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On the order of vanishing of modular L-functions at the critical point

par Henryk IWANIEC

1. Introduction

The nonvanishing of L-functions at special points is an attractive area of research in contemporary number theory, see [7]-[11]. One example is the Rankin-Selberg zeta-function $L(f \otimes g_j, s)$ associated with a holomorphic cusp form f of weight 2 and Maass cusp forms g_j of eigenvalue $\lambda_j = s_j(1 - s_j)$. In this case the nonvanishing of $L(f \otimes g_j, s)$ at $s = s_j$ plays a rôle in the work of R. Phillips and P.Sarnak [6] on deformations of groups and was proved to be true for infinitely many cusp forms g_j by J.-M. Deshouillers and H. Iwaniec [3]. Another example is the Birch-Swinnerton-Dyer conjecture which asserts that the rank of the group of rational points on an elliptic curve E defined over $\mathbb{Q}$ is equal to the order of vanishing of the associated Hasse-Weil L-function $L(s, E)$ at $s = 1$ (the center of the critical strip).

Recently V.A. Kolyvagin [4] has proved that the group of rational points on a modular elliptic curve E is finite if $L(1, E) \neq 0$ and that the L-function $L(s, E, \chi_d)$ twisted by a suitable real character χ_d has simple zero at $s = 1$. The latter condition was subsequently proved to hold true for infinitely many discriminants d by D. Bump, S. Friedberg and J. Hoffstein [2] and independently by K. Murty and R. Murty [5]. In these notes we establish (from scratch) quantitative results on Kolyvagin's condition.

2 - Statement of results

Let E be a modular elliptic curve defined over $\mathbb{Q}$ and

$$L(s, E) = \sum_1^\infty a_n n^{-s}$$

be the Hasse-Weil L-function associated with E. Thus

$$f(z) = \sum_1^\infty a_n e(nz)$$

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is a cusp form of weight 2 which is a newform of level N, where N is the conductor of E. The L-function is entire and it satisfies the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s,E) = w\left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s,E),$$

where $w = \pm 1$. We are interested in curves E for which $L(1,E) \neq 0$, so the functional equation holds with the sign $w = 1$. The twisted L-function

$$L(s,E,\chi_d) = \sum_1^\infty a_n \chi_d(n) n^{-s},$$

where χ_d is a real primitive character to modulus d prime to N is also entire and it satisfies the functional equation

$$(1) \quad \left(\frac{d\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s,E,\chi_d) = w_d\left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s,E,\chi_d)$$

with the sign $w_d = w\chi_d(-N)$. In the sequel we let d range over the set

$$\mathcal{D} = \{d : 0 < d \equiv -\nu^2 (\mathrm{mod} 4N) \text{ for some } \nu \text{ prime to } 4N\}$$

and we let $\chi_d(n) = (\frac{-d}{n})$ be the Kronecker symbol. Thus if d is squarefree χ_d is the primitive character to the modulus d which is associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Every prime dividing N splits in $\mathbb{Q}(\sqrt{-d})$. Moreover we have $w_d = -1$, so by (1) it follows that

$$(2) \quad L(1,E,\chi_d) = 0.$$

Our aim is to prove that $L(s,E,\chi_d)$ has a simple zero at $s = 1$, i.e. $L'(1,E,\chi_d) \neq 0$ for infinitely many d in $\mathcal{D}$. To this end we shall evaluate two sums of type

$$(3) \quad S_4(Y) = \sum_{d\in\mathcal{D}, d\leq Y}^{\flat} |L'(1,E,\chi_d)|^4$$

and

$$(4) \quad S_1(Y) = \sum_{d\in\mathcal{D}}^{\flat} L'(1,E,\chi_d)F(d/Y),$$

where $\sum^\flat$ means that the summation is restricted to squarefree numbers and F is a smooth function, compactly supported in $\mathbb{R}^+$ with positive mean value.

THEOREM. *For any $\epsilon > 0$ and $Y \geq 1$ we have*

(5) $$S_4(Y) << Y^{2+\epsilon}$$

and

(6) $$S_1(Y) = \alpha Y \log Y + \beta Y + O(Y^{13/14+\epsilon})$$

with some constants $\alpha \neq 0$ and β which depend on the curve E and the test function F.

COROLLARY. *Suppose $\epsilon > 0$ and $Y > c(\epsilon)$. Then $L'(1, E, \chi_d) \neq 0$ for at least $Y^{2/3-\epsilon}$ real primitive characters χ_d to modulus $d \in \mathcal{D}, d \leq Y$.*

3. Estimates for the coefficients of f

The Fourier coefficients a_n of the cusp form f are multiplicative. More exactly, for Re $s > 3/2$ we have the Euler product

(7) $$L(s, E) = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}$$

with $\alpha_p = 0, \pm 1, \beta_p = 0$ if $p|N$ and $|\alpha_p| = |\beta_p| = p^{1/2}$ if $p \nmid N$. In the latter case the result was proved by M. Eichler and P. Deligne. It yields the following bound for the coefficient a_n (known as the Ramanujan conjecture)

(8) $$|a_n| \leq n^{1/2}\tau(n),$$

where $\tau(n)$ denotes the divisor function, $\tau(n) << n^\epsilon$. This bound can be slightly improved on average. Indeed, arguing as G. Hardy and E. Hecke with Parseval's formula and using the boundedness of $yf(z)$ we get

(9) $$\sum_{m \leq M} |a_m|^2 \ll M^2.$$

Similarly we get

(10) $$\sum_{m \leq M} a_m e(\alpha m) << M \log M$$

for any real α and $M \geq 2$, the implied constant depending on f only. In this section we derive three variations on (10).

LEMMA I. *Let α be real and ψ be a periodic function of period r. We then have*

$$\sum_{m\leq M} a_m\psi(m)e(\alpha m) << \Psi M \log M, \tag{11}$$

where

$$\Psi = \frac{1}{r}\sum_{a(\mathrm{mod}\ r)} |\sum_{b(\mathrm{mod}\ r)} \psi(b)e(\frac{ab}{r})|.$$

Moreover, if $|\psi| \leq 1$ and s is a positive integer then we have

$$\sum_{m\leq M,(m,s)=1} a_m\psi(m)e(\alpha m) << \tau(s)r^{\frac{1}{2}} M \log M \tag{12}$$

and

$$\sum_{m\leq M,(m,s)=1}{}^{\flat}\, a_m\psi(m)e(\alpha m) << \tau(s)r^{\frac{1}{2}} M\ (\log M)^7 \tag{13}$$

PROOF: The sum on the left-hand side of (11) is equal to

$$\frac{1}{r}\sum_{a(\mathrm{mod}\ r)}\left(\sum_{b(\mathrm{mod}\ r)} \psi(b)e(\frac{ab}{r})\right)\sum_{m\leq M} a_m e((\alpha-\frac{a}{r})m),$$

whence the inequality (11) follows by (10). If $|\psi| \leq 1$ we obtain $\Psi \leq r^{1/2}$ by Cauchy's inequality. For the proof of (12) we can assume that $(r,s)=1$ by changing ψ suitably. Then we apply (11) for $\psi\chi_0$ in place of ψ, where χ_0 is the principal character to the modulus s. We obtain

$$\Psi = \frac{1}{rs}\sum_{a(\mathrm{mod}\ r)} |\sum_{b(\mathrm{mod}\ r)} \psi(b)e(\frac{ab}{r})|$$

$$\sum_{c(\mathrm{mod}\ s)} |\sum_{d(\mathrm{mod}\ s)} \chi_0(d)e(\frac{cd}{s})| << \frac{r^{\frac{1}{2}}}{s}\sum_{c(\mathrm{mod}\ s)}\sum_{d|(c,s)} d = \tau(s)r^{\frac{1}{2}},$$

which gives (12). Finally we derive (13) from (12). The sum on the left-hand side of (13) is equal to

$$\sum\sum_{\nu^2 m\leq M,(\nu m,s)=1} \mu(\nu)a_{\nu^2 m}\psi(\nu^2 m)e(\alpha\nu^2 m)$$

$$= \sum_{\substack{(\nu,s)=1\\ \nu^2\lambda\leq M}}\sum_{\lambda|\nu^\infty}\mu(\nu)a_{\nu^2\lambda}\sum_{\substack{m\leq M/\lambda\nu^2\\ (m,\nu s)=1}} a_m\psi(\nu^2\lambda m)e(\alpha\nu^2\lambda m)$$

$$\ll \tau(s)r^{\frac{1}{2}}M(\log M)\sum_{\substack{(\nu,s)=1\\ \nu^2\lambda\leq M}}\sum_{\lambda|\nu^\infty}|a_{\nu^2\lambda}|\ \frac{\tau(\nu)}{\nu^2\lambda}.$$

Hence (13) follows by (8).

4. Approximate formulas for $L'(1,E,\chi_d)$

We shall express $L'(1,E,\chi_d)$ in terms of the rapidly convergent sums

$$\mathcal{A}(X,\chi)=\sum_{1}^{\infty}a_n\chi(n)n^{-1}V(\frac{2\pi n}{X}),$$

where V is the incomplete gamma function defined by

$$V(X)=\int_X^\infty e^{-t}t^{-1}dt=\frac{1}{2\pi i}\int_{(3/4)}\frac{\Gamma(s)}{s}X^{-s}ds.$$

We have

$$\mathcal{A}(X,\chi_d)=\frac{1}{2\pi i}\int_{(3/4)}L(1+s,E,\chi_d)\frac{\Gamma(s)}{s}(\frac{2\pi}{X})^s ds.$$

Moving the integration to the line $\operatorname{Re} s=-3/4$ we pass a simple pole at $s=0$ with residuum $L'(1,E,\chi_d)$ by virtue of (2). On the other hand the integral over the line $\operatorname{Re} s=-3/4$ is equal to $-\mathcal{A}(d^2NX^{-1},\chi_d)$ by the functional equation (1). This gives

$$L'(1,E,\chi_d)=\mathcal{A}(X,\chi_d)+\mathcal{A}(d^2NX^{-1},\chi_d) \tag{14}$$

for any $X>0$ and d in $\mathcal{D}$ which is squarefree. In particular we have

$$L'(1,E,\chi_d)=2\mathcal{A}(d\sqrt{N},\chi_d). \tag{15}$$

By (9) we infer trivially that $\mathcal{A}(X,\chi_d)<<X^{1/2}$ for any $X>0$ and inserting this to (14) we obtain

$$L'(1,E,\chi_d)=\mathcal{A}(X,\chi_d)+O(dX^{-1/2}). \tag{16}$$

5. Estimation of the fourth moment of $L'(1,E,\chi_d)$

By the large sieve inequality (see [1]) together with (8) we get

$$\sum_{d\le Y}\ \sum_{\chi(\mathrm{mod}\ d)}{}^{*}|\mathcal{A}(X,\chi)|^4<<(X+Y)^{2+\epsilon}.$$

On the other hand by (14) we have for any $d \in \mathcal{D}, d \leq Y$, d squarefree that

$$|L'(1,E,\chi_d)|^4 \ll \int_1^{NY} |\mathcal{A}(X,\chi_d)|^4 X^{-1} dX.$$

Combining both results we infer the upper bound (5) for $S_4(Y)$.

6. An approximate formula for the first moment of $L'(1,E,\chi_d)$

By (15) we obtain

$$S_1(Y) = 2 \sum_{d\in\mathcal{D}}{}^{\flat} \mathcal{A}(d\sqrt{N},\chi_d) F(\frac{d}{Y}).$$

Now we relax the condition that d is squarefree by introducing the factor $\sum_{a^2|d}\mu(a)$, then we split the sum according to whether $a \leq A$ or $a > A$ and in the latter case we return to squarefree numbers by extracting square divisors of $a^{-2}d$. We obtain $S_1(Y) = S + R$, say, where

$$S = 2 \sum_{a\leq A,\ (a,4N)=1} \mu(a) \sum_{d\in\mathcal{D}} \mathcal{A}(a^2 d\sqrt{N},\ \chi_{a^2d}) F\left(\frac{a^2 d}{Y}\right)$$

and

$$R = 2 \sum_{(b,4N)=1} \left(\sum_{a|b,\ a>A} \mu(a) \right) \sum_{d\in\mathcal{D}}{}^{\flat} \mathcal{A}(b^2 d\sqrt{N},\ \chi_{b^2d}) F\left(\frac{b^2 d}{Y}\right).$$

Here A is a large number to be chosen later. In the term $\mathcal{A}(X,\chi_{b^2d})$ with $X = b^2 d\sqrt{N}$ we return to $L'(1,E,\chi_d)$ by reversing the arguments as follows

$$\begin{aligned}
\mathcal{A}(X,\chi_{b^2d}) &= \sum_{(n,b)=1} a_n \chi_d(n) n^{-1} V\left(\frac{2\pi n}{X}\right) \\
&= \sum_{k|b}\sum_{\ell|b} \alpha_k \beta_\ell \chi_d(k\ell) \frac{\mu(k)\mu(\ell)}{k\ell} \mathcal{A}\left(\frac{X}{k\ell},\chi_d\right) \\
&= L'(1,E,\chi_d) \prod_{p|b} \left(1-\chi_d(p)\frac{\alpha_p}{p}\right)\left(1-\chi_d(p)\frac{\beta_p}{p}\right) + O(\tau(b) d X^{-\frac{1}{2}})
\end{aligned}$$

the second line being obtained by (7) and the third line by (16). Finally applying (5) and the Hölder inequality we conclude that

$$(17)\quad R \ll \sum_b \Big(\sum_{a|b,\, a>A} 1\Big)\big(b^{-\frac{5}{2}}Y^{\frac{5}{4}} + b^{-4}Y^{\frac{3}{2}}\big)Y^{\epsilon} \ll \big(A^{-\frac{3}{2}}Y^{\frac{5}{4}} + A^{-3}Y^{\frac{3}{2}}\big)Y^{\epsilon}.$$

7. A transformation of S

It remains to evaluate S. For $(a, 4N) = 1$ and $d \in \mathcal{D}$ we have

$$\mathcal{A}(a^2 d\sqrt{N}, \chi_{a^2 d}) = \sum_{(n,a)=1} a_n n^{-1} \chi_d(n) V(2\pi n / a^2 d\sqrt{N}).$$

Every n can be written uniquely as the product $n = k\ell^2 m$, where k has prime factors in $4N$, ℓm is prime to $4N$ and m is squarefree. For n written this way and d in $\mathcal{D}$ we have $\chi_d(n) = \chi_d(m)$ subject to $(d, \ell) = 1$. The last condition is detected by the familiar formula of Möbius giving

$$S = 2\sum_{\substack{a\le A\\(a,4N)=1}} \mu(a) \sum_{\substack{n=k\ell^2 m\\(n,a)=1}} a_n n^{-1} \sum_{q|\ell} \mu(q) \sum_{dq\in\mathcal{D}} \chi_{dq}(m) F\left(\frac{a^2 dq}{Y}\right) V\left(\frac{2\pi n}{a^2 dq\sqrt{N}}\right).$$

Next, by means of Gauss sums we write

$$\chi_d(m) = \overline{\epsilon}_m m^{-\frac{1}{2}} \sum_{2|r|<m} \chi_{Nr}(m) e\Big(\frac{\overline{4N} rd}{m}\Big),$$

where $\epsilon_m = 1$ if $m \equiv 1 (mod\ 4)$, $\epsilon_m = i$ if $m \equiv -1 (\mathrm{mod}\ 4)$ and $4N\overline{4N} \equiv 1(\bmod m)$. This gives

$$S = 2\sum_{\substack{a\le A\\(a,4Nn)=1}} \sum_{n=k\ell^2 m} \mu(a) a_n n^{-1} \overline{\epsilon}_m m^{-\frac{1}{2}} \sum_{q|\ell} \mu(q) \sum_{2|r|<m} \chi_{Nrq}(m) \sum_d,$$

where

$$\sum_d = \sum_{dq\in\mathcal{D}} F\Big(\frac{a^2 dq}{Y}\Big) V\Big(\frac{2\pi n}{a^2 dq\sqrt{N}}\Big) e\Big(\frac{\overline{4N} rd}{m}\Big).$$

We put $\Delta = \min(1/2, a^2 q Y^{\epsilon - 1})$ and split $S = S_0 + S_1 + S_2$, where S_0, S_1, S_2 denote the partial sums restricted by the conditions $r = 0, 0 < |r| < \Delta m$, $\Delta m \le |r| < m/2$ respectively.

8. Estimates for S_2 and S_1

LEMMA 2. *Suppose $g(x)$ is a smooth and integrable function on $\mathbb{R}$ with derivatives $g^{(j)}(x) << (|x|+X)^{-j}$ for all $j \geq 1$ the implied constant depending on j only. Suppose α is real and q is a positive integer such that αq is not an integer. We then have*

$$\sum_{n\equiv v(\text{mod } q)} g(n)e(\alpha n) << \frac{X}{q}\left(\frac{q}{X\|\alpha q\|}\right)^j \tag{18}$$

for any $j \geq 2$, the implied constant depending on j only.

PROOF: By Poisson's formula the sum is equal to

$$\frac{1}{q}\sum_{u=-\infty}^{\infty} e\left(\frac{uv}{q}\right)\hat{g}\left(\alpha-\frac{u}{q}\right) ,$$

where $\hat{g}(y)$ denotes the Fourier transform of $g(x)$. We have $\hat{g}(y) << X(Xy)^{-j}$ by the partial integration j times, whence (18) follows by trivial summation over u.

To estimate S_2 we sum over d first by an appeal to (18). For any $j \geq 2$ we get $\sum_d << (n+Y)^{-j}$, whence $S_2 << 1$.

To estimate S_1 we sum over m first using (13) and partial summation together with the relation

$$e\left(\frac{\overline{4N}rd}{m}\right) = e\left(\frac{rd}{4Nm}-\frac{\overline{m}rd}{4N}\right)$$

and then we sum over r trivially getting

$$\sum_{0<|r|<\Delta m}\sum a_n\, n^{-1}\overline{\epsilon}_m\, m^{-\frac{1}{2}}\,\chi_{Nrq}(m)V\left(\frac{2\pi n}{a^2dq\sqrt{N}}\right)e\left(\frac{\overline{4N}rd}{m}\right)$$
$$<< k^{-\frac{3}{2}}\ell^{-3}a^3q^2Y^{\epsilon-\frac{1}{2}} .$$

Hence we conclude that

$$S_1 << \sum_{a\leq A}\sum_{k\ell^2}\sum_{q|\ell}\sum_{d} F\left(\frac{a^2dq}{Y}\right)k^{-\frac{3}{2}}\ell^{-3}a^3q^2Y^{\epsilon-\frac{1}{2}} << A^2Y^{\epsilon+\frac{1}{2}} .$$

9. Evaluation of S_0

Since $r = 0$ we have $\chi_{Nrq}(m) = 0$ for all $m > 1$ and the terms with $m = 1$ yield

$$S_0 = 2 \sum_{\substack{a \le A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=k\ell^2 \\ (n,a)=1}} a_n n^{-1} \sum_{q|\ell} \mu(q) \sum_d ,$$

where

$$\sum_d = \sum_{dq \in \mathcal{D}} F\left(\frac{a^2dq}{Y}\right) V\left(\frac{2\pi n}{a^2dq\sqrt{N}}\right) .$$

We split the summation over d into residue classes modulo 4N. Each class contributes

$$\frac{Y}{4Na^2q} \int F(t) V\left(\frac{2\pi n}{t\sqrt{N}Y}\right) dt \; + \; \mathrm{O}\left(\left(1 + \frac{n}{Y}\right)^{-j}\right)$$

for any $j \ge 2$, and the number of relevant classes is

$$\gamma(4N) = \#\{d (\bmod\ 4N) : d \equiv -\nu^2 (\bmod\ 4N), (\nu, 4N) = 1\}.$$

Hence

$$S_0 = \gamma(4N) Y \sum_{n=k\ell^2} \frac{a_n \varphi(\ell)}{2Nn\ell} \Big(\sum_{a \le A, (a,4N\ell)=1} \mu(a) a^{-2}\Big) \int F(t) V\left(\frac{2\pi n}{t\sqrt{N}Y}\right) dt + \mathrm{O}\left(AY^{\epsilon + \frac{1}{2}}\right)$$

$$= c_N Y \int F(t) \mathcal{B}(t\sqrt{N}Y) dt + \mathrm{O}((AY^{\frac{1}{2}} + A^{-1}Y)Y^{\epsilon}),$$

where

$$c_N = \frac{3\gamma(4N)}{\pi^2 N} \prod_{p|4N} (1 - \frac{1}{p^2})$$

and

$$\mathcal{B}(X) = \sum_{n=k\ell^2} \frac{b_n}{n} V\left(\frac{2\pi n}{X}\right)$$

with

$$b_n = a_n \prod_{p|n,\ p\nmid 4N} \left(1 + \frac{1}{p}\right) .$$

To evaluate the series $\mathcal{B}(X)$ we appeal to analytic properties of the zeta-function

$$L(s) = \sum_{n=k\ell^2} b_n n^{-s}.$$

The required properties are inherited from the properties of the Rankin-Selberg zeta-function

$$H(s) = \sum_{1}^{\infty} a_n^2 n^{-s} .$$

The Rankin-Selberg zeta-function is meromorphic on $\mathbb{C}$, holomorphic on Re $s \geq 1$ except for a simple pole at $s = 2$ with residuum

$$H = \operatorname*{res}_{s=2} H(s) > 0 ,$$

and it satisfies a functional equation which connects $H(s)$ with $H(2-s)$. Moreover, as shown by G. Shimura [12] the function

$$L(s, \mathrm{sym}^2) = \frac{\zeta(2s)}{\zeta(s)} H(s+1)$$

is entire. By the Phragmén-Lindelöf principle, using the functional equation, it follows that

$$L(s, \mathrm{sym}^2) << |s| \quad \text{if} \quad \mathrm{Re}\ s \geq 1/2 .$$

Since $L(s)$ agrees with $L(2s-1, \mathrm{sym}^2)/\zeta(4s-2) = H(2s)/\zeta(2s-1)$ up to an Euler product $P(s)$, say, which converges absolutely in Re $s \geq 3/4$ we conclude that $L(s)$ is holomorphic in Re $s \geq 3/4$, it satisfies

$$L(s) << |s|^2 \quad \text{if} \quad \mathrm{Re}\ s \geq 3/4$$

and that

(19) $$L(1) = HP(1) \neq 0 .$$

Now by the contour integration we get

$$\begin{aligned} \mathcal{B}(X) &= \frac{1}{2\pi i} \int_{(3/4)} L(s+1) \frac{\Gamma(s)}{s} \left(\frac{X}{2\pi}\right)^s ds \\ &= \operatorname*{res}_{s=0} L(s+1) \frac{\Gamma(s)}{s} \left(\frac{X}{2\pi}\right)^s + \frac{1}{2\pi i} \int_{(-1/4)} \\ &= L(1) \left(\log \frac{X}{2\pi} - \gamma\right) + L'(1) + O(X^{-1/4}) \end{aligned}$$

by the expansion $\Gamma(s) = s^{-1} - \gamma + \dots$, where γ is the Euler constant. Integrating against $F(t)$ we conclude that

$$S_0 = \alpha Y \log\ Y + \beta Y + O((AY^{\frac{1}{2}} + A^{-1}Y)Y^{\epsilon})$$

with

$$\alpha = c_N L(1) \int F(t)dt \neq 0 \tag{20}$$

and

$$\beta = c_N \int F(t) \left[L(1) \left(\log \frac{t\sqrt{N}}{2\pi} - \gamma \right) + L'(1) \right] dt\ . \tag{21}$$

10. Evaluation of the first moment of $L'(1, E, \chi_d)$. Conclusion

Collecting the established evaluations we infer that

$$\begin{aligned} S_1(Y) = S_0 + S_1 + S_2 + R = \alpha Y \log\ Y + \beta Y \\ + O((AY^{\frac{1}{2}} + A^{-1}Y + A^2 Y^{\frac{1}{2}} + A^{-\frac{3}{2}} Y^{\frac{5}{4}} + A^{-3} Y^{\frac{3}{2}}) Y^{\epsilon}) \end{aligned}$$

which gives (6) on taking $A = Y^{3/14}$.

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