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On elementary equivalence, isomorphism and isogeny

par Pete L. CLARK

Résumé. Motivé par un travail récent de Florian Pop, nous étudions les liens entre trois notions d'équivalence pour des corps de fonctions : isomorphisme, équivalence élémentaire et la condition que les deux corps puissent se plonger l'un dans l'autre, ce que nous appelons isogénie. Certains de nos résulats sont purement géométriques : nous donnons une classification par isogénie des variétiés de Severi-Brauer et des quadriques. Ces résultats sont utilisés pour obtenir de nouveaux exemples de "équivalence élémentaire entraine isomorphisme" : pour toutes les courbes de genre zéro sur un corps de nombres et pour certaine courbe de genre un sur un corps de nombres, incluant des courbes qui ne sont pas des courbes elliptiques.

Abstract. Motivated by recent work of Florian Pop, we study the connections between three notions of equivalence of function fields: isomorphism, elementary equivalence, and the condition that each of a pair of fields can be embedded in the other, which we call isogeny. Some of our results are purely geometric: we give an isogeny classification of Severi-Brauer varieties and quadric surfaces. These results are applied to deduce new instances of "elementary equivalence implies isomorphism": for all genus zero curves over a number field, and for certain genus one curves over a number field, including some which are not elliptic curves.

1. Introduction

Throughout this paper k shall stand for a perfect field, with fixed algebraic closure $\overline{k}$ and absolute Galois group $\mathfrak{g}_k$. Unless otherwise specified, by a variety V/k we shall mean a *nonsingular* geometrically irreducible projective k-variety.

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1.1. The elementary equivalence versus isomorphism problem. A fundamental problem in arithmetic algebraic geometry is to classify varieties over a field k up to birational equivalence, i.e., to classify finitely generated field extensions K/k up to isomorphism. On the other hand, there is the model-theoretic notion of elementary equivalence of fields – written as $K_1 \equiv K_2$ – i.e., coincidence of their first-order theories. Model theorists well know that elementary equivalence is considerably coarser than isomorphism: for any infinite field F there exist fields of all cardinalities elementarily equivalent to F as well as infinitely many isomorphism classes of countable fields elementarily equivalent to F.

However, the fields elementarily equivalent to a given field F produced by standard model-theoretic methods (Lowenheim-Skolem, ultraproducts) tend to be rather large: e.g., any field elementarily equivalent to $\mathbb{Q}$ has infinite absolute transcendence degree [10]. It is more interesting to ask about the class of fields elementarily equivalent to a given field and satisfying some sort of finiteness condition. This leads us to the following

Question 1. *Let $K_1,\ K_2$ be function fields with respect to a field k. Does $K_1 \equiv K_2 \implies K_1 \cong K_2$?*

On the model-theoretic side, we work in the language of fields and *not* in the language of k-algebras – i.e., symbols for the elements of $k \setminus \{0,1\}$ are not included in our alphabet. However, in the geometric study of function fields one certainly does want to work in the category of k-algebras. This turns out not to be a serious obstacle, but requires certain circumlocutions about function fields, which are given below.

By a **function field with respect to** k we mean a field K for which there exists a finitely generated field homomorphism $\iota : k \to K$ such that k is algebraically closed in K, but the *choice* of a particular ι is not given. Rather, such a choice of ι is said to give a k**-structure** on K, and we use the customary notation K/k to indicate a function field endowed with a particular k-structure. Suppose that $\varphi : K_1 \to K_2$ is a field embedding of function fields with respect to k. If k has the property that every field homomorphism $k \to k$ is an isomorphism – and fields of absolute transcendence degree zero have this property – then we can choose k-structures compatibly on K_1 and K_2 making φ into a morphism of k-algebras: indeed, take an arbitrary k-structure $\iota_1 : k \to K_1$ and *define* $\iota_2 = \varphi \circ \iota_1$.

Question 1 was first considered for one-dimensional function fields over an algebraically closed base field by Duret (with subsequent related work by Pierce), and for arbitrary function fields over a base field which is either algebraically closed or a finite extension of the prime subfield (i.e., a finite field or a number field) by Florian Pop. They obtained the following results:

Theorem 1. *(Duret* [6], *Pierce* [16]*) Let k be an algebraically closed field, and $K_1,\ K_2$ be one-variable function fields with respect to k such that $K_1 \equiv K_2$.*
a) If the genus of K_1 is different from 1, then $K_1 \cong K_2$.
b) If the genus of K_1 is one, then so also is the genus of K_2, and the corresponding elliptic curves admit two isogenies of relatively prime degrees.

The conclusion of part b) also allows us to deduce that $K_1 \cong K_2$ in most cases, e.g. when the corresponding elliptic curve E_1/k has $\mathrm{End}(E_1) = \mathbb{Z}$.

The **absolute subfield** of a field K is the algebraic closure of the prime subfield ($\mathbb{F}_p$ or $\mathbb{Q}$) in K. It is easy to see that two elementarily equivalent fields must have isomorphic absolute subfields.

Theorem 2. *(Pop* [17]*) Let $K_1,\ K_2$ be two function fields with respect to an algebraically closed field k such that $K_1 \equiv K_2$. Then:*
a) K_1 and K_2 have the same transcendence degree over k.
b) If K_1 is of ***general type****, $K_1 \cong K_2$.*

We recall that having general type means that for a corresponding projective model V/k with $k(V) = K$, the linear system given by a sufficiently large positive multiple of the canonical class gives a birational embedding of V into projective space. For curves, having general type means precisely that the genus is at least two, so Theorem 2 does not subsume but rather complements Theorem 1.

Pop obtains even stronger results (using the recent affirmative solution of the Milnor conjecture on K-theory and quadratic forms) in the case of finitely generated function fields.

Theorem 3. *(Pop* [17]*) Let $K_1,\ K_2$ be two finitely generated fields with $K_1 \equiv K_2$. Then there exist field homomorphisms $\iota : K_1 \to K_2$ and $\iota' : K_2 \to K_1$. In particular, K_1 and K_2 have the same absolute transcendence degree.*

Let K be a finitely generated field with absolute subfield isomorphic to k. Then, via a choice of k-structure, K/k is the field of rational functions of a variety V/k.

Corollary 4. *Let K_1/k be a function field of general type over either a number field or a finite field. Then any finitely generated field which is elementarily equivalent to K_1 is isomorphic to K_1.*

Proof: By Theorem 3, there are field homomorphisms $\varphi_1 : K_1 \to K_2$ and $\varphi_2 : K_2 \to K_1$, so $\varphi = \varphi_2 \circ \varphi_1$ gives a field homomorphism from K_1 to itself. If we choose k-structures $\iota_i : k \hookrightarrow K_i$, on K_1 and K_2, then it need not be true that φ gives a k-automorphism. But since k is a finite extension of its prime subfield k_0, $\mathrm{Aut}(k/k_0)$ is finite, and some power $\Phi = \varphi^k$ of φ induces the identity automorphism of k. In other words, there exists a dominant rational self-map $\Phi : V_1/k \to V_1/k$. By a theorem of Iitaka [9, §5.4], when V has general type such a map must be birational. Hence $(\varphi_2 \circ \varphi_1)^k$ is an isomorphism of fields, which implies that φ_2 is surjective, i.e., gives an isomorphism from K_2 to K_1. This completes the proof.

1.2. Isogeny of function fields. Thus Theorem 3 is "as good as" Theorem 2. But actually it is better, in that one can immediately deduce that elementary equivalence implies isomorphism from a weaker hypothesis than general type.

Definition: We say that two fields K_1 and K_2 are **field-isogenous** if there exist field homomorphisms $K_1 \to K_2$ and $K_2 \to K_1$ and denote this relation by $K_1 \sim K_2$. If for a field K_1 we have $K_1 \sim K_2 \implies K_1 \cong K_2$, we say K_1 is **field-isolated**. If K_1 and K_2 are function fields with respect to k, they are k-**isogenous**, denoted $K_1 \sim_k K_2$, if for some choice of k-structure ι_1 on K_1 and ι_2 on K_2, there exist k-algebra homomorphisms $\varphi_1 : K_1/k \to K_2/k$ and $\varphi_2 : K_2/k \to K_1/k$. We say K_1 is **k-isolated** if $K_1 \sim_k K_2 \implies K_1 \cong K_2$. Finally, if K_1/k and K_2/k are k-algebras, we say K_1 is **isogenous** to K_2 if there exist k-algebra homomorphisms $\varphi_1 : K_1 \to K_2$, $\varphi_2 : K_2 \to K_1$.

The distinction between field-isogeny and k-isogeny is a slightly unpleasant technicality. It is really the notion of isogeny of k-algebras which is the most natural (i.e., the most geometric), whereas for the problem of elementary equivalence versus isomorphism, Theorem 3 gives us field-isogeny. There are several ways around this dichotomy. The most extreme is to restrict attention to base fields k without nontrivial automorphisms, the so-called **rigid fields**. These include $\mathbb{F}_p$, $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}$ and "most" number fields. In this case, all k-structures are unique and we get the following generalization of Corollary 4.

Corollary 5. *Let K be a function field with respect to its absolute subfield k and assume that k is rigid. Then if K is k-isolated, any finitely generated field elementarily equivalent to K is isomorphic to K.*

The assumption of a rigid base is of course a loss of generality (which is not necessary, as will shortly become clear), but it allows us to concentrate on the purely geometric problem of classifying function fields K/k up to isogeny. In particular, which function fields are isolated? Which have finite isogeny classes?

We make some general comments on the notion of isogeny of function fields:

a) The terminology is taken from the theory of abelian varieties: indeed if K_1, K_2 are function fields of polarized abelian varieties A_1, A_2, then they are isogenous in the above sense if and only if there is a surjective homomorphism of group schemes with finite kernel $\varphi : A_1 \to A_2$ (the point being that in this case there is also an isogeny from A_2 to A_1).

b) By a **model** V/k for a function field K/k, we mean a *nonsingular* projective variety with $k(V) \cong K$. Thus the assertion that every function field has a model relies on resolution of singularities, which is known at present for transcendence degree at most two in all characteristics (Zariski, Abhyankar) and in arbitrary dimension in characteristic zero (Hironaka). None of our results – with the single exception of Proposition 6d), which is itself not used in any later result – are conditional on resolution of singularities.

We can express the notion of isogeny of two function fields K_1/k and K_2/k in terms of any models V_1 and V_2 by saying that there are generically finite *rational* maps $\iota : V_1 \to V_2$ and $\iota' : V_2 \to V_1$.

As usual in classification problems, the easiest way to show that two fields K_1 and K_2 are not isogenous is not to argue directly but rather to find some *invariant* that distinguishes between them. It turns out that the isogeny invariants we use are actually field-isogeny invariants.

Proposition 6. *Let k be a field. The following properties of a function field K/k are isogeny invariants. Moreover, when K is a function field with respect to its absolute subfield k, then they are also field-isogeny invariants.*
a) The transcendence degree of K/k.
b) When k has characteristic zero, the Kodaira dimension of a model V/k for K.
c) For any model V/k of K, the rational points $V(k)$ are Zariski-dense.
d) (assuming resolution of singularities) For any nonsingular *model V/k of K, there exists a k-rational point.*

Proof: Part a) follows from the basic theory of transcendence bases. As for part b), the first thing to say is that it is *false* in characteristic 2: there are unirational K3 surfaces [3]. However in characteristic zero, if $X \to Y$ is a generically finite rational map of algebraic varieties, then the Kodaira dimension of Y is at most the Kodaira dimension of X. Moreover, the Kodaira dimension is independent of the choice of k-structure.

For part c), If $X \to Y$ is a generically finite rational map of k-varieties and the rational points on X are Zariski-dense, then so too are the rational points on Y, so the Zariski-density of the rational points is an isogeny invariant. Moreover, if σ is an automorphism of k, then the natural map $V \to V^\sigma = V \times_\sigma k$ is an isomorphism of abstract schemes which induces a continuous bijection $V(k) \to V^\sigma(k)$. It follows that the Zariski-density of the rational points is independent of the choice of k-structure.

For the last part, we recall the theorem of Nishimura-Lang [14]: let $X \to Y$ be a rational map from an irreducible k-variety to a proper k-variety; if X has a smooth k-rational point, then Y has a k-rational point.

Let $\iota : K_1 \to K_2$ be a k-embedding of function fields (as in part c), the desired conclusion is independent of the choices of k-structure, so we may make choices such that ι is a k-map). Invoking resolution of singularities, let V_1 be a smooth proper k-model for K_1. From V_1 and ι we get an induced model V_2 of K_2, namely the normalization of V_1 in K_2, together with a k-morphism $V_2 \to V_1$. The variety V_2 is normal but not necessarily smooth; however, by resolution of singularities there exists an everywhere defined birational map $\tilde{V_2} \to V_2$, where $\tilde{V_2}/k$ is a smooth proper model. In all we get a k-morphism of nonsingular varieties $\tilde{V_2} \to V_1$. Our assumption is that $\tilde{V_2}(k) \neq \emptyset$ (Nishimura-Lang assures that this condition is independent of the choice of the model), so we have $V_1(k) \neq \emptyset$, which was to be shown.

The "invariants" of Proposition 6 are really only useful in analyzing the isogeny classes of varieties V/k without k-rational points. For instance, two elliptic function fields $\mathbb{Q}(E_1)$ and $\mathbb{Q}(E_2)$ have the same invariants a), b), c), d) if and only if the groups $E_1(\mathbb{Q})$ and $E_2(\mathbb{Q})$ are both finite or both infinite: this is a feeble way to try to show that two elliptic curves are not isogenous.

1.3. The Brauer kernel. In addition to the isogeny invariants of the previous subsection, we introduce another class of invariants of a function field $k(V)$, *a priori* trivial if $V(k) \neq \emptyset$, and having the advantage that their elementary nature is evident (rather than relying on the recent proof of the Milnor conjecture): the Brauer kernel.

Let V/k be a (complete nonsingular, geometrically irreducible, as always) variety over any field k, and recall the exact sequence

$$0 \to \mathrm{Pic}(V) \to \mathbf{Pic}(V/k)(k) \xrightarrow{\alpha} Br(k) \xrightarrow{\beta} Br(k(V)) \tag{1}$$

where $\mathrm{Pic}(V)$ denotes the Picard group of line bundles on the k-scheme V and $\mathbf{Pic}(V/k)$ denotes the group scheme representing the sheafified Picard group, so that in particular $\mathbf{Pic}(V)(k) = \mathrm{Pic}(V/\overline{k})^{\mathfrak{g}_k}$ gives the group of geometric line bundles which are linearly equivalent to each of their Galois conjugates. The map α gives the obstruction to a k-rational divisor class coming from a k-rational divisor, which lies in the Brauer group of k. One way to derive (1) is from the Leray spectral sequence associated to the étale sheaf $\mathbb{G}_m$ and the morphism of étale sites induced by the structure map $V \to \operatorname{Spec} k$. For details on this, see [2, Ch. IX].

We denote by $\kappa = \ker(\beta) = \mathrm{image}(\alpha)$ the **Brauer kernel** of V. Some of its useful properties are: since a k-rational point on V defines a section of β, $V(k) \neq \emptyset$ implies $\kappa = 0$. Moreover, since it is defined in terms of the function field $k(V)$, it is a birational invariant of V. The subgroup κ depends on the k-structure on $k(V)$ as follows: if σ is an automorphism of k, then the Brauer kernel of $V^\sigma = V \times_\sigma k$ is $\sigma(\kappa)$. If k is a finite field, $\kappa = 0$ (since $Br(k) = 0$).

If k is a number field, then κ is a finite group, being an image of the finitely generated group $\mathbf{Pic}(V)(k)$ in the torsion group $Br(k)$. Moreover the Galois conjugacy class of $\kappa \subset Br(k)$ is an elementary invariant of $K = k(V)$: knowing the conjugacy class of κ is equivalent to knowing which finite-dimensional central simple k-algebras B (up to conjugacy) become isomorphic to matrix algebras in K. But if $[B : k] = n$, $B \otimes_k K$ can be interpreted in K (up to $\mathfrak{g}_k$-conjugacy) via a choice of a k-basis $b_1, \dots, b_n$ of B and n^3 structure constants $c_{ij}^l \in k$ coming from the equations $b_i \cdot b_j = \sum_{l=1}^n c_{ij}^l b_l$ and the c_{ij}^l themselves represented in terms of the minimal polynomial for a generator of $k/\mathbb{Q}$. Then we can write down the statement that $B \otimes_k K \cong M_n(K)$ as the existence of an $n^2 \times n^2$ matrix A with nonzero determinant and such that $A(b_i \cdot b_j) = A(b_i) \cdot A(b_j)$ for all $1 \leq i,\ j \leq n$.[1]

Moreover, for any finite extension l/k, the conjugacy class of the Brauer kernel of V/l (which can be nontrivial even when $\kappa(V/k) = 0$) is again an elementary invariant of $k(V)$.

If $k(V_1) \to k(V_2)$ is an embedding of function fields, then clearly $\kappa(V_1) \subset \kappa(V_2)$. It follows that the Brauer kernel is an isogeny invariant, and the Galois-conjugacy class of the Brauer kernel is a field-isogeny invariant.

1.4. Statement of results. We begin with a result relating isomorphism, isogeny, Brauer kernels and elementary equivalence of function fields of certain geometrically rational varieties.

Theorem 7. *For any field k and any positive integer n, let SB_n be the set of function fields of Severi-Brauer varieties of dimension n over k and Q_n the class of function fields of quadric hypersurfaces of dimension n over k.*
a) Let $K_1,\ K_2 \in SB_n$ be cyclic elements.[2] *The following are equivalent:*
i) $K_1 \cong K_2$.
ii) K_1 and K_2 are isogenous.
iii) K_1 and K_2 have equal Brauer kernels.
b) If $K_1,\ K_2 \in Q_n$, $n \leq 2$ and the characteristic of k is not two, the following are equivalent:
i) $K_1 \cong K_2$.
ii) K_1 and K_2 are isogenous (as k-algebras).
iii) K_1 and K_2 have equal Brauer kernels, and for every quadratic extension l/k, lK_1 and lK_2 have equal Brauer kernels.
c) Let $K_1 \in SB_n$ and $K_2 \in Q_n, n > 1$. Assume the characteristic of k is not two. The following are equivalent:
i) $K_1 \cong K_2 \cong k(t_1, \dots, t_n)$ are rational function fields.
ii) $K_1 \cong K_2$.

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[1] In fact one can see that the conjugacy class of the Brauer kernel is an elementary invariant whenever k is merely algebraic over its prime subfield.

[2] A Severi-Brauer variety X/k is said to be cyclic if its corresponding division algebra D/k has a maximal commutative subfield l such that l/k is a cyclic (Galois) extension.

iii) K_1 and K_2 are isogenous.
iv) K_1 and K_2 have equal Brauer kernels.

Corollary 8. *Suppose k is algebraic over its prime subfield. Let $K_1 \equiv K_2$ be two function fields satisfying the hypotheses of part a), part b) or part c) of the theorem. Then $K_1 \cong K_2$.*

Proof of Corollary 8: By the discussion of Section 1.3, the elementary equivalence of K_1 and K_2 imply that their Brauer kernels are Galois conjugate. It follows that for any choice of k-structure on K_1, there exists a unique k-structure on K_2 such that we have $\kappa(K_1/k) = \kappa(K_2/k)$. The theorem then implies that $K_1 \cong_k K_2$ as k-algebras with this choice of k-structure; *a fortiori* they are isomorphic as abstract fields.

When $n = 1$, $SB_1 = Q_1$ and this class can be described equally well in terms of genus zero curves, quaternion algebras and ternary quadratic forms. The essential content of the theorem when $n = 1$ is that the Brauer kernel of a genus zero curve which is not $\mathbb{P}^1$ is cyclic of order two, generated by the corresponding quaternion algebra (Theorem 11). This fundamental result was first proved by Witt [21].

It is well-known that the cyclicity hypothesis is satisfied for all elements of the Brauer group of a local or global field and for any field when $n \leq 2$. Assuming a conjecture of Amitsur – see part c) of Theorem 12 – part a) of the theorem is valid for all Severi-Brauer function fields.

Corollary 9. *Let k be a number field and K a genus zero, one-variable function field with respect to k. Then any finitely generated field elementarily equivalent to K is isomorphic to K.*

Proof of Corollary 9: Let L be a finitely generated field such that $L \equiv K$. Theorem 3 applies to show that there exist field embeddings $\iota_1 : L \hookrightarrow K$ and $\iota_2 : K \hookrightarrow L$. By an appropriate choice of k-structures, we may view ι_2 as a k-algebra morphism, hence corresponding to a morphism of algebraic curves $C_K \to C_L$. By Riemann-Hurwitz, C_L has genus zero, so the result follows from Corollary 8.

Unfortunately the proof of Corollary 9 does not carry over to higher-dimensional rational function fields. Indeed, consider the case of $K = k(t_1, \ldots, t_n) = k(\mathbb{P}^n)$, a rational function field. Then the isogeny class of K is precisely the class of n-variable function fields which are unirational over k. When $n = 1$ every k-unirational function field is k-rational, as is clear from the Riemann-Hurwitz formula and the proof of Corollary 9 (and is well known in any case: Luroth's theorem). If k is algebraically closed of characteristic zero, then k-unirational surfaces are k-rational, an often-noted conseqeuence of the classification of complex algebraic surfaces [7, V.2.6.1]. However, for most non-algebraically closed fields this is false, as follows from work of Segre and Manin. Indeed, let $K = k(S)$ be the function field of a cubic hypersurface in $\mathbb{P}^3$. Then K is unirational over k if and only if for any model S, $S(k) \neq \emptyset$ [13, 12.11]; recall that all our varieties are smooth. So for all $a \in k^\times$, the cubic surface

$$S_a : x_0^3 + x_1^3 + x_2^3 + ax_3^3 = 0$$

is unirational over k. Segre showed that S_a is k-rational if and only if $a \in k^{\times 3}$; this was sharpened considerably by Manin [13, p. 184] to: $k(S_a) \cong k(S_b)$ if and only if $a/b \in k^{\times 3}$. Thus for any field in which the group of cube classes $k^\times/k^{\times 3}$ is infinite, the isogeny class of $k(\mathbb{P}^2)$ is infinite.

Among one-dimensional arithmetic function fields, Question 1 is open only for genus one curves. By exploring the notion of an "isogenous pair of genus one curves" and adapting the argument of Pierce [16] in our arithmetic context, we are able to show that elementary equivalence implies isomorphism for certain genus one function fields.

Theorem 10. *Let $K = k(C)$ be the function field of a genus one curve over a number field k, with Jacobian elliptic curve $J(C)$. Suppose all of the following hold:*

- *$J(C)$ does not have complex multiplication over $\overline{k}$.*
- *Either $J(C)$ is k-isolated* or *$J(C)(k)$ is a finite group.*
- *The order of C in $H^1(k, J(C))$ is* $1,\ 2,\ 3, 4,$ *or* 6.

Then any finitely generated field elementarily equivalent to K is isomorphic to K.

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2. Curves of genus zero

The key to the case $n = 1$ in Theorem 7 is the following classical (but still not easy) result of Witt [21] computing the Brauer kernel of a genus zero curve.

Theorem 11. *(Witt) Let C/k be a genus zero curve over an arbitrary field k. The Brauer kernel of $k(C)$ is trivial if and only if $C \cong \mathbb{P}^1$. Otherwise $\kappa(k(C)) = \{1, B_C\}$ with B_C a quaternion algebra over k. Moreover the assignment $C \mapsto B_C$ gives a bijection from the set of isomorphism classes* *of genus zero curves without k-rational points to the set of isomorphism classes of division quaternion algebras over k.*

If we grant this result of Witt, the proof of Theorem 7 for function fields of genus zero curves follows immediately: the Brauer kernel of a genus zero curve classifies the curve up to isomorphism (and hence its function field up to k-algebra isomorphism). Moreover, the Brauer kernel is an isogeny invariant, so genus zero curves are isogenous if and only if they are isomorphic.

We remark that Witt's theorem gives something a bit stronger than the k-isolation of the function field of a genus zero curve: it shows that a genus zero curve without k-rational point is not dominated by any nonisomorphic genus zero curve.

We give two "modern" approaches to Witt's theorem: via Severi-Brauer varieties and via quadratic forms. We admit that part of our goal is expository: we want to bring out the analogy between the Brauer group (of division algebras) and the Witt ring (of quadratic forms) of a field k and especially between two beautiful theorems, of Amitsur on the Brauer group side and of Cassels-Pfister on the Witt ring side.

3. Severi-Brauer varieties

Since the automorphism groups of $M_n(k)$ and $\mathbb{P}^{n-1}(k)$ are both $PGL_{n+1}(k)$, Galois descent gives a correspondence between twisted forms of $M_n(k)$ – the n^2-dimensional central simple k-algebras – and twisted forms of $\mathbb{P}^{n-1}$, the Severi-Brauer varieties of dimension $n-1$. In particular, to each Severi-Brauer variety V/k we can associate a class $[V]$ in the Brauer group of k, such that two Severi-Brauer varieties of the same dimension V_1 and V_2 are *isomorphic* over k if and only if $[V_1] = [V_2] \in Br(k)$.

As for the birational geometry of Severi-Brauer varieties, we have the following result [1].

Theorem 12. *(Amitsur) Let V_1, V_2 be two Severi-Brauer varieties of equal dimension over a field k, and for $i = 1, 2$ let $K_i = k(V_i)$ be the corresponding function field, the so-called* ***generic splitting field*** *of V_i.*
a) The subgroup $Br(K_1/k)$ of division algebras split by K_1 is generated by $[V_1]$.
b) It follows that if V_1 and V_2 are k-birational, then $[V_1]$ and $[V_2]$ generate the same cyclic subgroup of $Br(k)$.
c) If the division algebra representative for V_1 has a maximal commutative subfield which is a cyclic Galois extension of k, then the converse holds: if $[V_1]$ and $[V_2]$ generate the same subgroup of $Br(k)$, then V_1 and V_2 are k-birational.

Amitsur conjectured that the last part of this theorem should remain valid for all division algebras. As mentioned above, there has been some progress on this up to the present day [11], but the general case remains open.

Proof of Theorem 7 for cyclic Severi-Brauer varieties: let V_1/k and V_2/k be cyclic Severi-Brauer varieties of dimension n. By Amitsur's Theorem (Theorem 12), $\kappa(V_1) = \kappa(V_2)$ if and only if $k(V_1) \cong_k k(V_2)$. As in the one-dimensional case, it follows that each of these conditions is equivalent to $k(V_1)$ and $k(V_2)$ being isogenous. Finally, when k is the absolute subfield of $k(V_1)$ and $k(V_2)$, it follows from Theorem 3 that $k(V_1) \equiv k(V_2)$.

4. Quadric hypersurfaces

In this section the characteristic of k is different from 2. Our second approach to Witt's theorem (Theorem 11) is via the quadratic form(s) associated to a genus zero curve.

4.1. Background on quadratic forms. We are going to briefly review some vocabulary and results of quadratic forms; everything we need can be found in the wonderful books [12] and [18]. We assume familiarity with the notions of anisotropic, isotropic and hyperbolic quadratic forms, as well as with the Witt ring $W(k)$, which plays the role of the Brauer group here: it classifies quadratic forms up to a convenient equivalence relation so that the equivalence classes form a group, and every element of $W(k)$ has a unique "smallest" representative, an anisotropic quadratic form.

The correspondence between genus zero curves over k and quaternion algebras over a field of characteristic different from two is easy to make explicit: to a quaternion algebra B/k we associate the **ternary quadratic form** given by the reduced norm on the trace zero subspace (of "pure quaternions") of B. In coordinates, the correspondence is as follows:

$$\left(\frac{a,b}{k}\right) = 1 \cdot k \oplus i \cdot k \oplus j \cdot k \oplus ij \cdot k \mapsto \mathcal{C}_{a,b} : aX^2 + bY^2 - abZ^2 = 0.$$

By Witt cancellation, it would amount to the same to consider the quadratic form given by the reduced norm on all of B; this quaternary quadratic form has diagonal matrix $\langle 1,\ a,\ b,\ -ab\rangle$.

On the other hand, the equivalence class of the ternary quadratic form is *not* well-determined by the isomorphism class of the curve, for the simple reason that we could scale the defining equation of $\mathcal{C}_{a,b}$ by any $c \in k^\times$, which would change the ternary quadratic form to $\langle -ca,\ -cb, cab\rangle$. Thus at best the **similarity class** of the quadratic form is well-determined by the isomorphism class of $\mathcal{C}_{a,b}$, and, as we shall see shortly, this does turn out to be well-defined. Recall that the **discriminant** of a quadratic form is defined as the determinant of any associated matrix, and that this quantity is well-defined as an element of $k^\times/k^{\times 2}$. It follows that for any form q of odd rank, there is a unique form similar to q with any given discriminant $d \in k^\times/k^{\times 2}$. In particular, in odd rank each similarity class contains a unique form with discriminant 1, which we will call "normalized"; this leads us to consider the specific ternary form $q_B = \langle -a,\ -b, ab\rangle$. Moreover, to a quadratic form q of any rank we can associate its **Witt invariant** $c(q)$, which is a quaternion algebra over k. This is almost but not quite the **Hasse invariant**

$$s(\langle a_1, \ldots, a_n\rangle) = \sum_{i<j} (a_i, a_j) \in Br(k)$$

but rather a small variation, given e.g. by the following *ad hoc* modifications:[3]

$$\begin{aligned}
c(q) &= s(q), && \operatorname{rank}(q) \equiv 1,\ 2 \pmod 8,\\
c(q) &= s(q) + (-1, -d(q)), && \operatorname{rank}(q) \equiv 3,\ 4 \pmod 8,\\
c(q) &= s(q) + (-1, -1), && \operatorname{rank}(q) \equiv 5,\ 6 \pmod 8,\\
c(q) &= s(q) + (-1, d(q)), && \operatorname{rank}(q) \equiv 7,\ 8 \pmod 8.
\end{aligned}$$

For our purposes, the principal merit of $c(q)$ over $s(q)$ is that $c(q_B) = [B]$, the class of B in the Brauer group of k. In particular, c is a similarity invariant of three-dimensional forms.

As a consequence of our identification of genus zero curves with quaternion algebras, we conclude that over any field k, ternary quadratic forms up to similarity are classified by their Witt invariant, and ternary forms up to isomorphism are classified by their Witt invariant and their discriminant, cf. [18, Theorem 13.5].

Pfister forms: For $a_1, \ldots, a_n$, we define the **n-fold Pfister form**

$$\langle\langle a_1, \ldots, a_n\rangle\rangle = \bigotimes_{i=1}^n \langle 1,\ a_i\rangle = \perp \langle a_{i_1} \cdots a_{i_k}\rangle,$$

where the orthogonal sum extends over all 2^n subsets of $\{1, \ldots, n\}$. Notice that the full norm form on B is $\langle 1,\ -a\rangle \otimes \langle 1,\ -b\rangle$, a 2-fold Pfister form. This is good news, since the properties of Pfister forms are far better understood than those of arbitrary quadratic forms. As an important instance of this, a Pfister form is isotropic if and only if it is hyperbolic [18, Lemma 10.4]. As n increases, Pfister forms become increasingly sparse among all rank 2^n quadratic forms (and, obviously, among all quadratic forms), but observe that a quaternary quadratic form is similar to a Pfister form if and only if it has discriminant 1.

[3] Or more canonically by the theory of Clifford algebras; see [12, Ch. 5].

Quadric hypersurfaces: Finally, we need to link up the algebraic theory of quadratic forms with the geometric theory of quadric hypersurfaces, our second higher-dimensional analogue of the genus zero curves.

Let $q(x_1, \ldots, x_n) = a_0 x_1^2 + \ldots + a_n x_n^2$ be a nondegenerate quadratic form of rank $n \geq 3$. Let V_q be the corresponding hypersurface in $\mathbb{P}^n$ given by $q = 0$. V_q is geometrically irreducible and geometrically *rational.* More precisely, $k(V_q)$ is a k-rational function field if and only if q is isotropic: the "only if" is obvious, and the converse goes as above: if we have a single point $p \in V_q(k)$, then we can consider the family of lines in $\mathbb{P}^{n-1}$ passing through p; the generic line meets V_q transversely in two points, giving a birational map from $\mathbb{P}^{n-2}$ to V. However, if $n \geq 4$ then this need not be true for every line, i.e., V_q need not be isomorphic to $\mathbb{P}^{n-2}$.

Every isotropic quaternary quadratic form q can be written as $H \perp g$, where $H = \langle 1, -1 \rangle$ is the hyperbolic plane and g is an arbitrary binary quadratic form; by Witt cancellation, the equivalence classes of g parameterize the isotropic quaternary quadratic forms up to equivalence. Since for all $c \in k^\times$, $cH \cong H$, every isotropic quaternary form q is similar to $H \perp \langle 1, -d(q) \rangle$, and we conclude that isotropic quadric surfaces are classified up to isomorphism by their discriminant. The unique hyperbolic representative (with discriminant 1) is given by the equation $x_0^2 - x_1^2 + x_2^2 - x_3^2 = 0$, and on this quadric we find the lines $L_1 : [a : -a : b : -b]$ and $L_2 : [a : -b : -a : b]$ with intersection the single point $[a : -a : a : -a]$: we've shown that a hyperbolic quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 13. *Let q, q' be two quadratic forms over k. Then q is similar to q' if and only if $V_q \cong V_{q'}$.*

Proof: As above, it is clear that similar forms give rise to isomorphic quadrics. In rank 3 we saw that the Witt invariant, which gives the isomorphism class of the conic, classifies the quadratic form up to similarity. Since a quadric surface V is a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$, the class of the canonical bundle in $\mathrm{Pic}(V)$ is represented by $K_V = -2(e_1 + e_2)$, whereas the hyperplane class of $V \subset \mathbb{P}^3$ is represented by $e_1 + e_2$. If $\varphi : V_1 \cong V_2$ is an isomorphism of quadric surfaces, it must pull K_{V_2} back to K_{V_1}, which, since the Picard groups are torsionfree, implies that $e_1 + e_2$ on V_2 pulls back to $e_1 + e_2$ on V_1. That is, any isomorphism of quadrics extends to an automorphism of $\mathbb{P}^3$. Since $\mathrm{Aut}(\mathbb{P}^3) = PGL_4$, this gives a similitude on the corresponding spaces. In rank at least 5, the Picard group of V_q is infinite cyclic, generated by the canonical class K_V. Moreover $-K_V$ is very ample and embeds V into $\mathbb{P}^{n+1}$ as a quadric hypersurface, so again any isomorphism of quadrics extends to an automorphism of the ambient projective space.

If q is a rank n quadratic form, we denote by $k(q)$ the function field $k(V_q)$ of the associated quadric hypersurface.

If q/k is a quadratic form, we say a field extension l/k is a **field of isotropy** for q if q/l is isotropic, or equivalently if $l(q)$ is a rational function field.

On the other hand, we say l/k is a **splitting field** for q if q/l is hyperbolic, i.e., if q lies in the ideal $W(l/k)$ of $W(k)$ which is the kernel of the natural restriction map $W(k) \rightarrow W(l)$.

The analogy with Severi-Brauer varieties and the Brauer group is irresistible, but things are more subtle here. Of course the function field $k(q)$ is a field of isotropy for q: every variety has (generic) rational points over its function field. On the other hand it is not guaranteed that q becomes hyperbolic over $k(q)$. Indeed, this is obviously impossible unless q has even rank $n = 2m$, and then unless $d(q) = d(\mathbb{H}^m) = (-1)^m$ – since k is algebraically closed in $k(q)$, $d(q)/(-1)^m$ does not become a square in $k(q)$ unless it is already a square in k. On the other hand, if q is (similar to) a Pfister form, then isotropy implies hyperbolicity. So for quaternary quadratic forms, we've shown part a) of the following result, the analogue of Amitsur's theorem (Theorem 12) in the Witt ring.

Theorem 14. *(Cassels-Pfister)*
a) An anisotropic form q is similar to a Pfister form if and only if $q \in W(k(q)/k)$.
b) If q is similar to a Pfister form and q' is an anisotropic form, then $q' \in W(k(q)/k)$ if and only if $q' \cong g \otimes q$ for some quadratic form g. In particular, $W(k(q)/k)$ is the principal ideal of $W(k)$ generated by q.
c) Let q' be any quadratic form and q an anisotropic quadratic form. If $q \in W(k(q')/k)$, then q is similar to a subform of q' (We say that f is a subform of g if there exists h such that $g = f \perp h$).

For the proof, see e.g. [18, Theorem 4.5.4].

An immediate consequence is that if q_1 and q_2 are two anisotropic Pfister forms of equal rank such that $k(q_1)$ is a field of isotropy for q_2, then q_1 and q_2 are similar. Applying this to the normalized norm form of a genus zero curve, we get our second proof of Theorem 11.

We end this section by collecting a few more results that will be useful for the proof of Theorem 7b).

Theorem 15. *Let q, q' be quaternary quadratic forms over k with common discriminant d, and put $l = k(\sqrt{d})$.*
a) [18, 2.14.2] *The form q is anisotropic if and only if q_l is anistropic.*
b) (Wadsworth [20]*) If q'/l is similar to q/l, then q is similar to q'.*
c) (Wadsworth [20]*) If q is anistropic and $k(q) \cong k(q')$, then q is similar to q'.*

4.2. An algebraic proof of Ohm's theorem. We begin the proof of Theorem 7b) by explaining how the results we have recalled on quadratic forms can be used to deduce the theorem of Ohm on the isogeny classification of quadric surfaces. Indeed, thanks to the remarkable Theorem 15c), the classification result is more precise than we have let on.

Theorem 16. *(Ohm,* [15]*) Let q, q' be two nondegenerate quaternary quadratic forms over k with isogenous function fields. Then either:*
a) q and q' are both isotropic, so $k(q) \cong k(q') \cong k(t_1, t_2)$, or
b) q and q' are both anisotropic in which case $V_q \cong V_{q'}$, i.e., q and q' are similar*.*

That is, except in the case when both function fields are rational, quadric surfaces with isogenous function fields are not only birational but *isomorphic.*

Proof: Since isotropic quadric function fields are rational and the condition of being isotropic (i.e., of having a k-rational point) is an isogeny invariant, we need only consider the case when both q and q' are anisotropic quaternary quadratic forms. The proof divides into further cases according to the values of the discriminants $d = d(q)$, $d' = d(q')$.

The first case is $d = d' = 1$ (as elements of $k^\times/k^{\times 2}$). In this case q and q' are both similar to Pfister forms. If they are isogenous over k, *a fortiori* they are isogenous over $k(q')$, and since q' becomes isotropic over $k(q')$, so does q. Since q is similar to a Pfister form, this implies $q \in W(k(q')/k)$, and by Theorem 14c) we conclude that q and q' are similar.

Suppose $d = d' \neq 1$. Let $l = k(\sqrt{d})$. By Theorem 15a), q/l and q'/l remain anisotropic. Moreover they are now similar to Pfister forms, so the previous case applies to show that q/l and q'/l are similar. But now Theorem 15b) tells us that q and q' are already similar over k!

The last case is $d \neq d'$. Since the discriminant is a similarity invariant among quaternary quadratic forms, we must show that this case cannot occur, i.e., that two anisotropic quadratic forms with distinct discriminants cannot be isogenous. Let $l = k(\sqrt{d})$; it suffices to show that q/l and q'/l are nonisogenous. Again, Theorem 15a) implies that q/l remains anisotropic, whereas we may assume that q'/l is anistropic, for otherwise they could not be isogenous. We finish as in the first case: by construction q/l is (similar to) an anistropic Pfister form, so $q/l \in W(l(q')/l)$ and the Cassels-Pfister theorem implies that q/l and q'/l are similar, but their discriminants are different, a contradiction.

5. Geometry and Galois cohomology of quadric surfaces

Our strategy for proving Theorem 7b) in full is in fact to make the proof of §4.2 geometric: that is, we will use Brauer kernels to give proofs of Theorems 14 and 15 in the case of quaternary quadratic forms. The fact that two-dimensional quadric function fields are classified by their Brauer kernels over k and over all quadratic extensions of k will come as a byproduct of these proofs.

A convention: since we are working now with quadric surfaces, the associated quadratic forms are well-defined only up to similarity. Thus we will call a quadric surface "Pfister" if it can be represented by a Pfister form, and we call a quadratic form "Pfister" if it is similar to a Pfister form.

5.1. Preliminaries on twisted forms. The first step is to consider not just the quadric surfaces over k, but the larger set of all twisted forms of the hyperbolic surface $\mathbb{P}^1 \times \mathbb{P}^1$.

So let $\mathcal{T} = \mathcal{T}(\mathbb{P}^1 \times \mathbb{P}^1)$ be the set of all Galois twisted forms of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the set of all varieties X/k such that $X/\overline{k} \cong \mathbb{P}^1 \times \mathbb{P}^1$. We saw in the previous section that every quadric surface V_q is an element of $\mathcal{T}$. (More precisely, every quadric surface becomes isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ after an extension with Galois group 1, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and no anistropic quadric surface with nontrivial discriminant splits over a quadratic extension.)

By Galois descent, $\mathcal{T} = H^1(k, G)$, where G is the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$. G is a semidirect (or "wreath") product:

$$1 \rightarrow PGL_2^2 \rightarrow G \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1,$$

where the PGL_2^2 gives automorphisms of each factor separately, and a splitting of the sequence is given by the involution of the two $\mathbb{P}^1$ factors. Thus we have a split exact sequence of pointed sets

$$1 \rightarrow QA(k)^2 \rightarrow \mathcal{T} \xrightarrow{d} k^\times/k^{\times 2},$$

where $QA(k)$ stands for the set of all quaternion algebras over k. As we shall see shortly, this map d gives a generalization of the discriminant of a quadratic form to all twisted forms of $\mathbb{P}^1 \times \mathbb{P}^1$. The splitting just means that we have an injection $k^\times/k^{\times 2} \hookrightarrow \mathcal{T}$: we choose the embedding corresponding to the subset of all isotropic quadric surfaces (we have seen that these are parameterized by their discriminant).

The part of $\mathcal{T}$ in the kernel of d is easy to understand: we just take two different twisted forms $C_1,\ C_2$ of $\mathbb{P}^1$ – i.e., two genus zero curves over k – and put $X = C_1 \times C_2$. Using Witt's theorem, we can identify the Brauer kernel of such a surface: $\kappa(k(C_1 \times C_2)) = \langle B_{C_1},\ B_{C_2} \rangle$.

For any twisted form X, let $N = \mathbf{Pic}(X)(\overline{k})$ be the Picard group of $X/\overline{k}$ viewed as a G_k-module. As abelian group, N is isomorphic to $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = e_1^{\mathbb{Z}} \oplus e_2^{\mathbb{Z}}$, where e_1 and e_2 represent the two rulings. Write $N(k) := N^{G_k}$ for the G_k-equivariant line bundles on $X/\overline{k}$, so $N(k)$ is a free abelian group of rank at most 2. The rank is at least one, since $e_1 + e_2 \in N(k)$: the only two elements of the Néron-Severi lattice with self-intersection 2 are $\pm(e_1 + e_2)$, and $e_1 + e_2$ is distinguished from $-(e_1 + e_2)$ by being ample; both of these properties are preserved by the G_k-action. Moreover, since N is torsion free, for any $L \in N(\overline{k})$ and any $n \in \mathbb{Z}^+$, $L \in N(k) \iff nL \in N(k)$. In particular, the rank of $N(k)$ is 2 if and only if $N(\overline{k})$ is a trivial G_k-module.

Claim: $N(k)$ has rank 2 if and only if $d(X) = 1$.

Proof: If $d(X) = 1$, $X = C_1 \times C_2$, and choosing any point $p_2 \in C_2(\overline{k})$, for any $\sigma \in G_k$, $\sigma(C_1 \times p_2) = C_1 \times \sigma(p_2)$, so that the Galois action preserves the horizontal ruling; the same goes for the vertical ruling. The converse is similar: to say that $\sigma \in G_k$ acts trivially on the class of $[e_1]$ and $[e_2]$ is to say that it does not interchange the rulings, hence lies in the subgroup PGL_2^2 of G.

Look now at the rank one case, where $N(k)$ is an infinite cyclic group with generator $e_1 + e_2$. Recall from §1.3 the basic exact sequence (1) and especially the obstruction map $\alpha : \mathbf{Pic}(V/k)(k) \rightarrow \mathrm{Br}(k)$. From the exactness of this sequence it follows that the Brauer kernel of X is precisely the obstruction to $e_1 + e_2$ coming from a line bundle. Since $-2(e_1 + e_2)$ is represented by the canonical bundle, we get that for all $X \in \mathcal{T}$, $\kappa(X) \subset Br(k)[2]$.

Claim: $\alpha(e_1 + e_2) = 0$ if and only if X is a quadric surface.

Proof: On $\mathbb{P}^1 \times \mathbb{P}^1$, $H = e_1 + e_2$ is very ample and gives the embedding into $\mathbb{P}^3$ as a degree 2 hypersurface. It follows that as soon as the class of $[e_1 + e_2]$ is represented by a k-rational divisor, the same holds k-rationally, i.e., X is embedded in $\mathbb{P}^3$ as a degree 2 hypersurface. For the converse, just cut the quadric by a hyperplane to get a rational divisor in the class of $e_1 + e_2$.

Proposition 17. *Let X/k be a quadric surface. If $d(X) \neq 1$, then the Brauer kernel is trivial. If $d(X) = 1$, then $X \cong C \times C$ and is classified up to isomorphism by its Brauer kernel $\kappa(X) = \{1, B_C\}$.*

Proof: We just need to remark that when $d(X) = 1$, $X = C_1 \times C_2$, and since $\alpha(e_1) = B_{C_1}$, $\alpha(e_2) = B_{C_2}$ are 2-torsion elements of $Br(k)$, the fact that $\alpha(e_1 + e_2) = 0$ implies $\alpha(e_1) = \alpha(e_2)$, so that $C_1 \cong C_2$.

Claim: For a quadric surface $X \in \mathcal{T}$, the cohomologically defined quantity $d(X) \in k^\times/k^{\times 2}$ is just the discriminant $d(q_X)$ of any associated quadratic form.

Proof: It is enough to show that $d(X) = 1$ if and only if $d(q_X) = 1$, for then the general case follows by passage to $k(\sqrt{d(X)})$ (or to $k(\sqrt{d(q_X)})$). Now $d(X) = 1$ means $X \cong C \times C$. But this means that for any extension L/k such that $X(L) \neq \emptyset$, $X/L \cong \mathbb{P}^1 \times \mathbb{P}^1$. Taking $L = k(X)$ and applying Theorem 14a), we conclude that q_X is Pfister, i.e., has discriminant 1. Conversely, if q_X is Pfister, then $X/k(X) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and since k is algebraically closed in $k(X)$, this gives the triviality of the Galois action on $\{e_1, e_2\}$, hence the cohomological discriminant is 1.

This claim justifies our earlier remark that the cohomologically defined map $d : \mathcal{T} \to k^\times/k^{\times 2}$ is a generalization of the usual discriminant of a quaternary quadratic form.

5.2. The proof of Theorem 7b). First we give a geometric proof of Theorem 14 for quaternary quadratic forms: Since "Pfister quadrics" are just those isomorphic to $C \times C$, where C is a genus zero curve, we can turn our previous argument on its head and deduce part b) of the Cassels-Pfister theorem in rank 4 from Witt's theorem. (Recall from §4 that part a) is easy to show for quaternary quadratic forms.) Now let q' be a rank 4 quadratic form and q an anisotropic quadratic form, with associated quadric surfaces V and V', such that $q \in W(k(q')/k)$. But since k is algebraically closed in $k(q')$, this implies that $d(q) = 1$, so $V \cong C \times C$. If $d(q') = 1$ also, this reduces again to Theorem 11, so assume that $d(q') = d \neq 1$ and let $l = k(\sqrt{d})$. Consider the basic exact sequence

$$0 \to \mathrm{Pic}(V') \to \mathbf{Pic}(V')(k) \xrightarrow{\alpha} Br(k) \xrightarrow{\beta} Br(k(V')).$$

The hypothesis that q splits in $k(V')$ means that B_C is an element of the Brauer kernel of $k(V')$. But being a quadric surface with nontrivial discriminant, $\kappa(k(V')) = 0$, a contradiction.

Proof of Theorem 15a) for quaternary forms: let $q_1,\ q_2$ be quaternary forms with common discriminant d and corresponding quadrics $V_1,\ V_2$; put $l = k(\sqrt{d})$; and let σ be the nontrivial element of $\mathfrak{g}_{l/k}$.

First we must show that if V_1/l is isotropic, then V_1/k was isotropic. But if $X(l)$ is nonempty, then since the discriminant is 1 over l, then X splits over l. So we can choose rational curves C_1, C_2 over l such that $\sigma(C_1) = C_2$. But then $\sigma(C_1 \cap C_2) = \sigma(C_1) \cap \sigma(C_2) = C_2 \cap C_1 = C_1 \cap C_2$ gives a k-rational point.

We now give geometric proofs of Wadsworth's results, i.e., parts b) and c) of Theorem 15. The isotropic case of Theorem 15c) is easy, since isotropic quadric surfaces are classified by their discriminant. Since we know that two anisotropic Pfister quadrics are birational if and only if they are isomorphic, Theorem 15c) follows from Theorem 15b), and we are reduced to showing the following.

Proposition 18. *Let $V/k, W/k$ be two anisotropic quadric surfaces with common discriminant d; put $l = k(\sqrt{d})$. If $V_1/l \cong V_2/l$, then $V_1 \cong V_2$.*

Proof: We write σ for the nontrivial element of $G_{l/k}$. Let $\mathcal{S}$ be the set of all l/k twisted forms of V, and let $\mathcal{S}_d \subset \mathcal{S}$ be the subset of twisted forms W with $d(W) = d(V)$. We claim that $\mathcal{S}_d = \{V\}$, which gives the result we want. (In fact it is a stronger result, since we are *a priori* allowing twisted forms which are not quadric surfaces.)

To prove the claim we clearly may "replace" V by any element of $\mathcal{S}_d$. A convenient choice is the variety V_1/k constructed as follows: let B/k be the quaternion algebra whose Brauer class is $c(V)$, the Witt invariant of the quadric V, and let C/k denote the genus zero curve corresponding to B. Let $V_1 := \mathrm{Res}_{l/k}(C/l)$ be the k-variety obtained by viewing C as a curve over l and then taking the Weil restriction from l down to k.[4]

We have that $V_1/l \cong C \times C$. Let $G = \mathrm{Aut}(V_1)$. It is convenient (and correct!) to view G as an algebraic k-group scheme. In particular this gives the l-valued points $G(l)$ the structure of a $G_{l/k}$-module, and this Galois module structure is highly relevant, since $\mathcal{S} = H^1(l/k, G(l))$. Indeed we have a short exact sequence of k-group schemes

$$1 \to K \to G \to \mathbb{Z}/2\mathbb{Z} = \mathrm{Sym}\{e_1,\ e_2\} \to 1 \tag{2}$$

obtained by letting automorphisms of V_1 act on the $G_{l/k}$-set of rulings $\{e_1,\ e_2\}$; this exact sequence is of course a twisted analogue of the exact sequence considered in 5.1. In particular, we still have that K is the connected component of G, a linear algebraic group scheme; $K(l)$ is, as a group, isomorphic to $\mathrm{Aut}(C)(l)^2 = PGL(B)(l)^2$, where the group $PGL(B)$ is the twisted analogue of PGL_2 defined by the short exact sequence

$$1 \to \mathbb{G}_m \to B^\times \to PGL(B) \to 1.$$

However, the $G_{l/k}$-module structure on $K(l)$ is twisted: since σ interchanges e_1 and e_2, it also interchanges the two factors of $PGL(B)$. That is, $\sigma(x,y) = (\sigma(y), \sigma(x))$, so

$$K(k) = \{(x, \sigma(x)) \mid x \in PGL(B)(l)\} \cong PGL(B)(l),$$

and one finds that $K/k = \mathrm{Res}_{l/k}\,\mathrm{Aut}(C/l)$. But then Shapiro's Lemma implies

$$\#H^1(l/k, K(l)) = \#H^1(l/l, \mathrm{Aut}(C/l)) = 1.$$

Taking l-valued points and then $G_{l/k}$-invariants in (2), one gets an exact cohomology sequence, of which a piece is

$$H^1(l/k, K(l)) \to H^1(l/k, G(l)) \xrightarrow{d^*} H^1(l/k, \mathbb{Z}/2\mathbb{Z}) = \pm 1$$

where d^* is a twisted analogue of our cohomological discriminant map. The exactness means precisely that $H^1(l/k, K(l))$ is the subgroup of twisted forms X with $d^*(X) = 1$. Since our V_1 represents the basepoint of

[4] It is a byproduct of the proof that V_1 is a quadric surface. On the other hand, if we started with a genus zero curve C whose corresponding quaternion algebra was in $Br(l) \setminus Br(l)^{G_{l/k}}$, then the restriction of scalars construction would yield a twisted form V_1/k such that $V_1/l \cong C \times C^\sigma$ is not a quadric surface (even) over l.

$H^1(l/k, K(l))$, certainly $d^*(V_1) = 1$, so that forms with twisted discriminant 1 are precisely those whose discriminant in the former sense is equal to the discriminant of V_1. That is to say: $H^1(l/k, K(l)) = S_d$, so we are done.

End of the proof of Theorem 7b): Combining the results of this section with the argument of §4.2 we get a "geometric" proof of Ohm's Theorem 16. It remains to see that function fields of quadric surfaces $k(X)$ are classified by their Brauer kernels over k and over all quadratic extensions of k. Suppose $k(q)$ and $k(q')$ are *non*-isomorphic function fields of quadric surfaces. If one is isotropic and the other is anisotropic, then the isotropic one has trivial Brauer kernels over all extension fields of k, whereas the anisotropic one has a Brauer kernel of order two over $k(\sqrt{d})$. So suppose that both are anisotropic. If $d(q) \neq d(q')$, then over $k(\sqrt{d})$, q has nontrivial Brauer kernel and q' has trivial Brauer kernel. If their discriminants are the same, then by Proposition 18, $l(q)$ and $l(q')$ remain nonisomorphic, so have distinct nontrivial Brauer kernels. This shows the equivalence of the first three conditions in part b) of Theorem 7.

6. Comparing quadrics and Severi-Brauer varieties

6.1. The proof of Theorem 7c). For the proof of Theorem 7c), it suffices to show that for any $n > 1$, if $K_1 = k(V_1)$ is the function field of a nontrivial Severi-Brauer variety and $K_2 = k(V_2)$ is the function field of an anistropic quadric hypersurface, then $\kappa(k(V_1)) \neq \kappa(k(V_2))$.

But recall that the Picard group of a quadric hypersurface V_2/k in dimension at least 3 is *generated* by the canonical bundle (e.g. [7, Exercise II.6.5]), so the natural map $\mathrm{Pic}(V_2) \to \mathbf{Pic}(V_2)(k)$ is an isomorphism and $\kappa(k(V_2)) = 0$. On the other hand, a nontrivial Severi-Brauer variety has a nontrivial Brauer kernel, the cyclic subgroup generated by the corresponding Brauer group element.

When $n = 2$, the Brauer kernel of a nontrivial Severi-Brauer surface is cyclic of order 3, whereas the Brauer kernel of any quadric is 2-torsion.

6.2. Brauer kernels and the index. Earlier we mentioned the fact that if V has a k-rational point, $\kappa(k(V)) = 0$. This statement can be refined in terms of the **index** of a variety V/k, which is the least positive degree of a $\mathfrak{g}_k$-invariant zero-cycle on V; equivalently, it is the greatest common divisor over all degrees of finite field extensions l/k for which $V(l) \neq \emptyset$. Note then that the index is a (field-)isogeny invariant. Suppose l/k is a finite field extension of degree n such that $V(l) \neq \emptyset$. Then $\kappa(k(V)) = Br(k(V)/k) \subset Br(l/k)$. It follows that the index of V/k is an upper bound for the index of any element of the Brauer kernel of $k(V)$ (recall that the index of a Brauer group element is the square root of the k-vector space dimension of the corresponding division algebra D/k). In particular varieties with a k-rational zero-cycle of degree one have trivial Brauer kernel.

Notice that quadrics and Severi-Brauer varieties have a very special property among all varieties: namely the existence of a rational zero-cycle of degree one implies the existence of a rational point. For Severi-Brauer varieties, it is part of the basic theory of division algebras that the index of a division algebra is equal to the greatest common divisor over all degrees of splitting fields (and moreover the gcd is *attained*, by any maximal subfield of D/k). For quadrics – whose index is clearly at most 2 – this follows from Springer's theorem, that an anistropic quadratic form remains anisotropic over any finite field extension of odd degree.

To see how "special" this property is, observe that every variety over a finite field has index one, since the Weil bounds (it is enough to consider curves) imply that if $V/\mathbb{F}_q$ is a smooth projective variety, $V(\mathbb{F}_{q^n}) \neq \emptyset$ for all $n \gg 0$, and in particular there exists n such that $V/\mathbb{F}_q$ has rational zero cycles of coprime degrees n and $n+1$. This gives amusingly convoluted proofs of the familiar facts that the Brauer group of a finite field is trivial and that every quadratic form in at least three variables over a finite field is isotropic.

7. Curves of Genus One

In this section we suppose that all fields have characteristic zero.

7.1. Preliminaries on genus one curves. We shall begin by briefly recalling certain notions concerning genus one curves and their Jacobian elliptic curves. For a much more complete discussion of these matters, we highly recommend Cassels' survey article [4]. While hardly essential, we find it convenient to describe the Jacobian using the language of the Picard functor, a very careful treatment of which can be found in [2, Chapter 8]. For recent work on the relation between the period and the index of a genus one curve, the reader may consult [5] and the references therein.

Let $K = k(C)$ be the function field of a genus one curve. Recall that C can be given the structure of an elliptic curve if and only if $C(k) \neq \emptyset$. Moreover, if C is an arbitrary genus one curve, we can associate to it an elliptic curve, its **Jacobian** $J(C) = \mathbf{Pic}^0(C)$, the group scheme representing the subfunctor of $\mathbf{Pic}(C)$ consisting of divisor classes of degree zero. The Riemann-Roch theorem gives a canonical identification $C = \mathbf{Pic}^1(C)$; with this identification, C becomes a principal homogeneous space (or torsor) under $J(C)$. By Galois descent, the genus one curves C/k with Jacobian isomorphic to a given elliptic curve E are parameterized by the Galois cohomology group $H^1(k, E)$. There is a subtlety here: $H^1(k, E)$ parameterizes isomorphism classes of genus one curves endowed with the structure of a principal homogeneous space for E, so a genus one curve up to isomorphism corresponds to an orbit of $\mathrm{Aut}(E/k)$ on $H^1(k, E)$. We will assume that $\mathrm{Aut}(E/k) = \pm 1$ (this excludes only the notorious j-invariants 0 and 1728) – later we will exclude all elliptic curves with complex multiplication over the algebraic closure of k. So $[C]$ and $-[C]$ are in general distinct classes in $H^1(k, E)$ but represent isomorphic genus one curves.

If a genus one curve has a k-rational zero-cycle of degree one, then by Riemann-Roch it is an elliptic curve, i.e., index one implies the existence of rational points for genus one curves. Another important numerical invariant of a genus one curve C/k is its **period**, which is simply the order of $[C]$ in the torsion group $H^1(k, J(C))$. A useful alternative characterization is that the period is the least positive degree of a k-rational divisor *class* on C.

It turns out that the period of a genus one curve is also an isogeny invariant. At the request of the referee, we supply the proof. Let $f : C_1 \to C_2$ be a degree n morphism of curves (of any genus) over k, and let $[D] \in \mathbf{Pic}(C_1)(k)$ be a degree d rational divisor class on C_1. This means that $[D]$ is represented by a divisor D rational over $\overline{k}$, with the property that for all automorphisms $\sigma \in \mathfrak{g}_k$, $D^\sigma \sim D$ (i.e., the divisors are linearly equivalent). Let $D' = f_*(D)$, the pushforward of D to C_2, so D' is a degree d divisor on $C_2/\overline{k}$. From [7, Ex. IV.2.6], the pushforward of divisors respects linear equivalence, so descends to a map $f_* : \mathrm{Pic}(C_1/\overline{k}) \to \mathrm{Pic}(C_2/\overline{k})$. In particular, for any $\sigma \in \mathfrak{g}_k$, we have

$$(D')^\sigma = f_*(D)^\sigma = f_*(D^\sigma) \sim f_*(D) = D',$$

so D' represents a rational divisor class of degree d on C_2. Thus if C_1 and C_2 are isogenous genus one curves, the least degrees of k-rational divisor classes on C_1 and C_2 are equal, so their periods are equal.

Recall that an isogeny of elliptic curves (in the usual sense) is just a finite morphism of varieties $\varphi : (E_1, O_1) \to (E_2, O_2)$ preserving the distinguished points. But notice that if $f : E_1 \to E_2$ is any finite morphism of genus one curves with rational points, it can be viewed as an isogeny by taking $O_2 = f(O_1)$. Moreover, if $f : E_1 \to E_2$ is a finite morphism of elliptic curves, then there is an induced map $\mathrm{Pic}^0(f) = \mathrm{Pic}^0(E_2) \to \mathrm{Pic}^0(E_1)$. Since any elliptic curve is isomorphic to its Picard variety, this explains why our notion of an isogenous pair of elliptic function fields is consistent with the usual notion of isogenous elliptic curves: the morphism in the other direction is guaranteed.

But if $\varphi : C_1 \to C_2$ is a morphism of genus one curves without rational points, then since C_2 is not isomorphic to $\mathrm{Pic}^0(C_2)$, the existence of a finite map $\varphi' : C_2 \to C_1$ is not guaranteed. Indeed, it need not exist: let C be a genus one curve of period $n > 1$. Then the natural map $[n] : C = \mathbf{Pic}^1(C) \to \mathbf{Pic}^n(C) \cong J(C)$ gives a morphism of degree n^2 from C to its

Jacobian. Since $J(C)(k) \neq \emptyset$, there is no map in the other direction. So the classification of genus one curves up to isogeny is more delicate than the analogous classification of elliptic curves. We content ourselves here with the following result.

Proposition 19. *Let C, C'/k be two genus one curves with common Jacobian E, and assume that E does not have complex multiplication over $\overline{k}$. Then there exists a degree n^2 étale cover $C \to C'$ if and only if $[C'] = \pm n[C]$ as elements of the Weil-Chatelet group $H^1(k, E)$.*

Proof: As we saw above, there is a natural map

$$\psi_n : C = \mathrm{Pic}^1(C) \to \mathrm{Pic}^n(C)$$

induced by the map $D \mapsto nD$ on divisors. Upon base change to the algebraic closure and up to an isomorphism, this map can be identified with $[n]$ on $J(C)$, so it is an étale cover of degree n^2. Keeping in mind that n could be negative, corresponding to a twist of principal homogeneous structure by $[-1]$, we get the first half of the result.

For the converse, let $\pi : C \to C'$ be any finite étale cover. Choosing $P \in C(\overline{k})$ and its image $P' = \pi(P) \in C'(\overline{k})$, $\pi/\overline{k} : C/\overline{k} \to C/\overline{k}$ is an elliptic curve endomorphism. By assumption on E, $\pi/\overline{k} = [n]$ for some integer n, and its kernel $E[n]$ is the unique subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. It follows that the map $\pi : C \to C'$ factors as $C \to C/E[n] \to C'$, hence $C/E[n] \to C'$ is an isomorphism of varieties. It is not necessarily a morphism of principal homogeneous spaces: it will be precisely when $n > 0$ in $\pi/\overline{k}$ above. Taking into account again the possibility of $n < 0$ gives the stated result.

Corollary 20. *Let C/k be a genus one curve with non-CM Jacobian E/k. The number of genus one curves C'/k with $J(C) \cong J(C')$ which are k-isogenous to C is $N_C := \#(\mathbb{Z}/n\mathbb{Z})^\times/(\pm 1)$, where $C \in H^1(k, E)$ has exact order n. In particular $N_C = 1$ if and only if n one of:* 1, 2, 3, 4, 6*; and $N_C \to \infty$ with n.*

Proof: For any positive integer a and any element $C \in H^1(k, E)$, multiplication by a on the divisor group descends to give a map $C = \mathrm{Pic}^1(C) \to \mathrm{Pic}^a(C) = C'$, which is a twisted form of $[a]$ on the Jacobian elliptic curve. The class C' (together with its evident structure as a principal homogeneous space for $\mathrm{Pic}^0(C)$) represents the element $a[C] \in H^1(k, E)$. Moreover, if C has period n, then via a choice of rational divisor class of degree n we get an identification of $\mathrm{Pic}^n(C)$ with $\mathrm{Pic}^0(C)$. It follows that for any a prime to n, $C' = \mathrm{Pic}^a(C)$ is an isogenous genus one curve with the same Jacobian as C, and after quotienting out by $\langle \pm 1 \rangle$ to address the two possible principal homogeneous space structures on a given genus one curve, we get $\frac{\varphi(n)}{2}$ mutually isogenous, pairwise nonisomorphic genus one curves with

Jacobian isomorphic to E. That there are no more than this many curves in the isogeny class follows immediately from the previous proposition, so the proof is complete.

This result should be compared with Amitsur's theorem (Theorem 12): it is not true that two genus one curves, even with common Jacobian, which have the same splitting fields must be birational.

Proposition 21. *The field-isogeny class of a one-dimensional function field with respect to a number field is finite.*

Proof: When the genus is different from one, we have seen that field-isogeny implies isomorphism, so it remains to look at the case of K a genus one function field with respect to a number field k. Fix some k-structure on K (there are, of course, only finitely ways to do this). For the sake of clarity, let us first show that there are only finitely many function fields K'/k which are isogenous to K as k-algebras. It will then be easy to see that the proof actually gives finiteness of the field-isogeny class.

Let C/k be the genus one curve such that $K = k(C)$; let K'/k be a function field such that there exists a homomorphism $\iota : K' \to K$, which on the geometric side corresponds to a finite morphism
$\varphi : C \to C'$, where C'/k is another genus one curve with $K' = k(C')$.

By passing to the (covariant, i.e., Albanese) Jacobian of φ/k we get an isogeny of elliptic curves $J(C) \to J(C')$. By Shafarevich's theorem, the isogeny class of an elliptic curve over a number field is finite, so it is enough to bound the number of function fields $k(C')$ with a given Jacobian, say E'. Let n be the common period of C and C', so that $C' \in H^1(k, E')[n]$. Let S be the set places of k containing the infinite places and all finite places v such that $C(k_v) = \emptyset$; note that this is a finite set. But the existence of φ means that $C'(k_v) \neq \emptyset$ for all v outside of S, so that

$$[C'] \in \ker(H^1(k, E)[n] \to \prod_{v \notin S} H^1(k_v, E)[n]).$$

But the finiteness of this kernel is extremely well-known, using e.g. Hermite's discriminant bounds. (Indeed, this is the key step in the proof of the weak Mordell-Weil theorem; see e.g. [19].)

Notice that we actually showed the following: a given genus one function field K/k dominates only finitely many other genus one function fields. (In fact K dominates only finitely many function fields in all, the genus zero case being taken care of by the finiteness of the Brauer kernel $\kappa(C)$.) It follows that there are only finitely many isomorphism classes of fields L which admit a field isomorphic to K as a finite extension; this completes the proof.

7.2. The proof of Theorem 10. Let k be a number field and C_1/k a genus one curve of period 1, 2, 3, 4, or 6 whose Jacobian $J(C_1)$ has no complex multiplication over $\overline{k}$ and is isolated in its isogeny class. Let $K_1 = k(C_1)$ and K_2 be any finitely generated field such that $K_1 \equiv K_2$. By Theorem 3, K_1 and K_2 are isogenous as fields, so K_2 is isomorphic as a field to $k(C_2)$ where C_2/k is another genus one curve. By modifying if necessary the k-structure on C_2, we get a finite morphism $\varphi : C_1 \to C_2$ of k-schemes; passing to $J(\varphi)$ we deduce that $J(C_1) \sim J(C_2)$ and hence by hypothesis that $E = J(C_1) \cong J(C_2)$. By Proposition 19, $[C_2] = a[C_1]$ for some integer a. Applying the same argument with the roles of C_1 and C_2 interchanged, we get that $[C_1]$ and $[C_2]$ generate the same cyclic subgroup of $H^1(k, E)$, and by the hypothesis on the period of C_1 we conclude $C_1 \cong C_2$.

We remark that with hypotheses as above but C of arbitrary period n, we find that $k(C)$ could be elementarily equivalent only to one of $\#(\mathbb{Z}/n\mathbb{Z})^\times/(\pm 1)$ nonisomorphic function fields, but distinguishing between these isogenous genus one curves with common Jacobian seems quite difficult.

Finally, we must show that the assumption that $J(C)$ is isolated can be removed at the cost of assuming the finiteness of the Mordell-Weil group $J(C)(k)$. This is handled by the following result, which is a modification of the (clever, and somewhat tricky) argument of Pierce [16] to our arithmetic situation.

Proposition 22. *Let $K_1 = k(C_1)$ be the function field of a genus one curve over a number field. Assume that $J(C_1)$ does not have complex multiplication (even) over the algebraic closure of k, and that $J(C_1)(k)$ is finite. Let $K_2 \equiv K_1$ be any elementarily equivalent function field. Then $K_2 = k(C_2)$ is the function field of a genus one curve C_2 such that $J(C_1) \cong J(C_2)$.*

Proof: Let K_2 be a finitely generated function field such that $K_1 \equiv K_2$. By Theorem 3, we know that K_2 is field-isogenous to K_1. As above, this implies the existence of k-structures on K_1 and K_2 such that $K_1 = k(C_1)$, $K_2 = k(C_2)$ and $\iota/k : C_1 \to C_2$ is a finite morphism.

Step 1: In search of a contradiction, we assume that the greatest common divisor of the degrees of all finite maps $J(C_1) \to J(C_2)$ is divisible by some prime number p.

Step 2: We claim that the assumption of Step 1 and the finiteness of $J(C_2)(k)$ imply that there exists a finite set of étale maps $\{\Psi_i : C_i \to C_2\}$, each of degree p, such that for every finite morphism $\alpha : C_1 \to C_2$, there exists, for some i, a morphism $\lambda_i : C_1 \to C_i$ such that $\alpha = \Psi_i \circ \lambda_i$.

Proof of the claim: The induced map on the Jacobian $J(\alpha) : J(C_1) \to J(C_2)$ is of the form $J(C_1) \to J(C_1)/G$ for some finite subgroup scheme $G \subset_k J(C_1)$, of order $N = \deg(\alpha)$. Write $N = Mp^a$ with $(M,p) = 1$. Then $G[p^a](\overline{k})$ is of the form $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$ for $b < a$, since otherwise $J(\alpha)$ would factor through $[p^a]$ and yield a degree M isogeny $J(C_1) \to J(C_2)$, which would contradict Step 1. Put $H := G[Mp^{a-1}]$, so $H \subset_k G \subset_k J(C_1)$, and $\#G/H(\overline{k}) = p$. Then $J(\alpha)$ factors as

$$J(C_1) \to J(C_1)/H \xrightarrow{\Psi} J(C_1)/G.$$

Note that Ψ has degree p. Via dualization, the possible maps Ψ are in bijection with k-rational degree p isogenies with source $J(C_1)/G$, of which there are at most $p+1$. This establishes the claim for $J(\alpha)$ in place of α.

Next, since $J(C_1)$ acts on C_1 by automorphisms, it makes sense to take the quotients C_1/G and C_1/H. In fact any finite map $\alpha : C_1 \to C_2$ is "of the form" $q : C_1 \to C_1/G$, in the sense that there exists an isomorphism $\phi : C_1/G \to C_2$ as principal homogeneous spaces of $J(C_2) = J(C_1)/G$ such that $\alpha = \phi \circ q$. However, α is not determined by G, because after quotienting out by G one can perform a translation by any element of $J(C_2)(k)$. More precisely, the collection of finite maps whose source is a given genus one curve C_1 and whose "kernel" in the above sense is a given finite subgroup $G \subset_k J(C_1)$ is a principal homogeneous space for $J(C_2) = J(C_1)/G$. Since we have assumed that $J(C_1)$ has finitely many k-rational points, so then does the isogenous elliptic curve $J(C_2)(k)$, and in fact we can take for our list of Ψ_i's the at most $(p+1) \cdot \#J(C_2)(k)$ maps of the form $\tau \circ q$, where $q : C_1/H \to C_1/G$ with G and H as above and $\tau : C_2 \to C_2, Q \to Q + P$ for $P \in J(C_2)(k)$. This establishes the claim and completes Step 2.

Step 3: We choose a smooth affine model for C_2/k and let $\overline{x} = (x_1, \ldots, x_n)$ denote coordinates. The statement "$\overline{x} \in C_2$" can be viewed as first-order: let (P_j) be a finite set of generators for the ideal of C_2 in $k[\overline{x}]$; then $\overline{x} \in C_2$ is an abbreviation for "$\forall j \ P_j(\overline{x}) = 0$". For each i, choose $b_i \in k(C_i)$ such that $k(C_i) = \Psi_i^*(k(C_2))(b_i)$, and let $g_i(X, Y) \in k[\overline{X}, Y]$ be the minimal polynomial for b_i over $\Psi_i^*(k(C_2))$. Finally, we define a predicate $\overline{x} \in C_2 \wedge \neg \ \mathrm{Con}(\overline{x})$ with the meaning that $\overline{x}$ lies on C_2 and some coordinate is not in k. We must stress that this is to be regarded as a single symbol – we do not know how to define the constants in a function field over a number field, but since C_2 by assumption has only finitely many k-rational points, we can name them explicitly. Consider the sentence:

$$\forall \ \overline{x} \ \exists \ y \left(\overline{x} \in C_2 \wedge \neg \ \mathrm{Con}(\overline{x})) \implies \bigvee_i g_i(\overline{x}, y) = 0 \right)$$

Note well that $k(C_2)$ does not satisfy this sentence: take $\overline{x}$ to be any generic point of C_2. But $k(C_1)$ does: giving such an element $\overline{x} \in k(C_1)$ is equivalent to giving a field embedding $\iota : k(C_2) \to k(C_1)$, i.e. to a finite map $\iota : C_1 \to C_2$. So $\iota = \Psi_i \circ \lambda_i$ for some i, and we can take $y = \lambda_i^* b_i$:

$$g(\overline{x}, y) = g(\iota^* \overline{a}, \lambda^* b) = \lambda_i^* g(\Psi_i^* \overline{a}, b) = 0,$$

with $\overline{x} = \iota^*(\overline{a})$, $\overline{a}$ a generic point of C_2. So our sentence exhibits the elementary inequivalence of $k(C_1)$ and $k(C_2)$, a contradiction.

Step 4: Therefore the assumption of Step 1 is false, and it follows that there exist two isogenies between the non-CM elliptic curves $C_1/\overline{k}$, $C_2/\overline{k}$ of coprime degree, and (as in [16, Prop. 9]) this easily implies that $j(C_1) = j(C_2)$. In particular, the Jacobians $J(C_1)$ and $J(C_2)$ are isogenous elliptic curves with the same j-invariant and without complex multiplication. This implies that $J(C_1)$ and $J(C_2)$ are isomorphic over k: indeed, let $\iota : J(C_1) \to J(C_2)$ be any isogeny. Then, $\iota/\overline{\mathbb{Q}}$ must have Galois group $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ for some n, so that $J(C_2) = J(C_1)/ker(\iota) = J(C_1)/J(C_1)[n] \cong J(C_1)$.

The end of the proof is the same as in the first case of the theorem: since the period is $1,\ 2,\ 3,\ 4$, or 6, we may conclude $C_1 \cong C_2$.

8. Final Remarks

The purpose of this short final section (added in revision) is to collect some further remarks on the isogeny classification of function fields, and also to give an extension of Theorem 7b).

I. The "isogeny invariants" of Proposition 6 are sufficient to classify quadric hypersurfaces of any dimension up to isogeny. Indeed, if Q and Q' are quadric hypersurfaces defined over a field k, then $k(Q)$ can be embedded in $k(Q')$ if and only if $\dim(Q) \leq \dim(Q')$ and $k(Q')$ is a field of isotropy for Q [15, Thm. 1.1]. It is further known that quadric hypersurfaces of the same dimension are isogenous exactly when they are stably birationally equivalent.

II. The implication isogeny implies birational equivalence for quadrics is also known in dimensions 3 and 4 [8], but not in any higher dimension. For our logical applications, we are interested only in the number field case. The case when k is totally imaginary is trivial, since any form in at least 5 variables is isotropic. But taking, say, $k = \mathbb{Q}$, even the birational classification of quadrics seems to be quite subtle. E.g., it can be shown that there is a *unique* anistropic quadric function field in dimension 3, but not in dimensions 5 and 6.

III. The methods of Section 5 can be generalized to show that isogeny implies birationality for all twisted forms of $\mathbb{P}^1 \times \mathbb{P}^1$:

Theorem 23. *Let $X, X' \in \mathcal{T}$ be two $\overline{k}/k$-twisted forms of $\mathbb{P}^1 \times \mathbb{P}^1$. The following are equivalent:*
(i) For every finite extension l of k, the Brauer kernels of X/l and X'/l are equal.
(ii) $k(X)$ and $k(X')$ are isogenous.
(iii) $k(X)$ and $k(X')$ are isomorphic.

Proof: Clearly $(iii) \implies (ii) \implies (i)$, so we must show that $(i) \implies (iii)$.

Case 1: $d = d' = 1$, so $X \cong C_1 \times C_2$, $X' \cong C_3 \times C_4$, where for $1 \leq i \leq 4$, C_i is a genus zero curve. Setting the Brauer kernels equal gives $\kappa = \langle [C_1], [C_2] \rangle = \langle [C_3], [C_4] \rangle$. Note that $\kappa = 0$ if and only if X and X' are trivial. If $\#\kappa = 2$, then what we must show is that $C \times C$ is birational to $C \times \mathbb{P}^1$ (even though they are not isomorphic). However, they may both be viewed as genus one curves over $K = k(C)$, with corresponding Brauer classes $[C]|_K$ and 0; but since $[C]|_K = 0$, these are isomorphic as curves over K, so in particular they are birational.

Similar reasoning applies to the case where $X = C_1 \times C_2$, $X' = C_3 \times C_4$ and $\kappa(X)$ and $\kappa(X')$ are both isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Then, if $[C_1] + [C_2]$ happens to be isomorphic to the class $[C_0]$ of another genus zero curve C_0/k, equating the Brauer kernels (and relabelling if necessary) we get that either $[C_1] = [C_3]$, $[C_2] = [C_4]$ – in which case $X \cong X'$ – or $[C_3] = [C_1]$, $[C_4] = [C_0]$, so $X' \cong C_1 \times C_0$. But we can show as above that $C_1 \times C_0$ is birational to $C_1 \times C_2$. Indeed, both are isomorphic as genus zero curves over $K = k(C_1)$, since $[C_2] = [C_0]$ in $\mathrm{Br}(K)$.

Case 2: $d = d' \neq 1$; put $l = k(\sqrt{d})$. Equating Brauer kernels over k, we find that X and X' are either both quadric surfaces or neither one is; in view of what has already been proved, we may assume the latter, so their common Brauer kernel is $\kappa = \alpha(e_1 + e_2) = [C] \neq 0$. Since by assumption we also have equal Brauer kernels over l, from Case 1 we have $l(X) \cong l(X')$. Note that X/l and X'/l remain anisotropic (our argument for this in §5.2 did not use the fact that X was a quadric surface). Note that X/l is a quadric surface if and only if $[C]|_l = 0$ if and only if X'/l is a quadric surface.

Suppose first that they are both quadric surfaces. We have seen that anisotropic quadric surfaces with isomorphic function fields are in fact isomorphic. In other words, X and X' are l/k twisted forms of each other with the same discriminant, so the proof of Proposition 18 shows that they are already isomorphic over k.

If X/l and X'/l are not quadrics, we can write $X/l = C_1 \times C_2$ and $X'/l = C_3 \times C_4$, with

$$[C_1] + [C_2] = [C]_l = [C_3] + [C_4] \neq 0.$$

It is easy to check that this extra condition forces $[C_3] = [C_1]$ or $[C_3] = [C_2]$, so that once again X/l and X/l' are not merely birational but isomorphic, which implies that $X \cong X'$.

Case 3: $d \neq d'$. By extending the base, we may assume $d = 1$, $d' \neq 1$. So $X \cong C_1 \times C_2$; equating Brauer kernels gives $\kappa = \langle [C_1], [C_2] \rangle = \kappa(X')$. Since X' has nontrivial discriminant, $\kappa = \langle [C] \rangle$ is a cyclic group; in other words we may assume either that $[C_1] = [C]$ and $[C_2] = 1$ or $[C_1] = [C_2] = [C]$. Basechanging to $l = k(\sqrt{d'})$, X'/l is still anisotropic, so is of the form $[C_3] \times [C_4]$, where $\kappa(l) = \langle [C_3], [C_4] \rangle \neq 0$. Thus X/l must remain anistropic, so that $[C]_l \in \kappa(l)$. Moreover, $\kappa(l)$ must be stable under the action of $\mathfrak{g}_{l/k}$, so that either $[C_3] = [C_4] = [C]_l$ or $[C_4] = [C_3]^\sigma \neq [C_3]$ and $[C_3] + [C_3]^\sigma = [C]_l$, where as usual σ denotes the nontrivial element of $\mathfrak{g}_{l/k}$. But the Brauer kernel of X/l clearly is still cyclic, hence the only possibility is that $[C_3] = [C_3]^\sigma$. But this would imply that X'/l is a quadric surface, which is not possible either, since the obstruction to its being a quadric surface over k, namely $[C]$, did not become trivial in l. We conclude that this case cannot occur, completing the proof.

IV. It seems likely that similar methods would give an isogeny implies isomorphism result for twisted forms of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$.

V. It is natural to wonder what is the largest class of geometrically rational varieties for which isogeny implies birational isomorphism. Observe that all the varieties V/k studied here share the following property, which we will provisionally call **prerationality**: for any field extension l/k for which $V(l) \neq \emptyset$, $l(V)$ is a rational function field. Note that all twisted forms of products of projective spaces are prerational, as are all Del Pezzo surfaces of degree at least 5 [13, Thm. 29.4]. It follows easily from the results of this paper that isogeny implies birational isomorphism among Del Pezzo surfaces of degree at least 7 (indeed, in every case we have not considered, the function field must be rational [13, 29.4.4]). It would be interesting to know whether isogeny implies isomorphism also for Del Pezzo surfaces of degrees 5 and 6, or indeed for all prerational varieties.

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