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Kronecker-Weber via Stickelberger

par FRANZ LEMMERMEYER

RÉSUMÉ. Nous donnons une nouvelle démonstration du théorème de Kronecker et Weber fondée sur la théorie de Kummer et le théorème de Stickelberger.

ABSTRACT. In this note we give a new proof of the theorem of Kronecker-Weber based on Kummer theory and Stickelberger's theorem.

Introduction

The theorem of Kronecker-Weber states that every abelian extension of \mathbb{Q} is cyclotomic, i.e., contained in some cyclotomic field. The most common proof found in textbooks is based on proofs given by Hilbert [2] and Speiser [7]; a routine argument shows that it is sufficient to consider cyclic extensions of prime power degree p^m unramified outside p , and this special case is then proved by a somewhat technical calculation of differentials using higher ramification groups and an application of Minkowski's theorem, according to which every extension of \mathbb{Q} is ramified. In the proof below, this not very intuitive part is replaced by a straightforward argument using Kummer theory and Stickelberger's theorem.

In this note, ζ_m denotes a primitive m -th root of unity, and “unramified” always means unramified at all finite primes. Moreover, we say that a normal extension K/F

- is of type (p^a, p^b) if $\text{Gal}(K/F) \simeq (\mathbb{Z}/p^a\mathbb{Z}) \times (\mathbb{Z}/p^b\mathbb{Z})$;
- has exponent m if $\text{Gal}(K/F)$ has exponent m .

1. The Reduction

In this section we will show that it is sufficient to prove the following special case of the Kronecker-Weber theorem (it seems that the reduction to extensions of prime degree is due to Steinbacher [8]):

Proposition 1.1. *The maximal abelian extension of exponent p that is unramified outside p is cyclic: it is the subfield of degree p of $\mathbb{Q}(\zeta_{p^2})$.*

The corresponding result for the prime $p = 2$ is easily proved:

Proposition 1.2. *The maximal real abelian 2-extension of \mathbb{Q} with exponent 2 and unramified outside 2 is cyclic: it is the subfield $\mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}(\zeta_8)$.*

Proof. The only quadratic extensions of \mathbb{Q} that are unramified outside 2 are $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{2})$. \square

The following simple observation will be used repeatedly below:

Lemma 1.3. *If the compositum of two cyclic p -extensions K , K' is cyclic, then $K \subseteq K'$ or $K' \subseteq K$.*

Now we show that Prop. 1.1 implies the corresponding result for extensions of prime power degree:

Proposition 1.4. *Let K/\mathbb{Q} be a cyclic extension of odd prime power degree p^m and unramified outside p . Then K is cyclotomic.*

Proof. Let K' be the subfield of degree p^m in $\mathbb{Q}(\zeta_{p^{m+1}})$. If $K'K$ is not cyclic, then it contains a subfield of type (p, p) unramified outside p , which contradicts Prop. 1.1. Thus $K'K$ is cyclic, and Lemma 1.3 implies that $K = K'$. \square

Next we prove the analog for $p = 2$:

Proposition 1.5. *Let K/\mathbb{Q} be a cyclic extension of degree 2^m and unramified outside 2. Then K is cyclotomic.*

Proof. If $m = 1$ we are done by Prop. 1.2. If $m \geq 2$, assume first that K is nonreal. Then $K(i)/K$ is a quadratic extension, and its maximal real subfield M is cyclic of degree 2^m by Prop. 1.2. Since K/\mathbb{Q} is cyclotomic if and only if M is, we may assume that K is totally real.

Now let K' be the the maximal real subfield of $\mathbb{Q}(\zeta_{2^{m+2}})$. If $K'K$ is not cyclic, then it contains three real quadratic fields unramified outside 2, which contradicts Prop. 1.2. Thus $K'K$ is cyclic, and Lemma 1.3 implies that $K = K'$. \square

Now the theorem of Kronecker-Weber follows: first observe that abelian groups are direct products of cyclic groups of prime power order; this shows that it is sufficient to consider cyclic extensions of prime power degree p^m . If K/\mathbb{Q} is such an extension, and if $q \neq p$ is ramified in K/\mathbb{Q} , then there exists a cyclic cyclotomic extension L/\mathbb{Q} with the property that $KL = FL$ for some cyclic extension F/\mathbb{Q} of prime power degree in which q is unramified. Since K is cyclotomic if and only if F is, we see that after finitely many steps we have reduced Kronecker-Weber to showing that cyclic extensions of degree p^m unramified outside p are cyclotomic. But this is the content of Prop. 1.4 and 1.5.

Since this argument can be found in all the proofs based on the Hilbert-Speiser approach (see e.g. Greenberg [1] or Marcus [6]), we need not repeat the details here.

2. Proof of Proposition 1.1

Let K/\mathbb{Q} be a cyclic extension of prime degree p and unramified outside p . We will now use Kummer theory to show that it is cyclotomic. For the rest of this article, set $F = \mathbb{Q}(\zeta_p)$ and define $\sigma_a \in G = \text{Gal}(F/\mathbb{Q})$ by $\sigma_a(\zeta_p) = \zeta_p^a$ for $1 \leq a < p$.

Lemma 2.1. *The Kummer extension $L = F(\sqrt[p]{\mu})$ is abelian over \mathbb{Q} if and only if for every $\sigma_a \in G$ there is a $\xi \in F^\times$ such that $\sigma_a(\mu) = \xi^p \mu^a$.*

For the simple proof, see e.g. Hilbert [3, Satz 147] or Washington [9, Lemma 14.7].

Let K/\mathbb{Q} be a cyclic extension of prime degree p and unramified outside p . Put $F = \mathbb{Q}(\zeta_p)$ and $L = KF$; then $L = F(\sqrt[p]{\mu})$ for some nonzero $\mu \in \mathcal{O}_F$, and L/F is unramified outside p .

Lemma 2.2. *Let \mathfrak{q} be a prime ideal in F with $(\mu) = \mathfrak{q}^r \mathfrak{a}$, $\mathfrak{q} \nmid \mathfrak{a}$; if $p \nmid r$ and L/\mathbb{Q} is abelian, then \mathfrak{q} splits completely in F/\mathbb{Q} .*

Proof. Let σ be an element of the decomposition group $Z(\mathfrak{q}|q)$ of \mathfrak{q} . Since L/\mathbb{Q} is abelian, we must have $\sigma_a(\mu) = \xi^p \mu^a$. Now $\sigma_a(\mathfrak{q}) = \mathfrak{q}$ implies $\mathfrak{q}^r \parallel \xi^p \mu^a$, and this implies $r \equiv ar \pmod p$; but $p \nmid r$ show that this is possible only if $a = 1$. Thus $\sigma_a = 1$, and \mathfrak{q} splits completely in F/\mathbb{Q} . \square

In particular, we find that $(1 - \zeta) \nmid \mu$. Since L/F is unramified outside p , prime ideals $\mathfrak{p} \nmid p$ must satisfy $\mathfrak{p}^{bp} \parallel \mu$ for some integer b . This shows that $(\mu) = \mathfrak{a}^p$ is the p -th power of some ideal \mathfrak{a} . From $(\mu) = \mathfrak{a}^p$ and the fact that L/\mathbb{Q} is abelian we deduce that $\sigma_a(\mathfrak{a})^p = \mathfrak{a}^{pa} \xi^p$, where $\sigma_a(\zeta_p) = \zeta_p^a$. Thus $\sigma_a(c) = c^a$ for the ideal class $c = [\mathfrak{a}]$ and for every a with $1 \leq a < p$. Now we invoke Stickelberger’s Theorem (cf. [4] or [5, Chap. 11]) to show that \mathfrak{a} is principal:

Theorem 2.3. *Let $F = \mathbb{Q}(\zeta_p)$; then the Stickelberger element*

$$\theta = \sum_{a=1}^{p-1} a \sigma_a^{-1} \in \mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$$

annihilates the ideal class group $\text{Cl}(F)$.

From this theorem we find that $1 = c^\theta = \prod \sigma_a^{-1}(c)^a = c^{p-1} = c^{-1}$, hence $c = 1$ as claimed. In particular $\mathfrak{a} = (\alpha)$ is principal. This shows that $\mu = \alpha^p \eta$ for some unit η , hence $L = F(\sqrt[p]{\eta})$. Now write $\eta = \zeta^t \varepsilon$ for some unit ε in the maximal real subfield of F . Since ε is fixed by complex conjugation σ_{-1} and since L/\mathbb{Q} is abelian, we see that $\zeta^{-t} \varepsilon = \sigma_{-1}(\mu) = \xi^p \mu^{-1}$, hence $\zeta^{-t} \varepsilon = \xi^p \zeta^{-t} \varepsilon^{-1}$. Thus ε is a p -th power, and we find $\mu = \zeta^t$. But this implies that $L = \mathbb{Q}(\zeta_{p^2})$, and Prop. 1.1 is proved.

Since every cyclotomic extension is ramified, we get the following special case of Minkowski’s theorem as a corollary:

Corollary 2.4. *Every solvable extension of \mathbb{Q} is ramified.*

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References

- [1] M.J. GREENBERG, *An elementary proof of the Kronecker-Weber theorem*. Amer. Math. Monthly **81** (1974), 601–607; corr.: *ibid.* **82** (1975), 803
- [2] D. HILBERT, *Ein neuer Beweis des Kronecker'schen Fundamentalsatzes über Abel'sche Zahlkörper*. Gött. Nachr. (1896), 29–39
- [3] D. HILBERT, *Die Theorie der algebraischen Zahlkörper*. Jahresber. DMV 1897, 175–546; Gesammelte Abh. I, 63–363; Engl. Transl. by I. Adamson, Springer-Verlag 1998
- [4] K. IRELAND, M. ROSEN, *A Classical Introduction to Modern Number Theory*. Springer Verlag 1982; 2nd ed. 1990
- [5] F. LEMMERMEYER, *Reciprocity Laws. From Euler to Eisenstein*. Springer Verlag 2000
- [6] D. MARCUS, *Number Fields*. Springer-Verlag 1977
- [7] A. SPEISER, *Die Zerlegungsgruppe*. J. Reine Angew. Math. **149** (1919), 174–188
- [8] E. STEINBACHER, *Abelsche Körper als Kreisteilungskörper*. J. Reine Angew. Math. **139** (1910), 85–100
- [9] L. WASHINGTON, *Introduction to Cyclotomic Fields*. Springer-Verlag 1982

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