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Journal de Théorie des Nombres de Bordeaux, tome 15, n° 3 (2003),
p. 639-650.

http://www.numdam.org/item?id=JTNB_2003__15_3_639_0

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A representation theorem for a class of rigid analytic functions

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RÉSUMÉ. Soit p un nombre premier, \mathbb{Q}_p le corps des nombres p -adiques et \mathbb{C}_p la complétion d'une clôture algébrique de \mathbb{Q}_p . Dans cet article, nous obtenons un théorème de représentation pour les fonctions analytiques rigides sur $\mathbb{P}^1(\mathbb{C}_p) \setminus C(t, \varepsilon)$ qui sont équivariantes par le groupe de Galois $G = \text{Gal}_{\text{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$, où t désigne un élément Lipschitzien de \mathbb{C}_p et $C(t, \varepsilon)$ un ε -voisinage de la G -orbite de t .

ABSTRACT. Let p be a prime number, \mathbb{Q}_p the field of p -adic numbers and \mathbb{C}_p the completion of the algebraic closure of \mathbb{Q}_p . In this paper we obtain a representation theorem for rigid analytic functions on $\mathbb{P}^1(\mathbb{C}_p) \setminus C(t, \varepsilon)$ which are equivariant with respect to the Galois group $G = \text{Gal}_{\text{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$, where t is a Lipschitzian element of \mathbb{C}_p and $C(t, \varepsilon)$ denotes the ε -neighborhood of the G -orbit of t .

1. Introduction

Let p be a prime number, \mathbb{Q}_p the field of p -adic numbers, $\overline{\mathbb{Q}_p}$ a fixed algebraic closure of \mathbb{Q}_p and \mathbb{C}_p the completion of $\overline{\mathbb{Q}_p}$ with respect to the p -adic absolute value. Let $t \in \mathbb{C}_p$ and set $E(t) = \mathbb{P}^1(\mathbb{C}_p) \setminus C(t) = \mathbb{C}_p \cup \{\infty\} \setminus C(t)$ where $C(t)$ denotes the orbit of t with respect to the group G of all continuous automorphisms of \mathbb{C}_p over \mathbb{Q}_p . In this paper we are interested in the G -equivariant rigid analytic functions on $E(t)$ and their restrictions to affinoids of the form $E(t, \varepsilon) = \mathbb{C}_p \cup \{\infty\} \setminus C(t, \varepsilon)$ where $C(t, \varepsilon)$ stands for the ε -neighborhood of $C(t)$.

These functions are easily described in case t is algebraic over \mathbb{Q}_p . For instance, if $t \in \mathbb{Q}_p$ then one can use the equivariant transformation $z \mapsto \frac{1}{z-t}$ to send t to the point at infinity. Then the equivariant rigid analytic functions on $E(t)$ will correspond to the entire functions which are equivariant

and these are simply power series $f(z) = \sum_{n \geq 0} a_n z^n$ with $a_n \in \mathbb{Q}_p$ for any n and such that $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 0$.

If t is transcendental over \mathbb{Q}_p it is not obvious that there are any nonconstant equivariant rigid analytic functions on $E(t)$. For certain elements t (called Lipschitzian) such a function $z \mapsto F(t, z)$ is constructed in [APZ2]. In this paper we define for any Lipschitzian element t of \mathbb{C}_p and any natural numbers m, n an equivariant rigid analytic function $F_{m,n}(t, z)$ on $E(t)$, which is related to our basic trace series $F(t, z)$. Then in Theorem 4.2 below we express any equivariant rigid analytic function on an affinoid $E(t, \varepsilon)$ in terms of the above functions $F_{m,n}(t, z)$.

2. Background material

2.1. Let p be a prime number and \mathbb{Q}_p the field of p -adic numbers endowed with the p -adic absolute value $|\cdot|$, normalized such that $|p| = 1/p$. Let $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of \mathbb{Q}_p and denote by the same symbol $|\cdot|$ the unique extension of $|\cdot|$ to $\overline{\mathbb{Q}_p}$. Further, denote by $(\mathbb{C}_p, |\cdot|)$ the completion of $(\overline{\mathbb{Q}_p}, |\cdot|)$ (see [Am], [Ar]). Let $G = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ endowed with the Krull topology. The group G is canonically isomorphic with the group $\text{Gal}_{\text{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of \mathbb{C}_p over \mathbb{Q}_p . For any $x \in \mathbb{C}_p$ denote $C(x) = \{\sigma(x) \mid \sigma \in G\}$ the orbit of x and let $\overline{\mathbb{Q}_p[x]}$ be the closure of the ring $\mathbb{Q}_p[x]$ in \mathbb{C}_p . For any $x \in \overline{\mathbb{Q}_p}$ denote $\text{deg}(x) = [\mathbb{Q}_p(x) : \mathbb{Q}_p]$.

2.2. Let $x \in \mathbb{C}_p$. Given a real number $\varepsilon > 0$ let $B(x, \varepsilon) = \{y \in \mathbb{C}_p, |x - y| < \varepsilon\}$ the open ball of radius ε centered at x . If M is a compact subset of \mathbb{C}_p and $\varepsilon > 0$ is a real number, denote by $N(M, \varepsilon)$ the number of all disjoint balls of radius ε which have a non-empty intersection with M . We say that M is *Lipschitzian* if $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{N(M, \varepsilon)} = 0$. We call an element $x \in \mathbb{C}_p$ *Lipschitzian* if $C(x)$ is Lipschitzian.

According to [APZ2] if x is Lipschitzian then one can integrate Lipschitzian functions (see definition in 2.3 below) with respect to the p -adic Haar measure π_t induced by G on the set $C(x)$.

Let $G_x = \{\sigma \in G : \sigma(x) = x\}$ and P a closed subgroup of G which contains G_x . Then $C_P(x) = \{\sigma(x) : \sigma \in P\}$, the orbit of x with respect to P , is a compact subset of $C(x) = C_G(x)$. If x is Lipschitzian then $C_P(x)$ is a Lipschitzian compact set for any P with $G_x \subset P$. This follows from the fact that for any $\varepsilon > 0$, $N(C_P(x), \varepsilon)$ divides $N(C(x), \varepsilon)$.

Let $x \in \mathbb{C}_p$ and P a closed subgroup of G which contains G_x . For any $\varepsilon > 0$ let $H_P(x, \varepsilon) = \{\sigma \in P : |x - \sigma(x)| < \varepsilon\}$ and $N_P(x, \varepsilon) = N(C_P(x), \varepsilon)$. Then $H_P(x, \varepsilon)$ is an open subgroup of P and $N_P(x, \varepsilon) = [P : H_P(x, \varepsilon)]$. In particular $N(x, \varepsilon) = [G : H(x, \varepsilon)]$, where $H(x, \varepsilon) = H_G(x, \varepsilon)$.

2.3. The notion of rigid analytic function is defined in [FP] (see also [Am]). According to [APZ2], a rigid analytic function defined on a subset D of \mathbb{C}_p is said to be *equivariant* if for any $z \in D$ one has $C(z) \subset D$ and $f(\sigma(z)) = \sigma(f(z))$ for all $\sigma \in G$. A function $f : C(t) \rightarrow \mathbb{C}_p$, $t \in \mathbb{C}_p$ is *Lipschitzian* if there exists a real number $c > 0$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in C(t)$.

Let t be a Lipschitzian element of \mathbb{C}_p and $f : C(t) \rightarrow \mathbb{C}_p$ a Lipschitzian function. Then the integral

$$\int_{C(t)} f(x)d\pi_t(x)$$

is well defined (see [APZ2]). In particular for any polynomial $P(X) \in \mathbb{C}_p[X]$, any $z \in \mathbb{C}_p \cup \{\infty\} \setminus C(t)$ and any natural number n the function $z \mapsto f(x, z) = \frac{P(x)}{(z-x)^n}$ is Lipschitzian on $C(t)$ and we consider the integral $\int_{C(t)} f(x, z)d\pi_t(x)$. Let us denote

$$F_{m,n}(t, z) = \int_{C(t)} \frac{x^m}{(z-x)^n} d\pi_t(x), \quad m \geq 0, n \geq 0.$$

According to [APZ2] for any $m \geq 0$ one has

$$\int_{C(t)} x^m d\pi_t(x) = Tr(t^m).$$

This shows that $F_{m,0}(t, z) = Tr(t^m) \in \mathbb{Q}_p$, $F_{0,0}(t, z) = 1$ and $1 + F_{1,1}(t, \frac{1}{z}) = F(t, z)$, the trace function associated to t . Also by the equality

$$\frac{1}{(1-u)^m} = (1 + u + u^2 + \dots + u^n + \dots)^m = \sum_{s=0}^{\infty} \binom{m+s-1}{s} u^s$$

valid for any positive integer m and any u with $|u| < 1$, it follows that for $|z| > |x|$ one has

$$\frac{x^m}{(z-x)^n} = \sum_{s \geq 0} \binom{n+s-1}{s} \frac{x^{m+s}}{z^{n+s}}.$$

Then one may write:

$$F_{m,n}(t, z) = \sum_{s \geq 0} \binom{m+s-1}{s} \frac{Tr(t^{m+s})}{z^{n+s}}.$$

This formula represents the expansion of $F_{m,n}(t, z)$ in a suitable neighborhood of infinity. As in Theorem 6.1 of [APZ2] one shows that for all $m \geq 0, n \geq 0$, $F_{m,n}(t, z)$ is an equivariant rigid analytic function defined on $\mathbb{C}_p \cup \{\infty\} \setminus C(t)$.

Remark 2.1. $F'_{m,n}(t, z) = -nF_{m,n+1}(t, z)$ for any $m \geq 0, n \geq 1$. As a consequence one has $F_{m,n+1}(t, z) = \frac{(-1)^n}{n!} F_{m,1}^{(n)}(t, z)$, where the derivative is taken with respect to z .

2.4. The above considerations can be generalized as follows: Let $\varepsilon > 0$ be a real number and S a system of right representatives of G with respect to the subgroup $H(t, \varepsilon)$. Assume that the identity element e of G belongs to S and that t is Lipschitzian. For any $\sigma \in S$ the subset $C_\sigma(t, \varepsilon) = \{\tau(t) : \tau \in \sigma H(t, \varepsilon)\}$ is a compact subset of $C(t)$. For $m \geq 0, n \geq 0$ denote

$$F_{m,n}^\sigma(t, z) = \int_{C_\sigma(t, \varepsilon)} \frac{x^m}{(z-x)^n} d\pi_t(x).$$

It is clear that

$$F_{m,n}(t, z) = \sum_{\sigma \in S} F_{m,n}^\sigma(t, z).$$

In fact this formula represents the Mittag-Leffler decomposition of $F_{m,n}(t, z)$ viewed as a rigid analytic function in the connected affinoid $C_p \cup \{\infty\} \setminus \cup_{\sigma \in S} B(\sigma(t), \varepsilon) = E(t, \varepsilon)$. In what follows we try to obtain a similar decomposition for any element of the set $A(E(t, \varepsilon))$ of equivariant rigid analytic functions on $E(t, \varepsilon)$.

3. A combinatorial Lemma

Let $\alpha, x, y, \{a_m\}_{m \geq 1}$ be variables. For any $m \geq 1$, let us denote

$$(1) \quad h_m(\alpha) = a_1 \alpha^{m-1} + a_2 \binom{m-1}{1} \alpha^{m-2} + \dots + \binom{m-1}{m-1} a_m$$

where as usually $\binom{m}{k} = \frac{m(m-1)\dots(m-k+1)}{k!}$. For any integer $m \geq 1$ we set

$$(2) \quad A_m(x) = h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha)x + \dots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha)x^{m-1}$$

where $h_m^{(k)}(\alpha)$ denotes the formal k -th derivative of the polynomial $h_m(\alpha)$ with respect to α .

Lemma 3.1. *For any x, y and any $m \geq 1$ one has:*

$$(3) \quad A_m(x) = \sum_{r=1}^m a_r \binom{m-1}{r-1} (\alpha-x)^{m-r}$$

and

$$A_m(y) = \sum_{r=1}^m \binom{m-1}{r-1} A_r(x)(x-y)^{m-r}.$$

Proof. Equality (3) states that $A_m(x) = h_m(\alpha - x)$, which follows directly from the Taylor expansion (2). As for the second equality, by applying (3) and using the identity

$$\binom{m-1}{r-1} \binom{r-1}{j-1} = \binom{m-1}{j-1} \binom{m-j}{r-j}$$

one contains

$$\begin{aligned} & \sum_{r=1}^m \binom{m-1}{r-1} A_r(x)(x-y)^{m-r} \\ &= \sum_{r=1}^m \binom{m-1}{r-1} (x-y)^{m-r} \sum_{j=1}^r a_j \binom{r-1}{j-1} (\alpha-x)^{r-j} \\ &= \sum_{j=1}^m a_j \binom{m-1}{j-1} \sum_{r=j}^m \binom{m-j}{r-j} (x-y)^{m-r} (\alpha-x)^{r-j} \\ &= \sum_{j=1}^m a_j \binom{m-1}{j-1} (\alpha-y)^{m-j}. \end{aligned}$$

This equals $A_m(y)$ by (3) and so the lemma is proved. □

4. Equivariant rigid analytic functions on $E(t, \varepsilon)$

4.1. Let t be an element of \mathbb{C}_p , let $\varepsilon > 0$ be a real number and denote by $B(C(t), \varepsilon)$ the union of all disjoint open balls $B(x, \varepsilon)$ which have a nonempty intersection with $C(t)$. Choose $\alpha \in \overline{\mathbb{Q}_p}$ such that $|t - \alpha| < \varepsilon$. Then one has $H(t, \varepsilon) = H(\alpha, \varepsilon)$. Let S be a system of right representatives of G with respect to $H(t, \varepsilon)$ and assume $e \in S$. One has $B(C(t), \varepsilon) = \bigcup_{\sigma \in S} B(\sigma(\alpha), \varepsilon)$. Consider the affinoid $E(t, \varepsilon) = \mathbb{C}_p \cup \{\infty\} \setminus B(C(t), \varepsilon)$ and let $A(E(t, \varepsilon))$ be the set of equivariant rigid analytic functions on $E(t, \varepsilon)$. If t is Lipschitzian then the functions $F_{m,n}(t, z)$ defined in Section 2 are elements of $A(E(t, \varepsilon))$. In this section we shall prove that all the elements of $A(E(t, \varepsilon))$ can be expressed in terms of the functions $F_{m,n}(t, z)$, $m, n \geq 0$.

4.2. We have the following proposition.

Proposition 4.1. *Let t be an element of \mathbb{C}_p . Denote $K_t = \widetilde{\mathbb{Q}_p[t]} \cap \overline{\mathbb{Q}_p}$ and let $\varepsilon > 0$ and $\alpha \in K_t$ such that $|\alpha - t| < \varepsilon$. There exists a sequence $\{\alpha_n\}_{n \geq 1}$ of elements of K_t and a sequence $\{\varepsilon_n\}_{n \geq 1}$ of positive real numbers such that:*

- (i) $\varepsilon_1 = \varepsilon, \alpha_1 = \alpha,$
- (ii) *For any $n \geq 1$ one has $\varepsilon_{n+1} \leq \inf\{\varepsilon_n/2, |t - \alpha_n|\},$*
- (iii) *$|t - \alpha_n| < \varepsilon_n, n \geq 1,$ and $\deg \alpha_n$ is smallest with this property.*

The proof easily follows by induction on n since any ball $B(t, \varepsilon)$ contains elements of K_t (see [APZ1]).

In what follows we work with sequences $\{\alpha_n\}_n$ and $\{\varepsilon_n\}_n$ as in Proposition 4.1. It is clear that $\lim_n \varepsilon_n = 0$, and $t = \lim_n \alpha_n$. Note also that the ball $B(\alpha_{n+1}, \varepsilon_{n+1})$ is contained in $B(\alpha_n, \varepsilon_n)$ for all $n \geq 1$. Let us consider the subgroup $H(t, \varepsilon_n) = H(\alpha_n, \varepsilon_n)$ defined in Section 2. Denote $d_n = [G : H(T, \varepsilon_n)] = N(t, \varepsilon_n)$, and let S_n be a fixed system of representatives of right cosets of G with respect to $H(t, \varepsilon_n)$. We shall assume that the identity element e of G belongs to each S_n . We remark that $S_1 = S$ and d_n divides d_{n+1} for all $n \geq 1$.

4.3. Let $f \in A(E(t, \varepsilon))$. Then (see [FP], Ch I) f admits a Mittag-Leffler decomposition: $f(z) = \sum_{\sigma \in S} f_\sigma(z) + f(\infty)$ where $f(\infty)$ is the value of f at infinity and

$$(4) \quad f_\sigma(z) = \sum_{m \geq 1} \frac{a_{\sigma,m}}{(z - \sigma(\alpha))^m}, \quad \lim_m \frac{|a_{\sigma,m}|}{\varepsilon^m} = 0, \quad \sigma \in S.$$

Since f is equivariant then for any $z \in E(t, \varepsilon)$ and any $\tau \in G$ one has $\sum_{\sigma \in S} \tau(f_\sigma(z)) = \sum_{\sigma \in S} f_\sigma(\tau(z))$ and $\tau(f(\infty)) = f(\infty)$. Hence $f(\infty) \in \mathbb{Q}_p$ and for any $\sigma \in S$ one can write:

$$(5) \quad f_\sigma(\sigma(z)) = \sigma(f_e(z)), \quad a_{\sigma,m} = \sigma(a_{e,m}), \quad m \geq 1.$$

Next we remark that for any $\tau \in H(t, \varepsilon)$ and any $\sigma \in S$ the element $\sigma(\tau(\alpha))$ belongs to $B(\sigma(\alpha), \varepsilon)$, and so the function $f_\sigma(z) = \sum_{m \geq 1} \frac{a_{\sigma,m}}{(z - \sigma(\alpha))^m}$ can also be written as

$$f_\sigma(z) = \sum \frac{a_{\sigma\tau,m}}{(z - \sigma\tau(\alpha))^m} = f_{\sigma\tau}(z)$$

where

$$a_{\sigma\tau,m} = \sum_{i=1}^m \binom{m-1}{i-1} a_{\sigma,i} (\sigma(\alpha) - \sigma\tau(\alpha))^{m-i}.$$

In what follows we shall assume that $f(\infty) = 0$.

4.4. At this point we derive another convenient expression for $f(z)$, using the above elements α_n . Fix $n \geq 1$. Then $d = d_1$ divides $d_n = [G : H(t, \varepsilon_n)]$. Denote $q_n = d_n/d$ and let $B(\alpha_n^{(j)}, \varepsilon_n)$, $1 \leq j \leq q_n$ be all the balls of radius ε_n centered at suitable conjugates of α_n and such that these balls cover $C_{H(t,\varepsilon)}(T) = C_e(t, \varepsilon)$. Then

$$(6) \quad f_e(z) = \frac{d}{d_n} \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{e,m}^{(j)}}{(z - \alpha_n^{(j)})^m}$$

where

$$(7) \quad A_{e,m}^{(j)} = \sum_{i=1}^m \binom{m-1}{i-1} a_{e,i} (\alpha - \alpha_n^{(j)})^{m-i}.$$

According to (5) for all $\sigma \in S$ one has

$$f_\sigma(z) = \frac{d}{d_n} \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{\sigma,m}^{(j)}}{(z - \sigma(\alpha_n^{(j)}))^m},$$

where $A_{\sigma,m}^{(j)} = \sigma(A_{e,m}^{(j)})$. As a consequence of (4) there exists a positive real number M such that for any $m \geq 1$ one has:

$$(8) \quad |a_{\sigma,m}| \leq M \varepsilon^m.$$

It follows from (7) that

$$(9) \quad |A_{e,m}^{(j)}| \leq M \varepsilon^m$$

for any $n \geq 2$ and $1 \leq j \leq q_n$.

4.5. At this point we assume that t is a Lipschitzian element of \mathbb{C}_p , $\varepsilon > 0$ and $\alpha \in B(t, \varepsilon)$, $\alpha \in K_t$. Let $f \in A(E(t, \varepsilon))$, $f = \sum_{\sigma \in S} f_\sigma(z)$ with $f_\sigma(z)$ given by (4). For any $m \geq 1$ denote

$$h_m(\alpha) = a_{e,1} \alpha^{m-1} + \binom{m-1}{1} a_{e,2} \alpha^{m-2} + \dots + \binom{m-1}{m-1} a_{e,m}.$$

Also, for $\sigma \in S$ consider the function $F_{m,n}^\sigma(t, z)$ defined in Section 2.

Theorem 4.2. *Let t be a Lipschitzian element of \mathbb{C}_p , $\varepsilon > 0$, $\alpha \in B(t, \varepsilon) \cap K_t$ and $f \in A(E(t, \varepsilon))$. Then for any $z \in E(t, \varepsilon)$ one has*

$$f(z) = \sum_{\sigma \in S} \sum_{m \geq 1} \sum_{0 \leq j < m} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) F_{j,m}^\sigma(t, z).$$

Proof. For any $m \geq 1$ let $A_m(x) = \sum_{1 \leq i \leq m} a_{e,i} \binom{m-1}{i-1} (\alpha - x)^{m-i}$ and

$$(10) \quad A(x, z) = \sum_{m \geq 1} \frac{A_m(x)}{(z-x)^m}.$$

Step 1. Fix $z_0 \in E(t, \varepsilon)$. We assert that for any $z \in B(z_0, \varepsilon)$, the function $x \mapsto A(x, z)$ is defined and is Lipschitzian on $B(t, \varepsilon)$. Firstly we remark that for any $x \in B(t, \varepsilon)$ one has (see (8)):

$$(11) \quad \left| \frac{A_m(x)}{(z-x)^m} \right| \leq \frac{\left| \sum a_{e,i} \binom{m-1}{i-1} (\alpha - x)^{m-i} \right|}{\varepsilon^m}$$

$$\leq \max \left(\frac{\left| \sum_{i=1}^{\lfloor m/2 \rfloor} a_{e,i} \binom{m-1}{i-1} (\alpha-x)^{m-i} \right|}{\varepsilon^m}, \frac{\left| \sum_{i=\lfloor m/2 \rfloor+1}^m a_{e,i} \binom{m-1}{i-1} (\alpha-x)^{m-i} \right|}{\varepsilon^m} \right)$$

Notice that

$$\begin{aligned} \varepsilon^{-m} \left| \sum_{i=1}^{\lfloor m/2 \rfloor} a_{e,i} \binom{m-1}{i-1} (\alpha-x)^{m-i} \right| &\leq \max_{1 \leq i \leq \lfloor m/2 \rfloor} \left(M \left(\frac{|\alpha-x|}{\varepsilon} \right)^{m-i} \right) \\ &= M \left(\frac{|\alpha-x|}{\varepsilon} \right)^{m-\lfloor m/2 \rfloor} \end{aligned}$$

and

$$\varepsilon^{-m} \left| \sum_{i=\lfloor m/2 \rfloor+1}^m a_{e,i} \binom{m-1}{i-1} (\alpha-x)^{m-i} \right| \leq \max_{\lfloor m/2 \rfloor+1 \leq i \leq m} \frac{|a_{e,i}|}{\varepsilon^i}.$$

Since $\frac{|\alpha-x|}{\varepsilon} < 1$, by (4) and the above considerations it follows that $\left| \frac{A_m(x)}{(z-x)^m} \right| \rightarrow 0$ when $m \rightarrow \infty$. Then the function $A(x, z)$ is defined on $B(t, \varepsilon)$, as claimed. Now let $x, y \in B(t, \varepsilon)$. For any $m \geq 1$ we have

$$\frac{A_m(y)}{(z-y)^m} = \frac{A_m(y)}{(z-x)^m} \left(1 + \sum_{i \geq 1} D_i \left(\frac{y-x}{z-x} \right)^i \right)$$

where D_i are suitable natural numbers. Then one can write

$$\left| \frac{A_m(x)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m} \right| \leq \max_{i \geq 1} \left(\left| \frac{A_m(x) - A_m(y)}{(z-x)^m} \right|, \left| \frac{A_m(y)(y-x)^i}{(z-x)^{m+i}} \right| \right).$$

But (see (8)) for any $i \geq 1$ and $z \in B(z_0, \varepsilon)$ one has

$$\left| \frac{A_m(y)(y-x)^i}{(z-x)^{m+i}} \right| \leq \frac{|A_m(y)|}{|z-x|^m} \cdot \frac{|y-x|^i}{|z-x|^i} \leq M \frac{|y-x|}{\varepsilon}.$$

Also by an easy computation one sees that:

$$\left| \frac{A_m(x) - A_m(y)}{(z-x)^m} \right| \leq \frac{M|y-x|}{\varepsilon}.$$

Finally, one has $|A(x, y) - A(y, z)| \leq \frac{M}{\varepsilon}|x-y|$ i.e. $A(x, z)$ is Lipschitzian on $B(t, \varepsilon)$. The above considerations also show that for any $\delta > 0$ we have

$$(12) \quad \left| \frac{A_m(x)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m} \right| \leq \delta|x-y|$$

for all m large enough in terms of z and δ , uniformly for $x, y \in C_\varepsilon(t, \varepsilon)$.

Step 2. Let us denote $D = C_e(t, \varepsilon) = B(t, \varepsilon) \cap C(t)$. Then D is a compact Lipschitzian subset of \mathbb{C}_p and we consider the integral

$$F(z) = \int_D A(x, z) d\pi_t(x), \quad z \in B(z_0, \varepsilon).$$

Here we use the definition of the integral with respect the p-adic measure π_t as in [APZ2]. We assert that

$$f_e(z) = F(z), \quad z \in B(z_0, \varepsilon),$$

where e is the identity element of G .

To see this, consider the sequences $\{\varepsilon_n\}_n$ and $\{\alpha_n\}_n$ from Proposition 4.1. Let $H(t, \varepsilon_n)$, d_n , S_n be as above. In particular $\varepsilon_1 = \varepsilon$, $\alpha_1 = \alpha$, $d_1 = d$. For any $n \geq 1$ let $B(\alpha_n^{(i)}, \varepsilon_n)$, $1 \leq i \leq q_n$ be the open balls of radius ε_n which cover D . Then one has:

$$F(z) = \int_D A(x, z) d\pi_t(x) = \lim_n \Phi[A(x, z), \alpha_n^{(i)}, \varepsilon_n]$$

where

$$\Phi[A(x, z), \alpha_n^{(i)}, \varepsilon_n] = \frac{d}{d_n} \sum_{1 \leq i \leq q_n} A(\alpha_n^{(i)}, z)$$

is the Riemann sum associated to $(A, \alpha_n^{(i)}, \varepsilon_n)$ (see[APZ2]). We have

$$\frac{d}{d_n} A(\alpha_n^{(i)}, z) = \frac{d}{d_n} \sum_{m \geq 1} A_m(\alpha_n^{(i)}) (z - \alpha_n^{(i)})^m.$$

From (6) it now follows that

$$\Phi[A, \alpha_n^{(i)}, \varepsilon_n] = f_e(z).$$

Since this equality is valid for any n we conclude that

$$F(z) = \int_D A(x, z) d\pi_t(x) = f_e(z).$$

Step 3. We now apply formula (2) to obtain another expression for $A_m(x)$. One has:

$$\begin{aligned} A_m(x) &= h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha)x + \dots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha)x^{m-1} \\ &= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \frac{x^j}{(z-x)^m}. \end{aligned}$$

Therefore

$$(13) \quad \int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x) = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \int_D \frac{x^j}{(z-x)^m} d\pi_t(x) \\ = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) F_{j,m}^e(t, z).$$

We claim that

$$(14) \quad F(z) = f_e(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) F_{j,m}^e(t, z).$$

In order to prove this formula we need the following result:

Lemma 4.3. *Let t be a Lipschitzian element of \mathbb{C}_p , $\varepsilon > 0$ a real number, $g : B(C(t), \varepsilon) \rightarrow \mathbb{C}_p$ a Lipschitzian function, and let c be a real number such that $|g(x) - g(y)| \leq c|x - y|$ for all $x, y \in C(t)$. Then there exists a real number k independent of g such that:*

$$\left| \int_{C(t)} g(x) d\pi_t \right| \leq \max(\|g\|, ck)$$

when $\|g\| = \sup_{x \in C(t)} |g(x)|$.

Proof. Let $\{\varepsilon_n\}_{n \geq 1}$ be a decreasing sequence of positive real numbers such that $\lim_n \varepsilon_n = 0$, $\varepsilon_n/\varepsilon_{n+1} \leq 2$ and $C(t) \subseteq B(t, \varepsilon_1)$. Then one has:

$\int_{C(t)} g(x) d\pi_t = \lim_n \Phi(g, \tau(t), \varepsilon_n)$, where (see Section 2) $d_n = [G : H(t, \varepsilon_n)]$, S_n is a system of right cosets of G with respect $H(t, \varepsilon_n)$ and $\Phi(g, \tau(t), \varepsilon_n) = \frac{1}{d_n} \sum_{\tau \in S_n} g(\tau(t))$ is the Riemann sum associated to ε_n , S_n and g (see [APZ2]).

In particular $\Phi(g, \tau(t), \varepsilon_1) = g(t)$.

Let $n \geq 1$. Then d_n divides d_{n+1} and for any $\tau \in S_{n+1}$ there exists exactly one element $\sigma \in S_n$ such that $\tau(t) \in B(\sigma(t), \varepsilon_n)$. Then we have $|g(\sigma(t)) - g(\tau(t))| \leq c\varepsilon_n$, and so $\left| \frac{1}{d_n} \sum_{\sigma \in S_n} g(\sigma(t)) - \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right| \leq \frac{c\varepsilon_n}{|d_{n+1}|}$. Let n be large enough such that

$$\left| \int_{C(t)} g(x) d\pi_t \right| = \left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right|.$$

Then by the above considerations one has:

$$\left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right| = \left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) - \frac{1}{d_n} \sum_{\tau \in S_n} g(\sigma(t)) + \frac{1}{d_n} \sum_{\sigma \in S_n} g(\sigma(t)) + \dots \right. \\ \left. \dots + \frac{1}{d_2} \sum_{\chi \in S_2} g(\chi(t)) - g(t) + g(t) \right| \leq \max_{1 \leq i \leq n} \left| |g|, c \frac{\varepsilon_i}{|d_{i+1}|} \right|.$$

Now let us take $k = \sup_n \frac{\varepsilon_n}{|d_{n+1}|} = \sup_n \frac{\varepsilon_{n+1}}{|d_{n+1}|} \cdot \frac{\varepsilon_n}{\varepsilon_{n+1}} < \infty$ since $\lim_n \frac{\varepsilon_n}{|d_n|} = 0$, t being Lipschitzian by hypothesis. □

Let $\delta > 0$ be a real number. Then by (4), (11) and (12) it follows that for m large enough one has: $\left| \frac{A_m(x)}{(z-x)^m} \right| < \delta$ and $\left| \frac{A_m(x)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m} \right| < \delta|x-y|$ for any $x, y \in D$. Lemma 4.3 implies that $\left| \int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x) \right| \rightarrow 0$ as $m \rightarrow \infty$.

Therefore

$$F(z) = \int_D \sum_{m \geq 1} \frac{A_m(x)}{(z-x)^m} d\pi_t(x) = \sum_{m \geq 1} \int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x)$$

and using (13) one obtains (14).

Step 4. Let $\sigma \in S$ and denote $D^\sigma = B(\sigma(\alpha), \varepsilon) \cap C(t) = C_\sigma(t, \varepsilon)$. Working as above, one gets:

$$f_\sigma(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\sigma(\alpha)) F_{j,m}^\sigma(t, z).$$

Finally by adding these equalities for $\sigma \in S$ one obtains the expression of $f(z)$ stated in Theorem 4.2 □

Corollary 4.4. *The notations and hypothesis are as in Theorem 4.2 Assume $\alpha \in \mathbb{Q}_p$. Then $S = \{e\}$ and one has:*

$$f(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} h_m^{(j)}(\alpha) F_{j,m}(t, z).$$

References

- [APZ1] V. ALEXANDRU, N. POPESCU, A. ZAHARESCU, *On closed subfields of \mathbb{C}_p* . J. Number Theory **68** (1998), 131–150.
- [APZ2] V. ALEXANDRU, N. POPESCU, A. ZAHARESCU, *Trace on \mathbb{C}_p* . J. Number Theory **88** (2001), 13–48.
- [Am] Y. AMICE, *Les nombres p -adiques*, Presse Univ. de France, Collection Sup. 1975.
- [Ax] J. AX, *Zeros of Polynomials over Local Fields - The Galois Action*, J. Algebra **15** (1970), 417–428.
- [PZ] N. POPESCU, A. ZAHARESCU, *On the main invariant of an element over a local field*, Portugaliae Mathematica **54** (1997), 73–83.
- [Ar] E. ARTIN, *Algebraic Numbers and Algebraic Functions*, Gordon and Breach, New York/London/Paris, 1967.
- [FP] J. FRESNEL, M. VAN DER PUT, *Géometrie Analytique Rigide et Applications*, Birkhäuser. Boston. Basel. Stuttgart, 1981.

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