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# An Arakelov theoretic proof of the equality of conductor and discriminant

par SINAN ÜNVER

RÉSUMÉ. Nous donnons une preuve utilisant la théorie d'Arakelov de l'égalité du conducteur et du discriminant.

ABSTRACT. We give an Arakelov theoretic proof of the equality of conductor and discriminant.

## 1. Introduction

Let  $K$  be a number field,  $\mathcal{O}_K$  be the ring of integers of  $K$ , and  $S$  be  $\text{Spec}(\mathcal{O}_K)$ . Let  $f : X \rightarrow S$  be an arithmetic surface. By this we mean a regular scheme, proper and flat over  $S$ , of relative dimension one. We also assume that the generic fiber of  $X$  has genus  $\geq 1$ , and that  $X/S$  has geometrically connected fibers.

Let  $\omega_X$  be the dualizing sheaf of  $X/S$ . The Mumford isomorphism ([Mumf], Theorem 5.10)

$$\det Rf_*(\omega_X^{\otimes 2}) \otimes K \rightarrow (\det Rf_*\omega_X)^{\otimes 13} \otimes K,$$

which is unique up to sign, gives a rational section  $\Delta$  of

$$(\det Rf_*\omega_X)^{\otimes 13} \otimes (\det Rf_*(\omega_X^{\otimes 2}))^{\otimes -1}.$$

The discriminant  $\Delta(X)$  of  $X/S$  is defined as the divisor of this rational section ([Saito]). If  $\mathfrak{p}$  is a closed point of  $S$ , we denote the coefficient of  $\mathfrak{p}$  in  $\Delta(X)$  by  $\delta_{\mathfrak{p}}$ .

On the other hand  $X/S$  has an Artin conductor  $\text{Art}(X)$  (cf. [Bloch]), which is similarly a divisor on  $S$ . We denote the coefficient of  $\mathfrak{p}$  in  $\text{Art}(X)$  by  $\text{Art}_{\mathfrak{p}}$ . Let  $S'$  be the strict henselization of the complete local ring of  $S$  at  $\mathfrak{p}$ , with field of fractions  $K'$ . Let  $s$  be its special point,  $\eta$  be its generic point, and  $\bar{\eta}$  be a geometric generic point corresponding to an algebraic closure  $\bar{K}'$  of  $K'$ . Let  $\ell$  be a prime different from the residue characteristic

at  $\mathfrak{p}$ . Then

$$\begin{aligned} \text{Art}_{\mathfrak{p}}(X) &= \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Q}_\ell) - \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^i(X_s, \mathbb{Q}_\ell) \\ &\quad + \sum_{i \geq 0} (-1)^i \text{Sw}_{\bar{K}'/K'}(H_{\text{ét}}^i(X_{\bar{\eta}}, \mathbb{Q}_\ell)), \end{aligned}$$

where  $\text{Sw}_{\bar{K}'/K'}$  denotes the Swan conductor of the Galois representation of  $\bar{K}'/K'$ . Both of these divisors are supported on the primes of bad reduction of  $X$ . We give another proof of Saito’s theorem ([Saito], Theorem 1) in the number field case.

**Theorem 1.** *For any closed point  $\mathfrak{p} \in S$ , we have  $\delta_{\mathfrak{p}} = -\text{Art}_{\mathfrak{p}}$ .*

Fix a Kähler metric on  $X(\mathbb{C})$  invariant under complex conjugation, this gives metrics on  $\Omega_{X_\nu}^1$ ’s, for each  $\nu \in S(\mathbb{C})$ . For a hermitian coherent sheaf  $\mathcal{E}$ , we endow  $\det Rf_*(\mathcal{E})$  with its Quillen metric. The proof of the theorem has the following corollaries.

**Proposition 1.** *We have*

$$\begin{aligned} \deg \det Rf_*\omega_X &= \frac{1}{12} [\deg f_*(\widehat{c}_1(\omega_X)^2) + \log \text{Norm}(-\text{Art}(X))] \\ &\quad [K : \mathbb{Q}](g - 1)[2\zeta'(-1) + \zeta(-1)], \end{aligned}$$

with  $\zeta$  the Riemann zeta function.

Proposition 1 is an arithmetic analogue of Noether’s formula in which  $\det Rf_*\omega_X$  is endowed with the Quillen metric. Faltings [Falt] and Moret-Bailly [M-B] proved a similar formula for the Faltings metrics.

**Proposition 2.** *We have*

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\nu \in S(\mathbb{C})} \log \|\Delta_\nu\| = 12(1 - g)[2\zeta'(-1) + \zeta(-1)],$$

where  $\Delta_\nu$  is the section on  $X_\nu$  obtained by pulling back  $\Delta$  via the map from  $\text{Spec}(\mathbb{C})$  to  $\text{Spec}(\mathcal{O}_K)$  that corresponds to  $\nu \in S(\mathbb{C})$ . In particular, the norm of the Mumford isomorphism does not depend on the metric.

## 2. Proof

First we prove Proposition 1. By duality ([Deligne], Lemme 1.3),  $\deg \det Rf_*\omega_X = \deg \det Rf_*\mathcal{O}_X$ . By the arithmetic Riemann-Roch theorem of Gillet and Soulé ([G-S], Theorem 7), we get

$$\deg \det Rf_*\mathcal{O}_X = \deg f_*(\widehat{Td}(\Omega_X^1)^{(2)}) - \frac{1}{2} \sum_{\nu \in S(\mathbb{C})} \int_{X_\nu} Td(T_{X_\nu})R(T_{X_\nu}).$$

Here  $Td$  and  $R$  are the Todd and Gillet-Soulé genera respectively, and the superscript (2) denotes the degree 2 component. Applying the definitions of these characteristic classes we obtain

$$\deg \det Rf_* \mathcal{O}_X = \frac{1}{12} \deg f_*(\widehat{c}_1(\Omega_X^1)^2 + \widehat{c}_2(\Omega_X^1)) + [K : \mathbb{Q}](g - 1)[2\zeta'(-1) + \zeta(-1)].$$

Let  $Z$  denote the union of singular fibers of  $f$ , and let  $c_{2,X}^Z(\Omega_X^1)$  be the localized Chern class of  $\Omega_X^1$  with support in  $Z$  (cf. [Bloch], [Fulton]). Chinburg, Pappas, and Taylor ([CPT], Proposition 3.1) prove the formula

$$\deg f_*(\widehat{c}_2(\Omega_X^1)) = \log \text{Norm}(c_{2,X}^Z(\Omega_X^1)).$$

Combining this with the fundamental formula of Bloch ([Bloch], Theorem 1)

$$-\text{Art}_{\mathfrak{p}}(X) = \deg_{\mathfrak{p}} c_{2,X}^Z(\Omega_X^1),$$

we obtain the desired formula. Note that, since  $\det \Omega_X^1 = \omega_X$ ,  $\widehat{c}_1(\Omega_X^1) = \widehat{c}_1(\omega_X)$ . □

Taking degrees in the Mumford isomorphism gives

$$13 \deg \det Rf_* \omega_X = \deg \det Rf_*(\omega_X^{\otimes 2}) + \log \text{Norm}(\Delta(X)) - \sum_{\nu \in S(\mathbb{C})} \log \|\Delta_{\nu}\|.$$

The arithmetic Riemann-Roch theorem ([Falt], Theorem 3) gives

$$\deg \det Rf_*(\omega_X^{\otimes 2}) = \deg \det Rf_* \omega_X + \deg f_*(\widehat{c}_1(\omega_X)^2).$$

Therefore we get

$$(1) \quad \deg \det Rf_* \omega_X = \frac{1}{12} [\deg f_*(\widehat{c}_1(\omega_X)^2) + \log \text{Norm}(\Delta(X)) - \sum_{\nu \in S(\mathbb{C})} \log \|\Delta_{\nu}\|].$$

Subtracting (1) from the expression in the statement of Proposition 1, we obtain

$$(2) \quad \log \left( \frac{\text{Norm}(\Delta(X/S))}{\text{Norm}(-\text{Art}(X/S))} \right) = \sum_{\nu \in S(\mathbb{C})} \log \|\Delta_{\nu}\| + 12[K : \mathbb{Q}](g - 1)[2\zeta'(-1) + \zeta(-1)].$$

Now  $X_K$  has semistable reduction after a finite base change  $K'/K$ . For semistable  $X'/S'$ , both  $-\text{Art}_{\mathfrak{p}'}(X')$  ([Bloch]), and  $\delta_{\mathfrak{p}'}(X')$  ([Falt], Theorem 6) are equal to the number of singular points in the geometric fiber over  $\mathfrak{p}'$ . Therefore  $-\text{Art}_{\mathfrak{p}'} = \delta_{\mathfrak{p}'}$ , and hence

$$(3) \quad \text{Norm}(\Delta(X'/S')) = \text{Norm}(-\text{Art}(X'/S')).$$

Applying this to a semistable model  $X'$  of  $X \otimes_K K'$ , and noting that the base change multiplies the right hand side of (2) by  $[K' : K]$ , we see that the right hand side of (2) is equal to zero, and hence that the equality

$$(4) \quad \text{Norm}(\Delta(X/S)) = \text{Norm}(-\text{Art}(X/S))$$

holds for  $X$ .

To prove the equality  $\delta_{\mathfrak{p}} = -\text{Art}_{\mathfrak{p}}$  for an arbitrary closed point  $\mathfrak{p} \in S$ , we will use the following lemma.

**Lemma 1.** *Fix distinct closed points  $\beta_1, \dots, \beta_s \in S$ . For each  $i$  such that  $1 \leq i \leq s$ , let  $L_i$  be an extension of the completion  $K_i$  of  $K$  at  $\beta_i$  such that  $[L_i : K_i] = n$  is independent of  $i$ . Then there exists an extension  $L/K$  such that, for each  $1 \leq i \leq s$ , there is only one prime  $\gamma_i$  of  $L$  lying over  $\beta_i$ , and the completion of  $L$  at  $\gamma_i$  is isomorphic (over  $K_i$ ) to  $L_i$ .*

Proof. The proof is an application of Krasner's lemma, and the approximation lemma. Details are omitted.  $\square$

Take  $\mathfrak{p} = \beta_1$ , a prime of bad reduction. Denote the remaining primes of bad reduction by  $\beta_i$ ,  $2 \leq i \leq s$ . Choose extensions  $L_i$  of the local fields  $K_i$ , for all  $1 \leq i \leq s$ , such that  $L_1$  is unramified over  $K_1$ ,  $X$  has semistable reduction over  $L_i$ , for  $2 \leq i \leq s$ , and  $[L_i : K_i] = n$ , for some  $n$ . Applying the lemma to this data we obtain an extension  $L$  of  $K$ . Let  $T = \text{Spec}(\mathcal{O}_L)$ . The curve  $X \otimes_K L$  has a proper, regular model  $Y$  over  $T$  such that

- (i)  $Y \otimes_T T_{\gamma_1} \simeq X \otimes_S T_{\gamma_1}$ , and
- (ii)  $Y$  is semistable at  $\gamma_i$ , for  $2 \leq i \leq s$ .

Applying (4) to  $Y$  gives the equality

$$\sum_{1 \leq i \leq s} \delta_{\gamma_i} \log \text{Norm}(\gamma_i) = \sum_{1 \leq i \leq s} -\text{Art}_{\gamma_i} \log \text{Norm}(\gamma_i).$$

On the other hand because of semistability, we have  $\delta_{\gamma_i} = -\text{Art}_{\gamma_i}$ , for  $2 \leq i \leq s$ . Hence we get  $\delta_{\gamma_1} = -\text{Art}_{\gamma_1}$ . Since  $T/S$  is étale at  $\gamma_1$ , (i) implies

$$\delta_{\mathfrak{p}} = \delta_{\gamma_1} = -\text{Art}_{\gamma_1} = -\text{Art}_{\mathfrak{p}}.$$

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