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Binary quadratic forms and Eichler orders

par MONTSERRAT ALSINA

RÉSUMÉ. Pour tout ordre d'Eichler $\mathcal{O}(D, N)$ de niveau N dans une algèbre de quaternions indéfinie de discriminant D , il existe un groupe Fuchsien $\Gamma(D, N) \subseteq \mathrm{SL}(2, \mathbb{R})$ et une courbe de Shimura $X(D, N)$. Nous associons à $\mathcal{O}(D, N)$ un ensemble $\mathcal{H}(\mathcal{O}(D, N))$ de formes quadratiques binaires ayant des coefficients semi-entiers quadratiques et développons une classification des formes quadratiques primitives de $\mathcal{H}(\mathcal{O}(D, N))$ pour rapport à $\Gamma(D, N)$. En particulier nous retrouvons la classification des formes quadratiques primitives et entières de $\mathrm{SL}(2, \mathbb{Z})$. Un domaine fondamental explicite pour $\Gamma(D, N)$ permet de caractériser les $\Gamma(D, N)$ formes réduites.

ABSTRACT. For any Eichler order $\mathcal{O}(D, N)$ of level N in an indefinite quaternion algebra of discriminant D there is a Fuchsian group $\Gamma(D, N) \subseteq \mathrm{SL}(2, \mathbb{R})$ and a Shimura curve $X(D, N)$. We associate to $\mathcal{O}(D, N)$ a set $\mathcal{H}(\mathcal{O}(D, N))$ of binary quadratic forms which have semi-integer quadratic coefficients, and we develop a classification theory, with respect to $\Gamma(D, N)$, for primitive forms contained in $\mathcal{H}(\mathcal{O}(D, N))$. In particular, the classification theory of primitive integral binary quadratic forms by $\mathrm{SL}(2, \mathbb{Z})$ is recovered. Explicit fundamental domains for $\Gamma(D, N)$ allow the characterization of the $\Gamma(D, N)$ -reduced forms.

1. Preliminars

Let $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ be the quaternion \mathbb{Q} -algebra of basis $\{1, i, j, ij\}$, satisfying $i^2 = a, j^2 = b, ji = -ij, a, b \in \mathbb{Q}^*$. Assume H is an indefinite quaternion algebra, that is, $H \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}(2, \mathbb{R})$. Then the discriminant D_H of H is the product of an even number of different primes $D_H = p_1 \cdots p_{2r} \geq 1$ and we can assume $a > 0$. Actually, a discriminant D determines a quaternion algebra H such that $D_H = D$ up to isomorphism. Let us denote by $n(\omega)$ the reduced norm of $\omega \in H$.

Fix any embedding $\Phi : H \hookrightarrow \mathrm{M}(2, \mathbb{R})$. For simplicity we can keep in mind the embedding given at the following lemma.

Lemma 1.1. *Let $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ be an indefinite quaternion algebra with $a > 0$. An embedding $\Phi : H \hookrightarrow M(2, \mathbb{R})$ is obtained by:*

$$\Phi(x + yi + zj + tij) = \begin{pmatrix} x + y\sqrt{a} & z + t\sqrt{a} \\ b(z - t\sqrt{a}) & x - y\sqrt{a} \end{pmatrix}.$$

Given $N \geq 1$, $\gcd(D, N) = 1$, let us consider an Eichler order of level N , that is a \mathbb{Z} -module of rank 4, subring of H , intersection of two maximal orders. By Eichler's results it is unique up to conjugation and we denote it by $\mathcal{O}(D, N)$.

Consider $\Gamma(D, N) := \Phi(\{\omega \in \mathcal{O}(D, N)^* \mid \mathfrak{n}(\omega) > 0\}) \subseteq \mathrm{SL}(2, \mathbb{R})$ a group of quaternion transformations. This group acts on the upper complex half plane $\mathcal{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$. We denote by $X(D, N)$ the canonical model of the Shimura curve defined by the quotient $\Gamma(D, N) \backslash \mathcal{H}$, cf. [Shi67], [AAB01].

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ we denote by $\mathcal{P}(\gamma)$ the set of fixed points in \mathbb{C} of the transformation defined by $\gamma(z) = \frac{az+b}{cz+d}$.

Let us denote by $\mathcal{E}(H, F)$ the set of embeddings of a quadratic field F into the quaternion algebra H . Assume there is an embedding $\varphi \in \mathcal{E}(H, F)$. Then, all the quaternion transformations in $\Phi(\varphi(F^*)) \subset \mathrm{GL}(2, \mathbb{R})$ have the same set of fixed points, which we denote by $\mathcal{P}(\varphi)$. In the case that F is an imaginary quadratic field it yields to complex multiplication points, since $\mathcal{P}(\varphi) \cap \mathcal{H}$ is just a point, $z(\varphi)$.

Now, we take in account the arithmetic of the orders. Let us consider the set of optimal embeddings of quadratic orders Λ into quaternion orders \mathcal{O} ,

$$\mathcal{E}^*(\mathcal{O}, \Lambda) := \{\varphi \mid \varphi : \Lambda \hookrightarrow \mathcal{O}, \varphi(F) \cap \mathcal{O} = \varphi(\Lambda)\}.$$

Any group $G \leq \mathrm{Nor}(\mathcal{O})$ acts on $\mathcal{E}^*(\mathcal{O}, \Lambda)$, and we can consider the quotient $\mathcal{E}^*(\mathcal{O}, \Lambda)/G$. Put $\nu(\mathcal{O}, \Lambda; G) := \#\mathcal{E}^*(\mathcal{O}, \Lambda)/G$. We will also use the notation $\nu(D, N, d, m; G)$ for an Eichler order $\mathcal{O}(D, N) \subseteq H$ of level N and the quadratic order of conductor m in $F = \mathbb{Q}(\sqrt{d})$, which we denote $\Lambda(d, m)$.

Since further class numbers in this paper will be related to this one, we include next theorem (cf. [Eic55]). It provides the well-known relation between the class numbers of local and global embeddings, and collects the formulas for the class number of local embeddings given in [Ogg83] and [Vig80] in the case $G = \mathcal{O}^*$. Consider ψ_p the multiplicative function given by $\psi_p(p^k) = p^k(1 + \frac{1}{p})$, $\psi_p(a) = 1$ if $p \nmid a$. Put $h(d, m)$ the ideal class number of the quadratic order $\Lambda(d, m)$.

Theorem 1.2. *Let $\mathcal{O} = \mathcal{O}(D, N)$ be an Eichler order of level N in an indefinite quaternion \mathbb{Q} -algebra H of discriminant D . Let $\Lambda(d, m)$ be the quadratic order of conductor m in $\mathbb{Q}(\sqrt{d})$. Assume that $\mathcal{E}(H, \mathbb{Q}(\sqrt{d})) \neq \emptyset$*

and $\gcd(m, D) = 1$. Then,

$$\nu(D, N, d, m; \mathcal{O}^*) = h(d, m) \prod_{p|DN} \nu_p(D, N, d, m; \mathcal{O}^*).$$

The local class numbers of embeddings $\nu_p(D, N, d, m; \mathcal{O}^*)$, for the primes $p|DN$, are given by

- (i) If $p|D$, then $\nu_p(D, N, d, m; \mathcal{O}^*) = 1 - \left(\frac{D_F}{p}\right)$.
- (ii) If $p \parallel N$, then $\nu_p(D, N, d, m; \mathcal{O}^*)$ is equal to $1 + \left(\frac{D_F}{p}\right)$ if $p \nmid m$, and equal to 2 if $p|m$.
- (iii) Assume $N = p^r u_1$, with $p \nmid u_1$, $r \geq 2$. Put $m = p^k u_2$, $p \nmid u_2$.
 - (a) If $r \geq 2k + 2$, then $\nu_p(D, N, d, m; \mathcal{O}^*)$ is equal to $2\psi_p(m)$ if $\left(\frac{D_F}{p}\right) = 1$, and equal to 0 otherwise.
 - (b) If $r = 2k + 1$, then $\nu_p(D, N, d, m; \mathcal{O}^*)$ is equal to $2\psi_p(m)$ if $\left(\frac{D_F}{p}\right) = 1$, equal to p^k if $\left(\frac{D_F}{p}\right) = 0$, and equal to 0 if $\left(\frac{D_F}{p}\right) = -1$.
 - (c) If $r = 2k$, then $\nu_p(D, N, d, m; \mathcal{O}^*) = p^{k-1} \left(p + 1 + \left(\frac{D_F}{p}\right)\right)$.
 - (d) If $r \leq 2k - 1$, then $\nu_p(D, N, d, m; \mathcal{O}^*)$ is equal to $p^{k/2} + p^{k/2-1}$ if k is even, and equal to $2p^{k-1/2}$ if k is odd.

2. Classification theory of binary forms associated to quaternions

Given $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{R})$, we put $f_\alpha(x, y) := cx^2 + (d - a)xy - by^2$. It is called the binary quadratic form associated to α .

For a binary quadratic form $f(x, y) := Ax^2 + Bxy + Cy^2 = (A, B, C)$, we consider the associated matrix $A(f) = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$, and the determinants $\det_1(f) = \det A(f)$ and $\det_2(f) = 2^2 \det A(f) = -(B^2 - 4AC)$. Denote by $\mathcal{P}(f)$ the set of solutions in \mathbb{C} of $Az^2 + Bz + C = 0$. If f is (positive or negative) definite, then $\mathcal{P}(f) \cap \mathcal{H}$ is just a point which we denote by $\tau(f)$.

The proof of the following lemma is straightforward.

Lemma 2.1. *Let $\alpha \in M(2, \mathbb{R})$.*

- (i) *For all $\lambda, \mu \in \mathbb{Q}$, we have $f_{\lambda\alpha} = \lambda f_\alpha$ and $f_{\alpha+\mu\text{Id}} = f_\alpha$; in particular, $\mathcal{P}(f_{\lambda\alpha+\mu\text{Id}}) = \mathcal{P}(f_\alpha)$.*
- (ii) *$z \in \mathbb{C}$ is a fixed point of α if and only if $z \in \mathcal{P}(f_\alpha)$, that is, $\mathcal{P}(f_\alpha) = \mathcal{P}(\alpha)$.*
- (iii) *Let $\gamma \in \text{GL}(2, \mathbb{R})$. Then $A(f_{\gamma^{-1}\alpha\gamma}) = (\det \gamma^{-1})\gamma^t A(f_\alpha)\gamma$; in particular, if $\gamma \in \text{SL}(2, \mathbb{R})$, $z \in \mathcal{P}(f_\alpha)$ if and only if $\gamma^{-1}(z) \in \mathcal{P}(f_{\gamma^{-1}\alpha\gamma})$.*

Definition 2.2. For a quaternion $\omega \in H^*$, we define the binary quadratic form associated to ω as the binary quadratic form $f_{\Phi(\omega)}$.

Given a quaternion algebra H denote by H_0 the pure quaternions. By using lemma 2.1 it is enough to consider the binary forms associated to pure quaternions:

$$\mathcal{H}(a, b) = \{f_{\Phi(\omega)} : \omega \in H_0\}, \quad \mathcal{H}(\mathcal{O}) = \{f_{\Phi(\omega)} : \omega \in \mathcal{O} \cap H_0\}.$$

Definition 2.3. Let \mathcal{O} be an order in a quaternion algebra H . We define the denominator $m_{\mathcal{O}}$ of \mathcal{O} as the minimal positive integer such that $m_{\mathcal{O}} \cdot \mathcal{O} \subseteq \mathbb{Z}[1, i, j, ij]$. Then the ideal $(m_{\mathcal{O}})$ is the conductor of \mathcal{O} in $\mathbb{Z}[1, i, j, ij]$.

Properties for these binary forms are collected in the following proposition, easy to be verified.

Proposition 2.4. *Consider an indefinite quaternion algebra $H = \left(\frac{a, b}{\mathbb{Q}}\right)$, and an order $\mathcal{O} \subseteq H$. Fix the embedding Φ as in lemma 1.1. Then:*

(i) *There is a bijective mapping $H_0 \rightarrow \mathcal{H}(a, b)$ defined by $\omega \mapsto f_{\Phi(\omega)}$. Moreover $\det_1(f_{\Phi(\omega)}) = \mathfrak{n}(\omega)$.*

(ii) $\mathcal{H}(a, b) = \{(b(\lambda_2 + \lambda_3\sqrt{a}), \lambda_1\sqrt{a}, -\lambda_2 + \lambda_3\sqrt{a}) \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}\}$
 $= \{(b\beta', \alpha, -\beta) \mid \alpha, \beta \in \mathbb{Q}(\sqrt{a}), \text{tr}(\alpha) = 0\}.$

(iii) *the binary quadratic forms of $\mathcal{H}(\mathcal{O})$ have coefficients in $\mathbb{Z}\left[\frac{1}{m_{\mathcal{O}}}, \sqrt{a}\right]$*

Given a quaternion order \mathcal{O} and a quadratic order Λ , put

$$\mathcal{H}(\mathcal{O}, \Lambda) := \{f \in \mathcal{H}(\mathcal{O}) : \det_1(f) = -D_{\Lambda}\}.$$

Remark that an imaginary quadratic order yields to consider definite binary quadratic forms, and a real quadratic order yields to indefinite binary forms.

Given $\omega \in \mathcal{O} \cap H_0$, consider $F_{\omega} = \mathbb{Q}(\sqrt{d})$, $d = -\mathfrak{n}(\omega)$. Then $\varphi_{\omega}(\sqrt{d}) = \omega$ defines an embedding $\varphi_{\omega} \in \mathcal{E}(H, F_{\omega})$. By considering $\Lambda_{\omega} := \varphi_{\omega}^{-1}(\mathcal{O}) \cap F_{\omega}$, we have $\varphi_{\omega} \in \mathcal{E}^*(\mathcal{O}, \Lambda_{\omega})$. Therefore, by construction, it is clear that $\mathcal{P}(f_{\Phi(\omega)}) = \mathcal{P}(\Phi(\omega)) = \mathcal{P}(\varphi_{\omega})$. In particular, if we deal with quaternions of positive norm, we obtain definite binary forms, imaginary quadratic fields and a unique solution $\tau(f_{\Phi(\omega)}) = z(\varphi_{\omega}) \in \mathcal{H}$. The points corresponding to these binary quadratic forms are in fact the complex multiplication points.

Theorem 4.53 in [AB04] states a bijective mapping f from the set $\mathcal{E}(\mathcal{O}, \Lambda)$ of embeddings of a quadratic order Λ into a quaternion order \mathcal{O} onto the set $\mathcal{H}(\mathbb{Z} + 2\mathcal{O}, \Lambda)$ of binary quadratic forms associated to the orders $\mathbb{Z} + 2\mathcal{O}$ and Λ . By using optimal embeddings, a definition of primitivity for the forms in $\mathcal{H}(\mathbb{Z} + 2\mathcal{O}, \Lambda)$ was introduced. We denote by $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda)$ the corresponding subset of (\mathcal{O}, Λ) -primitive binary forms. Then equivalence of embeddings yields to equivalence of forms.

Corollary 2.5. *Given orders \mathcal{O} and Λ as above, for any $G \subseteq \mathcal{O}^*$ consider $\Phi(G) \subseteq \text{GL}(2, \mathbb{R})$. There is a bijective mapping between $\mathcal{E}^*(\mathcal{O}, \Lambda)/G$ and $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda)/\Phi(G)$.*

Fix $\mathcal{O} = \mathcal{O}(D, N)$, $\Lambda = \Lambda(d, m)$ and $G = \mathcal{O}^*$. We use the notation $h(D, N, d, m) := \#\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m))/\Gamma_{\mathcal{O}^*}$. Thus, $h(D, N, d, m) = \nu(D, N, d, m; \mathcal{O}^*)$, which can be computed explicitly by Eichler results (cf. theorem 1.2).

3. Generalized reduced binary forms

Fix an Eichler order $\mathcal{O}(D, N)$ in an indefinite quaternion algebra H . Consider the associated group $\Gamma(D, N)$ and the Shimura curve $X(D, N)$.

For a quadratic order $\Lambda(d, m)$, consider the set $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(2p, N), \Lambda)$ of binary quadratic forms. As above, for a definite binary quadratic form $f = Ax^2 + Bxy + Cy^2$, denote by $\tau(f)$ the solution of $Az^2 + Bz + C = 0$ in \mathcal{H} .

Definition 3.1. Fix a fundamental domain $\mathcal{D}(D, N)$ for $\Gamma(D, N)$ in \mathcal{H} . Make a choice about the boundary in such a way that every point in \mathcal{H} is equivalent to a unique point of $\mathcal{D}(D, N)$. A binary form $f \in \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda)$ is called $\Gamma(D, N)$ -reduced form if $\tau(f) \in \mathcal{D}(D, N)$.

Theorem 3.2. *The number of positive definite $\Gamma(D, N)$ -reduced forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m))$ is finite and equal to $h(D, N, d, m)$.*

Proof. We can assume $d < 0$, in order $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m))$ consists on definite binary forms. By lemma 2.1 (iii), we have that $\Gamma(D, N)$ -equivalence of forms yields to $\Gamma(D, N)$ -equivalence of points. Note that $\tau(f) = \tau(-f)$, but f is not $\Gamma(D, N)$ -equivalent to $-f$. Thus, in each class of $\Gamma(D, N)$ -equivalence of forms there is a unique reduced binary form.

Consider $G = \{\omega \in \mathcal{O}^* \mid \mathfrak{n}(\omega) > 0\}$ in order to get $\Phi(G) = \Gamma(D, N)$. The group G has index 2 in \mathcal{O}^* and the number of classes of $\Gamma(D, N)$ -equivalence in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m))$ is $2h(D, N, d, m)$. In that set, positive and negative definite forms were included, thus the number of classes of positive definite forms is exactly $h(D, N, d, m)$. \square

4. Non-ramified and small ramified cases

Definition 4.1. Let H be a quaternion algebra of discriminant D . We say that H is nonramified if $D = 1$, that is $H \simeq M(2, \mathbb{Q})$. We say H is small ramified if $D = pq$; in this case, we say it is of type A if $D = 2p$, $p \equiv 3 \pmod{4}$, and we say it is of type B if $D_H = pq$, $q \equiv 1 \pmod{4}$ and $\left(\frac{p}{q}\right) = -1$. It makes sense because of the following statement.

Proposition 4.2. For $H = \left(\frac{p,q}{\mathbb{Q}}\right)$, p, q primes, exactly one of the following statements holds:

- (i) H is nonramified.
- (ii) H is small ramified of type A.
- (iii) H is small ramified of type B.

We are going to specialize above results for reduced binary forms for each one of these cases.

4.1. Nonramified case. Consider $H = M(2, \mathbb{Q})$ and take the Eichler order

$$\mathcal{O}_0(1, N) := \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}.$$

Then $\Gamma(1, N) = \Gamma_0(N)$ and the curve $X(1, N)$ is the modular curve $X_0(N)$.

To unify results with the ramified case, it is also interesting to work with the Eichler order $\mathcal{O}(1, N) := \mathbb{Z} \left[1, \frac{j+ij}{2}, N \frac{-j+ij}{2}, \frac{1-i}{2} \right]$ in the nonramified quaternion algebra $\left(\frac{1,-1}{\mathbb{Q}}\right)$.

Proposition 4.3. Consider the Eichler order $\mathcal{O} = \mathcal{O}_0(1, N) \subseteq M(2, \mathbb{Q})$ and the quadratic order $\Lambda = \Lambda(d, m)$. Then:

- (i) $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda) \simeq \{f = (Na, b, c) \mid a, b, c \in \mathbb{Z}, \det_2(f) = -D_\Lambda\}$.
- (ii) The (\mathcal{O}, Λ) -primitivity condition is $\gcd(a, b, c) = 1$.
- (iii) If $d < 0$, the number of $\Gamma_0(N)$ -reduced positive definite primitive binary quadratic forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda)$ is equal to $h(1, N, d, m)$.

For $N = 1$, the well-known theory on reduced integer binary quadratic forms is recovered. In particular, the class number of $SL(2, \mathbb{Z})$ -equivalence is $h(d, m)$.

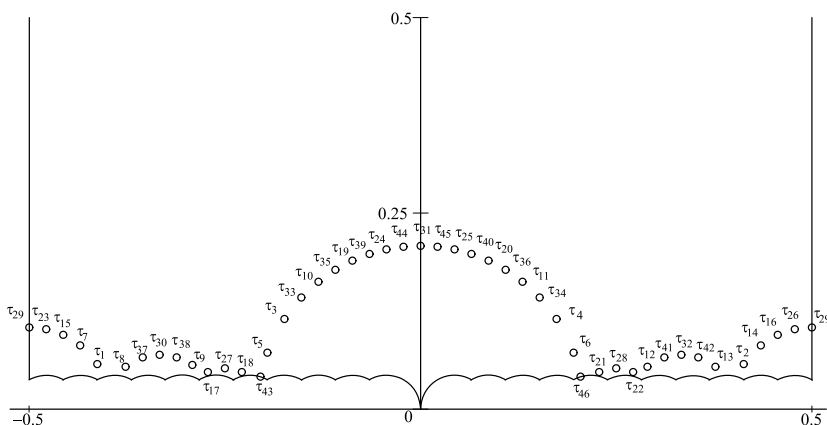
For $N > 1$, a general theory of reduced binary forms is obtained. For N equal to a prime, let us fix the symmetrical fundamental domain

$$\mathcal{D}(1, N) = \left\{ z \in \mathcal{H} \mid |\operatorname{Re}(z)| \leq 1/2, \left| z - \frac{k}{N} \right| > \frac{1}{N}, k \in \mathbb{Z}, 0 < |k| \leq \frac{N-1}{2} \right\}$$

given at [AB04]; a detailed construction can be found in [Als00]. Then a positive definite binary form $f = (Na, b, c)$, $a > 0$, is $\Gamma_0(N)$ -reduced if and only if $|b| \leq Na$ and $|\tau(f) - \frac{k}{N}| > \frac{1}{N}$ for $k \in \mathbb{Z}$, $0 < |k| \leq \frac{N-1}{2}$. Figure 4.1 shows the 46 points corresponding to reduced binary forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_0(1, 23), \Lambda)$ for $D_\Lambda = 7, 11, 19, 23, 28, 43, 56, 67, 76, 83, 88, 91, 92$, which occurs in an special graphical position. In fact these points are exactly the special complex multiplication points of $X(1, 23)$, characterized by the existence of elements $\alpha \in \Lambda(d, m)$ of norm DN (cf. [AB04]). The table describes the $n = h(1, 23, d, m)$ inequivalent points for each quadratic order $\Lambda(d, m)$.

Note that for these symmetrical domains it is easy to implement an algorithm to decide if a form in this set is reduced or not, by using isometric circles.

FIGURE 4.1. The points $\tau(f)$ for some f reduced binary forms corresponding to quadratic orders $\Lambda(d, m)$ in a fundamental domain for $X(1, 23)$.



(d, m)	n	$\tau(f)$
$(-7, 1)$	2	$\left\{ \tau_1 = \frac{-19+\sqrt{7}\ell}{46}, \tau_2 = \frac{19+\sqrt{7}\ell}{46} \right\}$
$(-7, 2)$	2	$\left\{ \tau_3 = \frac{-4+\sqrt{7}\ell}{23}, \tau_4 = \frac{4+\sqrt{7}\ell}{23} \right\}$
$(-11, 1)$	2	$\left\{ \tau_5 = \frac{-9+\sqrt{11}\ell}{46}, \tau_6 = \frac{9+\sqrt{11}\ell}{46} \right\}$
$(-14, 1)$	8	$\left\{ \tau_7 = \frac{-20+\sqrt{14}\ell}{46}, \tau_8 = \frac{-26+\sqrt{14}\ell}{69}, \tau_9 = \frac{-20+\sqrt{14}\ell}{69}, \tau_{10} = \frac{-3+\sqrt{14}\ell}{23}, \right.$ $\left. \tau_{11} = \frac{3+\sqrt{14}\ell}{23}, \tau_{12} = \frac{20+\sqrt{14}\ell}{69}, \tau_{13} = \frac{26+\sqrt{14}\ell}{69}, \tau_{14} = \frac{20+\sqrt{14}\ell}{46} \right\}$
$(-19, 1)$	2	$\left\{ \tau_{15} = \frac{-21+\sqrt{19}\ell}{46}, \tau_{16} = \frac{21+\sqrt{19}\ell}{46} \right\}$
$(-19, 2)$	6	$\left\{ \tau_{17} = \frac{-25+\sqrt{19}\ell}{92}, \tau_{18} = \frac{-21+\sqrt{19}\ell}{92}, \tau_{19} = \frac{-2+\sqrt{19}\ell}{23}, \right.$ $\left. \tau_{20} = \frac{2+\sqrt{19}\ell}{23}, \tau_{21} = \frac{21+\sqrt{19}\ell}{92}, \tau_{22} = \frac{25+\sqrt{19}\ell}{92} \right\}$
$(-22, 1)$	4	$\left\{ \tau_{23} = \frac{-22+\sqrt{22}\ell}{46}, \tau_{24} = \frac{-1+\sqrt{22}\ell}{23}, \tau_{25} = \frac{1+\sqrt{22}\ell}{23}, \tau_{26} = \frac{22+\sqrt{22}\ell}{46} \right\}$
$(-23, 1)$	3	$\left\{ \tau_{27} = \frac{-23+\sqrt{23}\ell}{92}, \tau_{28} = \frac{23+\sqrt{23}\ell}{92}, \tau_{29} = \frac{-23+\sqrt{23}\ell}{46} \sim \frac{23+\sqrt{23}\ell}{46} \right\}$
$(-23, 2)$	3	$\left\{ \tau_{30} = \frac{-23+\sqrt{23}\ell}{69}, \tau_{31} = \frac{\sqrt{23}\ell}{23}, \tau_{32} = \frac{23+\sqrt{23}\ell}{69} \right\}$
$(-43, 1)$	2	$\left\{ \tau_{33} = \frac{-7+\sqrt{43}\ell}{46}, \tau_{34} = \frac{7+\sqrt{43}\ell}{46} \right\}$
$(-67, 1)$	2	$\left\{ \tau_{35} = \frac{-5+\sqrt{67}\ell}{46}, \tau_{36} = \frac{5+\sqrt{67}\ell}{46} \right\}$
$(-83, 1)$	6	$\left\{ \tau_{37} = \frac{-49+\sqrt{83}\ell}{138}, \tau_{38} = \frac{-43+\sqrt{83}\ell}{138}, \tau_{39} = \frac{-3+\sqrt{83}\ell}{46}, \right.$ $\left. \tau_{40} = \frac{3+\sqrt{83}\ell}{46}, \tau_{41} = \frac{43+\sqrt{83}\ell}{138}, \tau_{42} = \frac{49+\sqrt{83}\ell}{138} \right\}$
$(-91, 1)$	4	$\left\{ \tau_{43} = \frac{-47+\sqrt{91}\ell}{230}, \tau_{44} = \frac{-1+\sqrt{91}\ell}{46}, \tau_{45} = \frac{1+\sqrt{91}\ell}{46}, \tau_{46} = \frac{47+\sqrt{91}\ell}{230} \right\}$

4.2. Small ramified case of type A. Let us consider $H_A(p) := \left(\frac{p, -1}{\mathbb{Q}}\right)$ and the Eichler order $\mathcal{O}_A(2p, N) := \mathbb{Z} \left[1, i, Nj, \frac{1+i+j+ij}{2}\right]$, for $N \mid \frac{p-1}{2}$, N square-free. The elements in the group $\Gamma_A(2p, N)$ are $\gamma = \frac{1}{2} \begin{pmatrix} \alpha & \beta \\ -\beta' & \alpha' \end{pmatrix}$ such that $\alpha, \beta \in \mathbb{Z}[\sqrt{p}]$, $\alpha \equiv \beta \equiv \alpha\sqrt{p} \pmod{2}$, $\det \gamma = 1$, $N \mid \left(\operatorname{tr}(\beta) - \frac{\beta - \beta'}{\sqrt{p}}\right)$. We denote by $X_A(2p, N)$ the Shimura curve of type A defined by $\Gamma_A(2p, N)$.

Proposition 4.4. *Consider the Eichler order $\mathcal{O}_A(2p, N)$ and the quadratic order $\Lambda = \Lambda(d, m)$.*

(i) *The set $\mathcal{H}(\mathbb{Z} + 2\mathcal{O}_A(2p, N), \Lambda)$ of binary forms is equal to*

$$\{f = (a + b\sqrt{p}, 2c\sqrt{p}, a - b\sqrt{p}) : a, b, c \in \mathbb{Z}, a \equiv b \equiv c \pmod{2}, \\ N \mid (a + b), \det_1(f) = -D_\Lambda\}.$$

- (ii) *The $(\mathcal{O}_A(2p, N), \Lambda)$ -primitivity condition for these binary quadratic forms is $\gcd\left(\frac{c+b}{2}, \frac{a+b}{2N}, b\right) = 1$.*
- (iii) *If $d < 0$, the number of $\Gamma_A(2p, N)$ -reduced positive definite primitive binary forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_A(2p, N), \Lambda)$ is equal to $h(2p, N, d, m)$.*

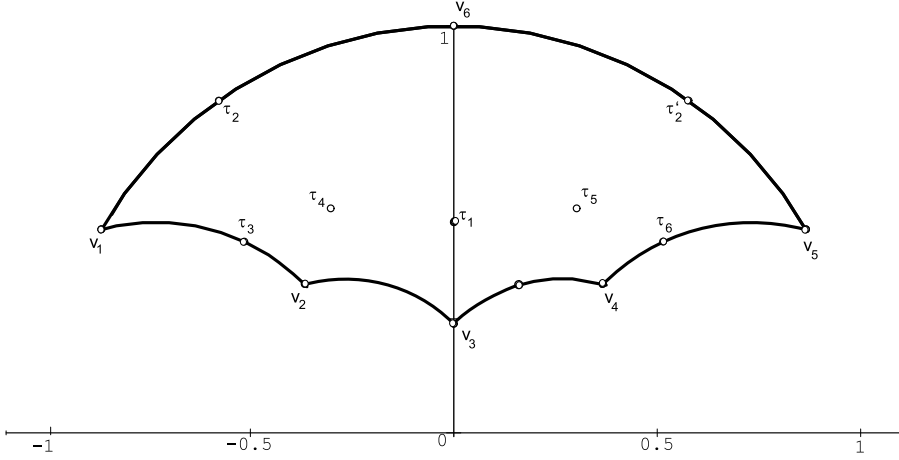
For example, consider the fundamental domain $\mathcal{D}(6, 1)$ for the Shimura curve $X_A(6, 1)$ in the Poincaré half plane defined by the hyperbolic polygon of vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ at figure 4.2 (cf. [AB04]). The table contains the corresponding reduced binary quadratic forms $f \in \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_A(6, 1), \Lambda(d, 1))$ and the associated points $\tau(f)$ for $\det_1(f) = 4, 3, 24, 40$, that is $d = -1, -3, -6, -10$. Since the vertices are elliptic points of order 2 or 3, they are the associated points to forms of determinant 4 or 3, respectively. We put $n = h(6, 1, d, 1)$ the number of such reduced forms for each determinant.

4.3. Small ramified case of type B. Consider $H_B(p, q) := \left(\frac{p, q}{\mathbb{Q}}\right)$ and the Eichler order $\mathcal{O}_B(pq, N) := \mathbb{Z} \left[1, Ni, \frac{1+i}{2}, \frac{i+ij}{2}\right]$, where $N \mid \frac{q-1}{4}$, N square-free and $\gcd(N, p) = 1$. Then the group of quaternion transformations is

$$\Gamma_B(pq, N) = \left\{ \gamma = \frac{1}{2} \begin{pmatrix} \alpha & \beta \\ q\beta' & \alpha' \end{pmatrix} : \alpha, \beta \in \mathbb{Z}[\sqrt{p}], \alpha \equiv \beta \pmod{2}, \\ N \mid \frac{\alpha - \alpha' - \beta + \beta'}{2\sqrt{p}}, \det \gamma = 1 \right\}.$$

We denote by $X_B(pq, N)$ the corresponding Shimura curve of type B.

FIGURE 4.2. Reduced binary forms in a fundamental domain for $X_A(6, 1)$.



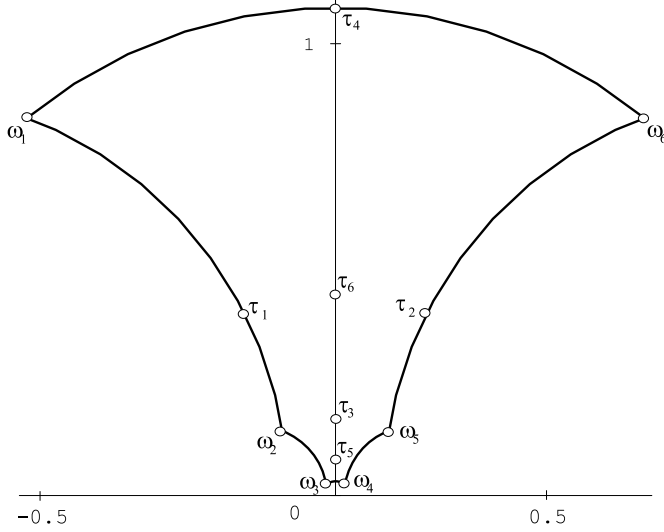
$\det_1(f)$	n	f	$\tau(f)$
3	2	$(3 + \sqrt{3})x^2 + 2\sqrt{3}xy + (3 - \sqrt{3})y^2$ $(3 + \sqrt{3})x^2 - 2\sqrt{3}xy + (3 - \sqrt{3})y^2$	$v_2 = \frac{1-\sqrt{3}}{2}(1-\iota)$ $v_4 = \frac{-1+\sqrt{3}}{2}(1-\iota)$
4	2	$4x^2 + 4\sqrt{3}xy + 4y^2$ $2x^2 + 2y^2$	$v_1 = \frac{-\sqrt{3}+\iota}{2} \sim v_3 \sim v_5$ $v_6 = \iota$
24	2	$(6 + 2\sqrt{3})x^2 - (-6 + 2\sqrt{3})y^2$ $6x^2 - 4\sqrt{3}xy + 6y^2$	$\tau_1 = \frac{(\sqrt{6}-\sqrt{2})\iota}{2}$ $\tau_2 = \frac{-\sqrt{3}+\sqrt{6}\iota}{3} \sim \tau_2'$
40	4	$(10 + 2\sqrt{3})x^2 + 8\sqrt{3}xy - (-10 + 2\sqrt{3})y^2$ $(8 + 2\sqrt{3})x^2 + 4\sqrt{3}xy - (-8 + 2\sqrt{3})y^2$ $(8 + 2\sqrt{3})x^2 - 4\sqrt{3}xy - (-8 + 2\sqrt{3})y^2$ $(10 + 2\sqrt{3})x^2 - 8\sqrt{3}xy - (-10 + 2\sqrt{3})y^2$	$\tau_3 = \frac{3-5\sqrt{3}}{11} + \frac{5\sqrt{10-\sqrt{30}}}{22}\iota$ $\tau_4 = \frac{3-4\sqrt{3}}{13} + \frac{4\sqrt{10-\sqrt{30}}}{22}\iota$ $\tau_5 = \frac{-3+4\sqrt{3}}{13} + \frac{4\sqrt{10-\sqrt{30}}}{22}\iota$ $\tau_6 = \frac{-3+5\sqrt{3}}{11} + \frac{5\sqrt{10-\sqrt{30}}}{22}\iota$

Proposition 4.5. Consider the Eichler order $\mathcal{O}_B(pq, N)$ in $H_B(p, q)$ and the quadratic order $\Lambda = \Lambda(d, m)$.

- (i) The set $\mathcal{H}(\mathbb{Z} + 2\mathcal{O}_B(pq, N), \Lambda)$ of binary forms contains precisely the forms $f = (q(a + b\sqrt{p}), 2c\sqrt{p}, -a + b\sqrt{p})$ where $a, b, c \in \mathbb{Z}$, $2N \mid (c - b)$ and $\det_1(f) = -D_\Lambda$.
- (ii) The $(\mathcal{O}_B(pq, N), \Lambda)$ -primitivity condition for these binary quadratic forms in (i) is $\gcd(a, b, \frac{c-b}{2N}) = 1$.
- (iii) If $d < 0$, the number of $\Gamma_B(pq, N)$ -reduced positive definite primitive binary forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_B(pq, N), \Lambda)$ is equal to $h(pq, N, d, m)$.

In figure 4.3 we show a fundamental domain for $\Gamma_B(10, 1)$ given by the hyperbolic polygon of vertices $\{w_1, w_2, w_3, w_4, w_5, w_6\}$. All the vertices are elliptic points of order 3; thus they are the associated points to binary

FIGURE 4.3. Reduced binary forms in a fundamental domain for $X_B(10, 1)$.



$\det_1(f)$	n	f	$\tau(f)$
3	4	$(-5 + 5\sqrt{2})x^2 + 2\sqrt{2}xy + (1 + \sqrt{2})y^2$ $(5 + 5\sqrt{2})x^2 + 2\sqrt{2}xy + (-1 + \sqrt{2})y^2$ $(35 + 25\sqrt{2})x^2 - 2\sqrt{2}xy + (-7 + 5\sqrt{2})y^2$ $(5 + 5\sqrt{2})x^2 + 2\sqrt{2}xy + (-1 + \sqrt{2})y^2$	$w_1 = \frac{-\sqrt{2} + \sqrt{3}\iota}{5(-1 + \sqrt{2})} \sim w_3$ $w_2 = \frac{-\sqrt{2} + \sqrt{3}\iota}{5(1 + \sqrt{2})}$ $w_4 = \frac{\sqrt{2} + \sqrt{3}\iota}{5(7 + 5\sqrt{2})} \sim w_6$ $w_5 = \frac{\sqrt{2} + \sqrt{3}\iota}{5(1 + \sqrt{2})}$
8	2	$5\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2$ $(5\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2)$	$\tau_1 = \frac{-1 + 2\iota}{5}$ $\tau_2 = \frac{1 + 2\iota}{5}$
20	2	$(10 + 10\sqrt{2})x^2 + (-2 + 2\sqrt{2})y^2$ $(-10 + 10\sqrt{2})x^2 + (2 + 2\sqrt{2})y^2$	$\tau_3 = \frac{(\sqrt{10} - \sqrt{5})\iota}{5}$ $\tau_4 = \frac{(\sqrt{10} + \sqrt{5})\iota}{5}$
40	2	$(40 + 30\sqrt{2})x^2 + (-8 + 6\sqrt{2})y^2$ $10\sqrt{2}x^2 + 2\sqrt{2}y^2$	$\tau_5 = \frac{(3\sqrt{5} - 2\sqrt{10})\iota}{5}$ $\tau_6 = \frac{\sqrt{5}\iota}{5}$

forms of determinant 3. We also represent the points corresponding to reduced binary quadratic forms f with $\det_1(f) = 40$, which correspond to special complex points. The table also contains the explicit reduced definite positive binary forms and the corresponding points for determinants 8 and 20.

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