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The distribution of the values of a rational function modulo a big prime

par Alexandru ZAHARESCU

Résumé. Étant donnés un grand nombre premier p et une fonction rationelle $r(X)$ définie sur $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, on évalue la grandeur de l'ensemble $\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\}$, où $\tilde{r}(x)$ et $\tilde{r}(x+1)$ sont les plus petits représentants de $r(x)$ et $r(x+1)$ dans $\mathbb{Z}$ modulo $p\mathbb{Z}$.

Abstract. Given a large prime number p and a rational function $r(X)$ defined over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, we investigate the size of the set $\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\}$, where $\tilde{r}(x)$ and $\tilde{r}(x+1)$ denote the least positive representatives of $r(x)$ and $r(x+1)$ in $\mathbb{Z}$ modulo $p\mathbb{Z}$.

1. Introduction

Several problems on the distribution of points satisfying various congruence constraints have been investigated recently. Given a large prime number p, for any $a \in \{1, 2, \dots, p-1\}$ let $\overline{a} \in \{1, 2, \dots, p-1\}$ be such that $a\,\overline{a} \equiv 1 \pmod{p}$. A question raised by D.H. Lehmer (see Guy [4, Problem F12]) asks to say something nontrivial about the number, call it $N(p)$, of those a for which a and $\overline{a}$ are of opposite parity. The problem was studied by Wenpeng Zhang in [8], [9] and [10] who proved that

$$N(p) = \frac{p}{2} + O\left(p^{1/2} \log^2 p\right) \tag{1}$$

and then generalized (1) to the case when p is replaced by any odd number q. In [2] it is obtained a generalization of (1), in which the pair $(a, \overline{a})$ is replaced by a point lying on a more general irreducible curve defined mod p. Zhang also studied the problem of the distribution of distances $|a - \overline{a}|$, where $a, \overline{a}$ run over the set of integers in $\{1, \dots, n-1\}$ which are relatively prime to n. He proved in [11] that for any integer $n \geq 2$ and any $0 < \delta \leq 1$ one has

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$$\left|\left\{a\colon\ 1 \le a \le n-1,\ (a,n)=1,\ |a-\overline{a}| < \delta n\right\}\right| \\ = \delta(2-\delta)\varphi(n) + O\left(n^{\frac{1}{2}} d^2(n) \log^3 n\right), \tag{2}$$

where $\varphi(n)$ is the Euler function and $d(n)$ denotes the number of divisors of n. In [12] Zhiyong Zheng investigated the same problem, with $(a,\overline{a})$ replaced by a pair (x,y) satisfying a more general congruence. Precisely, let p be a prime number and let $f(x,y)$ be a polynomial with integer coefficients of total degree $d \ge 2$, absolutely irreducible modulo p. Then it is proved in [12] that for any $0 < \delta \le 1$ one has:

$$\left|\left\{(x,y) \in \mathbb{Z}^2 : 0 \le x,y < p,\ f(x,y) \equiv 0 \pmod p, |x-y| < \delta p\right\}\right| \\ = \delta(2-\delta)p + O_d\left(p^{\frac{1}{2}} \log^2 p\right).$$

A generalization of this problem, where the pair (x,y) is replaced by a point lying on an irreducible curve in a higher dimensional affine space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, has been obtained in [3].

There are different ways to measure the randomness of the distribution of a given set. B. Z. Moroz showed in [5] that the squares (or the $l-$th powers, if l divides $p-1$) are randomly distributed among the values $\{i_p(f(0)), \dots, i_p(f(p-1))\}$ of a fixed irreducible polynomial $f(X)$ in $\mathbb{Z}[X]$ modulo a prime p, as $p \to \infty$ (here i_p stands for the reduction modulo p).

In the present paper we study what happens with the order of residue classes mod p when they are transformed through a rational function $r(X) \in \mathbb{F}_p(X)$. For any $y \in \mathbb{F}_p$ denote by $j(y)$ the least positive representative of y in $\mathbb{Z}$ modulo $p\mathbb{Z}$. To any rational function $r(X) \in \mathbb{F}_p(X)$ we associate the map $\tilde{r} : \mathbb{F}_p \to \{0,1,\dots,p-1\}$ given by $\tilde{r}(x) = j(r(x))$ if $x \in \mathbb{F}_p$ is not a pole of $r(X)$, and $\tilde{r}(x) = 0$ if x is a pole of $r(X)$. As the degree of $r(X)$ will be assumed to be small in terms of p in what follows, the contribution of the poles of $r(X)$ in our asymptotic results will be negligible. If we count those $x \in \mathbb{F}_p$ for which $\tilde{r}(x+1) < \tilde{r}(x)$, respectively those x for which $\tilde{r}(x+1) > \tilde{r}(x)$, there should be no bias towards any one of these inequalities. In other words one would expect that for about half of the elements $x \in \mathbb{F}_p$, $\tilde{r}(x+1)$ is larger than $\tilde{r}(x)$ and for about half of the elements $x \in \mathbb{F}_p$, $\tilde{r}(x+1)$ is smaller than $\tilde{r}(x)$.

In order to handle the above problem, we fix nonzero positive integers a, b and study the distribution of the set $\{b\tilde{r}(x+1) - a\tilde{r}(x) : x \in \mathbb{F}_p\}$. For any real number t consider the set $\mathcal{M}(a,b,p,r,t) = \{x \in \mathbb{F}_p : b\tilde{r}(x+1) - a\tilde{r}(x) < tp\}$ and denote by $D(a,b,p,r,t)$ the number of elements of $\mathcal{M}(a,b,p,r,t)$. Our aim is to provide an asymptotic formula for $D(a,b,p,r,t)$.

We now introduce a function $G(t,a,b)$ which will play an important role in the estimation of $D(a,b,p,r,t)$.

$$G(t,a,b)=\begin{cases} 0, & \text{if } t<-a \\ \frac{(t+a)^2}{2ab}, & \text{if } -a\le t\le W \\ \left(1-\frac{(W+a)^2}{ab}\right)\frac{t-W}{Z-W}+\frac{(W+a)^2}{2ab}, & \text{if } W<t<Z \\ 1-\frac{(t-b)^2}{2ab}, & \text{if } Z\le t<b \\ 1, & \text{if } b\le t \end{cases}$$

where $W=\min\{0,b-a\}$ and $Z=\max\{0,b-a\}$. We will prove the following

Theorem 1.1. *For any positive integers a,b,d, any prime number p, any real number t and any rational function $r(X)=\frac{f(X)}{g(X)}$ which is not a linear polynomial, with $f,g\in\mathbb{F}_p[X]$, $\deg f,\deg g\le d$, one has*

$$D(a,b,p,r,t)=pG(t,a,b)+O_{a,b,d}\left(p^{1/2}\log^2 p\right). \tag{3}$$

As a consequence of Theorem 1.1 we show that the inequality $\tilde r(x)>\tilde r(x+1)$ holds indeed for about half of the values of x in $\mathbb{F}_p$.

Corollary 1.2. *Let p be a prime number, d a positive integer and let $r(X)=\frac{f(X)}{g(X)}$ be a rational function which is not a linear polynomial, with $f,g\in\mathbb{F}_p[X]$ and $\deg f,\deg g\le d$. Then one has*

$$\#\{x\in\mathbb{F}_p:\tilde r(x)>\tilde r(x+1)\}=\frac{p}{2}+O_d\left(p^{1/2}\log^2 p\right).$$

As another application of Theorem 1.1 we obtain an asymptotic result for all the even moments of the distance between $\tilde r(x+1)$ and $\tilde r(x)$.

Corollary 1.3. *Let k be a positive integer and let $p,d,r(X)$ be as in the statement of Corollary 1. Then we have*

$$\begin{aligned} M(p,r,2k) &:= \sum_{x\in\mathbb{F}_p}\left(\tilde r(x+1)-\tilde r(x)\right)^{2k} \\ &= \frac{p^{2k+1}}{(k+1)(2k+1)}+O_{k,d}\left(p^{2k+1/2}\log^2 p\right). \end{aligned}$$

In particular, for $k=1$ one has

$$M(p,r,2)=\frac{p^3}{6}+O_d\left(p^{5/2}\log^2 p\right).$$

This says that in quadratic average $\left|\tilde r(x+1)-\tilde r(x)\right|$ is $\sim\frac{p}{\sqrt6}$.

2. Proof of Theorem 1.1

We will need the following lemma, which is a consequence of the Riemann Hypothesis for curves defined over a finite field (see [7], [6], [1]).

Lemma 2.1. *Let p be a prime number and $\mathbb{F}_p$ the field with p elements. Let ψ be a nontrivial character of the additive group of $\mathbb{F}_p$ and let $R(X)$ be a nonconstant rational function. Then*

$$\sum_{a\in\mathbb{F}_p}\psi(R(a))=O\left(\sqrt{p}\right),$$

where the poles of $R(X)$ are excluded from the summation, and the implicit $O-$constant depends at most on the degrees of the numerator and denominator of $F(X)$.

Let now p be a prime number, let a,b,d be positive integers less than p, let t be a real number and let $r(X)=\frac{f(X)}{g(X)}$, $r(X)$ not a linear polynomial, with $f(X),g(X)\in\mathbb{F}_p[X]$, $\deg f(X),\deg g(X)\le d$. For any $y,z\in\{0,1,\cdots,p-1\}$ we set

$$H(y,z)=H(t,y,z,a,b)=\begin{cases}1, & \text{if } \ bz-ay<tp\\ 0, & \text{if } \ bz-ay\ge tp\end{cases}\tag{4}$$

Then we may write $D(a,b,p,r,t)$ in the form

$$\begin{aligned}D(a,b,p,r,t)&=\sum_{x\in\mathbb{F}_p}H(\tilde r(x),\tilde r(x+1))\\&=\sum_{0\le y,z\le p-1}H(y,z)\#\{x\in\mathbb{F}_p:\tilde r(x)=y,\tilde r(x+1)=z\}.\end{aligned}$$

Next, we write $D(a,b,p,r,t)$ in terms of exponential sums mod p. Denote as usual $e_p(w)=e^{\frac{2\pi i w}{p}}$ for any w. Using the equalities

$$\sum_{0\le m\le p-1}e_p(m(y-\tilde r(x)))=\begin{cases}p, & \text{if } \ \tilde r(x)=y\\ 0, & \text{else}\end{cases}$$

and

$$\sum_{0\le n\le p-1}e_p(n(z-\tilde r(x+1)))=\begin{cases}p, & \text{if } \ \tilde r(x+1)=z\\ 0, & \text{else}\end{cases}$$

we find that

$$D(a,b,p,r,t)=\frac{1}{p^2}\sum_{0\le y,z\le p-1}H(y,z)\tag{5}$$

$$\times\sum_{x\in\mathbb{F}_p}\sum_{0\le m\le p-1}e_p(m(y-\tilde r(x)))\sum_{0\le n\le p-1}e_p(n(z-\tilde r(x+1)))$$

$$=\frac{1}{p^2}\sum_{0\le m,n\le p-1}\sum_{0\le y,z\le p-1}H(y,z)e_p(my+nz)\sum_{x\in\mathbb{F}_p}e_p(-m\tilde{r}(x)-n\tilde{r}(x+1))$$

$$=\frac{1}{p^2}\sum_{0\le m,n\le p-1}\check{H}(m,n)S(-m,-n,r,p),$$

where

$$\check{H}(m,n)=\sum_{0\le y,z\le p-1}H(y,z)e_p(my+nz) \tag{6}$$

and

$$S(-m,-n,r,p)=\sum_{x\in\mathbb{F}_p}e_p(-m\tilde{r}(x)-n\tilde{r}(x+1)). \tag{7}$$

Note that for $m=n=0$ one has

$$S(0,0,r,p)=p. \tag{8}$$

Next, we claim that if $(m,n)\neq(0,0)$ then the rational function $h(X)=mr(X)+nr(X+1)\in\mathbb{F}_p(X)$ is nonconstant. Indeed, if $n=0$ then $m\neq 0$ and $h(X)=mr(X)$ is nonconstant by the hypotheses from the statement of the theorem. The same conclusion holds if $m=0$ and $n\neq 0$. Let now $m\neq 0$, $n\neq 0$ and assume that

$$mr(X)+nr(X+1)=c \tag{9}$$

for some $c\in\mathbb{F}_p$. Suppose first that $r(X)$ is not a polynomial and choose a root $\alpha\in\overline{\mathbb{F}}_p$ of the denominator of $r(X)$, where $\overline{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$. Since α is a pole of $r(X)$, from (9) it follows that α is also a pole of $r(X+1)$, that is $\alpha+1$ is a pole of $r(X)$. By repeating the above reasoning with α replaced by $\alpha+1$ we see that $\alpha+2$, $\alpha+3,\dots,\alpha+p-1$ are poles of $r(X)$. This forces $\deg g(X)$ to be $\geq p$, so $d\geq p$, in which case (3) becomes trivial. Let us suppose now that $r(X)$ is a polynomial, say

$$r(X)=a_lX^l+a_{l-1}X^{l-1}+\cdots+a_1X+a_0$$

with $a_0,\dots,a_l\in\mathbb{F}_p$, $a_l\neq 0$. Then by the hypotheses of Theorem 1.1 it follows that $l\geq 2$. Looking at the coefficient of X^l in (9) we deduce that $m+n=0$ in $\mathbb{F}_p$. But then, the coefficient of X^{l-1} on the left side of (9) equals lna_l, which is nonzero in $\mathbb{F}_p$, contradicting (9). This proves our claim that $h(X)$ is nonconstant in $\mathbb{F}_p(X)$. By Lemma 2.1 it follows that

$$|S(-m,-n,r,p)|=O_d(\sqrt{p}) \tag{10}$$

for any $(m,n)\neq(0,0)$.

Next, we proceed to evaluate the coefficients $\check{H}(m,n)$. We calculate explicitly $\check{H}(0,0)$ and provide upper bounds for $|\check{H}(m,n)|$ for $(m,n)\neq(0,0)$. There are four cases.

I. $m = 0$, $n \neq 0$. We have

$$\check{H}(0,n) = \sum_{0 \leq y,z \leq p-1} H(y,z)e_p(nz).$$

By the definition of $H(y,z)$ it follows that for each $y \in \{0,1,\dots,p-1\}$ we have a sum of $e_p(nz)$ with z running over a subinterval of $\{0,1,\dots,p-1\}$, that is a sum of a geometric progression with ratio $e_p(n)$. The absolute value of such a sum is $\leq \frac{2}{|e_p(n)-1|}$ and consequently

$$(11) \qquad \left|\check{H}(0,n)\right| \leq \frac{2p}{|e_p(n)-1|} = \frac{p}{\sin\frac{n\pi}{p}} \leq \frac{p}{2\left\|\frac{n}{p}\right\|},$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

II. $m \neq 0$, $n = 0$. Similarly, as in case **I**, we have

$$(12) \qquad \left|\check{H}(m,0)\right| \leq \frac{p}{2\left\|\frac{m}{p}\right\|}.$$

III. $m \neq 0$, $n \neq 0$. We need the following lemma.

Lemma 2.2. *Let $h,k \not\equiv 0 \pmod p$, L, T and $u \geq 0$ be integers. Let $S = \sum_{y=0}^{L}\sum_{z=0}^{uy+T} e_p(hy)e_p(kz)$. Then one has*

$$|S| = O\left(\frac{1}{\left\|\frac{k}{p}\right\|}\min\left\{L, \frac{1}{\left\|\frac{h+uk}{p}\right\|}\right\} + \frac{1}{\left\|\frac{k}{p}\right\|}\cdot\frac{1}{\left\|\frac{h}{p}\right\|}\right).$$

Proof. One has

$$\begin{aligned} S &= \sum_{y=0}^{L} e_p(hy)\sum_{z=0}^{uy+T} e_p(kz) \\ &= \sum_{y=0}^{L} e_p(hy)\frac{1-e_p(k(uy+T+1))}{1-e_p(k)} \\ &= \frac{1}{1-e_p(k)}\sum_{y=0}^{L} e_p(hy) - \frac{e_p(k(T+1))}{1-e_p(k)}\sum_{y=0}^{L} e_p((h+ku)y). \end{aligned}$$

Thus

$$|S| \leq \frac{1}{|1-e_p(k)|}\left|\sum_{y=0}^{L} e_p(hy)\right| + \frac{1}{|1-e_p(k)|}\left|\sum_{y=0}^{L} e_p((h+ku)y)\right|.$$

Note that

$$\frac{1}{|1-e_p(k)|}=\frac{1}{\left|1-e^{\frac{2\pi ik}{p}}\right|}=\frac{1}{\left|e^{-\frac{\pi ik}{p}}-e^{\frac{\pi ik}{p}}\right|}=\frac{1}{\left|2\sin\frac{\pi k}{p}\right|}=O\left(\frac{1}{\left\|\frac{k}{p}\right\|}\right).$$

Also,

$$\left|\sum_{y=0}^{L}e_p(hy)\right|=\frac{|1-e_p(h(L+1))|}{|1-e_p(h)|}=O\left(\frac{1}{\left\|\frac{h}{p}\right\|}\right).$$

Lastly, if $h+ku$ is not a multiple of p, then

$$\left|\sum_{y=0}^{L}e_p((h+ku)y)\right|=\frac{|1-e_p((h+ku)(L+1))|}{|1-e_p(h+ku)|}=O\left(\frac{1}{\left\|\frac{h+ku}{p}\right\|}\right).$$

We also have the bound

$$\left|\sum_{y=0}^{L}e_p((h+ku)y)\right|\le L+1,$$

which is valid for any h, k and u. Putting the above bounds together, Lemma 2.2 follows.

We now return to the estimation of $\check{H}(m,n)$. Writing

$$\check{H}(m,n)=\sum_{\substack{0\le y,z\le p-1\\ bz-ay<tp}}e_p(my+nz)$$

as a sum of b sums according to the residue of y modulo b, one arrives at sums as in Lemma 2.2, with $h=mb$, $k=n$, $u=a$. It follows that

$$|\check{H}(m,n)|=O_{a,b}\left(\frac{1}{\left\|\frac{n}{p}\right\|}\min\left\{p,\frac{1}{\left\|\frac{mb+an}{p}\right\|}\right\}+\frac{1}{\left\|\frac{n}{p}\right\|}\cdot\frac{1}{\left\|\frac{mb}{p}\right\|}\right). \tag{13}$$

IV. $m,n=0$. By definition, we have

$$\check{H}(0,0)=\sum_{0\le y,z\le p-1}H(y,z).$$

Let $\mathcal{D}$ be the set of real points from the square $[0,p)\times[0,p)$ which lie below the line $bz-ay=tp$. Then $\check{H}(0,0)$ equals the number of integer points (y,z) from $\mathcal{D}$. Therefore

$$\check{H}(0,0)=Area(\mathcal{D})+O(length(\partial\mathcal{D})).$$

An easy computation shows that $Area(\mathcal{D})$ equals $p^2G(t,a,b)$ with $G(t,a,b)$ defined as in the Introduction, while the length of the boundary $\partial\mathcal{D}$ is $\le 4p$. Hence

$$\check{H}(0,0)=p^2G(t,a,b)+O(p).$$

By (5) we know that

$$\left|D(a,b,p,r,t)-\frac{1}{p^2}\check{H}(0,0)S(0,0,r,p)\right|\le D_1+D_2+D_3\,,$$

where

$$D_1=\frac{1}{p^2}\sum_{m=1}^{p-1}\left|\check{H}(m,0)\right|\left|S(-m,0,r,p)\right|\,,$$

$$D_2=\frac{1}{p^2}\sum_{n=1}^{p-1}\left|\check{H}(0,n)\right|\left|S(0,-n,r,p)\right|\,,$$

$$D_3=\frac{1}{p^2}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\left|\check{H}(m,n)\right|\left|S(-m,-n,r,p)\right|\,.$$

One has

$$\frac{1}{p^2}\check{H}(0,0)S(0,0,r,p)=\frac{\check{H}(0,0)}{p}=pG(t,a,b)+O(1).$$

By (11) and (10) we have

$$D_2=O_d\left(\frac{1}{p^2}\sum_{n=1}^{p-1}\frac{p}{\left\|\frac{n}{p}\right\|}\sqrt{p}\right)=O_d\left(\sqrt{p}\log p\right).$$

Similarly one has

$$D_1=O_d\left(\sqrt{p}\log p\right).$$

In order to estimate D_3 we first use (10) and (13) to obtain

$$D_3=O_{a,b,d}\left(\frac{1}{p^{3/2}}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\min\left\{p,\frac{1}{\left\|\frac{mb+an}{p}\right\|}\right\}\right.$$
$$\left.+\frac{1}{p^{3/2}}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\cdot\frac{1}{\left\|\frac{mb}{p}\right\|}\right) \tag{14}$$

The first double sum in (14) is

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\min\left\{p,\frac{1}{\left\|\frac{mb+an}{p}\right\|}\right\}$$

$$\le\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\sum_{\substack{m=1\\ mb+an\equiv0\,(\mathrm{mod}\ p)}}^{p-1}p+\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\sum_{\substack{m=1\\ mb+an\not\equiv0\,(\mathrm{mod}\ p)}}^{p-1}\frac{1}{\left\|\frac{mb+an}{p}\right\|}$$

$$\le p\sum_{n=1}^{\frac{p-1}{2}}\frac{p}{n}+\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\sum_{m'=1}^{p-1}\frac{1}{\left\|\frac{m'}{p}\right\|}\le p^2(1+\log p)+4p^2(1+\log p)^2 ,$$

while the second double sum is

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\frac{1}{\left\|\frac{n}{p}\right\|}\cdot\frac{1}{\left\|\frac{mb}{p}\right\|}=4\sum_{m=1}^{\frac{p-1}{2}}\frac{p}{m}\sum_{n=1}^{\frac{p-1}{2}}\frac{p}{n}\le 4p^2(1+\log p)^2 .$$

Hence $D_3 = O_{a,b,d}\left(\sqrt{p}\log^2 p\right)$. Putting all these together, Theorem 1.1 follows.

3. Proof of the Corollaries

For the proof of the first Corollary, let us notice that

$$\#\{x\in\mathbb{F}_p : \tilde{r}(x)>\tilde{r}(x+1)\}=D(1,1,p,r,0).$$

Here $W=Z=0$ and so

$$G(0,1,1)=\frac{(t+a)^2}{2ab}=\frac{1}{2}.$$

Thus

$$\#\{x\in\mathbb{F}_p : \tilde{r}(x)>\tilde{r}(x+1)\}=\frac{p}{2}+O_d(p^{\frac{1}{2}}\log^2 p)$$

which proves Corollary 1.2.

In order to prove Corollary 1.3 note that

$$\begin{aligned} M(p,r,2k) &= \sum_{x\in\mathbb{F}_p}(\tilde{r}(x+1)-\tilde{r}(x))^{2k}\\ &= \sum_{-p<m<p} m^{2k}\#\{x\in\mathbb{F}_p : \tilde{r}(x+1)-\tilde{r}(x)=m\}. \end{aligned}$$

This equals

$$\begin{aligned} \sum_{-p<m<p} m^{2k}(D(\frac{m+1}{p})-D(\frac{m}{p})) &= D(1)(p-1)^{2k}\\ &+\sum_{-p<m<p} D(\frac{m}{p})((m-1)^{2k}-m^{2k}) \end{aligned}$$

where for any t we denote $D(t) = D(1,1,p,r,t)$. From Theorem 1.1 it follows that

$$M(p,r,2k)=p^{2k+1}G(1,1,1)+p\sum_{-p<m<p}G(\frac{m}{p},1,1)((m-1)^{2k}-m^{2k})$$

$$+O_{k,d}\Big(p^{2k+\frac{1}{2}}\log^2 p\Big)+O_d\Big(p^{1/2}\log^2 p\sum_{-p<m<p}\left|(m-1)^{2k}-m^{2k}\right|\Big).$$

Since $(m-1)^{2k} - m^{2k} = -2km^{2k-1} + O_k(p^{2k-2})$ and $0 \le G(\frac{m}{p}, 1, 1) \le 1$ for any m, we derive

$$M(p,r,2k) = p^{2k+1}G(1,1,1) - 2kp \sum_{-p<m<p} m^{2k-1}G(\frac{m}{p},1,1) + O_{k,d}\Big(p^{2k+\frac{1}{2}}\log^2 p\Big).$$

From the definition of G we see that

$$G(\frac{m}{p},1,1) = \begin{cases} 0, & \text{if } m < -p \\ \frac{(1+\frac{m}{p})^2}{2}, & \text{if } -p \le m \le 0 \\ 1 - \frac{(1-\frac{m}{p})^2}{2}, & \text{if } 0 < m < p \\ 1, & \text{if } p \le m. \end{cases}$$

Using the fact that for any positive integer r one has $\sum_{-p<m<p} m^r = \frac{2p^{r+1}}{r+1} + O_r(p^r)$ if r is even and $\sum_{-p<m<p} m^r = 0$ if r is odd, the statement of Corollary 1.3 follows after a straightforward computation.

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