup_{m \in \mathbf{Z}} \bigcup_{(r, m_1, m_2, \ldots, m_r) \in H} \left(m x_{0l} + E(m_1, m_2, \ldots, m_r)\right),$$

where

$$H := \{(r, m_1, m_2, \ldots, m_r)\colon r \ge 1,\ m_1, m_2, \ldots, m_r \in \mathbf{Z},\ (4.2) \text{ is true}\},$$

then $e_i \in E$ for some $1 \le i \le 2$. Hence

$$f_2 \in (g + E) \bigcup (g + q + E).$$

Since $\mu\,(g + E) = \mu\,(g + q + E) = 0$ by (4.1), (4.2), so (4.3) is true, and the proposition is proved. □

Proof of Theorem 2. Assume that

$$G = \left\{\beta \in T :\ \lim_{n \in A} \|n\beta\| = 0\right\}$$

for some infinite $A \subseteq \mathbf{Z}$. For $x, y \in G$ let

$$d(x, y) = \|x - y\| + \max_{n \in A} \|n(x - y)\|. \tag{4.4}$$

It is clear that d is a metric on G, and (G, d) is a topological group. We show that d is complete. Let $\beta_j \in G$, $j \ge 1$ be a Cauchy sequence with respect to d. Then β_j is a Cauchy sequence also in T by (4.4), so there is a $\beta \in T$ such that $\|\beta_j - \beta\| \to 0$ as $j \to \infty$. Now, for $n \in A$, $j_1, j_2 \ge 1$ we have

$$\|n\,(\beta_{j_1} - \beta)\| \le \|n\,(\beta_{j_1} - \beta_{j_2})\| + \|n\,(\beta_{j_2} - \beta)\|. \tag{4.5}$$

Letting $j_2 \to \infty$ for fixed n and j_1 we get

$$\|n\beta\| \le \|n\beta_{j_1}\| + \limsup_{j_2 \to \infty} d(\beta_{j_1}, \beta_{j_2}),$$

and $\beta_{j_1} \in G$ gives

$$\limsup_{n \in A} \|n\beta\| \le \limsup_{j_2 \to \infty} d(\beta_{j_1}, \beta_{j_2})$$

for every $j_1 \geq 1$, which proves $\beta \in G$. Let $\epsilon > 0$, then we can take $j_2, N \geq 1$ so that

$$\|n(\beta_{j_2} - \beta)\| + \sup_{j_1 \geq j_2} d(\beta_{j_1}, \beta_{j_2}) < \epsilon$$

for every $n \in A$, $|n| \geq N$. Hence for $j_1 \geq j_2$, $n \in A$, $|n| \geq N$ we have $\|n(\beta_{j_1} - \beta)\| < \epsilon$ by (4.5). Since for any fixed $|n| < N$ we know that $\|n(\beta_{j_1} - \beta)\| \to 0$ as $j_1 \to \infty$, this proves $d(\beta_{j_1}, \beta) \to 0$, so d is complete.

Let X be a countable dense subset in T, and for $N, l \geq 1$ integers, $x \in X$ let

$$U_{N,l,x} = \left\{ \beta \in G : \begin{array}{r} \|\beta - x\| + \max_{n \in A, |n| \leq N} \|n(\beta - x)\| \\ + \max_{n \in A, |n| > N} \|n\beta\| < \frac{1}{l} \end{array} \right\}.$$

It is easy to check that if we take an element from each nonempty $U_{N,l,x}$, then we get a countable dense subset of (G, d). So the conditions of Proposition 1 are satisfied, hence $i : (G, d) \to T$ is not continuous. But this contradicts (4.4), so the theorem is proved. □

5. Some remarks and problems

If K is finite, it follows from [Bi2], Theorem 1 (ii) that Theorem 1 of the present paper would be false for $\epsilon = 0$. But we cannot decide the following

Problem 1. *Let K be uncountable. Is Theorem 1 true with $\epsilon = 0$?*

The following proposition is a consequence of [V], p.140, Theorem 2' (the quoted theorem of Varopoulos is stronger than this statement):

Proposition 2. *Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$, then there is an infinite subset $A \subseteq \mathbf{Z}$ such that*

$$G = \left\{ \beta \in G \cup L : \lim_{n \in A} \|n\beta\| = 0 \right\}.$$

Compare Proposition 2 with our Theorem 2. We do not know whether Proposition 2 can be strengthened in the following way:

Problem 2. *Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$. Is there an infinite subset $A \subseteq \mathbf{Z}$ such that*

$$G = \left\{ \beta \in G \cup L : \lim_{n \in A} \|n\beta\| = 0 \right\},$$

and

$$\sum_{n \in A} \|n\alpha\| < \infty$$

for every $\alpha \in G$?

We state without proof our following partial result in this direction.

Theorem 3. *Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$, and let v be a strictly increasing continuous function on the interva* $[0{,}1/2]$ *with* $v(0) = 0$. *Then there is an infinite subset $A \subseteq \mathbf{Z}$ such that we have*

$$\sum_{n \in A} l^{1+\epsilon}\left(\|n\alpha\|\right) < \infty$$

for every $\alpha \in G$, but

$$\sum_{n \in A} v\left(\|n\beta\|\right) = \infty$$

for every $\beta \in L$.

Remark that this theorem implies at once the result mentioned on p.40. of [H-M-P], namely that G is a saturated subgroup of T (for the definition of a saturated subgroup, see [H-M-P] or [N], Ch. 14). We note that the above-mentioned Theorem 2' on [V], p.140, also implies that G is saturated.

Finally, we mention that Theorem 2 and Proposition 2 together show that if K is uncountable, then G is a g-closed but not basic g-closed subgroup of T in the terminology of [D-M-T]. This answers the question of D. Dikranjan (oral communication) about the existence of such subgroups of T.

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András Biró
A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
1053 Budapest, Reáltanoda u. 13-15., Hungary
E-mail: biroand@renyi.hu