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Systems of quadratic diophantine inequalities

par WOLFGANG MÜLLER

RÉSUMÉ. Soient Q_1, \dots, Q_r des formes quadratiques avec des coefficients réels. Nous prouvons que pour chaque $\varepsilon > 0$ le système $|Q_1(x)| < \varepsilon, \dots, |Q_r(x)| < \varepsilon$ des inégalités a une solution entière non-triviale si le système $Q_1(x) = 0, \dots, Q_r(x) = 0$ a une solution réelle non-singulière et toutes les formes $\sum_{i=1}^r \alpha_i Q_i$, $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^s, \alpha \neq 0$ sont irrationnelles avec $\text{rang} > 8r$.

ABSTRACT. Let Q_1, \dots, Q_r be quadratic forms with real coefficients. We prove that for any $\varepsilon > 0$ the system of inequalities $|Q_1(x)| < \varepsilon, \dots, |Q_r(x)| < \varepsilon$ has a nonzero integer solution, provided that the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution and all forms in the real pencil generated by Q_1, \dots, Q_r are irrational and have $\text{rank} > 8r$.

1. Introduction

Let Q_1, \dots, Q_r be quadratic forms in s variables with real coefficients. We ask whether the system of quadratic inequalities

$$(1.1) \quad |Q_1(x)| < \varepsilon, \dots, |Q_r(x)| < \varepsilon$$

has a nonzero integer solution for every $\varepsilon > 0$. If some Q_i is rational¹ and ε is small enough then for $x \in \mathbb{Z}^s$ the inequality $|Q_i(x)| < \varepsilon$ is equivalent to the equation $Q_i(x) = 0$. Hence if all forms are rational then for sufficiently small ε the system (1.1) reduces to a system of equations. In this case W. SCHMIDT [10] proved the following result. Recall that the real pencil generated by the forms Q_1, \dots, Q_r is defined as the set of all forms

$$(1.2) \quad Q_\alpha = \sum_{i=1}^r \alpha_i Q_i$$

where $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r, \alpha \neq 0$. The rational and complex pencil are defined similarly. Suppose that Q_1, \dots, Q_r are rational quadratic forms. Then the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonzero integer solution provided that

¹A real quadratic form is called rational if its coefficients are up to a common real factor rational. It is called irrational if it is not rational.

- (i) the given forms have a common nonsingular real solution, and either
- (iia) each form in the complex pencil has rank $> 4r^2 + 4r$, or
- (iib) each form in the rational pencil has rank $> 4r^3 + 4r^2$.

Recently, R. DIETMANN [7] relaxed the conditions (iia) and (iib). He replaced them by the weaker conditions

- (iia') each form in the complex pencil has rank $> 2r^2 + 3r$, or
- (iib') each form in the rational pencil has rank $> 2r^3$ if r is even and rank $> 2r^3 + 2r$ if r is odd.

If $r = 2$ the existence of a nonsingular real solution of $Q_1(x) = 0$ and $Q_2(x) = 0$ follows if one assumes that every form in the real pencil is indefinite (cf. SWINNERTON-DYER [11] and COOK [6]). As noted by W. SCHMIDT [10] this is false for $r > 2$.

We want to consider systems of inequalities (1.1) without hidden equalities. A natural condition is to assume that *all* forms in the real pencil are irrational. Note that if Q_α is rational and ϵ is small enough, then (1.1) and $x \in \mathbb{Z}^s$ imply $Q_\alpha(x) = 0$. We prove

Theorem 1.1. *Let Q_1, \dots, Q_r be quadratic forms with real coefficients. Then for every $\epsilon > 0$ the system (1.1) has a nonzero integer solution provided that*

- (i) *the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution,*
- (ii) *each form in the real pencil is irrational and has rank $> 8r$.*

In the case $r = 1$ much more is known. G.A. MARGULIS [9] proved that for an irrational nondegenerate form Q in $s \geq 3$ variables the set $\{Q(x) \mid x \in \mathbb{Z}^s\}$ is dense in \mathbb{R} (*Oppenheim conjecture*). In the case $r > 1$ all known results assume that the forms Q_i are diagonal². For more information on these results see E.D. FREEMAN [8] and J. BRÜDERN, R.J. COOK [4].

In 1999 V. BENTKUS and F. GÖTZE [2] gave a completely different proof of the Oppenheim conjecture for $s > 8$. We use a multidimensional variant of their method to count weighted solutions of the system (1.1). To do this we introduce for an integer parameter $N \geq 1$ the weighted exponential sum

$$(1.3) \quad S_N(\alpha) = \sum_{x \in \mathbb{Z}^s} w_N(x) e(Q_\alpha(x)) \quad (\alpha \in \mathbb{R}^r).$$

Here Q_α is defined by (1.2), $e(x) = \exp(2\pi i x)$ as usual, and

$$(1.4) \quad w_N(x) = \sum_{n_1+n_2+n_3+n_4=x} p_N(n_1)p_N(n_2)p_N(n_3)p_N(n_4)$$

²Note added in proof: Recently, A. GORODNIK studied systems of nondiagonal forms. In his paper *On an Oppenheim-type conjecture for systems of quadratic forms*, Israel J. Math. **149** (2004), 125–144, he gives conditions (different from ours) that guarantee the existence of a nonzero integer solution of (1.1). His Conjecture 13 is partially answered by our Theorem 1.1.

denotes the fourfold convolution of p_N , the density of the discrete uniform probability distribution on $[-N, N]^s \cap \mathbb{Z}^s$. Since w_N is a probability density on \mathbb{Z}^s one trivially obtains $|S_N(\alpha)| \leq 1$. The key point in the analysis of BENTKUS and GÖTZE is an estimate of $S_N(\alpha + \epsilon)S_N(\alpha - \epsilon)$ in terms of ϵ alone. Lemma 2.2 gives a generalization of their estimate to the case $r > 1$. It is proved via the double large sieve inequality. It shows that for $N^{-2} < |\epsilon| < 1$ the exponential sums $S_N(\alpha - \epsilon)$ and $S_N(\alpha + \epsilon)$ cannot be simultaneously large. This information is almost sufficient to integrate $|S_N(\alpha)|$ within the required precision. As a second ingredient we use for $0 < T_0 \leq 1 \leq T_1$ the uniform bound

$$(1.5) \quad \lim_{N \rightarrow \infty} \sup_{T_0 \leq |\alpha| \leq T_1} |S_N(\alpha)| = 0.$$

Note that (1.5) is false if the real pencil contains a rational form. The proof of (1.5) follows closely BENTKUS and GÖTZE [2] and uses methods from the geometry of numbers.

2. The double large sieve bound

The following formulation of the double large sieve inequality is due to BENTKUS and GÖTZE [2]. For a vector $T = (T_1, \dots, T_s)$ with positive real coordinates write $T^{-1} = (T_1^{-1}, \dots, T_s^{-1})$ and set

$$(2.1) \quad B(T) = \{(x_1, \dots, x_s) \in \mathbb{R}^s \mid |x_j| \leq T_j \text{ for } 1 \leq j \leq s\}.$$

Lemma 2.1 (Double large sieve). *Let μ, ν denote measures on \mathbb{R}^s and let S, T be s -dimensional vectors with positive coordinates. Write*

$$(2.2) \quad J = \int_{B(S)} \left(\int_{B(T)} g(x)h(y)e(\langle x, y \rangle) d\mu(x) \right) d\nu(y),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^s and $g, h : \mathbb{R}^s \rightarrow \mathbb{C}$ are measurable functions. Then

$$|J|^2 \ll A(2S^{-1}, g, \mu)A(2T^{-1}, h, \nu) \prod_{j=1}^s (1 + S_j T_j),$$

where

$$A(S, g, \mu) = \int \left(\int_{y \in x+B(S)} |g(y)| d\mu(y) \right) |g(x)| d\mu(x).$$

The implicit constant is an absolute one. In particular, if $|g(x)| \leq 1$ and $|h(x)| \leq 1$ and μ, ν are probability measures, then

$$(2.3) \quad |J|^2 \ll \sup_{x \in \mathbb{R}^s} \mu(x + B(2S^{-1})) \sup_{x \in \mathbb{R}^s} \nu(x + B(2T^{-1})) \prod_{j=1}^s (1 + S_j T_j).$$

Remark. This is Lemma 5.2 in [1]. For discrete measures the lemma is due to E. BOMBIERI and H. IWANIEC [3]. The general case follows from the discrete one by an approximation argument.

Lemma 2.2. Assume that each form in the real pencil of Q_1, \dots, Q_r has rank $\geq p$. Then the exponential sum (1.3) satisfies

$$(2.4) \quad S_N(\alpha - \epsilon) S_N(\alpha + \epsilon) \ll \mu(|\epsilon|)^p \quad (\alpha, \epsilon \in \mathbb{R}^r),$$

where

$$\mu(t) = \begin{cases} 1 & 0 \leq t \leq N^{-2}, \\ t^{-1/2} N^{-1} & N^{-2} \leq t \leq N^{-1}, \\ t^{1/2} & N^{-1} \leq t \leq 1, \\ 1 & t \geq 1. \end{cases}$$

Proof. Set $S = S_N(\alpha - \epsilon) S_N(\alpha + \epsilon)$. We start with

$$\begin{aligned} S &= \sum_{x, y \in \mathbb{Z}^s} w_N(x) w_N(y) e(Q_{\alpha - \epsilon}(x) + Q_{\alpha + \epsilon}(y)) \\ &= \sum_{\substack{m, n \in \mathbb{Z}^s \\ m \equiv n(2)}} w_N(\frac{1}{2}(m - n)) w_N(\frac{1}{2}(m + n)) e(Q_{\alpha - \epsilon}(\frac{1}{2}(m - n)) + Q_{\alpha + \epsilon}(\frac{1}{2}(m + n))) \\ &= \sum_{\substack{m \equiv n(2) \\ |m|_\infty, |n|_\infty \leq 8N}} w_N(\frac{1}{2}(m - n)) w_N(\frac{1}{2}(m + n)) e(\frac{1}{2}Q_\alpha(m) + \frac{1}{2}Q_\alpha(n) + \langle m, Q_\epsilon n \rangle). \end{aligned}$$

To separate the variables m and n in the weight function write

$$(2.5) \quad w_N(x) = \int_B h(\theta) e(-\langle \theta, x \rangle) d\theta,$$

where $B = (-1/2, 1/2]^s$ and h denotes the (finite) Fourier series

$$h(\theta) = \sum_{k \in \mathbb{Z}^s} w_N(k) e(\langle \theta, k \rangle).$$

Since $w = p_N * p_N * p_N * p_N$ we find $h(\theta) = h_N(\theta)^2$, where

$$h_N(\theta) = \sum_{k \in \mathbb{Z}^s} p_N * p_N(k) e(\langle \theta, k \rangle).$$

Now set

$$a(m) = e\left(\frac{1}{2}(Q_\alpha(m) - \langle \theta_1 + \theta_2, m \rangle)\right),$$

$$b(n) = e\left(\frac{1}{2}(Q_\alpha(n) - \langle \theta_1 - \theta_2, n \rangle)\right).$$

Using (2.5) we find

$$|S| = \left| \int_B \int_B h(\theta_1)h(\theta_2) \sum_{\substack{m \equiv n(2) \\ |m|_\infty, |n|_\infty \leq 8N}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) d\theta_1 d\theta_2 \right|$$

$$\leq \left(\int_B |h(\theta)| d\theta \right)^2 \sup_{\theta_1, \theta_2 \in B} \left| \sum_{\substack{m \equiv n(2) \\ |m|_\infty, |n|_\infty \leq 8N}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \right|.$$

Note that $a(m)$ and $b(n)$ are independent of ϵ . Furthermore, by Bessel's inequality

$$\int_B |h(\theta)| d\theta = \int_B |h_N(\theta)|^2 d\theta \leq \sum_{k \in \mathbb{Z}^s} (p_N * p_N(k))^2$$

$$\leq (2N + 1)^{-s} \sum_{k \in \mathbb{Z}^s} p_N * p_N(k) \leq (2N + 1)^{-s}.$$

Hence

$$S \ll N^{-2s} \sum_{\omega \in \{0,1\}^s} \sup_{\theta_1, \theta_2 \in B} \left| \sum_{\substack{m \equiv n \equiv \omega(2) \\ |m|_\infty, |n|_\infty \leq 8N}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \right|.$$

We are now in the position to apply Lemma 2.1. Denote by $\lambda_1, \dots, \lambda_s$ the eigenvalues of Q_ϵ ordered in such a way that $|\lambda_1| \geq \dots \geq |\lambda_s|$. Then $Q_\epsilon = U^T \Lambda U$, where U is orthogonal and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_s)$. Set $\Lambda^{1/2} = \text{diag}(|\lambda_1|^{1/2}, \dots, |\lambda_s|^{1/2})$, $E = \text{diag}(\text{sgn}(\lambda_1), \dots, \text{sgn}(\lambda_s))$ and

$$\mathcal{M} = \{\Lambda^{1/2} U m \mid m \in \mathbb{Z}^s, m \equiv \omega(2), |m|_\infty \leq 8N\},$$

$$\mathcal{N} = \{E \Lambda^{1/2} U m \mid m \in \mathbb{Z}^s, m \equiv \omega(2), |m|_\infty \leq 8N\}.$$

Furthermore, let μ denote the uniform probability distribution on \mathcal{M} and ν the uniform probability distribution on \mathcal{N} . Choose $S_j = T_j = 1 + 8\sqrt{s}|\lambda_j|^{1/2}N$. Then $x \in \mathcal{M}$ implies $x \in B(T)$ and $y \in \mathcal{N}$ implies $y \in B(S)$. It follows by (2.3) that

$$\left| N^{-2s} \sum_{\substack{m \equiv n \equiv \omega(2) \\ |m|_\infty, |n|_\infty \leq 8N}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \right|^2$$

$$\ll N^{-2s} \left(\sup_{x \in \mathbb{R}^s} A(x) \right)^2 \prod_{j=1}^s (1 + |\lambda_j| N^2),$$

where

$$\begin{aligned} A(x) &= \#\{m \in \mathbb{Z}^s \mid |m|_\infty \leq 8N, m \equiv \omega(2), \Lambda^{1/2}Um - x \in B(2S^{-1})\} \\ &\ll \#\{z \in U\mathbb{Z}^s \mid |z|_\infty \ll N, \|\lambda_j\|^{1/2}z_j - x_j \ll S_j^{-1}\} \\ &\ll \prod_{j=1}^s \min(N, 1 + |\lambda_j|^{-1}N^{-1}). \end{aligned}$$

Hence

$$S \ll \prod_{j=1}^s \tilde{\mu}(|\lambda_j|)$$

with $\tilde{\mu}(t) = N^{-1}(1 + t^{1/2}N) \min(N, 1 + t^{-1}N^{-1})$. To prove (2.4) we have to consider the case $N^{-2} \leq |\epsilon| \leq 1$ only. Otherwise the trivial bound $|S_N(\alpha)| \leq 1$ is sufficient. Since $\lambda_j = \lambda_j(\epsilon)$ varies continuously on $\mathbb{R}^r \setminus \{0\}$ and $\lambda_j(c\epsilon) = c\lambda_j(\epsilon)$ for $c > 0$ there exist constants $0 < \underline{c}_j \leq \bar{c}_j < \infty$ such that

$$(2.6) \quad \begin{aligned} \lambda_j(\epsilon) &\leq \bar{c}_j|\epsilon| & (1 \leq j \leq s), \\ \underline{c}_j|\epsilon| &\leq \lambda_j(\epsilon) \leq \bar{c}_j|\epsilon| & (1 \leq j \leq p). \end{aligned}$$

If $N^{-2} \leq |\epsilon| \leq 1$ then $|\lambda_j| \ll 1$ and $\tilde{\mu}(|\lambda_j|) \ll 1$ for all $j \leq s$. Furthermore, for $j \leq p$ we find $|\lambda_j| \asymp |\epsilon|$ and $\tilde{\mu}(|\lambda_j|) \ll \max(|\epsilon|^{-1/2}N^{-1}, |\epsilon|^{1/2})$. Altogether this yields

$$S \ll \prod_{j=1}^p \tilde{\mu}(|\lambda_j|) \ll \max(|\epsilon|^{-1/2}N^{-1}, |\epsilon|^{1/2})^p \ll \mu(|\epsilon|)^p.$$

□

3. The uniform bound

Lemma 3.1 (H. DAVENPORT [5]). *Let $L_i(x) = \lambda_{i1}x_1 + \dots + \lambda_{is}x_s$ be s linear forms with real and symmetric coefficient matrix $(\lambda_{ij})_{1 \leq i, j \leq s}$. Denote by $\|\cdot\|$ the distance to the nearest integer. Suppose that $P \geq 1$. Then the number of $x \in \mathbb{Z}^s$ such that*

$$|x|_\infty < P \quad \text{and} \quad \|L_i(x)\| < P^{-1} \quad (1 \leq i \leq s)$$

is $\ll (M_1 \dots M_s)^{-1}$. Here M_1, \dots, M_s denotes the first s of the $2s$ successive minima of the convex body defined by $F(x, y) \leq 1$, where for $x, y \in \mathbb{R}^s$

$$F(x, y) = \max(P|L_1(x) - y_1|, \dots, P|L_s(x) - y_s|, P^{-1}|x_1|, \dots, P^{-1}|x_s|).$$

Lemma 3.2. *Assume that each form in the real pencil of Q_1, \dots, Q_r is irrational. Then for any fixed $0 < T_0 \leq T_1 < \infty$*

$$\lim_{N \rightarrow \infty} \sup_{T_0 \leq |\alpha| \leq T_1} |S_N(\alpha)| = 0.$$

Proof. We start with one Weyl step. Using the definition of w_N we find

$$\begin{aligned} |S_N(\alpha)|^2 &= \sum_{x, y \in \mathbb{Z}^s} w_N(x)w_N(y)e(Q_\alpha(y) - Q_\alpha(x)) \\ &= \sum_{\substack{z \in \mathbb{Z}^s \\ |z|_\infty \leq 8N}} \sum_{x \in \mathbb{Z}^s} w_N(x)w_N(x+z)e(Q_\alpha(z) + 2\langle z, Q_\alpha x \rangle) \\ &= (2N+1)^{-8s} \sum_{m_i, n_i, z} \sum_{x \in I(m_i, n_i, z)} e(Q_\alpha(z) + 2\langle z, Q_\alpha x \rangle). \end{aligned}$$

Here the first sum is over all $m_1, m_2, m_3, n_1, n_2, n_3, z \in \mathbb{Z}^s$ with $|m_i|_\infty \leq N, |n_i|_\infty \leq N, |z|_\infty \leq 8N$ and $I(m_i, n_i, z)$ is the set

$$\{x \in \mathbb{Z}^s \mid |x - n_1 - n_2 - n_3|_\infty \leq N, |x + z - m_1 - m_2 - m_3|_\infty \leq N\}.$$

It is an s -dimensional box with sides parallel to the coordinate axes and side length $\ll N$. By Cauchy's inequality it follows that

$$\begin{aligned} |S_N(\alpha)|^4 &\ll N^{-9s} \sum_{m_i, n_i, z} \left| \sum_{x \in I(m_i, n_i, z)} e(2\langle x, Q_\alpha z \rangle) \right|^2 \\ &\ll N^{-3s} \sum_{|z|_\infty \leq 8N} \prod_{i=1}^s \min(N, \|2\langle e_i, Q_\alpha z \rangle\|^{-1})^2. \end{aligned}$$

Here we used the well known bound

$$\sum_{x \in I_1 \times \dots \times I_s} e(\langle x, y \rangle) \ll \prod_{i=1}^s \min(|I_i|, \|\langle e_i, y \rangle\|^{-1}),$$

where I_i are intervals of length $|I_i| \gg 1$ and e_i denotes the i -th unit vector. Set

$$\mathcal{N}(\alpha) = \#\{z \in \mathbb{Z}^s \mid |z|_\infty \leq 16N, \|2\langle e_i, Q_\alpha z \rangle\| < 1/16N \text{ for } 1 \leq i \leq s\}.$$

We claim that

$$(3.1) \quad |S_N(\alpha)|^4 \ll N^{-s} \mathcal{N}(\alpha).$$

To see this set

$$\mathcal{D}_m(\alpha) = \#\{z \in \mathbb{Z}^s \mid |z|_\infty \leq 8N, \frac{m_i-1}{16N} \leq \{2\langle e_i, Q_\alpha z \rangle\} < \frac{m_i}{16N} \text{ for } i \leq s\},$$

where $\{x\}$ denotes the fractional part of x . Then $\mathcal{D}_m(\alpha) \leq \mathcal{N}(\alpha)$ for all $m = (m_1, \dots, m_s)$ with $1 \leq m_i \leq 16N$. Note that if z_1 and z_2 are counted

in $\mathcal{D}_m(\alpha)$ then $z_1 - z_2$ is counted in $\mathcal{N}(\alpha)$. It follows that

$$\begin{aligned} |S_N(\alpha)|^4 &\ll N^{-3s} \sum_{1 \leq m_i \leq 16N} \mathcal{D}_m(\alpha) \prod_{i=1}^s \min \left(N, \frac{16N}{m_i - 1} + \frac{16N}{16N - m_i} \right)^2 \\ &\ll N^{-3s} \mathcal{N}(\alpha) \sum_{1 \leq m_i \leq 8N} \prod_{i=1}^s \frac{N^2}{m_i^2} \\ &\ll N^{-s} \mathcal{N}(\alpha). \end{aligned}$$

To estimate $\mathcal{N}(\alpha)$ we use Lemma 3.1 with $P = 16N$ and $L_i(x) = 2\langle e_i, Q_\alpha x \rangle$ for $1 \leq i \leq s$. This yields

$$(3.2) \quad \mathcal{N}(\alpha) \ll (M_{1,\alpha} \dots M_{s,\alpha})^{-1},$$

where $M_{1,\alpha} \leq \dots \leq M_{s,\alpha}$ are the first s from the $2s$ successive minima of the convex body defined in Lemma 3.1.

Now suppose that there exists an $\epsilon > 0$, a sequence of real numbers $N_n \rightarrow \infty$ and $\alpha^{(n)} \in \mathbb{R}^r$ with $T_0 \leq |\alpha^{(n)}| \leq T_1$ such that

$$(3.3) \quad |S_{N_n}(\alpha^{(n)})| \geq \epsilon.$$

By (3.1) and (3.2) this implies

$$\epsilon^4 N_n^s \ll \left(\prod_{i=1}^s M_{i,\alpha^{(n)}} \right)^{-1}.$$

Since $(16N_n)^{-1} \leq M_{1,\alpha^{(n)}} \leq M_{i,\alpha^{(n)}}$ we obtain $\epsilon^4 N_n^s \ll N_n^{s-1} M_{s,\alpha^{(n)}}^{-1}$ and this proves

$$(16N_n)^{-1} \leq M_{1,\alpha^{(n)}} \leq \dots \leq M_{s,\alpha^{(n)}} \ll (\epsilon^4 N_n)^{-1}.$$

By the definition of the successive minima there exist $x_j^{(n)}, y_j^{(n)} \in \mathbb{Z}^s$ such that $(x_1^{(n)}, y_1^{(n)}), \dots, (x_s^{(n)}, y_s^{(n)})$ are linearly independent and $M_{j,\alpha^{(n)}} = F(x_j^{(n)}, y_j^{(n)})$. Hence for $1 \leq i, j \leq s$

$$\begin{aligned} |L_i(x_j^{(n)}) - y_{j,i}^{(n)}| &\ll N_n^{-2}, \\ |x_{j,i}^{(n)}| &\ll 1. \end{aligned}$$

Since $|\alpha^{(n)}| \leq T_1$ this inequalities imply $|y_{j,i}^{(n)}| \ll_{T_1} 1$. This proves that the integral vectors

$$W_n = (x_1^{(n)}, y_1^{(n)}, \dots, x_s^{(n)}, y_s^{(n)}) \quad (n \geq 1)$$

are contained in a bounded box. Thus there exists an infinite sequence $(n'_k)_{k \geq 1}$ with $W_{n'_k} = W_{n'_k}$ for $k \geq 1$. The compactness of $\{\alpha \in \mathbb{R}^s \mid T_0 \leq |\alpha| \leq T_1\}$ implies that there is a subsequence $(n_k)_{k \geq 1}$ of $(n'_k)_{k \geq 1}$ with

$\lim_{k \rightarrow \infty} \alpha^{(n_k)} = \alpha^{(0)}$ and $T_0 \leq |\alpha^{(0)}| \leq T_1$. Let $x_j = x_j^{(n_k)}$ and $y_j = y_j^{(n_k)}$ for $1 \leq j \leq s$. Then x_j and y_j are well defined and

$$(3.4) \quad y_j = (L_1(x_j), \dots, L_s(x_j)) = 2Q_{\alpha^{(0)}}x_j \quad (1 \leq j \leq s).$$

We claim that x_1, \dots, x_s are linearly independent. Indeed, suppose that there are q_j such that $\sum_{j=1}^s q_j x_j = 0$. Then $\sum_{j=1}^s q_j y_j = 0$ by (3.4). This implies $\sum_{j=1}^s q_j (x_j, y_j) = 0$ and the linear independence of (x_j, y_j) yields $q_j = 0$ for all j . The matrix equation $2Q_{\alpha^{(0)}}(x_1, \dots, x_s) = (y_1, \dots, y_s)$ implies that $Q_{\alpha^{(0)}}$ is rational. By our assumptions this is only possible if $\alpha^{(0)} = 0$, contradicting $|\alpha^{(0)}| \geq T_0 > 0$. This completes the proof of the Lemma. \square

Lemma 3.3. *Assume that each form in the real pencil of Q_1, \dots, Q_r is irrational and has rank ≥ 1 . Then there exists a function $T_1(N)$ such that $T_1(N)$ tends to infinity as N tends to infinity and for every $\delta > 0$*

$$\lim_{N \rightarrow \infty} \sup_{N^{\delta-2} \leq |\alpha| \leq T_1(N)} |S_N(\alpha)| = 0.$$

Proof. We first prove that there exist functions $T_0(N) \leq T_1(N)$ such that $T_0(N) \downarrow 0$ and $T_1(N) \uparrow \infty$ for $N \rightarrow \infty$ and

$$(3.5) \quad \lim_{N \rightarrow \infty} \sup_{T_0(N) \leq |\alpha| \leq T_1(N)} |S_N(\alpha)| = 0.$$

From Lemma 3.2 we know that for each $m \in \mathbb{N}$ there exist an N_m with

$$|S_N(\alpha)| \leq \frac{1}{m} \quad \text{for } N \geq N_m \quad \text{and} \quad \frac{1}{m} \leq |\alpha| \leq m.$$

Without loss of generality we assume that $(N_m)_{m \geq 1}$ is increasing. For $N_m \leq N < N_{m+1}$ define $T_0(N) = \frac{1}{m}$, $T_1(N) = m$ and for $N < N_1$ set $T_0(N) = T_1(N) = 1$. Obviously this choice satisfies (3.5). Replacing $T_0(N)$ by $\max(T_0(N), N^{-1})$ we can assume that $N^{-1} \leq T_0(N) \leq 1$. Finally, Lemma 2.2 with $p \geq 1$ yields

$$\begin{aligned} & \sup_{N^{\delta-2} \leq |\alpha| \leq T_0(N)} |S_N(\alpha)| \\ & \ll \sup_{N^{\delta-2} \leq |\alpha| \leq T_0(N)} \mu(|\alpha|)^p \ll \max(N^{-\delta/2}, T_0(N)^{1/2})^p \rightarrow 0. \end{aligned}$$

\square

4. The integration procedure

In this section we use Lemma 2.2 to integrate $|S_N(\alpha)|$. It is here where we need the assumption $p > 8r$.

Lemma 4.1. For $0 < U \leq T$ set $B(U, T) = \{\alpha \in \mathbb{R}^r \mid U \leq |\alpha| \leq T\}$ and define

$$\gamma(U, T) = \sup_{\alpha \in B(U, T)} |S_N(\alpha)|.$$

Furthermore, let h be a measurable function with $0 \leq h(\alpha) \leq (1 + |\alpha|)^{-k}$, $k > r$. If each form in the real pencil generated by Q_1, \dots, Q_r has rank $\geq p$ with $p > 8r$ and if $\gamma(U, T) \geq 4^{p/(8r)} N^{-p/4}$ then

$$\int_{B(U, T)} |S_N(\alpha)| h(\alpha) d\alpha \ll N^{-2r} \min(1, U^{-(k-r)}) \gamma(U, T)^{1 - \frac{8r}{p}}.$$

Proof. Set $B = B(U, T)$ and $\gamma = \gamma(U, T)$. For $l \geq 0$ define

$$B_l = \{\alpha \in B \mid 2^{-l-1} \leq |S_N(\alpha)| \leq 2^{-l}\}.$$

If L denotes the least non negative integer such that $\gamma \geq 2^{-L-1}$ then $|S_N(\alpha)| \leq \gamma \leq 2^{-L}$ and for any $M \geq L$

$$B = \bigcup_{l=L}^M B_l \cup D_M,$$

where $D_M = \{\alpha \in B \mid |S_N(\alpha)| \leq 2^{-M-1}\}$. By Lemma 2.2

$$|S_N(\alpha)S_N(\alpha + \epsilon)| \leq C\mu(|\epsilon|)^p$$

with some constant C depending on Q_1, \dots, Q_r . By considering $C^{-1/2}S_N(\alpha)$ instead of $S_N(\alpha)$ we may assume $C = 1$. If $\alpha \in B_l$ and $\alpha + \epsilon \in B_l$ it follows that

$$4^{-l-1} \leq |S_N(\alpha)S_N(\alpha + \epsilon)| \leq \mu(|\epsilon|)^p.$$

If $|\epsilon| \leq N^{-1}$ this implies $|\epsilon| \leq N^{-2}2^{4(l+1)/p} = \delta$, say, and if $|\epsilon| \geq N^{-1}$ this implies $|\epsilon| \geq 2^{-4(l+1)/p} = \rho$, say. Note that $\delta \leq \rho$ if $2^{8(l+1)/p} \leq N^2$, and this is true for all $l \leq M$ if

$$(4.1) \quad M + 1 \leq \log(N^{p/4}) / \log 2.$$

We choose M as the largest integer less or equal to $\log(N^{2r} \gamma^{\frac{8r}{p}-1}) / \log 2 - 1$. Then the assumption $\gamma \geq 4^{p/(8r)} N^{-p/4}$ implies $L \leq M$, (4.1) and

$$(4.2) \quad 2^{-M} \ll N^{-2r} \gamma^{1-8r/p}.$$

To estimate the integral over B_l we split B_l in a finite number of subsets. If $B_l \neq \emptyset$ choose any $\beta_1 \in B_l$ and set $B_l(\beta_1) = \{\alpha \in B_l \mid |\alpha - \beta_1| \leq \delta\}$. If $\alpha \in B_l \setminus B_l(\beta_1)$ then $|\alpha - \beta_1| \geq \rho$. If $B_l \setminus B_l(\beta_1) \neq \emptyset$ choose $\beta_2 \in B_l \setminus B_l(\beta_1)$ and set $B_l(\beta_2) = \{\alpha \in B_l \setminus B_l(\beta_1) \mid |\alpha - \beta_2| \leq \delta\}$. Then $|\alpha - \beta_1| \geq \rho$ and $|\alpha - \beta_2| \geq \rho$ for all $\alpha \in B_l \setminus \{B_l(\beta_1) \cup B_l(\beta_2)\}$. Especially $|\beta_1 - \beta_2| \geq \rho$. In this way we construct a sequence β_1, \dots, β_m of points in B_l with $|\beta_i - \beta_j| \geq \rho$ for $i \neq j$. This construction terminates after finitely many

steps. To see this note that the balls $K_{\rho/2}(\beta_i)$ with center β_i and radius $\rho/2$ are disjoint and contained in a ball with center 0 and radius $T + \rho/2$. Thus $m \text{vol}(K_{\rho/2}) \leq \text{vol}(K_{T+\rho/2})$ and this implies $m \ll (1 + T/\rho)^r$. Since $B_l \subseteq \biguplus_{i=1}^m B_l(\beta_i) \subseteq \biguplus_{i=1}^m K_\delta(\beta_i)$ we obtain

$$\begin{aligned} \int_{B_l} |S_N(\alpha)|h(\alpha) d\alpha &\leq 2^{-l} \sum_{i=1}^m \int_{K_\delta(\beta_i)} (1 + |\alpha|)^{-k} d\alpha \\ &\ll 2^{-l} \sum_{\substack{i \leq m \\ |\beta_i| \leq 1}} \delta^r + 2^{-l} \sum_{\substack{i \leq m \\ |\beta_i| > 1}} \left(\frac{\delta}{\rho}\right)^r \int_{K_{\rho/2}(\beta_i)} |\alpha|^{-k} d\alpha. \end{aligned}$$

Note that $|\alpha| \asymp |\beta_i|$ for $\alpha \in K_\rho(\beta_i)$ if $|\beta_i| \geq 1$. If $U > 1$ the first sum is empty and the second sum is $\ll (\delta/\rho)^r \int_{|\alpha| > U/2} |\alpha|^{-k} d\alpha \ll (\delta/\rho)^r U^{-(k-r)}$. If $U \leq 1$ then the first sum contains $\ll \rho^{-r}$ summands; Thus both sums are bounded by $(\delta/\rho)^r$. This yields

$$\int_{B_l} |S_N(\alpha)|h(\alpha) d\alpha \ll 2^{-l} \left(\frac{\delta}{\rho}\right)^r \min(1, U^{-(k-r)}).$$

Altogether we obtain by (4.2) and the definition of δ, ρ, L

$$\begin{aligned} \int_B |S_N(\alpha)|h(\alpha) d\alpha &\ll \sum_{l=L}^M 2^{-l} \left(\frac{\delta}{\rho}\right)^r \min(1, U^{-(k-r)}) + 2^{-M} \int_{|\alpha| \geq U} h(\alpha) d\alpha \\ &\ll \left(N^{-2r} \sum_{l=L}^M 2^{-l(1-8r/p)} + 2^{-M}\right) \min(1, U^{-(k-r)}) \\ &\ll \left(N^{-2r} 2^{-L(1-8r/p)} + 2^{-M}\right) \min(1, U^{-(k-r)}) \\ &\ll N^{-2r} \gamma^{1-8r/p} \min(1, U^{-(k-r)}). \end{aligned}$$

□

5. Proof of Theorem 1.1

We apply a variant of the Davenport-Heilbronn circle method to count weighted solutions of (1.1). Without loss of generality we may assume $\epsilon = 1$. Otherwise apply Theorem 1.1 to the forms $\epsilon^{-1}Q_i$. We choose an even probability density χ with support in $[-1, 1]$ and $\chi(x) \geq 1/2$ for $|x| \leq 1/2$. By choosing χ sufficiently smooth we may assume that its Fourier transform satisfies $\widehat{\chi}(t) = \int \chi(x)e(tx) dx \ll (1 + |t|)^{-r-3}$. Set

$$K(v_1, \dots, v_r) = \prod_{i=1}^r \chi(v_i).$$

Then $\widehat{K}(\alpha) = \prod_{i=1}^r \widehat{\chi}(\alpha_i)$. By Fourier inversion we obtain for an integer parameter $N \geq 1$

$$\begin{aligned} A(N) &:= \sum_{x \in \mathbb{Z}^s} w_N(x) K(Q_1(x), \dots, Q_r(x)) \\ &= \sum_{x \in \mathbb{Z}^s} w_N(x) \int_{\mathbb{R}^r} e(\alpha_1 Q_1(x) + \dots + \alpha_r Q_r(x)) \widehat{K}(\alpha) d\alpha_1 \dots d\alpha_r \\ &= \int_{\mathbb{R}^r} S_N(\alpha) \widehat{K}(\alpha) d\alpha. \end{aligned}$$

Our aim is to prove for $N \geq N_0$, say,

$$(5.1) \quad A(N) \geq cN^{-2r}$$

with some constant $c > 0$. This certainly implies the existence of a non-trivial solution of (1.1), since the contribution of the trivial solution $x = 0$ to $A(N)$ is $\ll N^{-s}$ and $s \geq p > 8r$. To prove (5.1) we divide \mathbb{R}^r in a major arc, a minor arc and a trivial arc. For $\delta > 0$ set

$$\begin{aligned} \mathfrak{M} &= \{\alpha \in \mathbb{R}^r \mid |\alpha| < N^{\delta-2}\}, \\ \mathfrak{m} &= \{\alpha \in \mathbb{R}^r \mid N^{\delta-2} \leq |\alpha| \leq T_1(N)\}, \\ \mathfrak{t} &= \{\alpha \in \mathbb{R}^r \mid |\alpha| > T_1(N)\}, \end{aligned}$$

where $T_1(N)$ denotes the function of Lemma 3.3. Using the bound $\widehat{K}(\alpha) \ll (1 + |\alpha|)^{-r-3}$, Lemma 4.1 (with the choice $U = T_1(N)$ and the trivial estimate $\gamma(T_1(N), \infty) \leq 1$) implies

$$\int_{\mathfrak{t}} S_N(\alpha) \widehat{K}(\alpha) d\alpha = O(N^{-2r} T_1(N)^{-3}) = o(N^{-2r}).$$

Furthermore, Lemma 4.1 with $U = N^{\delta-2}$ and $T = T_1(N)$, together with Lemma 3.3 yield

$$\int_{\mathfrak{m}} S_N(\alpha) \widehat{K}(\alpha) d\alpha = O(N^{-2r} \gamma(N^{\delta-2}, T_1(N))^{1-\frac{8r}{p}}) = o(N^{-2r}).$$

Thus (5.1) follows if we can prove that the contribution of the major arc is

$$(5.2) \quad \int_{\mathfrak{M}} S_N(\alpha) \widehat{K}(\alpha) d\alpha \gg N^{-2r}.$$

6. The major arc

Lemma 6.1. *Assume that each form in the real pencil of Q_1, \dots, Q_r has rank $\geq p$. Let $g, h : \mathbb{R}^s \rightarrow \mathbb{C}$ be measurable functions with $|g| \leq 1$ and $|h| \leq 1$. Then for $N \geq 1$*

$$N^{-2s} \int_{[-N, N]^s} \int_{[-N, N]^s} g(x) h(y) e(\langle x, Q_\alpha y \rangle) dx dy \ll (|\alpha|^{-1/2} N^{-1})^p.$$

Proof. Note that the bound is trivial for $|\alpha| \leq N^{-2}$. Hence we assume $|\alpha| \geq N^{-2}$. Denote by $\lambda_1, \dots, \lambda_s$ the eigenvalues of Q_α ordered in such a way that $|\lambda_1| \geq \dots \geq |\lambda_s|$. Then $Q_\alpha = U^T \Lambda U$, where U is orthogonal and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_s)$. Write $x = (\underline{x}, \bar{x})$, where $\underline{x} = (x_1, \dots, x_p)$ and $\bar{x} = (x_{p+1}, \dots, x_s)$. Then

$$\begin{aligned}
 & N^{-2s} \int_{[-N, N]^s} \int_{[-N, N]^s} g(x)h(y)e(\langle x, Q_\alpha y \rangle) dx dy \\
 &= N^{-2s} \int_{U[-N, N]^s} \int_{U[-N, N]^s} g(U^{-1}x)h(U^{-1}y)e(\langle x, \Lambda y \rangle) dx dy \\
 (6.1) \quad &= N^{-2(s-p)} \int_{\substack{|\bar{x}|_\infty \leq \sqrt{s}N \\ |\bar{y}|_\infty \leq \sqrt{s}N}} e\left(\sum_{i=p+1}^s \lambda_i x_i y_i\right) J(\bar{x}, \bar{y}) d\bar{x} d\bar{y},
 \end{aligned}$$

where

$$J(\bar{x}, \bar{y}) = N^{-2p} \int_{[-\sqrt{s}N, \sqrt{s}N]^p} \int_{[-\sqrt{s}N, \sqrt{s}N]^p} \tilde{g}(\underline{x})\tilde{h}(\underline{y})e\left(\sum_{i=1}^p \lambda_i x_i y_i\right) d\underline{x} d\underline{y}.$$

Here $\tilde{g}(\underline{x}) = g(U^{-1}x)I_{A(\bar{x})}(\underline{x})$ with

$$A(\bar{x}) = \{\underline{x} \in \mathbb{R}^p \mid (\underline{x}, \bar{x}) \in U[-N, N]^s\} \subseteq [-\sqrt{s}N, \sqrt{s}N]^p,$$

and \tilde{h} is defined similarly. If $|\alpha| \geq N^{-2}$ then by (2.6) $|\lambda_i| \asymp |\alpha| \gg N^{-2}$ for $i \leq p$. Now we apply the double large sieve bound (2.3). For $1 \leq j \leq p$ set $S_j = T_j = \sqrt{s|\lambda_j|}N$. Let $\mu = \nu$ be the continuous uniform probability distribution on $\prod_{j=1}^p [-T_j, T_j]$ and set $\bar{g}(\underline{x}) = \tilde{g}(|\lambda_1|^{-1/2}x_1, \dots, |\lambda_p|^{-1/2}x_p)$ and $\bar{h}(\underline{x}) = \tilde{h}(\text{sgn}(\lambda_1)|\lambda_1|^{-1/2}x_1, \dots, \text{sgn}(\lambda_p)|\lambda_p|^{-1/2}x_p)$. Then

$$\begin{aligned}
 |J(\bar{x}, \bar{y})|^2 &\ll \left| \int \int \bar{g}(\underline{x})\bar{h}(\underline{y}) d\mu(\underline{x}) d\nu(\underline{y}) \right|^2 \\
 &\ll \prod_{j=1}^p (1 + |\lambda_j|N^2)(|\lambda_j|^{-1}N^{-2})^2 \\
 &\ll |\alpha|^{-p}N^{-2p}.
 \end{aligned}$$

Together with (6.1) this proves the lemma. □

For $\alpha \in \mathfrak{M}$ we want to approximate $S_N(\alpha)$ by

$$(6.2) \quad G_0(\alpha) = \int \sum_{x \in \mathbb{Z}^s} w_N(x)e(Q_\alpha(x+z)) d\pi(z),$$

where $\pi = I_B * I_B * I_B * I_B$ is the fourfold convolution of the continuous uniform distribution on $B = (-1/2, 1/2]^s$. Set $g(u) = e(Q_\alpha(u))$. Denote by

g_{u_1} the directional derivative of g in direction u_1 , and set $g_{u_1 u_2} = (g_{u_1})_{u_2}$. We use the Taylor series expansions

$$\begin{aligned} f(1) &= f(0) + \int_0^1 f'(\tau) d\tau, \\ f(1) &= f(0) + f'(0) + \int_0^1 (1-\tau) f''(\tau) d\tau, \\ f(1) &= f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{2} \int_0^1 (1-\tau)^2 f'''(\tau) d\tau. \end{aligned}$$

Applying the third of these relations to $f(\tau) = g(x + \tau u_1)$, the second to $f(\tau) = g_{u_1}(x + \tau u_2)$ and the first to $f(\tau) = g_{u_1 u_i}(x + \tau u_3)$ we find for $u_1, u_2, u_3 \in \mathbb{R}^s$

$$\begin{aligned} g(x+u_1) &= g(x) + g_{u_1}(x) + \frac{1}{2} g_{u_1 u_1}(x) + \frac{1}{2} \int_0^1 (1-\tau)^2 g_{u_1 u_1 u_1}(x + \tau u_1) d\tau, \\ g_{u_1}(x+u_2) &= g_{u_1}(x) + g_{u_1 u_2}(x) + \int_0^1 (1-\tau) g_{u_1 u_2 u_2}(x + \tau u_2) d\tau, \\ g_{u_1 u_i}(x+u_3) &= g_{u_1 u_i}(x) + \int_0^1 g_{u_1 u_i u_3}(x + \tau u_3) d\tau. \end{aligned}$$

Together we obtain the expansion

$$\begin{aligned} g(x) &= g(x+u_1) - g_{u_1}(x+u_2) - \frac{1}{2} g_{u_1 u_1}(x+u_3) + g_{u_1 u_2}(x+u_3) \\ &\quad + \int_0^1 \left\{ -g_{u_1 u_2 u_3}(x + \tau u_3) + \frac{1}{2} g_{u_1 u_1 u_3}(x + \tau u_3) \right. \\ &\quad \left. + (1-\tau) g_{u_1 u_2 u_2}(x + \tau u_2) - \frac{1}{2} (1-\tau)^2 g_{u_1 u_1 u_1}(x + \tau u_1) \right\} d\tau. \end{aligned}$$

Multiplying with $w_N(x)$, summing over $x \in \mathbb{Z}^s$, and integrating u_1, u_2, u_3 with respect to the probability measure π yields

$$S_N(\alpha) = G_0(\alpha) + G_1(\alpha) + G_2(\alpha) + G_3(\alpha) + R(\alpha),$$

where $G_0(\alpha)$ is defined by (6.2),

$$\begin{aligned} G_1(\alpha) &= - \int \int \sum_{x \in \mathbb{Z}^s} w_N(x) g_u(x+z) d\pi(u) d\pi(z), \\ G_2(\alpha) &= - \frac{1}{2} \int \int \sum_{x \in \mathbb{Z}^s} w_N(x) g_{uu}(x+z) d\pi(u) d\pi(z), \\ G_3(\alpha) &= \int \int \int \sum_{x \in \mathbb{Z}^s} w_N(x) g_{uv}(x+z) d\pi(u) d\pi(v) d\pi(z), \end{aligned}$$

and

$$R(\alpha) \ll \sup_{|u|_\infty, |v|_\infty, |w|_\infty, |z|_\infty \leq 1} \left| \sum_{x \in \mathbb{Z}^s} w_N(x) g_{uvw}(x+z) \right|.$$

An elementary calculation yields

$$\begin{aligned}
 g_u(x) &= 4\pi i e(Q_\alpha(x)) \langle x, Q_\alpha u \rangle, \\
 g_{uv}(x) &= (4\pi i)^2 e(Q_\alpha(x)) \langle x, Q_\alpha u \rangle \langle x, Q_\alpha v \rangle + 4\pi i e(Q_\alpha(x)) \langle u, Q_\alpha v \rangle, \\
 g_{uvw}(x) &= (4\pi i)^3 e(Q_\alpha(x)) \langle x, Q_\alpha u \rangle \langle x, Q_\alpha v \rangle \langle x, Q_\alpha w \rangle + (4\pi i)^2 e(Q_\alpha(x)) \times \\
 &\quad \left(\langle x, Q_\alpha v \rangle \langle u, Q_\alpha w \rangle + \langle x, Q_\alpha u \rangle \langle v, Q_\alpha w \rangle + \langle x, Q_\alpha w \rangle \langle u, Q_\alpha v \rangle \right).
 \end{aligned}$$

Since g_u and g_{uv} are sums of odd functions (in at least one of the components of u) we infer $G_1(\alpha) = 0$ and $G_3(\alpha) = 0$. Furthermore, the trivial bound $g_{uvw}(x) \ll |\alpha|^3 N^3 + |\alpha|^2 N$ for $|x|_\infty \ll N$ yields

$$R(\alpha) \ll |\alpha|^3 N^3 + |\alpha|^2 N.$$

This is sharp enough to prove

$$\begin{aligned}
 \int_{\mathfrak{M}} |R(\alpha) \widehat{K}(\alpha)| d\alpha &\ll \int_{|\alpha| \leq N^{\delta-2}} (|\alpha|^3 N^3 + |\alpha|^2 N) d\alpha \\
 &\ll \int_0^{N^{\delta-2}} (u^{r+2} N^3 + u^{r+1} N) du \\
 &\ll N^{3-(2-\delta)(r+3)} + N^{1-(2-\delta)(r+2)} \\
 &\ll N^{-2r-3+\delta(r+3)} = o(N^{-2r}).
 \end{aligned}$$

To deal with G_0 and G_2 we need a bound for

$$\widetilde{G}_j(\alpha, u) = \int_{\mathbb{R}^s} \sum_{x \in \mathbb{Z}^s} w_N(x) L(x+z)^j e(Q_\alpha(x+z)) d\pi(z),$$

where $L(x) = \langle x, Q_\alpha u \rangle$ and $0 \leq j \leq 2$. Using the definition of w_N and π we find that $\widetilde{G}_j(\alpha, u)$ is equal to

$$\begin{aligned}
 &\int_{B^4} \sum_{x_1, \dots, x_4 \in \mathbb{Z}^s} \prod_{i=1}^4 p_N(x_i) L\left(\sum_{i=1}^4 (x_i + z_i)\right)^j e\left(Q_\alpha\left(\sum_{i=1}^4 (x_i + z_i)\right)\right) dz_1 \dots dz_4 \\
 &= (2N+1)^{-4s} \int_{|x_1|_\infty, \dots, |x_4|_\infty \leq N+1/2} L\left(\sum_{i=1}^4 x_i\right)^j e\left(Q_\alpha\left(\sum_{i=1}^4 x_i\right)\right) dx_1 \dots dx_4.
 \end{aligned}$$

Expanding $L(x_1 + x_2 + x_3 + x_4)$ and $Q_\alpha(x_1 + x_2 + x_3 + x_4)$ this can be bounded by

$$\begin{aligned}
 &\max_{l_1+l_2+l_3+l_4=j} N^{-4s} \left| \int \left\{ \prod_{i=1}^4 L(x_i)^{l_i} e(Q_\alpha(x_i)) \right\} e\left(2 \sum_{i < j} \langle x_i, Q_\alpha x_j \rangle\right) dx_1 \dots dx_4 \right| \\
 &\ll \max_{l_1+l_2+l_3+l_4=j} \frac{(|\alpha|N)^j}{N^{4s}} \left| \int \left\{ \prod_{i=1}^4 h_i(x_i) \right\} e\left(2 \sum_{i < j} \langle x_i, Q_\alpha x_j \rangle\right) dx_1 \dots dx_4 \right|.
 \end{aligned}$$

Here

$$h_i(x_i) = L(x_i)^{l_i} e(Q_\alpha(x_i)) (|\alpha|N)^{-l_i} I_{\{|x_i| \leq N+1/2\}} \ll 1.$$

Applying Lemma 6.1 to the double integral over x_1 and x_2 and estimating the integral over x_3 and x_4 trivially we obtain uniformly in $|u| \ll 1$

$$\tilde{G}_j(\alpha, u) \ll (|\alpha|N)^j |\alpha|^{-p/2} N^{-p}.$$

Setting

$$H_j(N) = \int_{\mathbb{R}^r} G_j(\alpha) \widehat{K}(\alpha) d\alpha$$

we conclude for sufficiently small $\delta > 0$ and $p > 8r$ ($G_0(\alpha) = \tilde{G}_0(\alpha, 0)$)

$$\begin{aligned} \int_{\mathfrak{M}} G_0(\alpha) \widehat{K}(\alpha) d\alpha &= H_0(N) - \int_{|\alpha| \geq N^{\delta-2}} \tilde{G}_0(\alpha, 0) \widehat{K}(\alpha) d\alpha \\ &= H_0(N) + O(N^{-p} (\int_{N^{\delta-2} \leq |\alpha| \leq 1} |\alpha|^{-p/2} d\alpha + 1)) \\ &= H_0(N) + O(N^{-p-(2-\delta)(r-p/2)}) + O(N^{-p}) \\ &= H_0(N) + o(N^{-2r}). \end{aligned}$$

Similarly, the explicit expression of $g_{uu}(x)$ and the definition of $\tilde{G}_j(\alpha, u)$ yield

$$\begin{aligned} &\int_{\mathfrak{M}} G_2(\alpha) \widehat{K}(\alpha) d\alpha \\ &= H_2(N) + O\left(\sup_{|u|_\infty \leq 2} \int_{|\alpha| \geq N^{\delta-2}} |\tilde{G}_2(\alpha, u) \widehat{K}(\alpha)| + |\alpha| |\tilde{G}_0(\alpha, u) \widehat{K}(\alpha)| d\alpha\right) \\ &= H_2(N) + O\left(N^{2-p} (\int_{N^{\delta-2} \leq |\alpha| \leq 1} |\alpha|^{2-p/2} d\alpha + 1)\right) \\ &\quad + O\left(N^{-p} (\int_{N^{\delta-2} \leq |\alpha| \leq 1} |\alpha|^{1-p/2} d\alpha + 1)\right) \\ &= H_2(N) + o(N^{-2r}). \end{aligned}$$

Hence

$$\int_{\mathfrak{M}} S_N(\alpha) \widehat{K}(\alpha) d\alpha = H_0(N) + H_2(N) + o(N^{-2r}).$$

Altogether we have proved that for $p > 8r$

$$(6.3) \quad A(N) = H_0(N) + H_2(N) + o(N^{-2r}).$$

7. Analysis of the terms $H_0(N)$ and $H_2(N)$

Lemma 7.1. Denote by π_N the fourfold convolution of the continuous uniform probability distribution on $B_N = (-N - 1/2, N + 1/2]^s$ and by f_N the density of π_N . Then

$$H_0(N) = \int K(Q_1(x), \dots, Q_r(x)) f_N(x) dx$$

and

$$H_2(N) = -\frac{1}{6} \int K(Q_1(x), \dots, Q_r(x)) \Delta f_N(x) dx,$$

where $\Delta f_N(x) = \sum_{i=1}^s \frac{\partial^2 f_N}{\partial x_i^2}(x)$. Furthermore, $\Delta f_N(x) \ll N^{-s-2}$.

Proof. By Fourier inversion and the definition of w_N and $\pi = \pi_0$ we find

$$\begin{aligned} H_0(N) &= \int_{\mathbb{R}^r} G_0(\alpha) \widehat{K}(\alpha) d\alpha \\ &= \int \sum_{x \in \mathbb{Z}^s} w_N(x) \int_{\mathbb{R}^r} e(Q_\alpha(x+z)) \widehat{K}(\alpha) d\alpha d\pi(z) \\ &= \int \sum_{x \in \mathbb{Z}^s} w_N(x) K(Q_1(x+z), \dots, Q_r(x+z)) d\pi(z) \\ &= \int K(Q_1(x), \dots, Q_r(x)) d\pi_N(x). \end{aligned}$$

This proves the first assertion of the Lemma. Similarly,

$$-2G_2(\alpha) = \iint g_{uu}(x) d\pi(u) d\pi_N(x).$$

This implies

$$-2H_2(N) = -2 \int G_2(\alpha) \widehat{K}(\alpha) d\alpha = \iint \int_{\mathbb{R}^r} g_{uu}(x) \widehat{K}(\alpha) d\alpha d\pi(u) d\pi_N(x).$$

With the abbreviations $L_m = 2\langle x, Q_mu \rangle$ and $\tilde{L}_m = 2\langle u, Q_mv \rangle$ the innermost integral can be calculated as

$$\begin{aligned} &\int_{\mathbb{R}^r} g_{uu}(x) \widehat{K}(\alpha) d\alpha \\ &= \int_{\mathbb{R}^r} e(Q_\alpha(x)) \left\{ \sum_{m,n=1}^r L_m L_n \widehat{\frac{\partial^2 K}{\partial v_m \partial v_n}}(\alpha) + \sum_{m=1}^r \tilde{L}_m \widehat{\frac{\partial K}{\partial v_m}}(\alpha) \right\} d\alpha \\ &= \sum_{m,n=1}^r L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n}(Q_1(x), \dots, Q_r(x)) + \sum_{m=1}^r \tilde{L}_m \frac{\partial K}{\partial v_m}(Q_1(x), \dots, Q_r(x)). \end{aligned}$$

Here we used the relations

$$\begin{aligned}\widehat{\frac{\partial K}{\partial v_m}}(\alpha) &= 2\pi i \alpha_m \widehat{K}(\alpha), \\ \widehat{\frac{\partial^2 K}{\partial v_m \partial v_n}}(\alpha) &= (2\pi i)^2 \alpha_m \alpha_n \widehat{K}(\alpha).\end{aligned}$$

Since

$$\begin{aligned}& \sum_{i,j=1}^s u_i u_j \frac{\partial^2}{\partial x_i \partial x_j} (K(Q_1(x), \dots, Q_r(x))) \\ &= \sum_{m,n=1}^r L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n} (Q_1(x), \dots, Q_r(x)) + \sum_{m=1}^r \tilde{L}_m \frac{\partial K}{\partial v_m} (Q_1(x), \dots, Q_r(x))\end{aligned}$$

we find

$$\int_{\mathbb{R}^r} g_{uu}(x) \widehat{K}(\alpha) d\alpha = \sum_{i,j=1}^s u_i u_j \frac{\partial^2}{\partial x_i \partial x_j} (K(Q_1(x), \dots, Q_r(x))).$$

Altogether we conclude

$$\begin{aligned}-2H_2(N) &= \int \int \sum_{i,j=1}^s u_i u_j \frac{\partial^2}{\partial x_i \partial x_j} (K(Q_1(x), \dots, Q_r(x))) d\pi(u) d\pi_N(x) \\ &= \sum_{i=1}^s \int \int u_i^2 \frac{\partial^2}{\partial x_i^2} (K(Q_1(x), \dots, Q_r(x))) d\pi(u) d\pi_N(x) \\ &= \left(\int u_1^2 d\pi(u) \right) \sum_{i=1}^s \int \frac{\partial^2}{\partial x_i^2} (K(Q_1(x), \dots, Q_r(x))) d\pi_N(x).\end{aligned}$$

Since π_N has compact support and f_N is two times continuously differentiable, partial integration yields

$$\int \frac{\partial^2}{\partial x_i^2} (K(Q_1(x), \dots, Q_r(x))) f_N(x) dx = \int K(Q_1(x), \dots, Q_r(x)) \frac{\partial^2 f_N}{\partial x_i^2}(x) dx.$$

This completes the proof of the second assertion of the Lemma, since $\int u_1^2 d\pi(u) = 1/3$.

Finally, we prove

$$\frac{\partial^2 f_N}{\partial x_i^2}(x) \ll N^{-s-2}.$$

Note that

$$\widehat{f_N}(t) = \prod_{i=1}^s \left(\frac{\sin(\pi t_i (2N+1))}{\pi t_i (2N+1)} \right)^4 = \widehat{f_0}((2N+1)t).$$

Hence, by Fourier inversion

$$\begin{aligned} \frac{\partial^2 f_N}{\partial x_i^2}(x) &= (-2\pi i)^2 \int \widehat{f}_N(t) t_i^2 e(-\langle t, x \rangle) dt \\ &= -(2\pi)^2 (2N + 1)^{-s-2} \int \widehat{f}_0(t) t_i^2 e(-(2N + 1)\langle t, x \rangle) dt \\ &\ll N^{-s-2}. \end{aligned}$$

This completes the proof of Lemma 7.1. We remark that we used the fourfold convolution in the definition of w_N, π_N, f_N for the above treatment of $H_2(N)$ only. At all other places of the argument a twofold convolution would be sufficient for our purpose. \square

Lemma 7.2. *Assume that the system $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution, then*

$$\lambda(\{x \in \mathbb{R}^s \mid |Q_i(x)| \leq N^{-2}, |x|_\infty \leq 1\}) \gg N^{-2r},$$

where λ denotes the s -dimensional Lebesgue measure.

Proof. This is proved in Lemma 2 of [10]. Note that if a system of homogeneous equations $Q_1(x) = 0, \dots, Q_r(x) = 0$ has a nonsingular real solution, then it has a nonsingular real solution with $|x|_\infty \leq 1/2$.

Now we complete the proof of Theorem 1.1 as follows. For $c > 0$ and $N > 0$ set

$$A(c, N) = \lambda(\{x \in \mathbb{R}^s \mid |Q_i(x)| \leq N^{-2}, |x|_\infty \leq c\}).$$

Then

$$A(c, N) = c^s A(1, cN).$$

By Lemma 7.1

$$\begin{aligned} H_0(N) &\gg N^{-s} \int_{|x|_\infty \leq 2N} K(Q_1(x), \dots, Q_r(x)) dx \\ &\gg \int_{|y|_\infty \leq 2} K(N^2 Q_1(y), \dots, N^2 Q_r(y)) dy \\ &\gg A(2, 2N) \\ &\gg A(1, 5N) \end{aligned}$$

and

$$\begin{aligned}
 H_2(N) &\ll N^{-s-2} \int_{|x|_\infty \leq 5N} K(Q_1(x), \dots, Q_r(x)) dx \\
 &\ll N^{-2} \int_{|y|_\infty \leq 5} K(N^2 Q_1(y), \dots, N^2 Q_r(y)) dy \\
 &\ll N^{-2} A(5, N) \\
 &\ll N^{-2} A(1, 5N).
 \end{aligned}$$

With Lemma 7.2 this yields

$$H_0(N) + H_2(N) \gg A(1, 5N) \gg N^{-2r}$$

for $N \geq N_0$, say. Together with (6.3) this completes the proof of Theorem 1.1. \square

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