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Extremal values of Dirichlet L -functions in the half-plane of absolute convergence

par JÖRN STEUDING

RÉSUMÉ. On démontre que, pour tout θ réel, il existe une infinité de $s = \sigma + it$ avec $\sigma \rightarrow 1+$ et $t \rightarrow +\infty$ tel que

$$\operatorname{Re} \{ \exp(i\theta) \log L(s, \chi) \} \geq \log \frac{\log \log \log t}{\log \log \log \log t} + O(1).$$

La démonstration est basée sur une version effective du théorème de Kronecker sur les approximations diophantiennes.

ABSTRACT. We prove that for any real θ there are infinitely many values of $s = \sigma + it$ with $\sigma \rightarrow 1+$ and $t \rightarrow +\infty$ such that

$$\operatorname{Re} \{ \exp(i\theta) \log L(s, \chi) \} \geq \log \frac{\log \log \log t}{\log \log \log \log t} + O(1).$$

The proof relies on an effective version of Kronecker's approximation theorem.

1. Extremal values

Extremal values of the Riemann zeta-function in the half-plane of absolute convergence were first studied by H. Bohr and Landau [1]. Their results rely essentially on the diophantine approximation theorems of Dirichlet and Kronecker. Whereas everything easily extends to Dirichlet series with real coefficients of one sign (see [7], §9.32) the question of general Dirichlet series is more delicate. In this paper we shall establish quantitative results for Dirichlet L -functions.

Let q be a positive integer and let χ be a Dirichlet character mod q . As usual, denote by $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, $i^2 = -1$, a complex variable. Then the Dirichlet L -function associated to the character χ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1},$$

where the product is taken over all primes p ; the Dirichlet series, and so the Euler product, converge absolutely in the half-plane $\sigma > 1$. Denote by

χ_0 the principal character mod q , i.e., $\chi_0(n) = 1$ for all n coprime with q . Then

$$(1) \quad L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Thus we may interpret the well-known Riemann zeta-function $\zeta(s)$ as the Dirichlet L -function to the principal character χ_0 mod 1. Furthermore, it follows that $L(s, \chi_0)$ has a simple pole at $s = 1$ with residue 1. On the other side, any $L(s, \chi)$ with $\chi \neq \chi_0$ is regular at $s = 1$ with $L(1, \chi) \neq 0$ (by Dirichlet's analytic class number-formula). Since $L(s, \chi)$ is non-vanishing in $\sigma > 1$, we may define the logarithm (by choosing any one of the values of the logarithm). It is easily shown that for $\sigma > 1$

$$(2) \quad \log L(s, \chi) = \sum_p \sum_{k \geq 1} \frac{\chi(p)^k}{kp^{ks}} = \sum_p \frac{\chi(p)}{p^s} + O(1).$$

Obviously, $|\log L(s, \chi)| \leq L(\sigma, \chi_0)$ for $\sigma > 1$. However

Theorem 1.1. *For any $\epsilon > 0$ and any real θ there exists a sequence of $s = \sigma + it$ with $\sigma > 1$ and $t \rightarrow +\infty$ such that*

$$\operatorname{Re} \{ \exp(i\theta) \log L(s, \chi) \} \geq (1 - \epsilon) \log L(\sigma, \chi_0) + O(1).$$

In particular,

$$\liminf_{\sigma > 1, t \geq 1} |L(s, \chi)| = 0 \quad \text{and} \quad \limsup_{\sigma > 1, t \geq 1} |L(s, \chi)| = \infty.$$

In spite of the non-vanishing of $L(s, \chi)$ the absolute value takes arbitrarily small values in the half-plane $\sigma > 1$!

The proof follows the ideas of H. Bohr and Landau [1] (resp. [8], §8.6) with which they obtained similar results for the Riemann zeta-function (answering a question of Hilbert). However, they argued with Dirichlet's *homogeneous* approximation theorem for growth estimates of $|\zeta(s)|$ and with Kronecker's *inhomogeneous* approximation theorem for its reciprocal. We will unify both approaches.

Proof. Using (2) we have for $x \geq 2$

$$(3) \quad \operatorname{Re} \{ \exp(i\theta) \log L(s, \chi) \} \\ \geq \sum_{p \leq x} \frac{\chi_0(p)}{p^\sigma} \operatorname{Re} \{ \exp(i\theta) \chi(p) p^{-it} \} - \sum_{p > x} \frac{\chi_0(p)}{p^\sigma} + O(1).$$

Denote by $\varphi(q)$ the number of prime residue classes mod q . Since the values $\chi(p)$ are $\varphi(q)$ -th roots of unity if p does not divide q , and equal to zero otherwise, there exist integers λ_p (uniquely determined mod $\varphi(q)$) with

$$\chi(p) = \begin{cases} \exp\left(2\pi i \frac{\lambda_p}{\varphi(q)}\right) & \text{if } p \nmid q, \\ 0 & \text{if } p|q. \end{cases}$$

Hence,

$$\operatorname{Re}\{\exp(i\theta)\chi(p)p^{-it}\} = \cos\left(t \log p - 2\pi \frac{\lambda_p}{\varphi(q)} - \theta\right).$$

In view of the unique prime factorization of the integers the logarithms of the prime numbers are linearly independent. Thus, Kronecker's approximation theorem (see [8], §8.3, resp. Theorem 3.2 below) implies that for any given integer ω and any x there exist a real number $\tau > 0$ and integers h_p such that

$$(4) \quad \left| \frac{\tau}{2\pi} \log p - \frac{\lambda_p}{\varphi(q)} - \frac{\theta}{2\pi} - h_p \right| < \frac{1}{\omega} \quad \text{for all } p \leq x.$$

Obviously, with $\omega \rightarrow \infty$ we get infinitely many τ with this property. It follows that

$$(5) \quad \cos\left(\tau \log p - 2\pi \frac{\lambda_p}{\varphi(q)} - \theta\right) \geq \cos\left(\frac{2\pi}{\omega}\right) \quad \text{for all } p \leq x,$$

provided that $\omega \geq 4$. Therefore, we deduce from (3)

$$\operatorname{Re}\{\exp(i\theta) \log L(\sigma + i\tau, \chi)\} \geq \cos\left(\frac{2\pi}{\omega}\right) \sum_{p \leq x} \frac{\chi_0(p)}{p^\sigma} - \sum_{p > x} \frac{\chi_0(p)}{p^\sigma} + O(1),$$

resp.

$$(6) \quad \operatorname{Re}\{\exp(i\theta) \log L(\sigma + i\tau, \chi)\} \geq \cos\left(\frac{2\pi}{\omega}\right) \log L(\sigma, \chi_0) - 2 \sum_{p > x} \frac{1}{p^\sigma} + O(1)$$

in view of (2). Obviously, the appearing series converges. Thus, sending ω and x to infinity gives the inequality of Theorem 1.1. By (1) we have

$$(7) \quad \log L(\sigma, \chi_0) = \log\left(\frac{1}{\sigma-1} + O(1)\right) = \log \frac{1}{\sigma-1} + o(1)$$

for $\sigma \rightarrow 1+$. Therefore, with $\theta = 0$, resp. $\theta = \pi$, and $\sigma \rightarrow 1+$ the further assertions of the theorem follow. \square

The same method applies to other Dirichlet series as well. For example, one can show that the Lerch zeta-function is unbounded in the half-plane

of absolute convergence:

$$\limsup_{\sigma > 1, t \geq 1} \sum_{n=0}^{\infty} \frac{\exp(2\pi i \lambda n)}{(n + \alpha)^s} = +\infty$$

if $\alpha > 0$ is transcendental; note that in the case of transcendental α the Lerch zeta-function has zeros in $\sigma > 1$ (see [3] and [4]).

In view of Theorem 1.1 we have to ask for quantitative estimates. Let $\pi(x)$ count the prime numbers $p \leq x$. By partial summation,

$$\sum_{x < p \leq y} \frac{1}{p^\sigma} = \frac{\pi(y)}{y^\sigma} - \frac{\pi(x)}{x^\sigma} + \sigma \int_x^y \frac{\pi(u)}{u^{\sigma+1}} du.$$

The prime number theorem implies for $x \geq 2$

$$\sum_{x < p \leq y} \frac{1}{p^\sigma} \sim \left(\frac{y^{1-\sigma}}{\log y} - \frac{x^{1-\sigma}}{\log x} \right) + \sigma \int_x^y \frac{du}{u^\sigma \log u}.$$

By the second mean-value theorem,

$$\int_x^y \frac{du}{u^\sigma \log u} du = \frac{1}{\log \xi} \int_x^y \frac{du}{u^\sigma} = \frac{x^{1-\sigma} - y^{1-\sigma}}{(\sigma - 1) \log \xi}$$

for some $\xi \in (x, y)$. Thus, substituting ξ by x and sending $y \rightarrow \infty$, we obtain the estimate

$$\sum_{x < p} \frac{1}{p^\sigma} \leq (1 + o(1)) \frac{x^{1-\sigma}}{(\sigma - 1) \log x}$$

as $x \rightarrow \infty$. This gives in (6)

$$\begin{aligned} (8) \quad & \operatorname{Re} \{ \exp(i\theta) \log L(\sigma + i\tau, \chi) \} \\ & \geq \cos \left(\frac{2\pi}{\omega} \right) \log L(\sigma, \chi_0) - (2 + o(1)) \frac{x^{1-\sigma}}{(\sigma - 1) \log x} + O(1). \end{aligned}$$

Substituting (7) in formula (8) yields

$$\begin{aligned} & \operatorname{Re} \{ \exp(i\theta) \log L(\sigma + i\tau, \chi) \} \\ & \geq (1 + O(\omega^{-2})) \log \frac{1}{\sigma - 1} - (2 + o(1)) \frac{x^{1-\sigma}}{(\sigma - 1) \log x} + O(1). \end{aligned}$$

Let

$$x = \exp \left(\frac{1}{\sigma - 1} \log \frac{1}{\sigma - 1} \right),$$

then x tends to infinity as $\sigma \rightarrow 1+$. We obtain for x sufficiently large

$$(9) \quad \operatorname{Re} \{ \exp(i\theta) \log L(\sigma + i\tau, \chi) \} \geq (1 + O(\omega^{-2})) \log \frac{\log x}{\log \log x} + O(1).$$

The question is how the quantities ω, x and τ depend on each other.

2. Effective approximation

H. Bohr and Landau [2] (resp. [8], §8.8) proved the existence of a τ with $0 \leq \tau \leq \exp(N^6)$ such that

$$\cos(\tau \log p_\nu) < -1 + \frac{1}{N} \quad \text{for } \nu = 1, \dots, N,$$

where p_ν denotes the ν -th prime number. This can be seen as a first effective version of Kronecker’s approximation theorem, with a bound for τ (similar to the one in Dirichlet’s approximation theorem). In view of (5) this yields, in addition with the easier case of bounding $|\zeta(s)|$ from below, the existence of infinite sequences $s_\pm = \sigma_\pm + it_\pm$ with $\sigma_\pm \rightarrow 1+$ and $t_\pm \rightarrow +\infty$ for which

$$(10) \quad |\zeta(s_+)| \geq A \log \log t_+ \quad \text{and} \quad \frac{1}{|\zeta(s_-)|} \geq A \log \log t_-,$$

where $A > 0$ is an absolute constant. However, for Dirichlet L -functions we need a more general effective version of Kronecker’s approximation theorem. Using the idea of Bohr and Landau in addition with Baker’s estimate for linear forms, Rieger [6] proved the remarkable

Theorem 2.1. *Let $v, N \in \mathbb{N}, b \in \mathbb{Z}, 1 \leq \omega, U \in \mathbb{R}$. Let $p_1 < \dots < p_N$ be prime numbers (not necessarily consecutive) and*

$$u_\nu \in \mathbb{Z}, \quad 0 < |u_\nu| \leq U, \quad \beta_\nu \in \mathbb{R} \quad \text{for } \nu = 1, \dots, N.$$

Then there exist $h_\nu \in \mathbb{Z}, 0 \leq \nu \leq N$, and an effectively computable number $C = C(N, p_N) > 0$, depending on N and p_N only, with

$$(11) \quad \left| h_0 \frac{u_\nu}{v} \log p_\nu - \beta_\nu - h_\nu \right| < \frac{1}{\omega} \quad \text{for } \nu = 1, \dots, N$$

and $b \leq h_0 \leq b + (2Uv\omega)^C$.

We need C explicitly. Therefore we shall give a sketch of Rieger’s proof and add in the crucial step a result on an explicit lower bound for linear forms in logarithms due to Waldschmidt [9].

Let \mathbb{K} be a number field of degree D over \mathbb{Q} and denote by $L_{\mathbb{K}}$ the set of logarithms of the elements of $\mathbb{K} \setminus \{0\}$, i.e.,

$$L_{\mathbb{K}} = \{\ell \in \mathbb{C} : \exp(\ell) \in \mathbb{K}\}.$$

If a is an algebraic number with minimal polynomial $P(X)$ over \mathbb{Z} , then define the absolute logarithmic height of a by

$$h(a) = \frac{1}{D} \int_0^1 \log |P(\exp(2\pi i\phi))| d\phi;$$

note that $h(a) = \log a$ for integers $a \geq 2$. Waldschmidt proved

Theorem 2.2. *Let $\ell_\nu \in L_{\mathbb{K}}$ and $\beta_\nu \in \mathbb{Q}$ for $\nu = 1, \dots, N$, not all equal zero. Define $a_\nu = \exp(\ell_\nu)$ for $\nu = 1, \dots, N$ and*

$$\Lambda = \beta_0 + \beta_1 \log a_1 + \dots + \beta_N \log a_N.$$

Let E, W and $V_\nu, 1 \leq \nu \leq N$, be positive real numbers, satisfying

$$W \geq \max_{1 \leq \nu \leq N} \{h(\beta_\nu)\},$$

$$\frac{1}{D} \leq V_1 \leq \dots \leq V_N,$$

$$V_\nu \geq \max \left\{ h(a_\nu), \frac{|\log a_\nu|}{D} \right\} \quad \text{for } \nu = 1, \dots, N$$

and

$$1 < E \leq \min \left\{ \exp(V_1), \min_{1 \leq \nu \leq N} \left\{ \frac{4DV_\nu}{|\log a_\nu|} \right\} \right\}.$$

Finally, define $V_\nu^+ = \max\{V_\nu, 1\}$ for $\nu = N$ and $\nu = N - 1$, with $V_1^+ = 1$ in the case $N = 1$. If $\Lambda \neq 0$, then

$$|\Lambda| > \exp \left(-c(N)D^{N+2}(W + \log(EDV_N^+)) \log(EDV_{N-1}^+) \times \right. \\ \left. \times (\log E)^{-N-1} \prod_{\nu=1}^N V_\nu \right)$$

with $c(N) \leq 2^{8N+51}N^{2N}$.

This leads to

Theorem 2.3. *With the notation of Theorem 2.1 and under its assumptions there exists an integer h_0 such that (11) holds and*

$$(12) \quad b \leq h_0 \leq b + 2 + ((3\omega U(N + 2) \log p_N)^4 + 2)^{N+2} \times \\ \times \exp \left(2^{8N+51}N^{2N}(1 + 2 \log p_N)(1 + \log p_{N-1}) \prod_{\nu=2}^N \log p_\nu \right);$$

if p_N is the N -th prime number, then, for any $\epsilon > 0$ and N sufficiently large,

$$(13) \quad b \leq h_0 \leq b + (\omega U)^{(4+\epsilon)N} \exp \left(N^{(2+\epsilon)N} \right).$$

Proof. For $t \in \mathbb{R}$ define

$$f(t) = 1 + \exp(t) + \sum_{\nu=1}^N \exp \left(2\pi i \left(t \frac{u_\nu}{v} \log p_\nu - \beta_\nu \right) \right).$$

With $\gamma_{-1} := 0, \beta_{-1} := 0, \gamma_0 := 1, \beta_0 := 0$ and $\gamma_\nu := \frac{u_\nu}{\nu} \log p_\nu, 1 \leq \nu \leq N$, we have

$$(14) \quad f(t) = \sum_{\nu=-1}^N \exp(2\pi i(t\gamma_\nu - \beta_\nu)).$$

By the multinomial theorem,

$$f(t)^k = \sum_{\substack{j_\nu \geq 0 \\ j_{-1} + \dots + j_N = k}} \frac{k!}{j_{-1}! \cdots j_N!} \exp\left(2\pi i \sum_{\nu=-1}^N j_\nu(t\gamma_\nu - \beta_\nu)\right).$$

Hence, for $0 < B \in \mathbb{R}$ and $k \in \mathbb{N}$

$$\begin{aligned} J &:= \int_b^{b+B} |f(t)|^{2k} dt \\ &= \sum_{\substack{j_\nu \geq 0 \\ j_{-1} + \dots + j_N = k}} \frac{k!}{j_{-1}! \cdots j_N!} \sum_{\substack{j'_\nu \geq 0 \\ j'_{-1} + \dots + j'_N = k}} \frac{k!}{j'_{-1}! \cdots j'_N!} \\ &\quad \int_b^{b+B} \exp\left(2\pi i \left(\sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu t - \sum_{\nu=-1}^N (j_\nu - j'_\nu)\beta_\nu\right)\right) dt. \end{aligned}$$

By the theorem of Lindemann

$$\sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu$$

vanishes if and only if $j_\nu = j'_\nu$ for $\nu = -1, 0, \dots, N$. Thus, integration gives

$$\int_b^{b+B} \exp\left(2\pi i \left(\sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu t - \sum_{\nu=-1}^N (j_\nu - j'_\nu)\beta_\nu\right)\right) dt = B$$

if $j_\nu = j'_\nu, \nu = -1, 0, \dots, N$, and

$$\begin{aligned} &\left| \int_b^{b+B} \exp\left(2\pi i \left(\sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu t - \sum_{\nu=-1}^N (j_\nu - j'_\nu)\beta_\nu\right)\right) dt \right| \\ &\leq \frac{1}{\pi} \left| \sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu \right|^{-1} \end{aligned}$$

if $j_\nu \neq j'_\nu$ for some $\nu \in \{-1, 0, \dots, N\}$. In the latter case there exists by Baker's estimate for linear forms an effectively computable constant A such that

$$\left| \sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu \right|^{-1} < A.$$

Setting $\beta_0 = j_0 - j'_0, \beta_\nu = \frac{u_\nu}{v}(j_\nu - j'_\nu)$ and $a_\nu = p_\nu$ for $\nu = 1, \dots, N$, we have, with the notation of Theorem 2.2,

$$\Lambda = \sum_{\nu=-1}^N (j_\nu - j'_\nu)\gamma_\nu.$$

We may take $E = 1, W = \log p_N, V_1 = 1$ and $V_\nu = \log p_\nu$ for $\nu = 2, \dots, N$. If $N \geq 2$, Theorem 2.2 gives

$$|\Lambda| > \exp \left(-2^{8N+51} N^{2N} (1 + 2 \log p_N)(1 + \log p_{N-1}) \prod_{\nu=2}^N \log p_\nu \right).$$

Thus we may take

$$(15) \quad A = \exp \left(2^{8N+51} N^{2N} (1 + 2 \log p_N)(1 + \log p_{N-1}) \prod_{\nu=2}^N \log p_\nu \right).$$

Hence, we obtain

$$(16) \quad J \geq B \sum_{\substack{j_\nu \geq 0 \\ j_{-1} + \dots + j_N = k}} \left(\frac{k!}{j_{-1}! \cdots j_N!} \right)^2 - \frac{A}{\pi} \sum_{\substack{j_\nu \geq 0 \\ j_{-1} + \dots + j_N = k}} \frac{k!}{j_{-1}! \cdots j_N!} \sum_{\substack{j'_\nu \geq 0 \\ j'_{-1} + \dots + j'_N = k}} \frac{k!}{j'_{-1}! \cdots j'_N!}.$$

Since

$$\sum_{\substack{j_\nu \geq 0 \\ j_{-1} + \dots + j_N = k}} 1 \leq (k + 1)^{N+2},$$

application of the Cauchy Schwarz-inequality to the first multiple sum and of the multinomial theorem to the second multiple sum on the right hand side of (16) yields

$$\begin{aligned} J &\geq \left(\frac{B}{(k + 1)^{N+2}} - \frac{A}{\pi} \right) \left(\sum_{\substack{j_\nu \geq 0 \\ j_{-1} + \dots + j_N = k}} \frac{k!}{j_{-1}! \cdots j_N!} \right)^2 \\ &\geq \left(\frac{B}{(k + 1)^{N+2}} - \frac{A}{\pi} \right) (N + 2)^{2k}. \end{aligned}$$

Setting $B = A(k + 1)^{N+2}$ and with $\tau \in [b, b + B]$ defined by

$$|f(\tau)| = \max_{t \in [b, b+B]} |f(t)|,$$

we obtain

$$\frac{B(N + 2)^{2k}}{2(k + 1)^{N+2}} \leq J \leq B|f(\tau)|^{2k}.$$

This gives

$$(17) \quad |f(\tau)| > N + 2 - 2\mu, \quad \text{where} \quad \mu := \frac{(N + 2)^2 \log k}{3k};$$

note that $\mu < 1$ for $k \geq 11$. By definition

$$f(t) = 1 + \exp(2\pi i(t\gamma_\nu - \beta_\nu)) + \sum_{\substack{m=0 \\ m \neq \nu}}^N \exp(2\pi i(t\gamma_m - \beta_m)).$$

Therefore, using the triangle inequality,

$$|f(t)| \leq N + |1 + \exp(2\pi i(\tau\gamma_\nu - \beta_\nu))| \quad \text{for} \quad \nu = 0, \dots, N,$$

and arbitrary $t \in \mathbb{R}$. Thus, in view of (17)

$$|1 + \exp(2\pi i(\tau\gamma_\nu - \beta_\nu))| > 2 - 2\mu \quad \text{for} \quad \nu = 0, \dots, N.$$

If h_ν denotes the nearest integer to $\tau\gamma_\nu - \beta_\nu$, then

$$|\tau\gamma_\nu - \beta_\nu - h_\nu| < \sqrt{\frac{\mu}{2}} \quad \text{for} \quad \nu = 0, \dots, N.$$

For $\nu = 0$ this implies $|\tau - h_0| < \sqrt{\mu}$. Replacing τ by h_0 yields

$$|h_0\gamma_\nu - \beta_\nu h_\nu| < \sqrt{\mu} \left(1 + \max_{\nu=1, \dots, N} |\gamma_\nu| \right) \quad \text{for} \quad \nu = 1, \dots, N.$$

Putting $k = [(3\omega U(N + 2) \log p_N)^4] + 1$ we get

$$b - 1 \leq h_0 \leq b + 1 + B = b + 1 + A([(3\omega U(N + 2) \log p_N)^4] + 2)^{N+2}.$$

Substituting (15) and replacing $b - 1$ by b , the assertion of Theorem 2.1 follows with the estimate (12) of Theorem 2.3; (13) can be proved by standard estimates. \square

3. Quantitative results

We continue with inequality (9). Let p_N be the N -th prime. Then, using Theorem 2.3 with $N = \pi(x)$, $v = u_\nu = 1$, and

$$\beta_\nu = \frac{\lambda_{p_\nu}}{\varphi(q)} + \frac{\theta}{2\pi} \quad \text{for} \quad \nu = 1, \dots, N,$$

yields the existence of $\tau = 2\pi h_0$ with

$$(18) \quad b \leq \frac{\tau}{2\pi} \leq b + \omega^{(4+\epsilon)N} \exp(N^{(2+\epsilon)N})$$

such that (4) holds, as N and x tend to infinity. We choose $\omega = \log \log x$, then the prime number theorem and (18) imply

$$\log x = \log N + O(\log \log N), \quad \log N \geq \log \log \log \tau + O(\log \log \log \log \tau).$$

Substituting this in (9) we obtain

Theorem 3.1. *For any real θ there are infinitely many values of $s = \sigma + it$ with $\sigma \rightarrow 1+$ and $t \rightarrow +\infty$ such that*

$$\operatorname{Re} \{ \exp(i\theta) \log L(s, \chi) \} \geq \log \frac{\log \log \log t}{\log \log \log \log t} + O(1).$$

Using the Phragmén-Lindelöf principle, it is even possible to get quantitative estimates on the abscissa of absolute convergence. We write $f(x) = \Omega(g(x))$ with a positive function $g(x)$ if

$$\liminf_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} > 0;$$

hence, $f(x) = \Omega(g(x))$ is the negation of $f(x) = o(g(x))$. Then, by the same reasoning as in [8], §8.4, we deduce

$$L(1 + it, \chi) = \Omega \left(\frac{\log \log \log t}{\log \log \log \log t} \right),$$

and

$$\frac{1}{L(1 + it, \chi)} = \Omega \left(\frac{\log \log \log t}{\log \log \log \log t} \right).$$

However, the method of Ramachandra [5] yields better results. As for the Riemann zeta-function (10) it can be shown that

$$L(1 + it, \chi) = \Omega(\log \log t), \quad \text{and} \quad \frac{1}{L(1 + it, \chi)} = \Omega(\log \log t),$$

and further that, assuming Riemann's hypothesis, this is the right order (similar to [8], §14.8). Hence, it is natural to expect that also in the half-plane of absolute convergence for Dirichlet L -functions similar growth estimates as for the Riemann zeta-function (10) should hold. We give a heuristical argument. Weyl improved Kronecker's approximation theorem by

Theorem 3.2. *Let $a_1, \dots, a_N \in \mathbb{R}$ be linearly independent over the field of rational numbers, and let γ be a subregion of the N -dimensional unit cube with Jordan volume Γ . Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \operatorname{meas} \{ \tau \in (0, T) : (a_1 t, \dots, a_N t) \in \gamma \pmod{1} \} = \Gamma.$$

Since the limit does not depend on translations of the set γ , we do not expect any *deep* influence of the inhomogeneous part to our approximation problem (4) (though it is a question of the speed of convergence). Thus, we may conjecture that we can find a suitable $\tau \leq \exp(N^c)$ with some positive constant c instead of (13), as in Dirichlet's *homogeneous* approximation theorem. This would lead to estimates similar to (10).

We conclude with some observations on the density of extremal values of $\log L(s, \chi)$. First of all note that if

$$|L(1 + i\tau, \chi)|^{\pm 1} \geq f(T)$$

holds for a subset of values $\tau \in [T, 2T]$ of measure μT , where $f(T)$ is any function which tends with T to infinity, then

$$\int_T^{2T} |L(1 + it, \chi)|^{\pm 2} dt \geq \mu T f(T)^2.$$

In view of well-known mean-value formulae we have $\mu = 0$, which implies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : |L(\sigma + i\tau)|^{\pm 1} \geq f(T)\} = 0.$$

This shows that the set on which extremal values are taken is rather thin.

The situation is different for fixed $\sigma > 1$. Let Q be the smallest prime p for which $\chi_0(p) \neq 0$. Then

$$|\log L(s, \chi)| \leq \log L(\sigma, \chi_0) = Q^{-\sigma} \left(1 + O\left(\left(\frac{Q}{Q+1}\right)^\sigma\right) \right);$$

note that the right hand side tends to $0+$ as $\sigma \rightarrow +\infty$, and that $Q \leq q + 1$.

Theorem 3.3. *Let $0 < \delta < \frac{1}{2}$. Then, for arbitrary θ and fixed $\sigma > 1$,*

$$\begin{aligned} \liminf_{M \rightarrow \infty} \frac{1}{M} \#\{m \leq M : (1 - \delta) \log L(\sigma, \chi_0) - \text{Re}\{\exp(i\theta) \log L(\sigma + 2\pi im, \chi)\}\} \\ \geq Q^{-2\sigma} \left(1 + \frac{24}{\sigma} \right) \geq \delta^{2Q^2+8} (2Q)^{-8Q^2-32} \exp\left(-2^3 Q^{2+51} Q^{4Q^2+2}\right). \end{aligned}$$

Proof. We omit the details. First, we may replace (2) by

$$\left| \log L(s, \chi) - \sum_p \frac{\chi(p)}{p^s} \right| \leq \sum_{p, k \geq 2} \frac{\chi_0(p)}{k p^{k\sigma}}.$$

This gives with regard to (8)

$$\text{Re}\{\exp(i\theta) \log L(\sigma + 2\pi im, \chi)\} \geq (1 - \delta) \log L(\sigma, \chi_0) - 2 \frac{x^{1-\sigma}}{\sigma - 1} - 8 \frac{Q^{2-2\sigma}}{2^\sigma(\sigma - 1)}$$

for some integer $h_0 = m$, satisfying (12), where $N = \pi(x)$ and $\cos \frac{2\pi}{\omega} = 1 - \delta$. Putting $x = Q^2$, proves (after some simple computation) the theorem. \square

For example, if χ is a character with odd modulus q , then the quantity of Theorem 3.3 is bounded below by

$$\geq \frac{\delta^{16}}{2^{128} \exp(2^{81})}.$$

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References

- [1] H. BOHR, E. LANDAU, *Über das Verhalten von $\zeta(s)$ und $\zeta^{(k)}(s)$ in der Nähe der Geraden $\sigma = 1$* . Nachr. Ges. Wiss. Göttingen Math. Phys. Kl. (1910), 303–330.
- [2] H. BOHR, E. LANDAU, *Nachtrag zu unseren Abhandlungen aus den Jahren 1910 und 1923*. Nachr. Ges. Wiss. Göttingen Math. Phys. Kl. (1924), 168–172.
- [3] H. DAVENPORT, H. HEILBRONN, *On the zeros of certain Dirichlet series I, II*. J. London Math. Soc. **11** (1936), 181–185, 307–312.
- [4] R. GARUNKŠTIS, *On zeros of the Lerch zeta-function II*. Probability Theory and Mathematical Statistics: Proceedings of the Seventh Vilnius Conf. (1998), B. Grigelionis et al. (Eds.), TEV/Vilnius, VSP/Utrecht, 1999, 267–276.
- [5] K. RAMACHANDRA, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$ - VII*. Ann. Acad. Sci. Fennicae **14** (1989), 27–40.
- [6] G.J. RIEGER, *Effective simultaneous approximation of complex numbers by conjugate algebraic integers*. Acta Arith. **63** (1993), 325–334.
- [7] E.C. TITCHMARSH, *The theory of functions*. Oxford University Press, 1939 2nd ed.
- [8] E.C. TITCHMARSH, *The theory of the Riemann zeta-function*. Oxford University Press, 1986 2nd ed.
- [9] M. WALDSCHMIDT, *A lower bound for linear forms in logarithms*. Acta Arith. **37** (1980), 257–283.
- [10] H. WEYL, *Über ein Problem aus dem Gebiete der diophantischen Approximation*. Göttinger Nachrichten (1914), 234–244.

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