

# JOURNAL

de Théorie des Nombres

# de BORDEAUX

*anciennement Séminaire de Théorie des Nombres de Bordeaux*

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Tome 16, n° 2 (2004), p. 373-375.

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## A monogenic Hasse-Arf theorem

par JAMES BORGER

RÉSUMÉ. On étend le théorème de Hasse–Arf de la classe des extensions résiduellement séparables des anneaux de valuation discrète complets à la classe des extensions monogènes.

ABSTRACT. I extend the Hasse–Arf theorem from residually separable extensions of complete discrete valuation rings to monogenic extensions.

Let  $B/A$  be a finite extension of henselian discrete valuation rings which is generically Galois with group  $G$ , that is, for which the corresponding extension of fraction fields is Galois with group  $G$ . For  $\sigma \in G - \{1\}$ , let  $I_B(\sigma)$  be the ideal of  $B$  generated by  $(\sigma - 1)B$  and let  $i_B(\sigma)$  be the length of the  $B$ -module  $B/I_B(\sigma)$ .

For any finite dimensional complex representation  $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$ , we define the naive Artin conductor exactly as we do when  $B/A$  is residually separable, i.e., when the extension of residue fields is separable:

$$\text{ar}_n(\rho) = e_{B/A}^{-1} \sum_{\sigma \neq 1} [\dim(V) - \text{trace}(\rho(\sigma))] i_B(\sigma).$$

By looking at real parts, it is immediate that this is a non-negative rational number, and when  $B/A$  is residually separable, the Hasse-Arf theorem [3, VI §2] tells us that it is also an integer.

In [4], De Smit shows that most of the classical ramification-theoretic properties of residually separable extensions  $B/A$  hold in the slightly more general, “monogenic” case where we require only that  $B$  is generated as an  $A$ -algebra by one element. The purpose of this note is to show that the Hasse–Arf theorem also holds in this context.

Partial results in this direction were obtained by Spriano [5]. A proof of the Hasse-Arf theorem in equal characteristic that is strong enough to cover monogenic extensions was outlined at the 1999 Luminy conference on ramification theory. It was based on a technical analysis of a refinement [2, 3.2.2] of Kato’s refined Swan conductor [1], but since then, an elementary reduction to the classical Hasse-Arf theorem has been found.

The contents of this paper are contained in my dissertation (U.C. Berkeley, 2000), which was written under the direction of Hendrik Lenstra.

**Proposition 1.** *Let  $B/A$  be a finite generically separable extension of henselian discrete valuation rings. Then the following are equivalent.*

(i) *There exists an  $x \in B$  such that  $B = A[x]$ .*

(ii) *The second exterior power  $\Omega_{B/A}^2$  of the module of relative Kähler differentials is zero.*

(iii) *There is a henselian discrete valuation ring  $A'$  that is finite over the maximal unramified subextension  $A^{\text{nr}}$  of  $B/A$  such that  $e_{A'/A^{\text{nr}}} = 1$  and  $B'/A'$  is a residually separable extension of discrete valuation rings, where  $B' = A' \otimes_{A^{\text{nr}}} B$ .*

*Proof.* De Smit [4, 4.2] shows that (i) follows from (ii). For any  $A'$  as in (iii), we have  $B' \otimes_B \Omega_{B/A}^2 \cong B' \otimes_B \Omega_{B/A^{\text{nr}}}^2 \cong \Omega_{B'/A'}^2 = 0$ , so (iii) implies (ii). Now we show (i) implies (iii).

Assume, as we may, that  $A = A^{\text{nr}}$ , and let  $l/k$  denote the residue extension of  $B/A$ . Take some  $x \in B$  such that  $B = A[x]$  and let  $\bar{x}$  denote the image of  $x$  in  $l$ . Let  $g(X) \in A[X]$  be a monic lift of the minimal polynomial  $X^q - a$  of  $\bar{x}$  over  $k$ . Since the maximal ideal of  $B$  is generated by that of  $A$  and  $g(x)$ , we may assume that  $g(x)$  generates the maximal ideal of  $B$ . Then modulo the maximal ideal of  $B$ , we have  $g(X + x) \equiv X^q + x^q - a \equiv X^q$ , so  $g(X + x)$  is an Eisenstein polynomial with coefficients in  $B$ . Now let  $A'$  be the discrete valuation ring  $A[X]/(g(X))$ . Then

$$B' = A' \otimes_A B \cong B[X]/(g(X)) \cong B[X]/(g(X + x))$$

is a discrete valuation ring which has the same residue field as  $B$  and, hence,  $A'$ . □

**Proposition 2.** *Let  $B/A$  be a finite extension of henselian discrete valuation rings that is generically Galois with group  $G$ , and let  $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  be a finite dimensional representation of  $G$ . If  $A'/A$  is a finite extension of henselian discrete valuation rings such that  $B' = A' \otimes_A B$  is a discrete valuation ring, then we have  $\text{ar}_n(\rho') = e_{A'/A} \text{ar}_n(\rho)$ , where  $\rho'$  is  $\rho$  viewed as a representation of the generic Galois group of the extension  $B'/A'$ .*

*Proof.* For  $\sigma \in G - \{1\}$ , we have  $I_{B'}(\sigma) = A' \otimes_A I_B(\sigma) = B' \otimes_B I_B(\sigma)$ , so

$$\begin{aligned} i_{B'}(\sigma) &= \text{length}_{B'}(B'/I_{B'}(\sigma)) = \text{length}_{B'}(B' \otimes_B B/I_B(\sigma)) \\ &= e_{B'/B} \text{length}_B(B/I_B(\sigma)) = e_{B'/B} i_B(\sigma). \end{aligned}$$

Thus

$$\text{ar}_n(\rho') = e_{B'/B} \frac{e_{B/A}}{e_{B'/A'}} \text{ar}_n(\rho) = e_{A'/A} \text{ar}_n(\rho).$$

□

**Corollary 3.** *Let  $B/A$  be a finite monogenic extension of henselian discrete valuation rings that is generically Galois with group  $G$ , and let  $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$  be a finite dimensional representation of  $G$ . Then  $\text{ar}_n(\rho)$  is an integer.*

*Proof.* Restricting to the maximal unramified subextension of  $B/A$  does not change the naive Artin conductor or the monogeneity of the extension. So assume  $B/A$  is residually purely inseparable. Now just apply the previous proposition with  $A'$  taken as in the first proposition and then use the classical Hasse-Arf theorem.  $\square$

**Remark.** One can define a naive Swan conductor [1, 6.7] as well. It also is an integer in the monogenic case but simply because it agrees with the naive Artin conductor whenever  $B/A$  is monogenic and not residually separable. It is not, however, a good invariant even in the monogenic case: it is a consequence of results outlined at the Luminy conference that in the (monogenic) equal-characteristic case, the naive Swan conductor of a faithful, one-dimensional representation agrees with Kato's Swan conductor if and only if either  $B/A$  is residually separable or  $e_{B/A} = 1$ , whereas for general monogenic extensions in equal-characteristic, the naive Artin conductor of a one-dimensional representation is equal to a non-logarithmic, "Artin-type" variant of Kato's Swan conductor.

## References

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