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## The metric simultaneous Diophantine approximations over formal power series

par KAE INOUE

RÉSUMÉ. Nous étudions les propriétés métriques de l'approximation diophantienne simultanée dans le cas non archimédien. Nous prouvons d'abord une loi du 0 – 1 de type Gallagher, que nous utilisons ensuite pour obtenir un résultat de type Duffin-Schaeffer.

ABSTRACT. We discuss the metric theory of simultaneous Diophantine approximations in the non-archimedean case. First, we show a Gallagher type 0-1 law. Then by using this theorem, we prove a Duffin-Schaeffer type theorem.

### Introduction

In the study of metric Diophantine approximations, one of the questions asks for conditions on a non-negative function  $\psi$  which guarantee that the inequality

$$(1) \quad \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}, \quad (p, q) = 1$$

has infinitely many solutions  $\frac{p}{q}$  for almost every  $x$ . In the one-dimensional case it is known that if the following two conditions hold then (1) has infinitely many solutions for almost every  $x$ , Duffin-Schaeffer [1]:

$$(i) \quad \sum \psi(q) = \infty$$
$$(ii) \quad \sum_{q \leq Q} \psi(q) < c \sum_{q \leq Q} \psi(q) \frac{\phi(q)}{q},$$

where  $c$  is a positive constant and  $\phi$  is the Euler function.

A similar result holds for  $s$ -dimensional Diophantine approximations.

In the previous paper [4], we discussed an analogue of Duffin-Schaeffer theorem for formal Laurent power series (one-dimensional case). As its continuation we consider the same question for higher dimensional case in this paper. To state our results we start with some fundamental definitions and notations.

We use the following notations:

$\mathbb{F}$  : a finite field with  $q$  elements,

$\mathbb{F}[X]$  : the set of polynomials with  $\mathbb{F}$ -coefficients,

$\mathbb{F}(X)$  : the fraction field of  $\mathbb{F}[X]$ ,

$\mathbb{F}((X^{-1}))$  : the set of formal Laurent power series.

We define the absolute value of  $f \in \mathbb{F}((X^{-1}))$  by  $|f| = q^{\deg f}$  and put

$$\mathbb{L} = \{f = a_{-1}X^{-1} + \cdots + a_{-i}X^{-i} + \cdots \mid a_i \in \mathbb{F} \text{ for } i \leq -1\}$$

which is a compact abelian group with the metric  $d(f, g) = |f - g|$ . We denote by  $m$  the normalized Haar measure on  $\mathbb{L}$ . This implies that for any  $b_i \in \mathbb{F}$  ( $1 \leq i \leq l$ ),

$$(2) \quad m\{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots \mid a_{-i} = b_i \text{ for } 1 \leq i \leq l\} = \frac{1}{q^l}.$$

In [4], we showed that

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|} \quad P, Q \in \mathbb{F}[X]$$

has either infinitely many solutions for  $m$ -a.e.  $f \in \mathbb{L}$  or only finitely many solutions for  $m$ -a.e.  $f \in \mathbb{L}$ . Here,  $\psi$  is a  $\{q^{-n} \mid n \geq 0\} \cup \{0\}$ -valued function and  $P$  and  $Q$  have no non-trivial common factor. We call this ‘‘Gallagher type theorem’’ since this property was proved by Gallagher in the classical case. Then, we showed ‘‘Duffin-Schaeffer type theorem’’ :

Let  $\psi$  be a  $\{q^{-n} \mid n \geq 0\} \cup \{0\}$ -valued function, which satisfies

$$\sum_{\substack{\deg Q=1 \\ Q:\text{monic}}}^{\infty} \psi(Q) = \infty.$$

Suppose for a positive constant  $C$ , there are infinitely many positive integers  $n$  such that

$$\sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi(Q) \frac{\Phi(Q)}{|Q|}$$

holds, then

$$\left|f - \frac{P}{Q}\right| < \frac{\psi(Q)}{|Q|}, \quad (P, Q) = 1$$

has infinitely many solutions  $\frac{P}{Q}$  for  $m$ -a.e.  $f \in \mathbb{L}$ . Here,

$$\Phi(Q) = \# \left\{ Q' \mid \begin{array}{l} \deg Q' < \deg Q, Q', Q : \text{monic polynomials} \\ Q' \text{ and } Q \text{ have no non-trivial common factor} \end{array} \right\}.$$

The purpose of this paper is to show the  $s$ -dimensional version of these theorems for formal Laurent power series ( $s \geq 2$ ). For a given  $\psi$  as above, we consider the following inequalities:

$$(3) \quad \left| f_1 - \frac{P_1}{Q} \right| < \frac{\psi(Q)}{|Q|}, \dots, \left| f_s - \frac{P_s}{Q} \right| < \frac{\psi(Q)}{|Q|}$$

$$(P_1, Q) = (P_2, Q) = \dots = (P_s, Q) = 1.$$

We shall give a sufficient condition on  $\psi$  so that (3) has infinitely many solutions  $(\frac{P_1}{Q}, \dots, \frac{P_s}{Q})$  for almost every  $(f_1, \dots, f_s) \in \mathbb{L}^s$  with respect to  $m^s$ , which is the  $s$ -fold product of  $m$ . For this purpose, let

$$E_Q = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|}, \text{ for some } P_i \text{ s.t. } \deg P_i < \deg Q, (P_i, Q) = 1, 1 \leq i \leq s \right\}$$

for a monic polynomial  $Q$  and

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\deg Q \geq n} E_Q,$$

we first prove the following theorem in §1.

**Theorem 1.** *For any  $\psi$ , either  $m^s(E) = 0$  or 1 holds, equivalently, (3) has infinitely many solutions of  $(Q, P_1, \dots, P_s)$  for  $m^s$ -a.e.  $(f_1, \dots, f_s) \in \mathbb{L}^s$  or has only finitely many solutions for  $m^s$ -a.e.  $(f_1, \dots, f_s) \in \mathbb{L}^s$ .*

We call this the  $s$ -dimensional version of the Gallagher type theorem. Secondly, making use of this theorem, we can show the  $s$ -dimensional version of the Duffin-Schaeffer type theorem in §2.

**Theorem 2.** *Let  $\psi$  be a  $\{q^{-n} \mid n \geq 0\} \cup \{0\}$ -valued function which satisfies*

$$\sum_{\substack{\deg Q=1 \\ Q:\text{monic}}}^{\infty} \psi^s(Q) = \infty.$$

*Suppose for a positive constant  $C$ , there are infinitely many positive integers  $n$  such that*

$$(4) \quad \sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi^s(Q) < C \sum_{\substack{\deg Q \leq n \\ Q:\text{monic}}} \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s}$$

*holds. Then (3) has infinitely many solutions  $(\frac{P_1}{Q}, \dots, \frac{P_s}{Q})$  for  $m^s$ -a.e.  $(f_1, \dots, f_s) \in \mathbb{L}^s$ .*

Finally, we explain a Duffin-Schaeffer type conjecture in formal Laurent power series case in §3.

### 1. Proof of Theorem 1

In the sequel, we study the metric property of  $s$ -dimensional simultaneous approximations (3) for  $s \geq 2$ . For  $f = a_l X^l + \cdots + a_1 X + a_0 + a_{-1} X^{-1} + \cdots \in \mathbb{F}((X^{-1}))$  ( $a_l \neq 0$ ), we put

$$[f] = a_l X^l + a_{l-1} X^{l-1} + \cdots + a_1 X + a_0.$$

For given  $h_i \in \mathbb{F}[X]$  such that

$$h_i = a_{i l_i} X^{l_i} + a_{i l_i - 1} X^{l_i - 1} + \cdots + a_{i 1} X + a_{i 0},$$

$$a_{i j} \in \mathbb{F}, \quad 1 \leq i \leq s, \quad 0 \leq j \leq l_i, \quad a_{i l_i} \neq 0,$$

we define the cylinder set  $\langle h_1, \dots, h_s \rangle$  as follows:

$$\langle h_1, \dots, h_s \rangle := \{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid [X^{l_1+1} \cdot f_1] = h_1, \dots, [X^{l_s+1} \cdot f_s] = h_s \}.$$

Then we see the following,

$$m^s(\langle h_1, \dots, h_s \rangle) = \frac{1}{q^{l_1+1}} \cdots \frac{1}{q^{l_s+1}}.$$

**Lemma 1.** *Let  $\{\langle h_{1k}, h_{2k}, \dots, h_{sk} \rangle \mid k \geq 1\}$  be a sequence of cylinder sets defined as above with*

$$\lim_{k \rightarrow \infty} \deg h_{ik} = \infty,$$

and  $\{E_k \mid k \geq 1\}$  be a sequence of measurable sets of  $\mathbb{L}^s$  for which  $E_k \subset \langle h_{1k}, \dots, h_{sk} \rangle$ . Suppose there exists  $\delta > 0$  such that  $m^s(E_k) \geq \delta m^s(\langle h_{1k}, \dots, h_{sk} \rangle)$  for any  $k \geq 1$ . Then

$$m^s\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_k\right) = m^s\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \langle h_{1k}, \dots, h_{sk} \rangle\right).$$

*Proof.* Let

$$H := \bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \langle h_{1k}, \dots, h_{sk} \rangle, \quad E_l^* = \bigcup_{k=l}^{\infty} E_k, \quad H_l^* := H \setminus E_l^*.$$

We show that  $m^s(H_l^*) = 0$  for any  $l \geq 1$ , which implies the assertion of this lemma. Suppose there exists a  $k_0 \in \mathbb{Z}_+$  such that  $m^s(H_k^*) > 0$  for  $k \geq k_0$ . There is a natural correspondence between cylinder sets defined for  $\mathbb{L}$  as in (2) and  $q$ -adic rational intervals, and so  $(\mathbb{L}, m)$  is isomorphic to  $[0, 1]$  with the Lebesgue measure. Similarly,  $(\mathbb{L}^s, m^s)$  is isomorphic to  $[0, 1]^s$  with the Lebesgue measure. So by using cylinder sets  $\langle h_1, \dots, h_s \rangle \subset \mathbb{L}^s$  instead of  $I_1 \times \cdots \times I_s \subset [0, 1]^s$ , we can apply Lebesgue density theorem. Then we get, since  $\{H_l^* \mid l \geq 1\}$  is an increasing sequence of sets,

$$(5) \quad \frac{m^s(H_k^* \cap \langle h_{1k}, \dots, h_{sk} \rangle)}{m^s(\langle h_{1k}, \dots, h_{sk} \rangle)} > 1 - \frac{\delta}{2}$$

for some  $k$ . On the other hand,

$$H_k^* \cap E_k^* = \emptyset.$$

From the assumption of this lemma,

$$\begin{aligned} m^s(\langle h_{1k}, \dots, h_{sk} \rangle) &\geq m^s(E_k) + m^s(H_k^* \cap \langle h_{1k}, \dots, h_{sk} \rangle) \\ &\geq \delta m^s(\langle h_{1k}, \dots, h_{sk} \rangle) + m^s(H_k^* \cap \langle h_{1k}, \dots, h_{sk} \rangle). \end{aligned}$$

That is

$$(1 - \delta) m^s(\langle h_{1k}, \dots, h_{sk} \rangle) \geq m^s(H_k^* \cap \langle h_{1k}, \dots, h_{sk} \rangle),$$

which contradicts (4). □

**Lemma 2.** For any polynomial  $h_i \in \mathbb{F}[X]$  ( $h_i \neq 0$ ) and  $g_i \in \mathbb{L}$ ,  $1 \leq i \leq s$ , the map  $T$  of  $\mathbb{L}^s$  onto itself defined by

$$T(f_1, \dots, f_s) = (h_1 f_1 + g_1 - [h_1 f_1 + g_1], \dots, h_s f_s + g_s - [h_s f_s + g_s])$$

for  $(f_1, \dots, f_s) \in \mathbb{L}^s$  is ergodic.

*Proof.* It is easy to see that each map

$$T_i(f_i) = h_i f_i + g_i - [h_i f_i + g_i], \quad 1 \leq i \leq s$$

is a Bernoulli transformation of  $\mathbb{L}$ . In other words, if we put

$$W_k(f_i) = [h_i \cdot T_i^{k-1} f_i + g_i], \quad \text{for } f_i \in \mathbb{L},$$

then  $\{W_k \mid k \geq 1\}$  gives a sequence of independent and identically distributed random variables. In particular,  $T_i$  is weak mixing. Since the  $s$ -fold product of weak mixing transformations is ergodic (see [5] Prop. 4.2.), this yields the assertion of the lemma. □

*Proof of Theorem 1.* In the case of formal power series, we mean by  $(P_i, Q) = 1$  that  $P_i$  and  $Q$  have no non-trivial common factor.

If

$$\lim_{\deg Q \rightarrow \infty} \frac{\psi(Q)}{q^{\deg Q}} > 0,$$

then we can find a sequence of polynomials  $Q_1, Q_2, Q_3, \dots$  and a positive integer  $l$  such that  $\frac{\psi(Q_k)}{q^{\deg Q_k}} > q^{-l}$  for any  $k \geq 1$ . In this case, for any  $f_i \in \mathbb{L}$  and a sufficiently large  $k$ , we can find  $P_i$  ( $\deg P_i < \deg Q_k$ ), such that

$$\left| f_i - \frac{P_i}{Q_k} \right| < \frac{1}{q^l} \left( < \frac{\psi(Q_k)}{q^{\deg Q_k}} \right), \quad (1 \leq i \leq s)$$

and  $P_i$  and  $Q_k$  have no non-trivial common factor. Indeed, the number of polynomials  $\hat{P}$  such that

$$\left| f_i - \frac{\hat{P}}{Q_k} \right| < \frac{1}{q^l}$$

is  $q^{\deg Q_k - l}$ . If all such polynomials  $\hat{P}$  are not relatively prime to  $Q_k$ , then  $Q_k$  has more than  $q^{\deg Q_k - l}$  factors, which is impossible if  $\deg Q_k$  is sufficiently large. This implies

$$E = \mathbb{L}^s.$$

Now we show the assertion of the theorem when

$$(6) \quad \lim_{\deg Q \rightarrow \infty} \frac{\psi(Q)}{q^{\deg Q}} = 0.$$

For fixed  $Q, P_1, \dots, P_{s-1}$  and  $P_s$ , there exist polynomials  $h_1, \dots, h_s$  such that  $\left\{ (f_1, \dots, f_s) \mid \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|} \right\} = \langle h_1, \dots, h_s \rangle$ . Then (6) implies  $\deg h_i \rightarrow \infty, 1 \leq i \leq s$  as  $\deg Q$  tends to  $\infty$ . Thus we can apply Lemma 1 when (6) holds. Then we evaluate the measure of  $\bigcap_{n=1}^{\infty} \bigcup_{\deg Q \geq n} E_Q$ . Let  $R$  be an irreducible polynomial and consider

$$(7) \quad \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)|R|^{n-1}}{|Q|}, \quad (P_i, Q) = 1$$

for  $n \geq 1$  and  $1 \leq i \leq s$ .

We put

$$E_0(n : R) = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid \begin{array}{l} (7) \text{ has infinitely many solutions} \\ P_i, Q \text{ with } R \nmid Q \text{ for } 1 \leq i \leq s \end{array} \right\}$$

and

$$E_1(n : R) = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid \begin{array}{l} (7) \text{ has infinitely many solutions} \\ P_i, Q \text{ with } R \parallel Q \text{ for } 1 \leq i \leq s \end{array} \right\}$$

Then we see

$$E_j(1 : R) \subset E_j(2 : R) \subset E_j(3 : R) \subset \dots$$

and

$$E_j(1 : R) \subset E$$

for  $j = 0, 1$ . From Lemma 1, we find that

$$m^s(E_j(n : R)) = m^s(E_j(1 : R)) = m^s\left(\bigcup_{n \geq 1} E_j(n : R)\right).$$

Let

$$T_j(f_1, \dots, f_s) = \begin{cases} (R \cdot f_1 - [R \cdot f_1], \dots, R \cdot f_s - [R \cdot f_s]) & j = 0 \\ (R \cdot f_1 + \frac{1}{R} - [R \cdot f_1 + \frac{1}{R}], \dots, \\ \quad R \cdot f_s + \frac{1}{R} - [R \cdot f_s + \frac{1}{R}]) & j = 1, \end{cases}$$

for  $(f_1, \dots, f_s) \in \mathbb{L}^s$ . Suppose (7), we have

$$\left| R \cdot f_i - \frac{R \cdot P_i}{Q} \right| < \frac{\psi(Q)|R|^n}{|Q|}$$

and see

$$(R \cdot P_i, Q) = 1.$$

Also, we have

$$\left| \left( R \cdot f_i + \frac{1}{R} \right) - \frac{R \cdot P_i + \frac{Q}{R}}{Q} \right| < \frac{\psi(Q)|R|^n}{|Q|}.$$

Here,

$$\left( R \cdot P_i + \frac{Q}{R}, Q \right) = 1.$$

These imply

$$T_j \left( \bigcup_{n \geq 1} E_j(n : R) \right) = \bigcup_{n \geq 2} E_j(n : R)$$

for  $j = 0, 1$ . Hence from Lemma 2, we have

$$m^s \left( \bigcup_{n \geq 1} E_j(n : R) \right) = 0 \text{ or } 1.$$

for  $j = 0, 1$ . Thus, if either  $m^s(E_0(1 : R))$  or  $m^s(E_1(1 : R)) > 0$  for some irreducible polynomial  $R$ , then we have  $m^s(E) = 1$ .

Now we assume that  $m^s(E_0(1 : R)) = m^s(E_1(1 : R)) = 0$  for any irreducible polynomial  $R$ . We put

$$F(R) = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid \begin{array}{l} \text{(8) has infinitely many solutions} \\ P_i, Q \text{ such that } R^2 \mid Q \end{array} \right\},$$

where (8) refers to:

$$(8) \quad \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|}, \quad (P_i, Q) = 1, \quad 1 \leq i \leq s.$$

Suppose (8), we have

$$\left| \left( f_i + \frac{U}{R} \right) - \frac{P_i + \frac{QU}{R}}{Q} \right| < \frac{\psi(Q)}{|Q|},$$

for any polynomial  $U$  with  $0 \leq \deg U < \deg R$ . Here, we see

$$\left( P_i + \frac{QU}{R}, Q \right) = 1,$$

which implies that if  $(f_1, \dots, f_s) \in F(R)$ , then  $(f_1 + \frac{U}{R}, \dots, f_s + \frac{U}{R}) \in F(R)$ . Also we put

$$S(U; R) = \{(f_1, \dots, f_s) \in \mathbb{L}^s \mid [Rf_1] = U, \dots, [Rf_s] = U\},$$



then its measure is  $\frac{1}{q^s \deg R}$  and

$$\bigcup_{U:0 \leq \deg U < \deg R} S(U; R) \cup \{(f_1, \dots, f_s) \in \mathbb{L}^s \mid \deg f_i < -\deg R\} = \mathbb{L}^s.$$

Since  $F(R)$  is  $(\cdot + \frac{U}{R})$ -invariant,  $S(U; R)$  ( $0 \leq \deg U < \deg R$ ) and  $\{(f_1, \dots, f_s) \in \mathbb{L}^s \mid \deg f_i < -\deg R\}$  have the same measure. Hence we have

$$m^s(F(R) \cap S(U; R)) = \frac{m^s(F(R))}{q^s \deg R},$$

which implies

$$\frac{m^s(F(R) \cap S(U; R))}{m^s(S(U; R))} = m^s(F(R)).$$

Suppose  $m^s(E) > 0$ , since  $E = F(R) \cup E_0(1, R) \cup E_1(1, R)$ , we see that  $m^s(F(R)) > 0$  for any irreducible  $R$ . By the density theorem, we have  $m^s(E) = m^s(F(R)) = 1$  where  $R$  is so chosen that  $\deg R$  is sufficiently large. Otherwise,  $m^s(E) = 0$ .  $\square$

### 2. Proof of Theorem 2

In the sequel, we always assume that  $Q, Q_1, Q'$  and  $Q'_1$  are monic. Recall

$$E_Q = \left\{ (f_1, \dots, f_s) \in \mathbb{L}^s \mid \begin{array}{l} \left| f_i - \frac{P_i}{Q} \right| < \frac{\psi(Q)}{|Q|}, \text{ for some } P_i \text{ s.t.} \\ \deg P_i < \deg Q, (P_i, Q) = 1, 1 \leq i \leq s \end{array} \right\}.$$

From (6), if  $\frac{\psi(Q)}{|Q|}$  is sufficiently small, then we have

$$(9) \quad m^s(E_Q) = \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s}.$$

Now consider the measure of  $E_Q \cap E_{Q'}$  ( $\deg Q' \leq \deg Q$ ). We let  $N(Q, Q')$  be the number of pairs of polynomials  $P$  and  $P'$  which satisfies

$$(10) \quad \left| \frac{P}{Q} - \frac{P'}{Q'} \right| < \frac{\psi(Q)}{|Q|} + \frac{\psi(Q')}{|Q'|},$$

$$(P, Q) = (P', Q') = 1, \deg P < \deg Q, \deg P' < \deg Q'$$

for given  $Q$  and  $Q'$ . Then we show that the measure is bounded as follows,

$$m^s(E_Q \cap E_{Q'}) \leq \left\{ \min \left( \frac{\psi(Q)}{|Q|}, \frac{\psi(Q')}{|Q'|} \right) \cdot N(Q, Q') \right\}^s.$$

Suppose

$$(11) \quad PQ' - P'Q = R$$

holds for some polynomial  $R$  and  $D = (Q, Q')$ . If  $D$  divides  $R$ , we may write

$$Q = DQ^*, Q' = DQ'^*, R = DR^*,$$

and have

$$(12) \quad PQ'^* - P'Q^* = R^* \quad (Q^*, Q'^*) = 1.$$

If  $P^*$  and  $P'^*$  also satisfy (12), then

$$(13) \quad P^*Q'^* - P'^*Q^* = R^*.$$

From (12),(13), we get

$$(14) \quad P = P^* + KQ^*, \quad K : \text{a polynomial.}$$

From (14), we see

$$|P - P^*| = |K||Q^*| < |Q| = |D||Q^*|,$$

which implies that

$$|K| < |D|$$

must hold. Thus the possible number of polynomials  $P$  satisfying (11) for a given  $R$  is no more than  $q^{\deg D}$ . Since (10) implies

$$0 \neq |R| < |Q'|\psi(Q) + |Q|\psi(Q')$$

and  $R$  is divisible by  $D$ , we find that

$$N(Q, Q') \leq \frac{|Q'|\psi(Q) + |Q|\psi(Q')}{|D|} \cdot |D| = |Q'|\psi(Q) + |Q|\psi(Q').$$

Then

$$\begin{aligned} m^s(E_Q \cap E_{Q'}) &\leq \left[ \min \left( \frac{\psi(Q)}{|Q|}, \frac{\psi(Q')}{|Q'|} \right) \cdot \{|Q'|\psi(Q) + |Q|\psi(Q')\} \right]^s \\ &= 2^s \psi^s(Q) \psi^s(Q'). \end{aligned}$$

Because we assume  $\sum_{\deg Q \leq n} \psi^s(Q) = \infty$ ,

$$\sum_{\deg Q \leq n} \psi^s(Q) \leq \left( \sum_{\deg Q \leq n} \psi^s(Q) \right)^2$$

holds for sufficiently large  $n$ . Therefore we have

$$\begin{aligned} \sum_{\deg Q, \deg Q' \leq n} m^s(E_Q \cap E_{Q'}) &\leq 2^s \sum_{\substack{\deg Q' \leq \deg Q \leq n \\ Q \neq Q'}} \psi^s(Q) \psi^s(Q') + \sum_{\deg Q \leq n} \psi^s(Q) \\ &< 2^s \left( \sum_{\deg Q \leq n} \psi^s(Q) \right)^2 \end{aligned}$$

for sufficiently large  $\deg Q$ . From (4) and (9), we have

$$(15) \quad \sum_{\deg Q, \deg Q' \leq n} m^s(E_Q \cap E_{Q'}) < 2^s C^2 \left( \sum_{\deg Q \leq n} m^s(E_Q) \right)^2$$

for infinitely many  $n$ . Then  $m^s(E) > (2^s C^2)^{-1}$ , by (15) and Lemma 5 of [8](pp. 17–18). Finally using to Theorem 1, we complete the proof of this theorem.  $\square$

**Example.** Put

$$\psi(Q) = \begin{cases} \frac{1}{|Q|^{\frac{1}{s}}} & \text{if } Q \text{ is irreducible,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, from Theorem 2.2 of [7], we have

$$\sum_{\deg Q \leq n} \psi^s(Q) \gg \sum_{k=1}^{\infty} \left( \frac{1}{q^k} \right)^s \cdot \frac{1}{k} \cdot q^k = \infty$$

and it is easy to see that

$$\sum_{\deg Q \leq n} \psi^s(Q) \leq C \sum_{\deg Q \leq n} \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s}$$

holds. Thus we see that there are infinitely many solutions  $(\frac{P_1}{Q}, \dots, \frac{P_s}{Q})$  with irreducible  $Q$ 's of

$$\left| f_i - \frac{P_i}{Q} \right| < \frac{1}{|Q|^{\frac{s+1}{s}}}, \quad \text{for } 1 \leq i \leq s$$

for a.e.  $(f_1, \dots, f_s) \in \mathbb{L}^s$ .

**Remark.** It is natural to ask whether we can get a necessary and sufficient condition instead of (4) in Theorem 2. In this sense, we give the  $s$ -dimensional Duffin-Schaeffer type conjecture in the following.

**Conjecture.** (3) has infinitely many solutions  $(\frac{P_1}{Q}, \dots, \frac{P_s}{Q})$  for  $m^s$ -a.e.  $(f_1, \dots, f_s) \in \mathbb{L}^s$  if and only if

$$\sum_{\substack{\deg Q=1 \\ Q:\text{monic}}}^{\infty} \psi^s(Q) \frac{\Phi^s(Q)}{|Q|^s}$$

*diverges.*

In the classical case, the  $s$ -dimensional Duffin-Schaeffer conjecture was proved Pollington and Vaughan [6] for  $s \geq 2$ . We may also prove this conjecture for the  $s$ -dimensional formal power series,  $s \geq 2$ , if we estimate the lower bound of  $\Phi(Q)$ . We will discuss it in another occasion.

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