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## On linear normal lattices configurations

par MORDECHAY B. LEVIN et MEIR SMORODINSKY

RÉSUMÉ. Dans cet article nous prolongeons la construction de Champernowne de nombres normaux dans la base  $b$  pour le cas  $\mathbb{Z}^d$ , et obtenons une construction explicite du point générique de la transformation de l'ensemble  $\{0, 1, \dots, b-1\}^{\mathbb{Z}^d}$  par  $\mathbb{Z}^d$  déplacement. Nous prouvons que l'intersection de la configuration de réseau considérée avec une droite arbitraire est une suite normale dans la base  $b$ .

ABSTRACT. In this paper we extend Champernowne's construction of normal numbers in base  $b$  to the  $\mathbb{Z}^d$  case and obtain an explicit construction of the generic point of the  $\mathbb{Z}^d$  shift transformation of the set  $\{0, 1, \dots, b-1\}^{\mathbb{Z}^d}$ . We prove that the intersection of the considered lattice configuration with an arbitrary line is a normal sequence in base  $b$ .

### 1. Introduction

A number  $\alpha \in (0, 1)$  is said to be *normal* to base  $b$ , if in the  $b$ -ary expansion of  $\alpha$ ,  $\alpha = .d_1 d_2 \dots$  ( $d_i \in \{0, 1, \dots, b-1\}, i = 1, 2, \dots$ ), each fixed finite block of digits of length  $k$  appears with an asymptotic frequency of  $b^{-k}$  along the sequence  $(d_i)_{i \geq 1}$ . Normal numbers were introduced by Borel (1909). Champernowne [2] gave an explicit construction of such a number, namely,

$$\theta = .1 2 3 4 5 6 7 8 9 10 11 12 \dots,$$

obtained by successively concatenating the digital expansions all the natural numbers.

We shall call the sequence of digits obtained from a normal number a “normal sequence”.

Champernowne's construction is associated with the independent identical distributed process of variables having uniform distribution over  $b$  states. In [1], [14], and [15], constructions of normal sequences for various stationary stochastic processes, similar to Champernowne's were introduced.

Our goal is to extend such constructions to  $\mathbb{Z}^d$ -arrays ( $d > 1$ ) of random variables, which we shall call  $\mathbb{Z}^d$ -processes. We shall deal with stationary  $\mathbb{Z}^d$ -processes, that is, processes with distribution, invariant under the  $\mathbb{Z}^d$  action. We shall call a specific realization of a  $\mathbb{Z}^d$ -process a “configuration” (“lattice configuration”). To begin with, the very definition of a “normal configuration” is subject to various generalizations from the 1-dimensional case. In this paper we continue investigations beginning in [11] and [12] (see also [3] and [8]).

*Rectangular normality.* We denote by  $\mathbb{N}$  the set of non-negative integers. Let  $d, b \geq 2$  be two integers,  $\mathbb{N}^d = \{(n_1, \dots, n_d) \mid n_i \in \mathbb{N}, i = 1, \dots, d\}$ ,  $\Delta_b = \{0, 1, \dots, b - 1\}$ ,  $\Omega = \Delta_b^{\mathbb{N}^d}$ .

We shall call  $\omega \in \Omega$  a *configuration (lattice configuration)*. A configuration is a function  $\omega : \mathbb{N}^d \rightarrow \Delta_b$ .

Given a subset  $F$  of  $\mathbb{N}^d$ ,  $\omega_F$  will be the restriction of the function  $\omega$  to  $F$ . Let  $\mathbf{N} \in \mathbb{N}^d$ ,  $\mathbf{N} = (N_1, \dots, N_d)$ . We denote a *rectangular block* by

$$F_{\mathbf{N}} = \{(f_1, \dots, f_d) \in \mathbb{N}^d \mid 0 \leq f_i < N_i, i = 1, \dots, d\},$$

$\mathbf{h} = [0, h_1) \times \dots \times [0, h_d)$ ,  $h_i \geq 1, i = 1, \dots, d$ ;  $G = G_{\mathbf{h}}$  is a fixed block of digits  $G = (g_{\mathbf{i}})_{\mathbf{i} \in F_{\mathbf{h}}}$ ,  $g_{\mathbf{i}} \in \Delta_b$ ;  $\chi_{\omega, G}(\mathbf{f})$  is a characteristic function of the block of digits  $G$  shifted by vector  $\mathbf{f}$  in the configuration  $\omega$

$$(1) \quad \chi_{\omega, G}(\mathbf{f}) = \begin{cases} 1 & \text{if } \omega(\mathbf{f} + \mathbf{i}) = g_{\mathbf{i}}, \forall \mathbf{i} \in F_{\mathbf{h}} \\ 0 & \text{otherwise .} \end{cases}$$

**Definition 1.**  $\omega \in \Omega$  is said to be *rectangular normal* if for any  $\mathbf{h} \subset \mathbb{N}^d$  and block  $G_{\mathbf{h}}$

$$(2) \quad \#\{\mathbf{f} \in F_{\mathbf{N}} \mid \chi_{\omega, G_{\mathbf{h}}}(\mathbf{f}) = 1\} - b^{-h_1 \dots h_d} N_1 \dots N_d = o(N_1 \dots N_d),$$

when  $\max(N_1, \dots, N_d) \rightarrow \infty$ .

We shall say that  $\omega$  is *square normal* in case we consider only square blocks i.e.,  $N_1 = N_2 = \dots = N_d$ . For the sake of clarity, we shall carry the proof only for the case  $d = 2$ .

*Linear normality.* Let

$$\omega = \{f_{ij} \mid f_{ij} \in \Delta_b, i, j = 0, 1, 2, \dots\}$$

be a 2-dimensional configuration;  $\alpha \geq 0, \beta$  be real. Let the tiling of the plane by unit squares be given. We label the squares of the tiles of the positive quadrant of the plane by  $\omega_{ij}$  where  $(i, j)$  are the coordinates of the left, lower vertex of the tile. Consider the line  $y = \alpha x + \beta$  (or the vertical line  $x = \beta$ ). The line is partitioned into successive intervals of the intersections with tiles. Therefore, to each line corresponds a sequence of digits

$$(u_{\alpha, \beta}(n))_{n \geq 0}.$$

Let

(3)

$$R_N(\omega, \alpha, \beta, m) = \#\{0 \leq n < N \mid u_{\alpha, \beta}(n+i) = g_i, i = 1, \dots, m\} - b^{-m}N.$$

**Definition 2.**  $\omega$  is said to be linear normal if for all reals  $\alpha \geq 0$ ,  $\beta$  the sequence  $(u_{\alpha, \beta}(n))_{n \geq 0}$  is normal in base  $b$ , i.e.,

$$R_N(\omega, \alpha, \beta, m) = o(N)$$

for all integer  $s \geq 1$  and all blocks of digits  $(g_1, \dots, g_m)$  ( $g_i \in \Delta_b, i = 1, \dots, m$ ).

**Construction.** Let

$$(4) \quad L(f_1, f_2) = \begin{cases} f_1^2 + f_2, & \text{if } f_2 < f_1 \\ f_2^2 + 2f_2 - f_1 & \text{if } f_2 \geq f_1. \end{cases}$$

This map is clearly a bijection between  $\mathbb{N}$  and  $\mathbb{N}^2$ , inducing a total order on  $\mathbb{N}^2$  from the usual one on  $\mathbb{N}$ . We define the configuration  $\omega_n$  on  $F_{(2nb^{2n^2}, 2nb^{2n^2})}$  as the concatenation of  $b^{4n^2}$  blocks of digits of size  $2n \times 2n$  with the left lower corner  $(2nx, 2ny)$ ,  $0 \leq x, y < b^{2n^2}$ . To each of these blocks we set in correspondence the number  $L(x, y)$ . Next we use  $b$ -expansion of the number  $L(x, y)$  according to the order  $L$  to obtain digits of the considered  $2n \times 2n$  block. It is easy to obtain the analytic expression for digits of the configuration  $\omega_n$ :

$$(5) \quad \omega_n(2nx + s, 2ny + t) = \begin{cases} a_{s^2+t}(u), & \text{if } t < s \\ a_{t^2+2t-s}(u) & \text{if } t \geq s, \end{cases}$$

where

$$(6) \quad u = u(x, y) = \begin{cases} x^2 + y, & \text{if } y < x \\ y^2 + 2y - x & \text{if } y \geq x, \end{cases}$$

$s, t, x, y$  are integers,  $0 \leq x, y < b^{2n^2}$ ,  $0 \leq s, t < 2n$ , and

$$(7) \quad n = \sum_{i \geq 0} a_i(n)b^i, \quad (a_i(n) \in \{0, 1, \dots, b-1\})$$

is the  $b$ -expansion of integer  $n$ .

Next we define inductively a sequence of increasing configurations  $\omega_n$  on  $F_{(2nb^{2n^2}, 2nb^{2n^2})}$ . Put  $\omega'_1 = \omega_1$ ,  $\omega'_{n+1}(\mathbf{f}) = \omega'_n(\mathbf{f})$  for  $\mathbf{f} \in F_{(2nb^{2n^2}, 2nb^{2n^2})}$  and  $\omega'_{n+1}(\mathbf{f}) = \omega_{n+1}(\mathbf{f})$  otherwise. Put

$$(8) \quad \begin{aligned} \omega_\infty &= \lim \omega'_n, \\ (\omega_\infty)_{F_{(2nb^{2n^2}, 2nb^{2n^2})}} &= \omega'_n, \quad n = 1, 2, \dots \end{aligned}$$

**Theorem 1.**  $\omega_\infty$  is a rectangular and linear normal configuration with

$$R_N(\omega_\infty, \alpha, \beta, m) = O(N/(\log N)^{1/4})$$

with  $O$ -constant depending on  $\alpha, \beta$ , and  $m$ .

The rectangular normality of  $\omega_\infty$  follows from [11] or [12]. Proof of the linear normality property is given in Section 3.

**Theorem 2** Let  $\mu$  be a Bernoulli measure on  $\Omega$ , then  $\mu$ -almost all  $\omega$  are linear normal.

*Proof.* We consider first the lines with  $0 \leq \beta < 1$  and the finite sequences that are determined by these lines confined to the square  $\{(x, y) : 0 \leq x, y \leq N\}$ . There are at most  $3N^2$  different finite sequences  $(u_{\alpha, \beta}(n))_{n \geq 0}$  whose length can vary between  $N - 1$  and  $2N$ . Given  $\epsilon > 0$ , (using the Chebishev inequality or the central limit theorem) the  $\mu$  measure of the average along any of these sequences to exceed the expected value by more than  $\epsilon$  is less than  $C/N^4$ . Using Borel-Cantelli will prove the claim for these lines. To conclude the claim for lines of all values of  $\beta$ , observe that  $\mu$  is invariant under translations.

**Generalizations, examples, and problems.**

**1.1. Polynomial normality.**

Let

$$\omega = \{g_{ij} \mid g_{ij} \in \Delta_b = \{0, 1, \dots, b - 1\}, i, j = 0, 1, 2, \dots\}$$

be a 2-dimensional configuration;  $\alpha_i$  ( $i = 0, 1, \dots, d$ ) be reals,  $\phi(x) = \alpha_d x^d + \dots + \alpha_1 x + \alpha_0$ ,  $\alpha_d > 0$ . Let the tiling of the plane by unit squares be given. We label the squares of the tiles of the positive quadrant of the plane by  $\omega_{ij}$  where  $(i, j)$  are the coordinates of the left, lower vertex of the tile. Consider the curve  $y = \phi(x)$  (or the curve  $x = \phi(y)$ ). The curve is partitioned into successive intervals of the intersections with tiles. Therefore, to each curve corresponds a sequence of digits

$$(u_\phi(n))_{n \geq 0}.$$

**Definition 3.** The configuration  $\omega$  is said to be polynomial normal if for all real polynomial  $\phi$  the sequence  $(u_\phi(n))_{n \geq 0}$  is normal in base  $b$ .

In [13] we proved that the configuration  $\omega_\infty$  is polynomial normal.

Now we note that notions of linear, polynomial, square, and rectangular normal configurations define different sets in  $\Omega$ . These differences have null measure subsets, but are not empty:

**Example 1.** A configuration which is linear normal, but not square normal: Let  $x(n)$  be an arbitrary normal sequence in base  $b$ , put  $\omega(i, j) = x(i + j)$ ,  $i, j = 1, 2, \dots$ . This is not a square normal configuration since the  $2 \times 2$  block of digits  $\begin{pmatrix} 10 \\ 00 \end{pmatrix}$  in  $\omega$  does not appear in this configuration.

Bearing in mind that  $\omega(i, j) = \omega(j, i)$ , we obtain it is sufficient to consider only the case  $0 \leq \alpha \leq 1$ . It is easy to see that  $u_{\alpha, \beta}(n) = x(f(n))$  with  $f(n) \in [n + [\alpha n + \beta], n + [\alpha(n + 1) + \beta]]$ . Observe that  $(f(n))_{n \leq 0}$  is a *completely deterministic* sequence in the sense of B. Weiss [16] (see also [5, p. 147-148]) and thus  $(u_{\alpha, \beta}(n))_{n \geq 0}$  is a normal sequence.

**Example 2.** *A configuration which is rectangle normal, but not linear normal:* Given  $\omega$  is a rectangle normal configuration, change the configuration so that  $\omega'(i, i) = 0, i, = 1, 2, \dots$

**Example 3.** *A configuration which is rectangle and linear normal, but not polynomial normal:* Similar to Example 2. Change the configuration along non-linear polynomial line.

**Example 4.** *A configuration which is square and linear normal, but not rectangular normal:*

*Construction:* Again we begin by constructing for each natural  $n$ , a block configuration  $\omega_n(u, v)_{0 \leq u, v < 2nb^{2n^2}}$ . We put

$$(9) \quad \omega_n(2nx + s, 2ny + t) = \begin{cases} a_{2nt+s}(x), & \text{if } t < n \\ a_{ns+t-n}(y) & \text{if } t \geq n, \end{cases}$$

where  $s, t, x, y$  are integers,  $0 \leq x, y < b^{2n^2}, 0 \leq s, t < 2n$ .

Let us describe the structure of the configuration  $\omega_n$  in words.  $\omega_n$  is composed of configurations of  $2n \times 2n$  blocks. Each such block is composed of two  $2n \times n$  lower and upper rectangles. Each rectangle can store the expansion of an integer  $m, 0 \leq m < b^{2n^2}$ . Given a number  $m$ , we store the digits in lower rectangles by first using the bottom row (left to right) then going to the next upper row etc.

For each row of bottom rectangles we store all the different  $b^{2n^2}$  numbers in their natural order. So all the rows of bottom rectangles look the same. For the upper rectangles we use a different rule to store a number  $m$ . We first use the first column (from top to bottom) then the second row etc. We proceed by storing for each column of top rectangles all the different numbers. Notice that at the end if we observe the  $2n \times 2n$  configurations of  $\omega_n$  all the possible configurations appear.

This choice of  $\omega_n$  will lead to a configuration that will guarantee the normality of all lines  $y = \alpha x + \beta$ . To guarantee the normality of all vertical lines too, we shall have to introduce a variant of the definition of  $\omega_n$ . Namely: apply (9) for all  $x > 0$  or  $x = 0$  and  $s > n$ ; for  $x = 0$  and  $s \in [0, n)$  use

$$\omega_n(s, 2ny + t) = a_{2ns+t}(y)$$

instead of (2), where  $0 \leq y < b^{2n^2}, 0 \leq t < 2n$ . We now proceed with the construction of the desired configuration. We construct the sequence of

block configurations (in a similar way as in the first construction see (8)) inductively.

These considerations lead to the following problems:

**Problem 1.** Is the intersection of  $\omega_\infty$  with all monotone, increasing convex curves also normal?

**Problem 2.** Is the Davenport-Erdős like rectangular normal configuration from [11] also linear (polynomial) normal?

**1.2.  $s$ -dimensional surfaces in  $\mathbb{R}^d$ .**

Consider a function  $\psi : \mathbb{R}^s \rightarrow \mathbb{R}^d$ . Let  $G_\psi = \{\psi(\mathbf{x}) \in \mathbb{R}^d \mid \mathbf{x} \in \mathbb{R}^s\}$ ,  $s \leq d$ ,

$$G'_\psi = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{n} + [0, 1)^d \cap G_\psi \neq \emptyset\}, \quad H_\psi : G'_\psi \rightarrow \mathbb{Z}^s,$$

and  $\Psi = \{\psi\}$  is a set of function  $\psi$  (a set of  $s$ -dimensional surfaces), such that  $H_\psi$  is a bijection with  $H_\psi(G'_\psi) = \mathbb{Z}^s$ .

**Definition 4.** The configuration  $\omega \in \{0, 1, \dots, b - 1\}^{\mathbb{Z}^d}$  is said to be  $\Psi$ -normal if  $\omega(G'_\psi(H_\psi^{-1}))$  is rectangular normal in  $\mathbb{Z}^s$  for all  $\psi \in \Psi$ .

**Problem 3.** Let  $\omega$  be a  $d$ -dimensional configuration, constructed similarly to (3) - (7);  $\Psi_p$  be a set of all  $s$ -dimensional polynomial surfaces in  $\mathbb{R}^d$ . Is  $\omega$  a  $\Psi_p$ -normal configuration ?

**1.3. Connection with uniform distribution.**

Let  $(\mathbf{x}_n)_{n \geq 1}$  be an infinite sequence of points in an  $s$ -dimensional unit cube  $[0, 1)^s$ ; let  $v = [0, \gamma_1) \times \dots \times [0, \gamma_s)$  be a box in  $[0, 1)^s$ ; and  $A_v(N)$  be the number of indexes  $n \in [1, N]$  such that  $\mathbf{x}_n$  lies in  $v$ . The quantity

$$(10) \quad D_N = D((\mathbf{x}_n)_{n=1}^N) = \sup_{v \in (0,1)^s} \left| \frac{1}{N} A_v(N) - \gamma_1 \dots \gamma_s \right|$$

is called the *discrepancy* of  $(\mathbf{x}_n)_{n=1}^N$ . The sequence  $(\mathbf{x}_n)_{n \geq 1}$  is said to be *uniformly distributed* in  $[0, 1)^s$  if  $D_N \rightarrow 0$ .

It is known (Wall, 1949, see [7, p. 70]), that a number  $\alpha$  is normal to base  $b$  if and only if the sequence  $\{\alpha b^n\}_{n \geq 1}$  is uniformly distributed (abbreviated u.d.) in  $[0, 1)$ .

Let  $\omega = (a_{i,j})_{i,j \geq 1}$  ( $a_{i,j} \in \{0, 1, \dots, b - 1\}$ ) be a configuration

$$\alpha_m = \sum_{i=1}^{\infty} a_{m,i} / b^i, \quad m = 1, 2, \dots,$$

$s \geq 1$  be an integer. The following statement is proved in [9]:

The lattice configuration  $\omega$  is normal to base  $b$  if and only if for all  $s \geq 1$  the double sequence

$$(11) \quad (\{\alpha_m b^n\}, \dots, \{\alpha_{m+s-1} b^n\})_{m,n \geq 1}$$

is uniformly distributed in  $[0, 1)^s$ , i.e.,

$$(12) \quad D\left(\left(\{\alpha_m b^n\}, \dots, \{\alpha_{m+s-1} b^n\}\right)_{1 \leq n \leq N, 0 \leq m < M}\right) = o(1),$$

when  $\max(M, N) \rightarrow \infty$ . Hence we have another definition of normal configuration (of *normal sequence*  $\alpha = (\alpha_1, \alpha_2, \dots) \in [0, 1)^\infty$  to the base  $b$ ). It is evident that a.e. sequences  $\alpha$  are normal for all bases  $b \geq 2$  (*absolutely normal*). It is easy to see that the u.d. definition of normality permits to transport a large set of problems on normal numbers to the  $\mathbb{Z}^d$  case, for example, normality in different bases. Another example: in [9] we prove that the lower bound of the discrepancy of the double sequence (11)-(12) coincides (with the exactitude of the logarithmic multiplier) with the lower bound of the discrepancy of ordinary sequences  $(x_n)_{n=1}^{MN}$  in an  $s$ -dimensional unit cube ( $s, M, N = 1, 2, \dots$ ).

**1.4. Connection with completely uniform distribution.**

Now let  $(v_n)_{n \geq 1}$  be an arbitrary sequence of real numbers. Starting with the sequence  $(v_n)_{n \geq 1}$ , we construct for every integer  $s \geq 1$  the  $s$ -dimensional sequence  $(x_n^{(s)}) = (\{v_{n+1}\}, \dots, \{v_{n+s}\})$ , where  $\{x\}$  is the fractional part of  $x$ . The sequence  $(v_n)_{n \geq 1}$  is said to be *completely uniformly distributed* (abbreviated c.u.d.) if for any integer  $s \geq 1$  the sequence  $(x_n^{(s)})$  is u.d. in  $[0, 1)^s$  (Korobov, 1949, see [6]). The c.u.d. sequence is the universal sequence for computing multidimensional integrals, modeling Markov chains, random numbers, and for other problems [4, 6, 7]. Let  $b \geq 2$  be an integer,  $(v_n)$  be a c.u.d. sequence,  $a_n = [b\{v_n\}]$ ,  $n = 1, 2, \dots$ . Then  $\alpha = .a_1 a_2 \dots$  is normal to base  $b$  (Korobov, [6]). In [10] we constructed a c.u.d. double sequence  $(v_{n,m})_{n,m \geq 1}$ , such that for all integers  $s, t \geq 1$

$$MND\left(\left(\left(v_{n+i,m+j}\right)_{i=1,j=1}^{s,t}\right)_{n=1,m=1}^{N,M}\right) = O((\log(MN + 1))^{st+4})$$

for all  $M, N \geq 1$ . Similarly to [6], we get from here the estimate of error term in (2) as  $O((\log(N_1 N_2 + 1))^{st+4})$  for the configuration  $(a_{n,m})_{n,m \geq 1}$ , where  $a_{n,m} = [b\{v_{n,m}\}]$ ,  $n, m \geq 1$ .

Let  $\phi = \{x, y \mid x, y \in \mathbb{R}, y = \phi(x)\}$  be a curve in the plane. Define a sequence of reals by  $(v_\phi(n))_{n \geq 1} = \{v_{i,j} \mid (i, j) \in \mathbb{N}^d, (i, j) + [0, 1)^2 \cap \phi \neq \emptyset\}$ .

**Problem 4.** Find explicitly the c.u.d. double sequence  $(v_{n,m})_{n,m \geq 1}$ , such that the sequence  $(v_\phi(n))_{n \geq 1}$  is c.u.d. for all linear (polynomial) curves  $\phi$ .



**2. Auxiliary notations and results.**

To estimate the discrepancy we use the Erdős-Turan-Koksma inequality (see, for example, [4] p.18)

$$(13) \quad ND_N \leq \left(\frac{3}{2}\right)^k \left(\frac{2N}{M+1} + \sum_{\substack{0 < \max_{1 \leq i \leq k} (|m_i|) \leq M}} \frac{|\sum_{x=0}^{N-1} e(m_1 u_{1x} + \dots + m_k u_{kx})|}{\bar{m}_1 \dots \bar{m}_k}\right)$$

where  $e(y) = e^{2\pi iy}$ ,  $\bar{m} = \max(1, |m|)$ ,  $M \geq 1$  - arbitrary.

We shall use the following estimates (see, for example, [6] p.1)

$$(14) \quad \left| \sum_{x=A}^{A+q-1} e(\theta x) \right| \leq \min \left( P, \frac{1}{2\|\theta\|} \right),$$

where  $\|\theta\| = \min(\{\theta\}, 1 - \{\theta\})$ .

**3. Proof of Theorem 1.**

We consider separately the following cases: when the line is horizontal, vertical, rational ( $\alpha \in \mathbb{Q}$ ), and irrational ( $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ ). We start with the difficult case:

**3.1. The case of irrational  $\alpha$ .**

We shall find the number of occurrences of a fixed block of digits in the considered configuration in term of Diophantine inequalities.

Since  $\alpha$  is irrational,  $\{\alpha i\} \neq \{\alpha j\}$  ( $i \neq j$ ). Let  $(\xi_i)_{i=1}^m$  be the set  $\{1 - \{\alpha i\} \mid i = 1, \dots, m\}$  arranged in increasing order. Put  $\xi_0 = 0$ ,  $\xi_{m+1} = 1$ . Let  $f$  be the map of the set  $\{1, \dots, m\}$  so that

$$(15) \quad \xi_{f(i)} = 1 - \{\alpha i\}, \quad i = 1, \dots, m.$$

For a fixed integer  $\nu \in [0, m]$  consider the sequence  $(l_i(\nu))_{i=1}^m$ :

$$(16) \quad l_i(\nu) = \begin{cases} [\alpha i], & \text{if } \nu < f(i) \\ [\alpha i] + 1 & \text{otherwise.} \end{cases}$$

**Lemma 1.** *Let*

$$(17) \quad \left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\} \in \left[ \frac{t + \xi_\nu}{2n}, \frac{t + \xi_{\nu+1}}{2n} \right), \quad \text{with } \nu \in [0, m].$$

*Then*

$$(18) \quad \left\{ \frac{\alpha(2nx + s + i) + \beta}{2n} \right\} \in \left[ \frac{t + l_i(\nu)}{2n}, \frac{t + l_i(\nu) + 1}{2n} \right), \quad i = 1, \dots, m.$$

*Proof.* Using (17), we have

$$\begin{aligned} z_i &= \left\{ \frac{\alpha(2nx + s + i) + \beta}{2n} \right\} = \left\{ \frac{\alpha(2nx + s) + \beta}{2n} + \frac{\alpha i}{2n} \right\} \\ &= \left\{ \frac{t + \xi_\nu + \epsilon + [\alpha i] + \{\alpha i\}}{2n} \right\}, \text{ with } 0 \leq \epsilon < \xi_{\nu+1} - \xi_\nu. \end{aligned}$$

From (15), it follows that

$$z_i = \left\{ \frac{t + [\alpha i] + \xi_\nu + 1 - \xi_{f(i)} + \epsilon}{2n} \right\}.$$

It is evident that if  $f(i) > \nu$ , then

$$\xi_\nu + \epsilon - \xi_{f(i)} < \xi_{\nu+1} - \xi_{f(i)} \leq 0.$$

Applying (16), we get (18) with  $l_i(\nu) = [\alpha i]$ . Now let  $f(i) \leq \nu$ ; then

$$0 \leq \xi_\nu + \epsilon - \xi_{f(i)} < \xi_{\nu+1}.$$

By (16), we obtain (18) with  $l_i(\nu) = [\alpha i] + 1$ . □

Let us examine the conditions for the line  $(z, \alpha z + \beta)_{z \geq 0}$  to intersect the square  $(2nx + s, 2ny' + t') + [0, 1]^2$  where  $s, t', x, y'$  are integers,  $0 \leq x, y' < b^{2n^2}, 0 \leq s < 2n, 0 \leq t' < 2n - \sqrt{n}$ :

$$(19) \quad [\alpha(2nx + s) + \beta] \leq 2ny' + t' < \alpha(2nx + s + 1) + \beta.$$

Consider the pair  $(y, t)$  such that

$$(20) \quad \begin{aligned} [\alpha(2nx + s) + \beta] &= 2ny + t, \\ 2ny + t &\leq \alpha(2nx + s) + \beta < 2ny + t + 1. \end{aligned}$$

Hence

$$y + \frac{t}{2n} \leq \frac{\alpha(2nx + s) + \beta}{2n} < y + \frac{t + 1}{2n}.$$

We get that (20) is valid for some integer  $y$  if and only if

$$(21) \quad \left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\} \in \left[ \frac{t}{2n}, \frac{t + 1}{2n} \right).$$

Let  $U_n(N)$  ( $U_n^*(N)$ ) be the subset of all lattice points  $(2nx + s, 2ny' + t')$  such that the line  $(z, \alpha z + \beta)_{z \geq 0}$  intersects the square  $(2nx + s, 2ny' + t') + [0, 1]^2$  where  $0 \leq x, y' < N, 0 \leq s < 2n$ , and  $0 \leq t' < 2n$  (respectively  $0 \leq t' < 2n - \sqrt{n}$ ).

Using (16) and (19), we have

$$(22) \quad (1 + [\alpha])2nN \leq \#U_n(N) \leq (2 + [\alpha])2nN.$$

It is easy to see that

$$(23) \quad \#U_n(N) = \#U_n^*(N) + O(N\sqrt{n}).$$

Fix  $x$  and  $s$ . We get from (19) that a lower bound of  $2ny' + t'$  is equal to

$$(24) \quad [\alpha(2nx + s) + \beta] = 2ny_0 + t_0$$

for some  $y_0, t_0$ . Since  $\alpha$  is irrational, for given  $\beta$  there exists an  $n(\beta)$  such that for any greater  $n$ ,  $\{\alpha n + \beta\} > 0$ .

Again from (19) we have that for  $n > n(\beta)$  a upper bound of  $2ny' + t'$  is equal to

$$(25) \quad [\alpha(2nx + s + 1) + \beta] = 2ny_1 + t_1$$

for some  $y_1, t_1$ . Let  $n > \max(n(\beta), \alpha)$  and  $t \in [0, 2n - \sqrt{n}]$  satisfy (20). We have from (20), (24), and (25) that

$$(26) \quad y_1 = y_0 = y \quad \text{and} \quad t_0 = t.$$

To determine  $t_1$ , we use (20), (21), and (25):

$$\left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\} \in \left[ \frac{t}{2n}, \frac{t + 1}{2n} \right)$$

and

$$(27) \quad \left\{ \frac{\alpha(2nx + s + 1) + \beta}{2n} \right\} \in \left[ \frac{t_1}{2n}, \frac{t_1 + 1}{2n} \right).$$

Clearly, there exists a unique  $\nu_0 = \nu_0(x, s, t) \in [0, m]$  such that

$$\left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\} \in \left[ \frac{t + \xi_{\nu_0}}{2n}, \frac{t + \xi_{\nu_0+1}}{2n} \right).$$

Applying Lemma 1 with  $i = 1$  we get that

$$(28) \quad t_1 = t + l_1(\nu_0).$$

From (19), (24)-(26), and (28), we get that  $U_n^*(N)$  decomposes as follows.

$$(29) \quad U_n^*(N) = \bigcup_{0 \leq x < N} \bigcup_{0 \leq s < 2n} B(x, s)$$

where

$$(30) \quad B(x, s) = \bigcup_{0 \leq t < 2n - \sqrt{n}} \bigcup_{0 \leq j \leq l_1(\nu_0)} \{z(x, s, j) \mid \alpha_{x,s} \in \Delta_{t,\nu_0}\},$$

and

$$(31) \quad \alpha_{x,s} = \left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\}, \quad \Delta_{t,\nu} = \left[ \frac{t + \xi_\nu}{2n}, \frac{t + \xi_{\nu+1}}{2n} \right),$$

$$(32) \quad z(x, s, j) = (2nx + s, [\alpha(2nx + s) + \beta] + j).$$

Now, it follows from (31) that for given  $x$  and  $s$  there is a unique pair  $(t, \nu)$  with  $t \in [0, 2n)$ ,  $\nu \in [0, m]$  such that

$$\alpha_{x,s} \in \Delta_{t,\nu}.$$

By (30), we obtain that

$$B(x, s) = \bigcup_{0 \leq t < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{z(x, s, j) \mid \alpha_{x,s} \in \Delta_{t,\nu}\}.$$

Using (29), we get

(33)

$$U_n^*(N) = \bigcup_{0 \leq x < N} \bigcup_{0 \leq s < 2n} \bigcup_{0 \leq t < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{z(x, s, j) \mid \alpha_{x,s} \in \Delta_{t,\nu}\}$$

or

(34)

$$U_n^*(N) = \bigcup_{0 \leq s < 2n} \bigcup_{0 \leq t < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{z(x, s, j) \mid x \in E_{\nu s}(t, N)\},$$

where

(35)

$$E_{\nu s}(t, N) = \{x \in [0, N] \mid \alpha_{x,s} \in \Delta_{t,\nu}\}.$$

We consider the case  $\alpha < 1$  (hence  $x \geq y$ ) and  $s \geq t$  first. We examine the interval  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ . For  $s \in [0, \sqrt{n})$  and  $s \in [2n - \sqrt{n}, 2n)$ , we use a trivial estimate.

Let

(36)

$$U'_n(N) = \bigcup_{0 \leq t \leq s < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{z(x, s, j) \mid x \in E_{\nu s}(t, N)\},$$

and

(37)

$$E_{\nu s}(t, Q_1, Q_2) = \{x \in [Q_1, Q_1 + Q_2) \mid \alpha_{x,s} \in \Delta_{t,\nu}\}.$$

For  $z \in U_n(N)$  denote by  $z^{(1)}, \dots, z^{(m)}$  the sequence of  $z^{(i)} \in U_n(N)$ , ( $i = 1, \dots, m$ ) stretching in succession, such that  $z^{(1)} = z$ . For  $z = z(x, s, j) = (2nx + s, [\alpha(2nx + s) + \beta] + j)$  with  $j \in [0, l_1(\nu)]$ , denote by  $z^{(i)}(x, s, j)$  a vector  $z^{(i)}$ :

(38)

$$z^{(i)}(x, s, j) = z^{(i)}, \quad i = 1, \dots, m.$$

Let  $G_m = (g_1, \dots, g_m)$  be a block of digits i.e.,  $g_i \in \Delta_b$ ,  $i = 1, \dots, m$ ,

(39)

$$V_n(N, G_m) = \{z \in U_n(N) \mid G_m = (\omega_n(z^{(1)}), \dots, \omega_n(z^{(m)}))\}$$

is the set of the linear indices of beginning of a block  $G_m$  in the linear configuration of digits, confined to the  $\omega_n$ -square configuration.

Define also

(40)

$$V_n^*(N, G_m) = \{z \in U_n^*(N) \mid G_m = (\omega_n(z^{(1)}), \dots, \omega_n(z^{(m)}))\}.$$

Let

(41)

$$\chi_{\omega_n G_m}(z(x, s, j)) = \begin{cases} 1 & \text{if } z^{(i)}(x, s, j) = g_i, \quad i = 1, \dots, m \\ 0 & \text{otherwise.} \end{cases}$$

Applying (33) and (39)-(41), we obtain that

$$V_n^*(N, G_m) = \bigcup_{0 \leq x < N} \bigcup_{0 \leq s < 2n} \bigcup_{0 \leq t < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{z(x, s, j) \mid \chi_{\omega_n G_m}(z(x, s, j)) = 1, \alpha_{x,s} \in \Delta_{t,\nu}\}.$$

Analogously to (34) and (35), we have that

$$(42) \quad V_n^*(N, G_m) = \bigcup_{0 \leq s < 2n} \bigcup_{0 \leq t < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{z(x, s, j) \mid x \in A_{j\nu s}(t, N)\},$$

where

$$(43) \quad A_{j\nu s}(t, N) = \{x \in [0, N] \mid \alpha_{x,s} \in \Delta_{t,\nu}, \text{ and } \chi_{\omega_n G_m}(z(x, s, j)) = 1\}.$$

Let

$$(44) \quad V'_n(N, G_m) = \bigcup_{0 \leq t \leq s < 2n - \sqrt{n}} \bigcup_{0 \leq \nu \leq m} \bigcup_{0 \leq j \leq l_1(\nu)} \{(2nx + s, [\alpha(2nx + s) + \beta] + j) \mid x \in A_{j\nu s}(t, N)\},$$

and

$$(45) \quad A_{j\nu s}(t, Q_1, Q_2) = \{x \in [Q_1, Q_1 + Q_2) \mid \alpha_{x,s} \in \Delta_{t,\nu}, \text{ and } \chi_{\omega_n G_m}(z(x, s, j)) = 1\}.$$

Next we write an analytic expression for the sequence  $(z^{(i)}(x, s, j))_{1 \leq i \leq m}$ : For given integers  $t \in [0, 2n - \sqrt{n})$ ,  $x, s, \nu$ , and  $j \in [0, l_1(\nu))$  put

$$(46) \quad \alpha_{x,s} \in \Delta_{t,\nu}.$$

Using (19)-(21) and Lemma 1 we have similarly to (24)-(28) that the part of  $U'_n(N)$  corresponding to the interval  $[2nx + s + i', 2nx + s + i' + 1)$  (with condition (46)) is equal to

$$((2nx + s + i', [\alpha(2nx + s) + \beta] + l_{i'}(\nu) + v))$$

where  $v = j\delta_{i'}, j\delta_{i'} + 1, \dots, l_{i'+1}(\nu) - l_{i'}(\nu) - 1$ ,  $i' = 0, 1, \dots, m - 1$ , with  $l_0(\nu) = 0$ ,

$$(47) \quad \delta_{i'} = \begin{cases} 1 & \text{if } i' = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the sequence  $(z^{(i)}(x, s, j))_{1 \leq i \leq m}$  (38) coincides with the first  $m$  terms of the following sequence

$$(48) \quad (((2nx + s + i', [\alpha(2nx + s) + \beta] + l_{i'}(\nu) + v))_{j\delta_{i'} \leq v < l_{i'+1}(\nu) - l_{i'}(\nu)})_{i' \geq 0}.$$

Now let  $(k_n)_{n \geq 1}$  be an increasing sequence of integers from the set

$$(49) \quad \{(s + i')^2 + t + l_{i'}(\nu) + v + 1 \mid v = j\delta_{i'}, j\delta_{i'} + 1, \dots, l_{i'+1}(\nu) - l_{i'}(\nu) - 1, \\ i' = 0, 1, \dots, m - 1\}.$$

Observe that the equality

$$\chi_{\omega_n G_m}(z(x, s, j)) = 1$$

(with condition (46) or (21)) is equivalent to the assertion (found in (5), (6), (48), and (49)) that the  $(k_i - 1)$ th digit of the  $b$ -expansion of  $x^2 + y$  is equal to  $g_i$  ( $i = 1, \dots, m$ ):

$$(50) \quad x^2 + y = x_1 + g_1 b^{k_1 - 1} + x_2 b^{k_1} + g_2 b^{k_2 - 1} + \dots + x_m b^{k_m - 1} + g_m b^{k_m - 1} + x_{m+1} b^{k_m},$$

where  $x_i \in [0, b^{k_i - k_{i-1} - 1})$  are integers ( $i = 1, \dots, m$ ),  $k_0 = 0$ , and  $y = y(x, s)$  is a function of  $(x, s)$  (see (20)).

In the sequel we consider  $y$  as  $y = y(x, s)$ . Using (43), (45), and (50), we have that

$$(51) \quad A_{j\nu s}(t, N) = \{x \in [0, N) \mid \alpha_{x,s} \in \Delta_{t,\nu}, \text{ and } x^2 + y = x_1 \\ + g_1 b^{k_1 - 1} + x_2 b^{k_1} + g_2 b^{k_2 - 1} + \dots + x_m b^{k_m - 1} + g_m b^{k_m - 1} + x_{m+1} b^{k_m}\},$$

and

$$(52) \quad A_{j\nu s}(t, Q_1, Q_2) = \{x \in [Q_1, Q_1 + Q_2) \mid \alpha_{x,s} \in \Delta_{t,\nu}, \text{ and } x^2 + y = x_1 \\ + g_1 b^{k_1 - 1} + x_2 b^{k_1} + g_2 b^{k_2 - 1} + \dots + x_m b^{k_m - 1} + g_m b^{k_m - 1} + x_{m+1} b^{k_m}\}.$$

Let

$$(53) \quad B_N(s, t, j, \nu, g) = \#\{x \in [0, N) \mid \alpha_{x,s} \in \Delta_{t\nu}, \text{ and} \\ \{(x^2 + y)b^{-k_m}\} \in [\frac{g}{b^{k_m - k_1 + 1}}, \frac{g + 1}{b^{k_m - k_1 + 1}})\}.$$

**Lemma 2.** Let  $N \in [b^{2n^2 - 5n}, b^{2n^2})$ ,  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ . Then

$$(54) \quad \#A_{j\nu s}(t, N) = \sum_{x_2=0}^{b^{k_2 - k_1 - 1} - 1} \dots \sum_{x_m=0}^{b^{k_m - k_{m-1} - 1} - 1} B_N(s, t, j, \nu, g),$$

where

$$(55) \quad g = g(x_2, \dots, x_m) = g_1 + x_2 b + g_2 b^{k_2 - k_1} + \dots + x_m b^{k_m - 1 - k_1 + 1} + g_m b^{k_m - k_1}.$$

*Proof.* Using (51) and (55) we get

$$A_{j\nu s}(t, N) = \bigcup_{x_2=0}^{b^{k_2-k_1-1}-1} \dots \bigcup_{x_m=0}^{b^{k_m-k_{m-1}-1}-1} \{x \in [0, N) \mid \alpha_{x,s} \in \Delta_{t\nu}, \text{ and } x^2 + y = x_1 + gb^{k_1-1} + x_{m+1}b^{k_m}\}.$$

Bearing in mind that the condition

$$x^2 + y = x_1 + gb^{k_1-1} + x_{m+1}b^{k_m}$$

is equivalent to the condition

$$\{(x^2 + y)b^{-k_m}\} \in [\frac{g}{b^{k_m-k_1+1}}, \frac{g+1}{b^{k_m-k_1+1}})$$

with any integer  $x_1 \in [0, b^{k_1-1})$ ,  $x_{m+1} \geq 0$  (integer  $g \in [0, b^{k_m-k_1+1})$  is fixed) we obtain from (53) the assertion of the lemma.  $\square$

Let  $q_1 < q_2 \dots < q_n < \dots$  be a sequence of denominators of the continued fraction of  $\alpha$ ,

$$(56) \quad \alpha = \frac{p_r}{q_r} + \frac{\theta_r}{q_r q_{r+1}}, \text{ with } (p_r, q_r) = 1, \quad |\theta_r| \leq 1$$

and

$$(57) \quad q_r \leq b^{(s+6m)^2/2} < q_{r+1}$$

( $r = r(s, m)$  is the function of  $s, m$  with fixed integer  $m$ , and  $s = 1, 2, \dots$ ).

We will consider two cases. In the first case  $q_r$  is large ( $q_r \geq n^6 b^{2sm}$ , the growth of  $q_r$  is small, case of almost all  $\alpha$ ). In the second case  $q_r$  is small ( $q_r \leq n^6 b^{2sm}$ , the growth of  $q_r$  is large), case of almost no  $\alpha$  (the complicated case).

**3.1.1. case  $q_r$  large.**

In this subsection, we prove that the frequency of the appearance of the block of digits  $G_m$  in the sequence  $(u_{\alpha,\beta}(n))_{n=1}^N$  tends to  $b^{-m}$ . According to (42) and (51)-(54), it is sufficient to get the estimate of  $B_N(s, t, j, \nu, g)$  (see (53)) to prove the above statement. We use the estimate of exponential sums (Lemma 3) and the Erdős-Turan-Koksma inequality (13) to compute  $B_N(s, t, j, \nu, g)$ .

**Lemma 3.** Let  $q_r \geq n^6 b^{2sm}$ ,  $N \in [b^{2n^2-5n}, b^{2n^2})$ ,  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ ,  $0 < \max(|m_1|, |m_2|) \leq M = n^4 b^{2sm}$ ,

$$(58) \quad S_N(m_1, m_2) = \sum_{x=0}^{N-1} e(m_1(x^2 + \alpha x + \gamma)b^{-k_m} + m_2(\alpha(2nx + s) + \beta)/2n).$$

Then

$$(59) \quad |S_N(0, m_2)| \leq q_r,$$

$$(60) \quad |S_N(m_1, m_2)| = O(Nn^{-6}b^{-2sm}), \quad m_1 \neq 0.$$

*Proof.* Let  $m_1 = 0$ . It is easy to see that

$$(61) \quad \sigma_1 = |S_N(0, m_2)| = \left| \sum_{x=0}^{N-1} e(m_2\alpha x) \right|.$$

Using (14) we have

$$(62) \quad \sigma_1 \leq \min\left(N, \frac{1}{2\|m_2\alpha\|}\right).$$

Bearing in mind (56) and the condition of Lemma 3, we obtain that

$$0 < |m_2| \leq n^4 b^{2sm} < q_r/2$$

and

$$\|m_2\alpha\| = \left\| m_2 \left( \frac{p_r}{q_r} + \frac{\theta_r}{q_r q_{r+1}} \right) \right\| \geq \left| \frac{1}{q_r} - \frac{1}{2q_{r+1}} \right| \geq \frac{1}{2q_r}.$$

We have from (62) that  $\sigma_1 \leq q_r$ . Hence (59) follows from (61). Assume  $m_1 \neq 0$ . It is easy to see that

$$(63) \quad \begin{aligned} \sigma_2^2 &= |S_N(m_1, m_2)|^2 \\ &= \sum_{x_1, x_2=0}^{N-1} e(m_1(x_1^2 - x_2^2 + \alpha(x_1 - x_2))b^{-k_m} + m_2\alpha(x_1 - x_2)). \end{aligned}$$

Let  $x_1 - x_2 = u$ . Then  $u \in [-N+1, N-1]$ ,  $x_2 = x_1 - u \in [0, N-1]$ ,  $x_1 + x_2 = 2x_1 - u$ ,  $x_1^2 - x_2^2 = u(2x_1 - u)$ ,  $x_1 \in [\max(0, u), \min(N-1, N-1+u)]$ . Hence

$$\begin{aligned} \sigma_2^2 &= \sum_{u=-N+1}^{N-1} \sum_{x_1=\max(0,u)}^{\min(N-1, N-1+u)} e(m_1(2ux_1 - u^2 + \alpha u)b^{-k_m} + m_2\alpha u) \\ &\leq \sum_{u=-N+1}^{N-1} \left| \sum_{x_1=\max(0,u)}^{\min(N-1, N-1+u)} e(2m_1 u x_1 b^{-k_m}) \right|. \end{aligned}$$

By (14) we get

$$(64) \quad \sigma_2^2 \leq \sum_{u=-N+1}^{N-1} \min\left(2N, \frac{1}{2\|2m_1 u b^{-k_m}\|}\right).$$



Let  $(2m_1, b^{k_m}) = f$ ,  $u = u_1 + u_2 b^{k_m}/f$  with  $u_1 \in [0, b^{k_m}/f - 1]$  and  $u_2 \in [-N_1, N_1]$ , where  $N_1 = [f(N - 1)b^{-k_m}] + 1$ . It is easy to see that

$$\begin{aligned}
 \sigma_2^2 &\leq \sum_{u_2=-N_1}^{N_1} (2N + \sum_{u_1=1}^{b^{k_m}/f-1} \frac{1}{2\|fu_1 b^{-k_m}\|}) \\
 &\leq (\frac{2N}{b^{k_m}/f} + 3)(2N + (b^{k_m}/f) \log b^{k_m}) \\
 (65) \quad &\leq 4N^2|m_1|b^{-k_m} + 6N + 2Nk_m \log b + 3b^{k_m}k_m \log b.
 \end{aligned}$$

Hence

$$\sigma_2 = O(Nb^{-k_m/2}\sqrt{|m_1|} + \sqrt{k_m}b^{k_m/2}).$$

Using (49) we obtain that  $s^2 + 1 \leq k_m \leq (s + m)^2 + \alpha m + 2$  and

$$(66) \quad \sigma_2 = O(Nb^{-s^2/2}n^2b^{sm} + (s + m)b^{(s+m)^2/2}).$$

Taking into account that  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ , we get

$$(67) \quad Nn^2b^{sm-s^2/2} = O(Nn^{-6}b^{-2sm}),$$

and

$$\begin{aligned}
 (s + m)^2b^{(s+m)^2/2} &= O(n^2b^{(s+4m)^2/2-2sm}) = O(n^2b^{(2n-\sqrt{n}+4m)^2/2-2sm}) \\
 (68) \quad &= o(n^{-6}b^{2n^2-5n-2sm}) = O(Nn^{-6}b^{-2sm}).
 \end{aligned}$$

Using (63), and (66)-(68), we obtain (60). □

**Lemma 4.** Let  $q_r \geq n^6b^{2sm}$ ,  $N \in [b^{2n^2-5n}, b^{2n^2})$ ,  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ . Then

$$\begin{aligned}
 ND_N &= ND_N(\{(x^2 + \alpha x + \gamma)b^{-k_m}\}, \{(\alpha(2nx + s) + \beta)/2n\}_{x=0}^{N-1}) \\
 &= O(Nn^{-4}b^{-k_m+k_1}).
 \end{aligned}$$

*Proof.* Using the Erdős-Turan-Koksma inequality (13) with  $k = 2$ ,  $M = n^4b^{2sm}$ , and (58), we obtain that

$$(69) \quad ND_N = O(Nn^{-4}b^{-2sm} + \sigma_3 + \sigma_4)$$

where

$$(70) \quad \sigma_3 = \sum_{0 < |m_2| \leq M} \frac{|S(0, m_2)|}{\bar{m}_2}$$

and

$$(71) \quad \sigma_4 = \sum_{0 < \max(|m_1|, |m_2|) \leq M, m_1 \neq 0} \frac{|S(m_1, m_2)|}{\bar{m}_1 \bar{m}_2}.$$

Applying (59) we have that

$$\sigma_3 \leq 2 \sum_{m_2=1}^M \frac{q_r}{\bar{m}_2} \leq 2q_r(1 + \log M).$$

Using (57) and the conditions of Lemma 4, we obtain that

$$\begin{aligned} \sigma_3 &= O(b^{(s+6m)^2/2} sm) = O(nb^{(2n-\sqrt{n}+6m)^2/2}) \\ (72) \quad &= O(n^{-4}b^{2n^2-5n-4nm}) = O(n^{-4}b^{2n^2-5n-2sm}) = O(n^{-4}Nb^{-2sm}). \end{aligned}$$

From (60) and (71), we get

$$\begin{aligned} \sigma_4 &= O(n^{-6}Nb^{-2sm} \sum_{|m_1|, |m_2| \leq M} \frac{1}{\bar{m}_1 \bar{m}_2}) \\ (73) \quad &= O(n^{-6}Nb^{-2sm}(\log M)^2) = O(s^2n^{-6}Nb^{-2sm}) = O(n^{-4}Nb^{-2sm}). \end{aligned}$$

Combining (73) with (69)-(72), we have that

$$(74) \quad ND_N = O(n^{-4}Nb^{-2sm}).$$

By (49), and (16) we obtain

$$(75) \quad -k_m + k_1 \geq s^2 - (s + m)^2 - \alpha m - 1 = -2sm - m^2 - \alpha m - 1.$$

The assertion of the lemma follows from (74). □

Let

$$(76) \quad \begin{aligned} &B_N(s, t, \nu, \gamma, \gamma_1, \gamma_2) \\ &= \#\{x \in [0, N) \mid \alpha_{xs} \in \Delta_{t\nu}, \{(x^2 + \alpha x + \gamma)b^{-k_m}\} \in [\gamma_1, \gamma_2]\}. \end{aligned}$$

**Corollary 1.** *Let  $0 \leq \gamma_1 < \gamma_2 \leq 1$ . Under the assumptions of Lemma 4,*

$$B_N(s, t, \nu, \gamma, \gamma_1, \gamma_2) = N \frac{\xi_{\nu+1} - \xi_\nu}{2n} (\gamma_2 - \gamma_1) + O(Nn^{-4}b^{-k_m+k_1}).$$

*Proof.* The proof follows from (10), (31), and Lemma 4. □

Apply (20) to get a formula for the number  $x^2 + y$  (where  $y = y(x, s)$ ):

$$(77) \quad [\alpha(2nx + s) + \beta] = 2ny + t.$$

Let also

$$\begin{aligned} \alpha_{x,s} &= \left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\} \in \Delta_{t,\nu} = \left[ \frac{t + \xi_\nu}{2n}, \frac{t + \xi_{\nu+1}}{2n} \right), \\ \alpha_{x,s} &= \frac{t + \epsilon}{2n}, \quad \text{where } \epsilon \in [0, 1). \end{aligned}$$

Then

$$\frac{\alpha(2nx + s) + \beta}{2n} = i + \frac{t + \epsilon}{2n}$$

for an integer  $i$ . Hence

$$\{\alpha(2nx + s) + \beta\} = \{2ni + t + \epsilon\} = \epsilon.$$

Using (77) we obtain

$$y = \frac{[\alpha(2nx + s) + \beta] - t}{2n} = \frac{\alpha(2nx + s) + \beta - t - \{\alpha(2nx + s) + \beta\}}{2n} \\ = \frac{\alpha(2nx + s) + \beta - t}{2n} - \frac{\epsilon}{2n}.$$

Therefore

$$(78) \quad x^2 + y = x^2 + \alpha x + \frac{\alpha s + \beta - t}{2n} - \frac{\epsilon}{2n}.$$

**Lemma 5.** *Let  $0 \leq g < b^{k_m - k_1 + 1}$ . Under the assumptions of Lemma 4,*

$$(79) \quad B_N(s, t, j, \nu, g) = N \frac{\xi_{\nu+1} - \xi_\nu}{2n} b^{k_1 - k_m - 1} (1 + O(n^{-3})),$$

where  $B_N(s, t, j, \nu, g)$  is defined in (53).

*Proof.* We have from (78) that if

$$\{(x^2 + \alpha x + \gamma_0)b^{-k_m}\} \in [\frac{g}{b^{k_m - k_1 + 1}} + \theta_0, \frac{g + 1}{b^{k_m - k_1 + 1}})$$

with  $\theta_0 = \frac{1}{2nb^{k_m}}$ , and  $\gamma_0 = \frac{\alpha s + \beta - t}{2n}$ , then

$$(80) \quad \{(x^2 + y)b^{-k_m}\} \in [\frac{g}{b^{k_m - k_1 + 1}}, \frac{g + 1}{b^{k_m - k_1 + 1}}).$$

Hence and from (53) and (76) we get

$$(81) \quad B_N(s, t, \nu, \gamma_0, \frac{g}{b^{k_m - k_1 + 1}} + \theta_0, \frac{g + 1}{b^{k_m - k_1 + 1}}) \leq B_N(s, t, j, \nu, g).$$

Conversely, assume (80), then

$$\{(x^2 + \alpha x + \gamma_0)b^{-k_m}\} \in [\frac{g}{b^{k_m - k_1 + 1}} - \theta_0, \frac{g + 1}{b^{k_m - k_1 + 1}}) \text{ for } g \geq 1$$

and

$$\{(x^2 + \alpha x + \gamma_0)b^{-k_m}\} \in [0, \frac{1}{b^{k_m - k_1 + 1}}) \cup [1 - \theta_0, 1) \text{ for } g = 0.$$

Hence

$$(82) \quad B_N(s, t, j, \nu, g) \leq B_N(s, t, \nu, \gamma_0, \max(0, \frac{g}{b^{k_m - k_1 + 1}} - \theta_0), \frac{g + 1}{b^{k_m - k_1 + 1}}) \\ + B_N(s, t, \nu, \gamma_0, 1 - \theta_0, 1).$$

Using Corollary 1, we have

$$(83) \quad B_N(s, t, j, \nu, g) = N \frac{\xi_{\nu+1} - \xi_\nu}{2n} (\frac{1}{b^{k_m - k_1 + 1}} + \epsilon_1 \theta_0) + O(Nn^{-4}b^{-k_m + k_1}),$$

with  $|\epsilon_1| < 2$ . Applying (49) and (80), we get that

$$b^{k_1 - k_m - 1} + \epsilon \theta_0 = b^{k_1 - k_m - 1} (1 + O(n^{-1}b^{-k_1})) = b^{k_1 - k_m - 1} (1 + O(n^{-3})).$$

Bearing in mind (83) and that  $\xi_{\nu+1} - \xi_\nu \geq c > 0$  ( $\nu = 0, 1, \dots, m$ ), we obtain (79). □

**Corollary 2.** *Let  $0 \leq t \leq s$ ,  $j \in [0, l_1(\nu)]$ . Under the assumptions of Lemma 4,*

$$(84) \quad \#A_{j\nu s}(t, N) = b^{-m} N \frac{\xi_{\nu+1} - \xi_\nu}{2n} (1 + O(1/n^3)).$$

The proof follows from Lemma 2 and Lemma 5. □

**Lemma 6.** *Let  $0 \leq t \leq s$ . Under the assumptions of Lemma 4,*

$$\#A_{j\nu s}(t, N) = b^{-m} \#E_{\nu s}(t, N) (1 + O(1/n)).$$

*Proof.* We get from (31), (37), and (56) that

$$E_{\nu s}(t, Q_1, q_r) = \{x \in [0, q_r) \mid \{ \frac{xp_r}{q_r} + \frac{\alpha(2nQ_1 + s) + \beta}{2n} + \frac{\theta_r x}{q_r q_{r+1}} \} \in \Delta_{t,\nu} \}.$$

Using (57) and that  $(p_r, q_r) = 1$ , we obtain

$$E_{\nu s}(t, Q_1, q_r) = \{x \in [0, q_r) \mid \{ \frac{x}{q_r} + \gamma + \frac{\theta(x)}{q_r} \} \in \Delta_{t,\nu} \}$$

with  $\gamma = \frac{\alpha(2nQ_1 + s) + \beta}{2n}$ ,  $|\theta(x)| \leq 1$ .

Recalling the definition (31) of  $\Delta_{t,\nu}$ , get

$$\#E_{\nu s}(t, Q_1, q_r) = q_r \left( \frac{\xi_{\nu+1} - \xi_\nu}{2n} \right) + 4\epsilon, \quad \text{where } |\epsilon| \leq 1.$$

Taking into account that  $\xi_{\nu+1} - \xi_\nu \geq c > 0$  ( $\nu = 0, 1, \dots, m$ ) and the condition of the lemma, we obtain

$$\#E_{\nu s}(t, Q_1, q_r) = q_r \left( \frac{\xi_{\nu+1} - \xi_\nu}{2n} \right) (1 + O(1/n)).$$

From (35), (37), and the condition of the lemma, we get

$$\begin{aligned} \#E_{\nu s}(t, N) &= \sum_{0 \leq i < [N/q_r]} \#E_{\nu s}(t, iq_r, q_r) + \epsilon q_r \\ &= [N/q_r] q_r \left( \frac{\xi_{\nu+1} - \xi_\nu}{2n} \right) (1 + O(1/n)) + \epsilon q_r \\ &= N \left( \frac{\xi_{\nu+1} - \xi_\nu}{2n} \right) (1 + O(1/n)). \end{aligned}$$

Applying Corollary 2, the assertion of Lemma 6 follows. □

**3.1.2. Small  $q_r$  (the complicated case).**

The main idea of this subsection is the partition of the interval of summation into  $q_r$  arithmetical progressions  $(x + vq_r)_{v \geq 0}$  with  $x \in [0, q_r)$ . Next we fix the integer  $x$  and compute (similarly to Lemma 3 - Lemma 5) the number of occurrences of the block of digits  $G_m$  in the sequence  $(u_{\alpha,\beta}(n))_{n=1}^N$ .

We obtain in Lemma 10 the main estimate of this subsection for the special Diophantine condition. We find in Lemma 11 the set of parameters, for which the above Diophantine condition is true. In Lemma 12 we partition the interval of summation (according to Lemma 11) to essential intervals, and to auxiliary intervals. We use the estimate of Lemma 10 for the essential intervals and the trivial estimate for auxiliary intervals. We collect all the gotten results in Lemma 13 - Lemma 15.

**Lemma 7.** *Let  $q_r \leq n^6 b^{2sm}$ ,  $P \geq n^4 b^{s^2/2+3sm}$ ,  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ ,  $0 < |m| \leq M = n^4 b^{2sm}$ ,  $x \in [H, (H + 1)q_r]$ ,  $H \geq 0$ ,*

$$(85) \quad S_P(m) = \sum_{v=0}^{P-1} e(m((x + vq_r)^2 + \alpha(x + vq_r) + \gamma)b^{-km}).$$

Then

$$|S_P(m)| = O(Pn^{-3}b^{-2sm}).$$

*Proof.* Similarly to (63)-(64), we have that

$$|S_P(m)|^2 \leq \sum_{u=-P+1}^{P-1} \min(2P, \frac{1}{2\|2mq_r^2 b^{-km} u\|}).$$

Let  $(2mq_r^2, b^{km}) = f$ . We note that

$$1 \leq f \leq 2mq_r^2 = O(n^{16}b^{6sm}).$$

By (49) and by the conditions of the lemma, we get similarly to (65) that

$$\begin{aligned} |S_P(m)|^2 &= O(P^2 f b^{-km} + Pk_m + b^{km} k_m) \\ &= O(P^2 n^{16} b^{6sm-s^2} + s^2 P + s^2 b^{(s+m)^2}) \\ &= O(P^2 b^{-10sm} + s^2 n^{-8} P^2 b^{(s+m)^2-s^2-6sm}) \\ &= O(n^{-6} P^2 b^{-4sm}). \end{aligned} \quad \square$$

**Lemma 8.** *Let  $q_r \leq n^6 b^{2sm}$ ,  $P \geq n^4 b^{s^2/2+3sm}$ ,  $s \in [\sqrt{n}, 2n - \sqrt{n}]$ ,  $x \in [H, (H + 1)q_r]$ ,  $H \geq 0$ . Then*

$$PD_P = PD_P(\{((x+vq_r)^2 + \alpha(x+vq_r) + \gamma)b^{-km}\}_{v=0}^{P-1}) = O(Pn^{-4}b^{-km+k_1}).$$

*Proof.* Using (13) with  $k = 1$ ,  $M = n^4 b^{2sm}$ , Lemma 7, and (85), we obtain that

$$\begin{aligned} PD_P &= O(Pn^{-4}b^{-2sm} + \sum_{0 < |m| \leq M} \frac{|S_P(m)|}{\bar{m}}) = O(Pn^{-4}b^{-2sm} \\ &+ Pn^{-3}b^{-2sm} \log(n^4 b^{2sm})) = O(Pn^{-2}b^{-2sm}) = O(Pn^{-2}b^{-km+k_1}). \end{aligned} \quad \square$$

Let

$$B_P(x, s, \nu, \gamma, \gamma_1, \gamma_2) = \#\{v \in [0, P) \mid \{((x + \nu q_r)^2 + \alpha(x + \nu q_r) + \gamma)b^{-k_m}\} \in [\gamma_1, \gamma_2]\}.$$

**Corollary 3.** *Let  $0 \leq \gamma_1 < \gamma_2 \leq 1$ . Under the assumptions of Lemma 8,*

$$B_P(x, s, t, \nu, \gamma, \gamma_1, \gamma_2) = P(\gamma_2 - \gamma_1) + O(Pn^{-2}b^{-k_m+k_1}).$$

The proof follows from (10), (86), and Lemma 8. □

Let

$$B_P(x, s, t, j, \nu, g) = \#\{v \in [0, P) \mid \{((x + \nu q_r)^2 + y)b^{-k_m}\} \in [\frac{g}{b^{k_m-k_1+1}}, \frac{g+1}{b^{k_m-k_1+1}}]\},$$

where  $g \in [0, b^{k_m-k_1+1})$  be integer,  $x \in [H, (H+1)q_r)$ , and  $y = y(x + \nu q_r, s)$  satisfy (20).

**Lemma 9.** *Under the assumptions of Lemma 8,*

$$B_P(x, s, t, j, \nu, g) = Pb^{k_1-k_m-1}(1 + O(n^{-2})).$$

*Proof.* Using Corollary 3 and repeating the proof of Lemma 5, we obtain (89). □

**Lemma 10.** *Under the assumptions of Lemma 8, let*

$x \in E_{\nu s}(t, Hq_r, q_r)$  if and only if  $x + (Q - H)q_r \in E_{\nu s}(t, Qq_r, q_r)$  for all  $Q \in [H, H + P]$ . Then

$$\#A_{j\nu s}(t, Hq_r, Pq_r) = b^{-m}P\#E_{\nu s}(t, Hq_r, q_r)(1 + O(1/n)).$$

*Proof.* It follows from (52), (37), and the condition of the lemma that

$$\#A_{j\nu s}(t, Hq_r, Pq_r) = \sum_{x \in E_{\nu s}(t, Hq_r, q_r)} \#A_{j\nu s x}(P)$$

with

$$A_{j\nu s x}(P) = \{v \in [0, P) \mid (x + \nu q_r)^2 + y = x_1 + g_1b^{k_1-1} + x_2b^{k_1} + \dots + g_mb^{k_m-1} + x_{m+1}b^{k_m}\}.$$

Hence

$$A_{j\nu s x}(P) = \sum_{x_2=0}^{b^{k_2-k_1-1}-1} \dots \sum_{x_m=0}^{b^{k_m-k_{m-1}-1}-1} \{v \in [0, P) \mid (x + \nu q_r)^2 + y = x_1 + gb^{k_1-1} + x_{m+1}b^{k_m}\},$$

where

$$g = g(x_2, \dots, x_m) = g_1 + x_2b + g_2b^{k_2-k_1} + \dots + x_mb^{k_{m-1}-k_1+1} + g_mb^{k_m-k_1}.$$

Bearing in mind that the condition

$$x^2 + y = x_1 + gb^{k_1-1} + x_{m+1}b^{k_m}$$

is equivalent to the condition

$$\{(x^2 + y)b^{-k_m}\} \in [\frac{g}{b^{k_m-k_1+1}}, \frac{g+1}{b^{k_m-k_1+1}})$$

with any integer  $x_1 \in [0, b^{k_1-1})$ ,  $x_2 \geq 0$  (the integer  $g \in [0, b^{k_m-k_1+1})$  is fixed), we obtain from (88) that

$$\begin{aligned} \#A_{j\nu sx}(P) &= \sum_{x_2=0}^{b^{k_2-k_1-1}-1} \dots \sum_{x_m=0}^{b^{k_m-k_{m-1}-1}-1} \#\{v \in [0, P) \mid \\ &\quad \{((x + vq_r)^2 + y)b^{-k_m}\} \in [\frac{g}{b^{k_m-k_1+1}}, \frac{g+1}{b^{k_m-k_1+1}})\} \\ &= \sum_{x_2=0}^{b^{k_2-k_1-1}-1} \dots \sum_{x_m=0}^{b^{k_m-k_{m-1}-1}-1} B_P(x, s, t, j, \nu, g). \end{aligned}$$

Using Lemma 9 we obtain

$$\begin{aligned} \#A_{j\nu sx}(P) &= \sum_{x_2=0}^{b^{k_2-k_1-1}-1} \dots \sum_{x_m=0}^{b^{k_m-k_{m-1}-1}-1} Pb^{k_1-k_m-1}(1 + O(n^{-2})) \\ (92) \quad &= b^{-m}P(1 + O(n^{-2})). \end{aligned}$$

Applying (90) the assertion of Lemma 10 follows. □

Let

$$(93) \quad \beta = b_r/q_r + \beta_1/q_r, \quad \text{with } |\beta_1| \leq 1/2$$

where  $b_r$  is an integer,

$$(94) \quad Q' = \lceil \frac{-q_{r+1}\beta_1}{2nq_r\theta_r} \rceil - 2, \quad P_1 = \lfloor n^{-7}b^{s^2/2+4sm} \rfloor, \quad Q_0 = \begin{cases} Q' & \text{if } Q' \in [0, P_1) \\ P_1 & \text{otherwise} \end{cases}$$

where  $\lceil x \rceil = x + 1$  for integer  $x$ ; otherwise  $\lceil x \rceil = \lfloor x \rfloor$ .

**Lemma 11.** *Let  $q_r \leq n^6b^{2sm}$ ,  $N \in [b^{2n^2-5n}, b^{2n^2})$ ,  $\sqrt{n} \leq t \leq s \leq 2n - \sqrt{n}$ ,  $x \in [0, q_r)$ ,  $n > m^2$ . Then  $\forall Q \in [0, Q_0)$*

$$(95) \quad x \in E_{\nu s}(t, q_r) \text{ if and only if } x + Qq_r \in E_{\nu s}(t, Qq_r, q_r),$$

and  $\forall Q_1 \in (Q_0 + 2, P_1]$

$$(96) \quad x + P_1q_r \in E_{\nu s}(t, P_1q_r, q_r) \text{ if and only if } x + Q_1q_r \in E_{\nu s}(t, Q_1q_r, q_r).$$

*Proof.* Consider the case of  $\theta_r > 0$ . (For  $\theta_r < 0$  the proof is similar.) Let

$$(97) \quad z(x) \equiv p_r(2nx + s) + b_r \pmod{2nq_r}, \quad \text{with } z(x) \in [0, 2nq_r)$$

and

$$(98) \quad \gamma = \gamma(x) = \beta_1 + \frac{\theta_r(2nx + s)}{q_{r+1}}.$$

Using (57), (93) and the condition of lemma, we have that for sufficiently large  $n$ .

$$(99) \quad |\gamma(x)| < 3/4.$$

It follows from (31), (56), (93), (97), and (98) that

$$\begin{aligned} \alpha_{x+Qq_r, s} &= \left\{ \left( \frac{p_r}{q_r}(2n(x + Qq_r) + s) + \frac{\theta_r}{q_r q_{r+1}}(2n(x + Qq_r) + s) \right) / 2n + \frac{\beta}{2n} \right\} \\ &= \left\{ \frac{p_r(2nx + s) + b_r + \beta_1}{2nq_r} + \frac{\theta_r(2n(x + Qq_r) + s)}{2nq_r q_{r+1}} \right\} \\ (100) \quad &= \left\{ \frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r Q}{q_{r+1}} \right\}. \end{aligned}$$

Applying (37) we have

$$x + Qq_r \in E_{\nu s}(t, Qq_r, q_r) \text{ if and only if } \alpha_{x+Qq_r, s} \in \Delta_{t, \nu}$$

or

$$(101) \quad \left\{ \frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r Q}{q_{r+1}} \right\} \in \left[ \frac{t + \xi_\nu}{2n}, \frac{t + \xi_{\nu+1}}{2n} \right).$$

Now let

$$(102) \quad x + Qq_r \in E_{\nu s}(t, Qq_r, q_r) \text{ and } x + (Q + 1)q_r \notin E_{\nu}(t, (Q + 1)q_r, q_r)$$

with any  $Q \in [0, Q_0]$ . Using (101) and that  $\theta_r > 0$  and  $t \leq 2n - \sqrt{n}$ , we get that the transition  $Q \rightarrow Q + 1$  causes a shift in the fractional part of (101) to the right of the interval  $[\frac{t+\xi_\nu}{2n}, \frac{t+\xi_{\nu+1}}{2n})$  to the interval  $[\frac{t+\xi_{\nu+1}}{2n}, \frac{t+\xi_{\nu+2}}{2n})$ :

$$(103) \quad \left\{ \frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r Q}{q_{r+1}} \right\} < \frac{t + \xi_{\nu+1}}{2n} \leq \left\{ \frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r(Q + 1)}{q_{r+1}} \right\}.$$

We have from (57), (94), (99) and the condition of lemma that

$$(104) \quad 0 \leq Q \leq P_1 \leq n^{-7} b^{s^2/2+4sm} < \frac{q_{r+1}}{8nq_r}$$

and

$$(105) \quad \left| \frac{\gamma(x)}{2nq_r} + \frac{\theta_r Q}{q_{r+1}} \right| < \frac{1}{2nq_r}.$$



Bearing in mind (97), (101) and so that  $t \in [\sqrt{n}, 2n - \sqrt{n}]$ , we get

$$\left\{ \frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r Q}{q_{r+1}} \right\} \in \left[ \frac{\sqrt{n}}{2n}, \frac{2n - \sqrt{n} + 2}{2n} \right],$$

and we can eliminate the sign of the fractional part in (103):

$$\frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r Q}{q_{r+1}} < \frac{t + \xi_{\nu+1}}{2n} \leq \frac{z(x) + \gamma(x)}{2nq_r} + \frac{\theta_r(Q + 1)}{q_{r+1}}.$$

Hence

$$(106) \quad Q = \left\lceil \frac{q_{r+1}}{\theta_r} \left( \frac{t + \xi_{\nu+1}}{2n} - \frac{z(x) + \gamma(x)}{2nq_r} \right) \right\rceil - 1.$$

Let  $\xi_{\nu+1} = 1 - \{\alpha i\} = 1 - \left\{ \frac{p_r i}{q_r} + \frac{\theta_r i}{q_r q_{r+1}} \right\} = \frac{i_1}{q_r} - \frac{\theta_r i}{q_r q_{r+1}}$  with the same  $i \in [0, m]$ ,  $i_1 \in [1, q_r]$ . It follows from (106), (97) and (98) that

$$(107) \quad \begin{aligned} Q &= \left\lceil \frac{q_{r+1}}{\theta_r} \left( \frac{q_r t + i_1}{2nq_r} - \frac{z(x) + \beta_1}{2nq_r} - \frac{\theta_r(2nx + s + i)}{2nq_r q_{r+1}} \right) \right\rceil - 1 \\ &= \left\lceil \frac{q_{r+1}}{2nq_r \theta_r} (v - \beta_1) - \frac{(2nx + s + i)}{2nq_r} \right\rceil - 1, \end{aligned}$$

with  $v = q_r t - z(x) + i_1$ . From the condition of the lemma we have that  $0 \leq 2nx + s + i < 2nq_r - \sqrt{n} + m < 2nq_r$ . Using (93) we get, that if  $v \neq 0$ , then

$$|Q| \geq \frac{q_{r+1}}{4nq_r \theta_r} - 3 \geq \frac{q_{r+1}}{4nq_r} - 3.$$

This is in contradiction with (104).

Let  $v = 0$ . We get from (94) and (107) that

$$(108) \quad Q = Q(x) = \left\lceil \frac{-q_{r+1} \beta_1}{2nq_r \theta_r} - \frac{2nx + s + i}{2nq_r} \right\rceil - 1 \in [Q', Q' + 2],$$

where  $i \in [0, m]$ . Hence if  $Q' < -2$  or  $Q' > P_1$ , then there is no  $Q \in [0, P_1]$  satisfying (102). Now, let

$$(109) \quad x + Q_1 q_r \in E_\nu(t, Q_1 q_r, q_r) \text{ and } x(Q_1 - 1)q_r \notin E_\nu(t, (Q_1 - 1)q_r, q_r),$$

with  $0 \leq Q_1 = Q_1(x) \leq P_1$ . Similarly to (94)-(108) we obtain for all  $x \in [0, q_r]$ , that

$$(110) \quad Q_1(x) \in [Q', Q' + 2].$$

Hence if  $Q' < -2$  or  $Q' > P_1$ , then there is no  $Q_1$  satisfying (109). Now from (94), (102), (108), (109), and (110), the assertion of Lemma 11 follows.  $\square$

**Lemma 12.** *Under the assumptions of Lemma 11,*

$$\#A_{j\nu s}(t, N) = b^{-m} \#E_{\nu s}(t, N)(1 + O(1/n)) + O(N/n^4).$$

*Proof.* Let

$$(111) \quad P_2 = \lceil n^4 b^{s^2/2+3sm} \rceil + 1.$$

We have from (94) that

$$(112) \quad P_2 = O(n^{-2}P_1).$$

If  $Q_0 \leq P_2$ , or  $Q_0 \in [P_1 - P_2, P_1]$ , then we use the trivial estimate

$$(113) \quad \#A_{j\nu s}(t, Q_0q_r) \leq \#E_{\nu s}(t, Q_0q_r) \leq P_2q_r$$

for  $Q_0 \leq P_2$ , and

$$(114) \quad \#A_{j\nu s}(t, Q_0q_r, (P_1 - Q_0)q_r) \leq \#E_{\nu s}(t, Q_0q_r, (P_1 - Q_0)q_r) \leq P_2q_r$$

for  $Q_0 \in [P_1 - P_2, P_1]$ . Let  $Q_0 \leq P_2$ .

Applying (94), (111), (112), Lemma 11, and Lemma 10 with  $H = P_2$  and  $P = P_1 - P_2$ , we obtain that

$$\#E_{\nu, s}(t, P_2q_r, (P_1 - P_2)q_r) = (P_1 - P_2)\#E_{\nu, s}(t, P_2q_r, q_r)$$

and

$$\#A_{j\nu s}(t, P_2q_r, (P_1 - P_2)q_r) = b^{-m}(P_1 - P_2)\#E_{\nu s}(t, P_2q_r, q_r)(1 + O(1/n)).$$

Hence

$$\#A_{j\nu s}(t, P_2q_r, (P_1 - P_2)q_r) = b^{-m}\#E_{\nu s}(t, P_2q_r, (P_1 - P_2)q_r)(1 + O(1/n)).$$

Using (45) and (113) we have

$$(115) \quad \#A_{j\nu s}(t, P_1q_r) = b^{-m}\#E_{\nu s}(t, P_1q_r)(1 + O(1/n)) + O(P_2q_r).$$

By (114) and Lemma 10 with  $H = 0$  and  $P = P_1 - P_2$ , we get similarly that (115) is valid for the case  $Q_0 \in [P_1 - P_2, P_1]$ . Now we prove that (115) is valid for  $Q_0 \in (P_2, P_1 - P_2)$ . Let  $Q_0 \in (P_2, P_1 - P_2)$ . We have from Lemma 11 that

$$\#E_{\nu s}(t, Q_0q_r) = Q_0\#E_{\nu s}(t, q_r),$$

and

$$\begin{aligned} \#E_{\nu s}(t, (Q_0 + 3)q_r, (P_1 - Q_0 - 3)q_r) &= (P_1 - Q_0 - 3)\#E_{\nu s}(t, P_1q_r, q_r) \\ &= (P_1 - Q_0 - 3) \\ &\quad \times \#E_{\nu s}(t, (Q_0 + 3)q_r, q_r). \end{aligned}$$

Hence

$$(116) \quad \begin{aligned} \#E_{\nu s}(t, P_1q_r) &= Q_0\#E_{\nu s}(t, q_r) \\ &+ (P_1 - Q_0 - 3)\#E_{\nu s}(t, (Q_0 + 3)q_r, q_r) + 3\epsilon q_r. \end{aligned}$$

Applying (94), (96) and Lemma 10 twice, we have that

$$\#A_{j\nu s}(t, Q_0q_r) = b^{-m}Q_0\#E_{\nu s}(t, q_r)(1 + O(1/n)),$$

and

$$\begin{aligned} \#A_{j\nu s}(t, (Q_0 + 3)q_r, (P_1 - Q_0 - 3)q_r) \\ = b^{-m}(P_1 - Q_0 - 3)\#E_{\nu s}(t, (Q_0 + 3)q_r, q_r)(1 + O(1/n)). \end{aligned}$$

Hence, and from (116), we get

$$\begin{aligned} \#A_{j\nu s}(t, P_1q_r) &= b^{-m}(Q_0\#E_{\nu s}(t, q_r) + (P_1 - Q_0 - 3)\#E_{\nu s}(t, P_1q_r, q_r)) \\ &\times (1 + O(1/n)) + 3\epsilon q_r = b^{-m}\#E_{\nu s}(t, P_1q_r)(1 + O(1/n)) + O(q_r). \end{aligned}$$

Therefore (115) is valid for all  $Q_0 \in [0, P_1]$ . Bearing in mind that (115) is valid for all  $\beta$ , we get (replacing  $\beta$  by  $\beta + iP_1q_r\alpha$ )

$$\#A_{j\nu s}(t, iP_1q_r, P_1q_r) = b^{-m}\#E_{\nu s}(t, iP_1q_r, P_1q_r)(1 + O(1/n)) + O(P_2q_r),$$

with  $i = 0, 1, \dots$ . Hence

$$\begin{aligned} \#A_{j\nu s}(t, N) &= \sum_{0 \leq i < [N/P_1q_r]} \#A_{j\nu s}(t, iP_1q_r, P_1q_r) + O(P_1q_r) \\ &= \sum_{0 \leq i < [N/P_1q_r]} (b^{-m}\#E_{\nu s}(t, iP_1q_r, P_1q_r) \\ &\quad \times (1 + O(1/n)) + O(P_2q_r)) + O(P_1q_r). \end{aligned}$$

By (94), (112) and condition of the lemma, we obtain

$$\begin{aligned} A_{j\nu s}(t, N) &= b^{-m}E_{\nu s}(t, N)(1 + O(1/n)) + O(P_1q_r + N/n^4) \\ &= b^{-m}E_{\nu s}(t, N)(1 + O(1/n)) + O(N/n^4). \end{aligned} \quad \square$$

**Lemma 13.** *Let  $s_0, s_0 + s_1 \in [0, 2n)$ . Then*

$$\sum_{s_0 \leq s < s_0 + s_1} \sum_{0 \leq t < 2n} \sum_{0 \leq \nu \leq m} \#E_{\nu s}(t, N) \leq s_1 N.$$

*Proof.* From (15), (31), and (35) we have that

$$\sum_{0 \leq t < 2n} \sum_{0 \leq \nu \leq m} \#E_{\nu s}(t, N) = N. \quad \square$$

**Lemma 14.** *Let  $[t_0, t_0 + t_1) \in [0, 2n)$ . Then*

$$\mu = \sum_{0 \leq s < 2n} \sum_{t_0 \leq t < t_0 + t_1} \sum_{0 \leq \nu \leq m} \#E_{\nu s}(t, N) = O(t_1 N).$$

*Proof.* From (15), (31), and (35) we have that

$$\begin{aligned} \sum_{t_0 \leq t < t_0 + t_1} \sum_{0 \leq \nu \leq m} \#E_{\nu s}(t, N) \\ = \#\{x \in [0, N) \mid \{\frac{\alpha(2nx + s) + \beta}{2n}\} \in [\frac{t_0}{2n}, \frac{t_0 + t_1}{2n})\}, \end{aligned}$$

and

$$(117) \quad \mu = \sum_{x=0}^{N-1} \#\{s \in [0, 2n) \mid \{\frac{\alpha(2nx + s) + \beta}{2n}\} \in [\frac{t_0}{2n}, \frac{t_0 + t_1}{2n})\}.$$

Since  $\alpha \in (0, 1)$ , we have that

$$\#\{s \in [0, 2n) \mid \{\frac{\alpha(2nx + s) + \beta}{2n}\} \in [\frac{t_0}{2n}, \frac{t_0 + t_1}{2n})\} \leq \frac{t_1}{\alpha} + 2.$$

Hence from (117) we get the assertion of the lemma. □

**Lemma 15.** *Let  $N \in [b^{2n^2-5n}, b^{2n^2})$ . Then*

$$\#V'_n(N, G_m) = b^{-m} \#U'_n(N)(1 + O(1/\sqrt{n})) + O(N\sqrt{n}).$$

*Proof.* Using (44) we have that

$$(118) \quad \#V'_n(N, G_m) = \mu_1 + \epsilon_2\mu_2 + \epsilon_3\mu_3,$$

with  $0 \leq \epsilon_2, \epsilon_3 \leq 1$ , and

$$\begin{aligned} \mu_1 &= \sum_{\sqrt{n} \leq t \leq s < 2n - \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} \#A_{j\nu s}(t, N) \\ \mu_2 &= \sum_{0 \leq s < \sqrt{n}} \sum_{0 \leq t < 2n - \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} \#A_{j\nu s}(t, N) \\ \mu_3 &= \sum_{0 \leq s < 2n - \sqrt{n}} \sum_{0 \leq t < \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} \#A_{j\nu s}(t, N) \end{aligned}$$

Now let

$$(119) \quad \sigma_2 = \sum_{0 \leq s < \sqrt{n}} \sum_{0 \leq t < 2n - \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} \#E_{j\nu s}(t, N),$$

$$(120) \quad \sigma_3 = \sum_{0 \leq s < 2n - \sqrt{n}} \sum_{0 \leq t < \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} \#E_{\nu s}(t, N).$$

By Lemma 6 and Lemma 12 we get

$$(121) \quad \begin{aligned} \mu_1 &= \sum_{\sqrt{n} \leq t \leq s < 2n - \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} (b^{-m} \#E_{\nu s}(t, N) \\ &\quad \times (1 + O(1/n)) + O(Nn^{-4})). \end{aligned}$$

Using (16) and Lemma 13, we get

$$(122) \quad \mu_2 \leq \sigma_2 \leq 4\sqrt{n}(\alpha + 1)N.$$

Applying Lemma 14, we obtain

$$(123) \quad \mu_3 \leq \sigma_3 \leq (\alpha + 1)(\sqrt{n} + 3)\alpha^{-1}N.$$

From (118)-(121), we have

$$\begin{aligned} \#V'_n(N, G_m) = & \sum_{0 \leq t \leq s < 2n - \sqrt{n}} \sum_{0 \leq \nu \leq m} \sum_{0 \leq j \leq l_1(\nu)} (b^{-m} \#E_{\nu s}(t, N)(1 + O(1/n)) \\ & + O(Nn^{-4})) + \epsilon_2\mu_2 + \epsilon_3\mu_3 + \epsilon_4\sigma_2 + \epsilon_5\sigma_3, \end{aligned}$$

with  $0 \leq |\epsilon_\nu| \leq 1$ ,  $\nu \in [2, 5]$ . By (36), (122), and (123) we obtain the assertion of Lemma 15. □

**Main Lemma.** *Let  $N \in [b^{2n^2-5n}, b^{2n^2})$ . Then*

$$(124) \quad \#V_n(N, G_m) = b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})).$$

*Proof.* Using (5), (23), (34), (39), (40), (42) and repeating the proof of Lemmas 2-15 for the case  $t > s$ , we obtain

$$\begin{aligned} \#U_n(N) &= \#U_n^*(N) + O(N\sqrt{n}) = \#U'_n(N) + O(N\sqrt{n}), \\ \#V_n(N, G_m) &= \#V_n^*(N, G_m) + O(N\sqrt{n}) = \#V'_n(N, G_m) + O(N\sqrt{n}), \end{aligned}$$

and

$$\#V_n(N, G_m) = b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})) + O(N\sqrt{n}).$$

Applying (22) we get (124). □

**Remark.** For the case  $\alpha > 1$ , we consider the line  $(z, \alpha z + \beta)$  as the line  $((w - \beta)/\alpha, w)$ ; the variable  $y$  as an independent variable (instead of  $x$ ); and in (5), (6) we choose the case of  $y > x$ . (Hence we use the formula  $u = y^2 + 2y - x$  instead of  $u = x^2 + y$ ). Now repeating the proof of Lemmas 2-15, we get the proof of the Main Lemma for the case  $\alpha > 1$ .

**3.2. The case of the rational line with  $\alpha > 0$ .**

In this subsection we slightly modify partition of Section 3.1. Next we prove that the estimate of Lemma 10 can be applied for rational  $\alpha$ .

Let  $\alpha = p/q > 0$ ,  $(p, q) = 1$ . We use the notation of (15), (16):

$$(125) \quad \xi_j = j/q, \quad j = 0, \dots, q; \quad f(i) = q(1 - \{\alpha i\}), \quad i = 1, \dots, m.$$

We fix an integer  $\nu \in [0, q - 1]$  and consider the sequence  $(l_i(\nu))_{i=1}^m$ :

$$l_i(\nu) = \begin{cases} [\alpha i], & \text{if } \nu < f(i) \\ [\alpha i] + 1. & \text{otherwise.} \end{cases}$$

**Lemma 16.** *Let*

$$(126) \quad \left\{ \frac{\alpha(2nx + s) + \beta}{2n} \right\} \in \left[ \frac{t + \nu/q}{2n}, \frac{t + (\nu + 1)/q}{2n} \right), \quad \text{with } \nu \in [0, q - 1].$$

*Then*

$$(127) \quad \left\{ \frac{\alpha(2nx + s + i) + \beta}{2n} \right\} \in \left[ \frac{t + l_i(\nu)}{2n}, \frac{t + l_i(\nu) + 1}{2n} \right), \quad i = 1, \dots, m.)$$

*Proof.* Using (126) we get

$$\begin{aligned} z_i &= \left\{ \frac{\alpha(2nx + s + i) + \beta}{2n} \right\} = \left\{ \frac{\alpha(2nx + s) + \beta}{2n} + \frac{\alpha i}{2n} \right\} \\ &= \left\{ \frac{t + \nu/q + \epsilon + [\alpha i] + \{\alpha i\}}{2n} \right\}, \quad \text{with } 0 \leq \epsilon < 1/q. \end{aligned}$$

From here and from (125), we have that

$$z_i = \left\{ \frac{t + [\alpha i] + \nu/q + 1 - f(i)/q + \epsilon}{2n} \right\}.$$

It is evident that if  $f(i) > \nu$ , then

$$\nu/q + \epsilon - f(i)/q < (\nu + 1)/q - f(i)/q \leq 0.$$

Applying (126) we get (127) with  $l_i(\nu) = [\alpha i]$ .

Now let  $f(i) \leq \nu$ . Then

$$0 \leq \nu/q + \epsilon - f(i)/q < (\nu + 1)/q.$$

By (126) we obtain (127) with  $l_i(\nu) = [\alpha i] + 1$ . □

We use a variant of notations (31) - (43) as follows :

$$\begin{aligned} \alpha_{xs} &= \{p(2nx + s)/q + \beta\}, \\ (128) \quad \Delta_{t,\nu} &= \left[ \frac{t + \xi_\nu}{2n}, \frac{t + \xi_{\nu+1}}{2n} \right), \end{aligned}$$

$$(129) \quad w_{xs} = \begin{cases} 1 & \text{if } \alpha_{xs} = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and in the summation of  $y$  in formulas (34), (36), (42), and (44), we apply  $l'_1(\nu) = l_1(\nu) - w_{xs}$  instead of  $l_1(\nu)$ . (Note that  $w_{xs} = 0$  for all  $\beta \neq c/q$  with integer  $c$ .) For fixed integers  $x, s, t, \nu$  and  $j$ , we denote:  $(k_n)_{n \geq 1}$  as an increasing sequence of integers from the set

$$(130) \quad \{(s + i')^2 + t + l_{i'}(\nu) + v + 1 \mid v = j\delta_{i'}, j\delta_{i'} + 1, \dots, l_{i'+1}(\nu) - l_{i'}(\nu) - w_{xs}, i' = 0, 1, \dots, m - 1\},$$

and  $(z^{(r)}(x, s, j))_{r=1}^m$  as the sequence of the first  $m$  vectors from the set

$$((2nx + s + i', [\alpha(2nx + s) + \beta] + l_{i'}(\nu) + v))$$

with  $i' = 0, 1, \dots, m$ , and

$$(131) \quad v = j\delta_{i'}, j\delta_{i'} + 1, \dots, l'_{i'+1}(\nu) - l'_{i'}(\nu) - w_{xs}, \quad (\text{see (47)});$$

$$(132) \quad E_{\nu s}(x', t, N) = \{x \in [0, N] \mid x \equiv x' \pmod{q}, \alpha_{x,s} \in \Delta_{t,\nu}\},$$

$$(133) \quad \begin{aligned} A_{j\nu s}(x', t, N) &= \{x \in [0, N] \mid x \equiv x' \pmod{q}, \\ &\chi_{\omega_n G_m}(z(x, s, j)) = 1, \alpha_{x,s} \in \Delta_{t,\nu}\} \end{aligned}$$

$$(134) \quad U_n(N) = \bigcup_{0 \leq x' < q} \bigcup_{0 \leq s < 2n} \bigcup_{0 \leq t < 2n} \bigcup_{0 \leq \nu < q} \bigcup_{0 \leq j \leq l'_1(\nu)} E_{\nu s}(x', t, N),$$

where  $l'_1(\nu) = l_1(\nu) - w_{x's}$ ,

$$(135) \quad V_n(N, G_m) = \bigcup_{0 \leq x' < q} \bigcup_{0 \leq s < 2n} \bigcup_{0 \leq t < 2n} \bigcup_{0 \leq \nu < q} \bigcup_{0 \leq j \leq l'_1(\nu)} A_{j\nu s}(x', t, N).$$

**Lemma 17.** *Let  $N \in [b^{2n^2-5n}, b^{2n^2})$ ,  $\sqrt{n} \leq t \leq s \leq 2n - \sqrt{n}$ . Then*

$$(136) \quad \#A_{j\nu s}(x', t, N) = b^{-m} \#E_{\nu s}(x', t, N)(1 + O(1/n)).$$

*Proof.* Let

$$\delta(x, s, t, \nu) = \begin{cases} 1 & \text{if } \alpha_{x,s} \in \Delta_{t,\nu} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the sequences  $\alpha_{x,s}$ ,  $\delta(x, s, t, \nu)$  ( $x = 1, 2, \dots$ ) (128)-(129) have period  $q$ . Hence we have from (132)

$$(137) \quad \#E_{\nu s}(x', t, N) = \delta(x', s, t, \nu)(N/q + \epsilon), \quad \text{with } |\epsilon| \leq 1.$$

Consider Lemma 10 with  $q_r = q$  ( $r = 1$ ). Then the condition of Lemma 10

$$x \in E_{\nu s}(t, Hq_r, q_r) \text{ if and only if } x + Qq_r \in E_{\nu s}(t, Qq_r, q_r)$$

for all  $Q \in [H, H + P)$  is valid for all  $H \geq 0$  and all  $P \geq 1$ . Applying (133), and (91), (92) for the case  $P = [N/q]$ ,  $x \equiv x' \pmod{q}$ , we obtain

$$\#A_{j\nu s}(x', t, N) = b^{-m} \delta(x', s, t, \nu)[N/q](1 + O(1/n)).$$

(136) follows from (137) and the condition of the lemma □

(Def.  $V'_n$  ( $U'_n$ ) be the part of  $V_n$  ( $U_n$ ) with  $t < s$ )

**Lemma 18.** *Let  $N \in [b^{2n^2-5n}, b^{2n^2})$ . Then*

$$(138) \quad \#V'_n(N, G_m) = b^{-m} \#U'_n(N)(1 + O(1/n)) + O(N\sqrt{n}).$$

*Proof.* Using Lemmas 13, 14, 17 and repeating the proof of Lemma 15, we obtain (138). □

Now we consider the case of  $t \geq s$  and (similarly to (138)), we get:

**Corollary.** *Let  $N \in [b^{2n^2-5n}, b^{2n^2})$ . Then*

$$(139) \quad \#V_n(N, G_m) = b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})).$$

The proof is the same as that of the Main Lemma.

**3.3. The case of the horizontal lines ( $\alpha = 0$ ).**

Using (19)-(24) we obtain that  $\beta \geq 0, y = 0, t = [\beta], U_n(N) = 2nN$ . Let  $N \in [b^{2n^2-5n}, b^{2n^2})$ . It follows from (5), (6) that for  $s \in [t, 2n - 4m)$  the block of digits  $G_m = (g_1, \dots, g_m)$  appears at the point  $(2nx + s, t)$  if and only if

$$x^2 = x_1 + g_1 b^{s^2+t} + x_2 b^{s^2+t+1} + g_2 b^{(s+1)^2+t} + \dots + g_m b^{(s+m-1)^2+t} + x_{m+1} b^{(s+m-1)^2+t+1},$$

where  $x_i \in [0, b^{k_i - k_{i-1} - 1})$  are integers ( $i = 1, \dots, m$ ),  $k_0 = 0$ , and  $k_i = (s + i - 1)^2 + t + 1, i = 1, 2, \dots$ . We have from (39) that

$$\begin{aligned} \#V_n(N, G_m) = & \sum_{s \in [t, 2n-4m)} \sum_{x_2=0}^{b^{k_2 - k_1 - 1 - 1}} \dots \sum_{x_m=0}^{b^{k_m - k_{m-1} - 1 - 1}} \#\{x \in [0, N) \mid \\ & \{x^2 b^{-k_m}\} \in [\frac{g}{b^{k_m - k_1 + 1}}, \frac{g+1}{b^{k_m - k_1 + 1}})\} + O(N), \end{aligned}$$

with

$$g = g(x_2, \dots, x_m) = g_1 + x_2 b + g_2 b^{k_2 - k_1} + \dots + x_m b^{k_{m-1} - k_1 + 1} + g_m b^{k_m - k_1}.$$

Using (88) and Lemma 9 with  $q_r = 1, x = 0, H = 0$  and  $P = N$ , we have that

$$\#V_n(N, G_m) = \sum_{s \in [t, 2n-4m)} \sum_{x_2=0}^{b^{k_2 - k_1 - 1 - 1}} \dots \sum_{x_m=0}^{b^{k_m - k_{m-1} - 1 - 1}} N b^{k_1 - k_m - 1} (1 + O(n^{-1})).$$

Bearing in mind that  $U_n(N) = 2nN$ , we have that

$$(140) \quad \#V_n(N, G_m) = b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})).$$

**3.4. The case of vertical lines.**

Using (19)-(24), we get  $x = 0, s = [\beta]$ , and  $U_n(N) = 2nN$ . Let  $t \in [s, 2n - m), N \in [b^{2n^2-5n}, b^{2n^2})$ . The block of digits  $G_m = (g_1, \dots, g_m)$  appears at the point  $(s, 2ny + t)$  if and only if

$$y^2 + 2y = y_1 + g_1 b^{t^2+2t-s} + y_2 b^{t^2+2t-s+1} + g_2 b^{(t+1)^2+2t-s+2} + \dots + g_m b^{(t+m-1)^2+2(t+m-1)-s} + y_{m+1} b^{(t+m-1)^2+2(t+m-1)-s+1},$$



where  $y_{i+1} \in [0, b^{2t+2i})$  are integers  $i = 1, 2, \dots$ . Repeating the previous proof, we get

$$(141) \quad \#V_n(N, G_m) = b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})).$$

**3.5. Completion of the proof of Theorem 1.**

In this subsection we go from the configuration  $\omega_n$  to the configuration  $\omega_\infty$ .

Let  $P \geq 1$  be an integer. There exists an integer  $n \geq 1$  such that

$$(142) \quad 2(n - 1)b^{2(n-1)^2} \leq P < 2nb^{2n^2}.$$

Let  $N = [P/2n]$ ,  $P = 2nN + N_1$ , where  $N_1 \in [0, 2n)$ ,  $\alpha > 0$  is irrational. Let  $U(P)$  be the subset of all lattice points  $(u, v)$  such that the line  $(z, \alpha z + \beta)_{z \in [0, P]}$  intersects the square  $(u, v) + [0, 1)^2$ . Applying the definition of  $U_n(N)$  (see (21)-(24)), we have that

$$(143) \quad \#U(P) = \#U_n(N) + O(n).$$

Let

$$V(P, G_m) = \{z \in U(P) \mid G_m = (\omega(z^{(1)}), \dots, \omega(z^{(m)}))\}$$

be the set of the linear indices of the beginning of a block  $G_m$  in the linear configuration of digits, confined to the  $P$ -square configuration. It is evident, that

$$(144) \quad \#V(P, G_m) = \#V(2nN, G_m) + O(n),$$

with

$$(145) \quad V(2nN, G_m) = \{z \in U_n(N) \mid G_m = (\omega(z^{(1)}), \dots, \omega(z^{(m)}))\}.$$

To compute  $\#V(2nN, G_m)$ , consider the configuration  $\omega$  as the union of the parts of the configurations  $\omega_n, \omega_{n-1}$  and  $\omega'_{n-2}$  (where  $\omega'_{n-2}$  is the part of configuration  $\omega$  corresponding to  $\omega_{n-2}$ ). We use a trivial estimate for  $\omega'_{n-2}$ :

By (8) (definition of  $\omega$ ), (34), (39), and (145), we have

$$\begin{aligned} \#V(2nN, G_m) &= \#V_n(N, G_m) - \#V_n([(n - 1)b^{2(n-1)^2} / n], G_m) \\ &\quad + \#V_{n-1}(b^{2(n-1)^2}, G_m) + O(nb^{2(n-2)^2}), \end{aligned}$$

and

$$(146) \quad \#U_{n-1}(b^{2(n-1)^2}) = \#U_n([(n - 1)b^{2(n-1)^2} / n]) + O(n).$$

Let  $\alpha$  be irrational. Using the Main Lemma, we get

$$\begin{aligned} \#V(2nN, G_m) &= b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})) \\ &\quad - b^{-m} \#U_n([(n - 1)b^{2(n-1)^2} / n])(1 + O(1/\sqrt{n})) \\ &\quad + b^{-m} \#U_{n-1}(b^{2(n-1)^2})(1 + O(1/\sqrt{n})) + O(nb^{2(n-2)^2}). \end{aligned}$$

From here, (22), and (146), we obtain

$$\#V(2nN, G_m) = b^{-m} \#U_n(N)(1 + O(1/\sqrt{n})).$$

Hence from (144), (143), and (142), we obtain

$$\#V(P, G_m) = b^{-m} \#U(P)(1 + O(1/(\log P)^{1/4})).$$

For the case  $\alpha > 0$  rational (the horizontal line, and vertical lines) we use (139) (respectively (140), (141)) instead of the Main Lemma. Now by (3) we obtain the assertion of Theorem 1.  $\square$

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