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Florian LUCA et Igor E. SHPARLINSKI

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On the largest prime factor of $n! + 2^n - 1$

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Résumé. Pour un entier $n \geq 2$, notons $P(n)$ le plus grand facteur premier de n. Nous obtenons des majorations sur le nombre de solutions de congruences de la forme $n! + 2^n - 1 \equiv 0 \pmod q$ et nous utilisons ces bornes pour montrer que

$$\limsup_{n\to\infty} P(n! + 2^n - 1)/n \geq (2\pi^2 + 3)/18.$$

Abstract. For an integer $n \geq 2$ we denote by $P(n)$ the largest prime factor of n. We obtain several upper bounds on the number of solutions of congruences of the form $n! + 2^n - 1 \equiv 0 \pmod q$ and use these bounds to show that

$$\limsup_{n\to\infty} P(n! + 2^n - 1)/n \geq (2\pi^2 + 3)/18.$$

1. Introduction

For any positive integer $k > 1$ we denote by $P(k)$ the largest prime factor of k and by $\omega(k)$ the number of distinct prime divisors of k. We also set $P(1) = 1$ and $\omega(1) = 0$.

It is trivial to see that $P(n! + 1) > n$. Erdős and Stewart [4] have shown that

$$\limsup_{n\to\infty} \frac{P(n!+1)}{n} > 2.$$

This bound is improved in [7] where it is shown that the above upper limit is at least $5/2$, and that it also holds for $P(n! + f(n))$ with a nonzero polynomial $f(X) \in \mathbb{Z}[X]$.

Here we use the method of [7], which we supplement with some new arguments, to show that

$$\limsup_{n\to\infty} \frac{P(n! + 2^n - 1)}{n} > (2\pi^2 + 3)/18.$$

We also estimate the total number of distinct primes which divide at least one value of $n! + 2^n - 1$ with $1 \leq n \leq x$.

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These results are based on several new elements, such as bounds for the number of solutions of congruences with $n! + 2^n - 1$, which could be of independent interest.

Certainly, there is nothing special in the sequence $2^n - 1$, and exactly the same results can be obtained for $n! + u(n)$ with any nonzero binary recurrent sequence $u(n)$.

Finally, we note that our approach can be used to estimate $P(n! + u(n))$ with an arbitrary linear recurrence sequence $u(n)$ (leading to similar, albeit weaker, results) and with many other sequences (whose growth and the number of zeros modulo q are controllable).

Throughout this paper, we use the Vinogradov symbols $\gg$, $\ll$ and $\asymp$ as well as the Landau symbols O and o with their regular meanings. For $z > 0$, $\log z$ denotes the natural logarithm of z.

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2. Bounding the number of solutions of some equations and congruences

The following polynomial

$$F_{k,m}(X) = (2^k - 1)\prod_{i=1}^{m}(X+i) - (2^m - 1)\prod_{i=1}^{k}(X+i) + 2^m - 2^k \tag{2.1}$$

plays an important role in our arguments.

Lemma 2.1. *The equation*

$$F_{k,m}(n) = 0$$

has no integer solutions (n, k, m) *with* $n \geq 3$ *and* $m > k \geq 1$.

Proof. One simply notices that for any $n \geq 3$ and $m > k \geq 1$

$$\begin{aligned}(2^k - 1)\prod_{i=1}^{m}(n+i) &\geq 2^{k-1}(n+1)^{m-k}\prod_{i=1}^{k}(n+i)\\ &\geq (n+1)2^{m-2}\prod_{i=1}^{k}(n+i) \geq 2^m\prod_{i=1}^{k}(n+i)\\ &> (2^m - 1)\prod_{i=1}^{k}(n+i).\end{aligned}$$

Hence, $F_{k,m}(n) > 0$ for $n \geq 3$. □

Let $\ell(q)$ denote the multiplicative order of 2 modulo an odd integer $q \geq 3$.

For integers $y \geq 0$, $x \geq y+1$, and $q \geq 1$, we denote by $\mathcal{T}(y,x,q)$ the set of solutions of the following congruence

$$\mathcal{T}(y,x,q) = \{n \mid n! + 2^n - 1 \equiv 0 \pmod q,\ y+1 \leq n \leq x\},$$

and put $T(y,x,q) = \#\mathcal{T}(y,x,q)$. We also define

$$\mathcal{T}(x,q) = \mathcal{T}(0,x,q) \qquad \text{and} \qquad T(x,q) = T(0,x,q).$$

Lemma 2.2. *For any prime* p *and integers* x *and* y *with* $p > x \geq y+1 \geq 1$, *we have*

$$T(y,x,p) \ll \max\{(x-y)^{3/4}, (x-y)/\ell(p)\}.$$

Proof. We assume that $p \geq 3$, otherwise there is nothing to prove. Let $\ell(p) > z \geq 1$ be a parameter to be chosen later.

Let $y+1 \leq n_1 < \ldots < n_t \leq x$ be the complete list of $t = T(y,x,p)$ solutions to the congruence $n! + 2^n - 1 \equiv 0 \pmod p$, $y+1 \leq n \leq x$. Then

$$\mathcal{T}(y,x,p) = \mathcal{U}_1 \cup \mathcal{U}_2,$$

where

$$\mathcal{U}_1 = \{n_i \in \mathcal{T}(y,x,p) \mid |n_i - n_{i+2}| \geq z,\ i = 1,\ldots,t-2\},$$

and $\mathcal{U}_2 = \mathcal{T}(y,x,p)\backslash\mathcal{U}_1$.

It is clear that $\#\mathcal{U}_1 \ll (x-y)/z$. Assume now that $n \in \mathcal{U}_2\backslash\{n_{t-1}, n_t\}$. Then there exists a nonzero integers k and m with $0 < k < m \leq z$, and such that

$$n! + 2^n - 1 \equiv (n+k)! + 2^{n+k} - 1 \equiv (n+m)! + 2^{n+m} - 1 \equiv 0 \pmod p.$$

Eliminating 2^n from the first and the second congruence, and then from the first and the third congruence, we obtain

$$n!\left(\prod_{i=1}^{k}(n+i) - 2^k\right) + 2^k - 1 \equiv n!\left(\prod_{i=1}^{m}(n+i) - 2^m\right) + 2^m - 1 \equiv 0 \pmod p.$$

Now eliminating $n!$, we derive

$$(2^m - 1)\left(\prod_{i=1}^{k}(n+i) - 2^k\right) - (2^k - 1)\left(\prod_{i=1}^{m}(n+i) - 2^m\right) \equiv 0 \pmod p,$$

or $F_{k,m}(n) \equiv 0 \pmod p$, where $F_{k,m}(X)$ is given by (2.1). Because $\ell(p) > z$, we see that for every $0 < k < m \leq z$ the polynomial $F_{k,m}(X)$ has a nonzero coefficient modulo p and $\deg F_{k,m} = m \leq z$, thus for every $0 < k < m < z$ there are at most z suitable values of n (since $p > x \geq y+1 \geq 1$).

Summing over all admissible values of k and m, we derive $\#\mathcal{U}_2 \ll z^3 + 1$. Therefore

$$T(y,x,p) \le \#\mathcal{U}_1 + \#\mathcal{U}_2 \ll (x-y)/z + z^3 + 1.$$

Taking $z = \min\{(x-y)^{1/4}, \ell(p) - 1\}$ we obtain the desired inequality. □

Obviously, for any $n \ge p$ with $n! + 2^n - 1 \equiv 0 \pmod p$, we have $2^n \equiv 1 \pmod p$. Thus

$$T(p,x,p) \ll x/\ell(p). \tag{2.2}$$

Lemma 2.3. *For any integers $q \ge 2$ and $x \ge y + 1 \ge 1$, we have*

$$T(y,x,q) \le \left(2 + O\left(\frac{1}{\log x}\right)\right) \frac{(x-y)\log x}{\log q} + O(1).$$

Proof. Assume that $T(y,x,q) \ge 6$, because otherwise there is nothing to prove. We can also assume that q is odd. Then, by the Dirichlet principle, there exist integers $n \ge 4$, $m > k \ge 1$, satisfying the inequalities

$$1 \le k < m \le 2\frac{x-y}{T(y,x,q)-4}, \qquad y+1 \le n < n+k < n+m \le x,$$

and such that

$$n! + 2^n - 1 \equiv (n+k)! + 2^{n+k} - 1 \equiv (n+m)! + 2^{n+m} - 1 \equiv 0 \pmod q.$$

Arguing as in the proof of Lemma 2.2, we derive $F_{m,k}(n) \equiv 0 \pmod q$. Because $F_{m,k}(n) \ne 0$ by Lemma 2.1, we obtain $|F_{m,k}(n)| \ge q$. Obviously, $|F_{m,k}(n)| = O(2^k x^m) = O((2x)^m)$. Therefore,

$$\log q \le m(\log x + O(1)) \le 2\frac{(x-y)(\log x + O(1))}{T(y,x,p)-4},$$

and the result follows. □

Certainly, Lemma 2.2 is useful only if $\ell(p)$ is large enough.

Lemma 2.4. *For any x the inequality $\ell(p) \ge x^{1/2}/\log x$ holds for all except maybe $O(x/(\log x)^3)$ primes $p \le x$.*

Proof. Put $L = \lfloor x^{1/2}/\log x \rfloor$. If $\ell(p) \le L$ then $p|R$, where

$$R = \prod_{i=1}^{L}(2^i - 1) \le 2^{L^2}.$$

The bound $\omega(R) \ll \log R/\log\log R \ll L^2/\log L$ concludes the proof. □

We remark that stronger results are known, see [3, 6, 9], but they do not seem to be of help for our arguments.

3. Main Results

Theorem 3.1. *The following bound holds:*

$$\limsup_{n\to\infty} \frac{P(n! + 2^n - 1)}{n} \geq \frac{2\pi^2 + 3}{18} = 1.2632893\ldots .$$

Proof. Assuming that the statement of the above theorem is false, we see that there exist two constants $\lambda < (2\pi^2 + 3)/18$ and μ such that the inequality $P(n! + 2^n - 1) < \lambda n + \mu$ holds for all integer positive n.

We let x be a large positive integer and consider the product

$$W = \prod_{1\leq n\leq x} (n! + 2^n - 1).$$

Let $Q = P(W)$ so we have $Q \leq \lambda x + \mu$. Obviously,

$$\log W = \frac{1}{2}x^2 \log x + O(x^2). \tag{3.1}$$

For a prime p, we denote by s_p the largest power of p dividing at least one of the nonzero integers of the form $n! + 2^n - 1$ for $n \leq x$. We also denote by r_p the p-adic order of W. Hence,

$$r_p = \sum_{1\leq s\leq s_p} T(x, p^s), \tag{3.2}$$

and therefore, by (3.1) and (3.2), we deduce

$$\sum_{\substack{p|W\\ p\leq Q}} \log p \sum_{1\leq s\leq s_p} T(x, p^s) = \log W = \frac{1}{2}x^2 \log x + O(x^2). \tag{3.3}$$

We let $\mathcal{M}$ be the set of all possible pairs (p, s) which occur on the left hand side of (3.3), that is,

$$\mathcal{M} = \{(p, s) \mid p|W,\ p \leq Q,\ 1 \leq s \leq s_p\},$$

and so (3.3) can be written as

$$\sum_{(p,s)\in\mathcal{M}} T(x, p^s) \log p = \frac{1}{2}x^2 \log x + O(x^2). \tag{3.4}$$

As usual, we use $\pi(y)$ to denote the number of primes $p \leq y$, and recall that by the Prime Number Theorem we have $\pi(y) = (1 + o(1))y/\log y$.

Now we introduce subsets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \in \mathcal{M}$, which possibly overlap, and whose contribution to the sums on the left hand side of (3.4) is $o(x^2 \log x)$. After this, we study the contribution of the remaining set $\mathcal{L}$.

- Let $\mathcal{E}_1$ be the set of pairs $(p,s)\in\mathcal{M}$ with $p\le x/\log x$. By Lemma 2.3, we have

$$\begin{aligned}\sum_{(p,s)\in\mathcal{E}_1} T(x,p^s)\log p &\ll x\log x \sum_{(p,s)\in\mathcal{E}_1}\frac{1}{s}+\sum_{(p,s)\in\mathcal{E}_1}\log p\\ &\ll x\log x\sum_{p\le x/\log x}\log(s_p+1)\\ &\quad+\sum_{p\le x/\log x} s_p\log p\ll x^2,\end{aligned}$$

because obviously $s_p\ll x\log x$.

- Let $\mathcal{E}_2$ be the set of pairs $(p,s)\in\mathcal{M}$ with $s\ge x/(\log x)^2$. Again by Lemma 2.3, and by the inequality

$$s_p\ll x\frac{\log x}{\log p},$$

we have

$$\begin{aligned}\sum_{(p,s)\in\mathcal{E}_2} T(x,p^s)\log p &\ll x\log x \sum_{(p,s)\in\mathcal{E}_2}\frac{1}{s}+\sum_{(p,s)\in\mathcal{E}_2}\log p\\ &\ll x\log x\sum_{p\le Q}\sum_{x/(\log x)^2\le s\le s_p}\frac{1}{s}+\sum_{p\le Q}s_p\log p\\ &\ll x\pi(Q)\log x\log\log x\ll x^2\log\log x,\end{aligned}$$

because $Q=O(x)$ by our assumption.

- Let $\mathcal{E}_3$ be the set of pairs $(p,s)\in\mathcal{M}$ with $\ell(p)\le x^{1/2}/\log x$. Again by Lemmas 2.3 and 2.4, and by the inequality $s_p\ll x\log x$, we have

$$\begin{aligned}\sum_{(p,s)\in\mathcal{E}_3} T(x,p^s)\log p &\ll x\log x \sum_{(p,s)\in\mathcal{E}_2}\frac{1}{s}+\sum_{(p,s)\in\mathcal{E}_3}\log p\\ &\ll x\log x\sum_{\substack{p\le Q\\ \ell(p)\le x^{1/2}/\log x}}\sum_{1\le s\le s_p}\frac{1}{s}\\ &\quad+\sum_{\substack{p\le Q\\ \ell(p)\le x^{1/2}/\log x}}s_p\log p\\ &\ll x(\log x)^2\sum_{\substack{p\le Q\\ \ell(p)\le x^{1/2}/\log x}}1\ll x^2/\log x.\end{aligned}$$

- Let $\mathcal{E}_4$ be the set of pairs $(p,s)\in\mathcal{M}\backslash(\mathcal{E}_1\cup\mathcal{E}_3)$ with $s<x^{1/4}$. By Lemma 2.2 and by (2.2), we have

$$\begin{aligned}\sum_{(p,s)\in\mathcal{E}_3} T(x,p^s)\log p &\ll x^{1/4}\sum_{p\le Q} T(x,p)\log p\\ &\ll x^{1/4}\sum_{p\le Q}\Big(p^{3/4}+x/\ell(p)\Big)\log p\\ &\ll x^{1/4}Q^{3/4}\sum_{p\le Q}\log p\\ &\ll x^{1/4}Q^{7/4}\ll x^2.\end{aligned}$$

We now put $\mathcal{L}=\mathcal{M}\backslash(\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_3\cup\mathcal{E}_4)$.

The above estimates, together with (3.4), show that

$$\sum_{(p,s)\in\mathcal{L}} T(x,p^s)\log p=\frac{1}{2}x^2\log x+O(x^2\log\log x). \tag{3.5}$$

The properties of the pairs $(p,s)\in\mathcal{L}$ can be summarized as

$$p>\frac{x}{\log x},\qquad \ell(p)\ge\frac{x^{1/2}}{\log x},\qquad \frac{x}{(\log x)^2}\ge s\ge x^{1/4}.$$

In what follows, we repeatedly use the above bounds.

We now remark that because by our assumption $P(n!+2^n-1)\le\lambda n+\mu$ for $n\le x$, we see that $T(x,p^s)=T(\lfloor(p-\mu)/\lambda\rfloor,x,p^s)$.

Thus, putting $x_p=\min\{x,p\}$, we obtain

$$T(x,p^s)=T(\lfloor(p-\mu)/\lambda\rfloor,x,p^s)=T(\lfloor(p-\mu)/\lambda\rfloor,x_p,p^s)+T(x_p,x,p^s).$$

Therefore,

$$\sum_{(p,s)\in\mathcal{L}} T(x,p^s)\log p=U+V, \tag{3.6}$$

where

$$U=\sum_{(p,s)\in\mathcal{L}} T(\lfloor(p-\mu)/\lambda\rfloor,x_p,p^s)\log p,$$

and

$$V=\sum_{(p,s)\in\mathcal{L}} T(x_p,x,p^s)\log p.$$

To estimate U, we observe that, by Lemma 2.3,

$$U \le (2+o(1))\log x \sum_{p\le Q}\left(\left(x_p - \frac{p-\mu}{\lambda}\right)\sum_{x/\log x > s \ge x^{1/4}} \frac{1}{s} + O(1)\right)$$
$$\le (3/2+o(1))\,(\log x)^2 \sum_{p\le Q}\left(x_p - \frac{p-\mu}{\lambda}\right) + O(x^2).$$

Furthermore,

$$\sum_{p\le Q}\left(x_p - \frac{p-\mu}{\lambda}\right) = \sum_{p\le x}\left(p - \frac{p-\mu}{\lambda}\right) + \sum_{x<p\le Q}\left(x - \frac{p-\mu}{\lambda}\right)$$
$$= \left(\frac{\lambda-1}{2\lambda}+o(1)\right)\frac{x^2}{\log x} + \left(\frac{(\lambda-1)^2}{2\lambda}+o(1)\right)\frac{x^2}{\log x}$$
$$= \left(\frac{\lambda-1}{2}+o(1)\right)\frac{x^2}{\log x}.$$

Hence

$$U \le \left(\frac{3(\lambda-1)}{4}+o(1)\right) x^2 \log x. \tag{3.7}$$

We now estimate V. For an integer $\alpha \ge 1$ we let $\mathcal{P}_\alpha$ be the set of primes $p \le Q$ with

$$\ell(p) = \ldots = \ell(p^\alpha) \ne \ell(p^{\alpha+1}).$$

Thus, $\ell(p^{\alpha+1}) = \ell(p)p$.

Accordingly, let $\mathcal{L}_\alpha$ be the subset of pairs $(p,s) \in \mathcal{L}$ for which $p \in \mathcal{P}_\alpha$.

We see that if $(p,s) \in \mathcal{L}$ and $n \le x$, then $p^2 > n$, and therefore the p-adic order of $n!$ is

$$\mathrm{ord}_p n! = \left\lfloor \frac{n}{p} \right\rfloor.$$

For $p \in \mathcal{P}_\alpha$ we also have

$$\mathrm{ord}_p(2^{\ell(p)} - 1) = \alpha.$$

Clearly, if $n \ge p$ then $\mathrm{ord}_p(n! + 2^n - 1) > 0$ only for $n \equiv 0 \pmod{\ell(p)}$. Because $\ell(p^{\alpha+1}) = p\ell(p) \gg x^{3/2}/(\log x)^2 > x$, we see that, for $p \le n \le x$,

$$\mathrm{ord}_p(2^n - 1) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{\ell(p)}, \\ \alpha, & \text{if } n \equiv 0 \pmod{\ell(p)}. \end{cases}$$

Therefore, for $n \le \alpha p - 1$ and $n \equiv 0 \pmod{\ell(p)}$, we have

$$\mathrm{ord}_p(n! + 2^n - 1) \le \mathrm{ord}_p n! < n/(p-1) \ll \log x.$$

Thus, $T(x_p, \alpha p - 1, p^s) = 0$ for $(p,s) \in \mathcal{L}_\alpha$.

On the other hand, for $n \ge (\alpha+1)p$, we have $\mathrm{ord}_p(n!) > n/p - 1 \ge \alpha$. Hence, for $n \equiv 0 \pmod{\ell(p)}$, we derive

$$\mathrm{ord}_p(n! + 2^n - 1) = \mathrm{ord}_p(2^n - 1) = \alpha < n/p \ll \log x.$$

As we have mentioned $\mathrm{ord}_p(n! + 2^n - 1) = 0$ for every $n \ge p$ with $n \equiv 0 \pmod{\ell(p)}$. Thus, $T((\alpha+1)p, x, p^s) = 0$ for $(p, s) \in \mathcal{L}_\alpha$.

For $\alpha = 1, 2, \ldots$, let us define

$$Y_{\alpha,p} = \min\{x, \alpha p - 1\} \qquad \text{and} \qquad X_{\alpha,p} = \min\{x, (\alpha+1)p\}.$$

We then have

$$V = \sum_{\alpha=1}^{\infty} V_\alpha,$$

where

$$V_\alpha = \sum_{(p,s)\in\mathcal{L}_\alpha} T(x_p, x, p^s) \log p.$$

For every $\alpha \ge 1$, and $(p, s) \in \mathcal{L}_\alpha$, as we have seen,

$$T(x_p, x, p^s) = T(Y_{\alpha,p}, X_{\alpha,p}, p^s).$$

We now need the bound,

$$T(Y_{\alpha,p}, X_{\alpha,p}, p^s) \le \frac{X_{\alpha,p} - Y_{\alpha,p}}{\ell(p)} + 1, \tag{3.8}$$

which is a modified version of (2.2). Indeed, if $Y_{\alpha,p} = x$ then $X_{\alpha,p} = x$ and we count solutions in an empty interval. If $Y_{\alpha,p} = \alpha p - 1$ (the other alternative), we then replace the congruence modulo p^s by the congruence modulo p and remark that because $n > Y_{\alpha,p} \ge p$ we have $n! + 2^n - 1 \equiv 2^n - 1 \pmod p$ and (3.8) is now immediate.

We use (3.8) for $x^{1/2}/(\log x)^2 \ge s \ge x^{1/4}$, and Lemma 2.3 for $x/(\log x)^2 > s \ge x^{1/2}/(\log x)^2$. Simple calculations lead to the bound

$$V_\alpha \le (1 + o(1)) (\log x)^2 \sum_{p\in\mathcal{P}_\alpha} (X_{\alpha,p} - Y_{\alpha,p}) + O(x^2).$$

We now have

$$\sum_{p\in\mathcal{P}_\alpha} (X_{\alpha,p} - Y_{\alpha,p}) = \sum_{\substack{p\in P_\alpha \\ p \le x/(\alpha+1)}} (p+1) + \sum_{\substack{p\in P_\alpha \\ x/(\alpha+1) < p \le (x+1)/\alpha}} (x - \alpha p + 1).$$

Thus, putting everything together, and taking into account that the sets $\mathcal{P}_\alpha$, $\alpha = 1, 2, \ldots$, are disjoint, we derive

$$\begin{aligned} V &\le (1+o(1)) (\log x)^2 \left(\sum_{p \le x/2} p + \sum_{\alpha=1}^{\infty} \sum_{x/(\alpha+1) < p \le (x+1)/\alpha} (x - \alpha p) \right) \\ &= (1+o(1)) (\log x)^2 \\ &\qquad \times \left(\frac{x^2}{8 \log x} + \frac{x^2}{\log x} \sum_{\alpha=1}^{\infty} \left(\frac{1}{\alpha(\alpha+1)} - \frac{2\alpha+1}{2\alpha(\alpha+1)^2} \right) \right) \\ &= (1+o(1)) (\log x)^2 \left(\frac{x^2}{8 \log x} + \frac{x^2}{2 \log x} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha(\alpha+1)^2} \right) \\ &= (1+o(1)) (\log x)^2 \\ &\qquad \times \left(\frac{x^2}{8 \log x} + \frac{x^2}{2 \log x} \sum_{\alpha=1}^{\infty} \left(\frac{1}{\alpha(\alpha+1)} - \frac{1}{(\alpha+1)^2} \right) \right) \\ &= (1+o(1)) (\log x)^2 \left(\frac{x^2}{8 \log x} + \frac{x^2}{2 \log x} \left(2 - \frac{\pi^2}{6} \right) \right). \end{aligned}$$

Hence

$$V \le \left(\frac{27 - 2\pi^2}{24} + o(1) \right) x^2 \log x. \tag{3.9}$$

Substituting (3.7) and (3.9) in (3.6), and using (3.5), we derive

$$\frac{3(\lambda - 1)}{4} + \frac{27 - 2\pi^2}{24} \ge \frac{1}{2},$$

which contradicts the assumption $\lambda < (2\pi^2 + 3)/18$, and thus finishes the proof. □

Theorem 3.2. *For any sufficiently large x, we have:*

$$\omega \left(\prod_{1 \le n \le x} (n! + 2^n - 1) \right) \gg \frac{x}{\log x}.$$

Proof. In the notation of the proof of Theorem 3.1, we derive from (3.2) and Lemma 2.3, that

$$r_p \ll \sum_{1 \le s \le s_p} \frac{x \log x}{s \log p} + 1 \ll \frac{x \log x \log(s_p + 1)}{\log p} + s_p.$$

Obviously $s_p \ll x \log x / \log p$, therefore $r_p \ll x (\log x)^2 / \log p$. Thus, for any prime number p,

$$p^{r_p} = \exp\left(O\left(x (\log x)^2 \right) \right),$$

which together with (3.1) finishes the proof. □

4. Remarks

We recall the result of Fouvry [5], which asserts that $P(p-1) \geq p^{0.668}$ holds for a set of primes p of positive relative density (see also [1, 2] for this and several more related results). By Lemma 2.4, this immediately implies that $\ell(p) \geq p^{0.668}$ for a set of primes p of positive relative density. Using this fact in our arguments, one can easily derive that actually

$$\limsup_{n\to\infty} \frac{P(n! + 2^n - 1)}{n} > \frac{2\pi^2 + 3}{18}.$$

However, the results of [5], or other similar results like the ones from [1, 2], do not give any effective bound on the relative density of the set of primes with $P(p-1) \geq p^{0.668}$, and thus cannot be used to get an explicit numerical improvement of Theorem 3.1.

We also remark that, as in [7], one can use lower bounds on linear forms in p-adic logarithms to obtain an "individual" lower bound on $P(n!+2^n-1)$. The *ABC-conjecture* can also used in the same way as in [8] for $P(n!+1)$.

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Florian Luca
Instituto de Matemáticas
Universidad Nacional Autónoma de México
C.P. 58089, Morelia, Michoacán, México
E-mail : fluca@matmor.unam.mx

Igor E. Shparlinski
Department of Computing
Macquarie University
Sydney, NSW 2109, Australia
E-mail : igor@ics.mq.edu.au