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An almost-sure estimate for the mean of generalized Q -multiplicative functions of modulus 1

par JEAN-LOUP MAUCLAIRE

RÉSUMÉ. Soit $Q = (Q_k)_{k \geq 0}$, $Q_0 = 1$, $Q_{k+1} = q_k Q_k$, $q_k \geq 2$, $k \geq 0$, une échelle de Cantor, Z_Q le groupe compact $\prod_{0 \leq j} \mathbf{Z}/q_j \mathbf{Z}$, et μ sa mesure de Haar normalisée. A un élément x of Z_Q écrit $x = \{a_0, a_1, a_2, \dots\}$, $0 \leq a_k \leq q_{k+1} - 1$, $k \geq 0$, on associe la suite $x_k = \sum_{0 \leq j \leq k} a_j Q_j$. On montre que si g est une fonction Q -multiplicative unimodulaire, alors

$$\lim_{x_k \rightarrow x} \left(\frac{1}{x_k} \sum_{n \leq x_k - 1} g(n) - \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(a Q_j) \right) = 0 \quad \mu\text{-p.s.}$$

ABSTRACT Let $Q = (Q_k)_{k \geq 0}$, $Q_0 = 1$, $Q_{k+1} = q_k Q_k$, $q_k \geq 2$, be a Cantor scale, Z_Q the compact projective limit group of the groups $\mathbf{Z}/Q_k \mathbf{Z}$, identified to $\prod_{0 \leq j \leq k-1} \mathbf{Z}/q_j \mathbf{Z}$, and let μ be its normalized Haar measure. To an element $x = \{a_0, a_1, a_2, \dots\}$, $0 \leq a_k \leq q_{k+1} - 1$, of Z_Q we associate the sequence of integral valued random variables $x_k = \sum_{0 \leq j \leq k} a_j Q_j$. The main result of this article is that, given a complex Q -multiplicative function g of modulus 1, we have

$$\lim_{x_k \rightarrow x} \left(\frac{1}{x_k} \sum_{n \leq x_k - 1} g(n) - \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(a Q_j) \right) = 0 \quad \mu\text{-a.e.}$$

1. INTRODUCTION

Let \mathbf{N} be the set of non-negative integers, and let $Q = (Q_k)_{k \geq 0}$, $Q_0 = 1$, be an increasing sequence of positive integers. Using the greedy algorithm, to every element n of \mathbf{N} , one can associate a representation

$$n = \sum_{k=0}^{+\infty} \varepsilon_k(n) Q_k$$

which is unique if for every K ,

$$\sum_{k=0}^{K-1} \varepsilon_k(n) Q_k < Q_K.$$

The simplest examples are the q -adic scale, q integer, $q \geq 2$, and its generalization, the Cantor scale $Q_{k+1} = q_k Q_k$, $Q_0 = 1$, $q_k \geq 2$, $k \geq 0$. In this article, we are concerned with the Cantor scale. For a given integer $n \geq 1$, we denote by $k(n)$ the maximal index k for which $\varepsilon_k(n)$ is different from zero. The integers $\varepsilon_k(n)$ are the digits from n in the basis Q . We recall that if G is an abelian group, a G -valued arithmetical function f such that

$$f(n) = \sum_{k=0}^{k(n)} f(\varepsilon_k(n) Q_k) \quad \text{for } n \geq 1 \quad \text{and} \quad f(0) = 0_G,$$

is called a Q -additive function, an extension of the notion of q -additive function introduced by A. O. Gelfond in the q -adic case [4]. We recall that a real-valued sequence $f(n)$ has an asymptotic distribution if there exists a distribution function F such that for all continuity points x of F , the probability measures defined by $\mu_N(x) = N^{-1} \text{card}\{n \leq N; f(n) \leq x\}$ tends to $F(x)$ as N tends to infinity. In the case of the q -adic scale, necessary and sufficient conditions for the existence of an asymptotic distribution for a real-valued q -additive function have been given by H. Delange in 1972 [3]. J. Coquet [2] considered in 1975 the same kind of problem in cases of Cantor scales and obtained mainly sufficient conditions. In both cases, it appears essential to have information on the difference

$$\left(\frac{1}{x} \sum_{0 \leq n < x} g(n) - \prod_{0 \leq j \leq k(x)} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(a Q_j) \right),$$

where $g(\cdot)$ is any Q -multiplicative function of modulus 1, and more precisely, to get a characterization of

$$(1) \quad \lim_{x \rightarrow +\infty} \left(\frac{1}{x} \sum_{0 \leq n < x} g(n) - \prod_{0 \leq j \leq k(x)} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(a Q_j) \right) = 0.$$

In fact, if the sequence $\{q_j\}_{j \geq 0}$ is bounded, the relation 1 is always true. But if $\{q_j\}_{j \geq 0}$ is unbounded, the situation is quite different. In [1], G. Barat constructs a Q -multiplicative function h with values 1 or -1 such that

$$\lim_{x \rightarrow +\infty} \prod_{0 \leq j \leq k(x)} \frac{1}{q_j} \sum_{0 \leq a < q_j} h(a Q_j)$$

exists and is a positive number while

$$\liminf_{x \rightarrow +\infty} \frac{1}{x} \sum_{n < x} h(n)$$

is less than or equal to zero. This difference is due to the existence of a *first digit phenomenon*, unavoidable for unbounded sequence $\{q_j\}_{0 \leq j}$, as remarked by E. Manstavičius in a recent article [6].

Let \mathbf{Z}_Q denote the group of Q -adic integers, considered as the compact projective limit group of $\mathbf{Z}/Q_k\mathbf{Z}$ and identified to $\prod_{0 \leq k} \mathbf{Z}/q_k\mathbf{Z}$ (see [5], p. 109). The products

$$\prod_{0 \leq j \leq k(n)} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(aQ_j)$$

are clearly related to this group in the following way: an element a of \mathbf{Z}_Q can be written $a = (a_0, a_1, \dots)$, $0 \leq a_k \leq q_k - 1$, $0 \leq k$, and we may identify an element of \mathbf{N} with an element of \mathbf{Z}_Q which has only a finite number of digits different from zero. For all $a = (a_0, a_1, \dots)$ belonging to \mathbf{Z}_Q , we define on \mathbf{Z}_Q the sequence of \mathbf{N} -valued random variables $x_k(\cdot)$ given by $x_k(a) = \sum_{j=0}^k a_j Q_j$, the compact group \mathbf{Z}_Q being endowed with its normalized Haar measure μ , and clearly

$$\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(aQ_j) = \int_{\mathbf{Z}_Q} g(x_k) d\mu.$$

In this article, we show roughly speaking that although the relation 1 is not always true according to the example of G. Barat (for unbounded sequence $\{q_j\}_{j \geq 0}$), it is almost surely true for a path chosen at random.

2. RESULTS

2.1. Main theorem.

Theorem 1. *Let g be a unimodular Q -multiplicative function and set*

$$m_j(g) = \frac{1}{q_j} \sum_{0 \leq a < q_j} g(aQ_j).$$

Then, the relation

$$\lim_{k \rightarrow \infty} \left(\frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} g(n) - \prod_{0 \leq j \leq k} m_j(g) \right) = 0$$

holds μ -a.e.

2.2. Consequence of Theorem 1.

Theorem 2. *Let G be a metrizable locally compact abelian group with group law denoted by $+$. Γ denotes the dual group of G endowed with its Haar measure m , and let $f(n)$ be a G -valued Q -additive function. Given a sequence $A(k)$ in G , we denote by F_k^A the distribution of the G -valued*

function defined on \mathbf{Z}_Q by $t \mapsto (f(x_k(t)) - A(k))$, and by $\delta_{(a)}$ the measure consisting in a unit mass at the point a .

The following assertions are equivalent:

i) there exists a sequence $A(k)$ in G and a probability measure ν on G such that the sequence of distributions F_k^A converges vaguely to ν (i.e., $\lim_k \int_G \varphi dF_k^A = \int_G \varphi d\nu$ for all continuous maps $\varphi : G \rightarrow \mathbf{C}$ with compact support);

ii) there exists a sequence $A(k)$ in G and a probability measure ν on G such that μ -a.e., the sequence of random measures $\frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} \delta_{(f(n) - A(k))}$ converges vaguely to ν as k tends to infinity;

iii) the set X of characters g of Γ for which there exists an integer $N(g)$ such that $\prod_{j \geq N(g)} m_j(g) \neq 0$ is not m -negligible.

Remarks. 1) Assertion iii) is always satisfied if G is a compact metrizable group, for X is not empty (it contains the trivial character), and consequently is not m -negligible.

2) Necessary and sufficient conditions for the continuity of ν can be easily found, since ν appears as a convolution of measures on \mathbf{Z}_Q : in fact, the same method as in [7] (p. 84–87), gives that X is a closed and open subgroup. Denoting by H the orthogonal of X and by T_H the canonical projection $G \mapsto G/H$, the measure ν is not continuous if and only if H is finite and

$$\lim_{k \rightarrow \infty} \sum_{0 \leq j \leq k} \frac{1}{q_j} \sum_{\substack{0 \leq a < q_j \\ T_H(f(aQ_j)) \neq 0}} 1 < +\infty.$$

2.3. Proof of Theorem 2. A straightforward adaptation of the argument given in [7] (p 84–87) leads to, *primo* if one of the assumptions i), ii), iii), holds, then, X is a closed and open subgroup of Γ ; and *secundo*, there exists a probability measure ν on G and a G -valued sequence $\{A(k)\}_k$ such that for all g in Γ , the sequence

$$\{\bar{g}(A(k)) \prod_{0 \leq j \leq k} m_j(g \circ f)\}_k$$

tends to $\hat{\nu}(g)$ where $\hat{\nu}$ is the Fourier transform of ν . This is due to the fact that for g in X there exists an $N(g)$ for which the relation $\prod_{j \geq N(g)} m_j(g) \neq 0$,

holds. Hence we get by Theorem 1 that for all g , the sequence

$$\left\{ \frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} g(f(n) - A(k)) \right\}_k$$

converges to $\hat{\nu}(g)$ μ -a.e. Next, we use the Fubini theorem on the measured space $(\Gamma \times \mathbf{Z}_Q, m \otimes \mu)$ in an essential way, by saying that since Γ is countable at infinity and \mathbf{Z}_Q is compact, both of the measures m and μ are σ -finite

and so, μ -a.e., the sequence $\{\frac{1}{x_k} \sum_{n < x_k(\cdot)} g(f(n) - A(k))\}_k$ converges to $\hat{\nu}(g)$ m -a.e.. In order to prove that μ -a.e., the sequence

$$\left\{ \frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} \delta_{(f(n) - A(k))} \right\}_k$$

converges vaguely to ν , it suffices to show that for any real-valued continuous function F defined on G whose support is compact, the sequence

$$\left\{ \frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} F(f(n) - A(k)) \right\}_k$$

converges to $\nu(F)$. This can be done as follows. Take any $\varepsilon > 0$; by assumption on F , there exists V , a symmetric neighborhood of the origin in G , such that for all t in G and all u in V , one has $|F(t + u) - F(t)| \leq \varepsilon$. Denoting by M the Haar measure on G normalized with respect to m , we have

$$\begin{aligned} \left| F(t) - \frac{1}{M(V)} \int_V F(t + u) dM(u) \right| &= \left| \frac{1}{M(V)} \int_V (F(t + u) - F(t)) dM(u) \right| \\ &\leq \frac{1}{M(V)} \int_V |F(t + u) - F(t)| dM(u) \\ &\leq \frac{1}{M(V)} \int_V \varepsilon dM(u) \leq \varepsilon. \end{aligned}$$

The function $F_V(t)$ defined by

$$F_V(t) = \frac{1}{M(V)} \int_V F(t + u) dM(u)$$

is the convolution product of F with the characteristic function of V normalized by the constant $M(V)^{-1}$. Therefore, the Fourier transform \widehat{F}_V is integrable and we get

$$\begin{aligned} \frac{1}{x_k} \sum_{n < x_k} F(f(n) - A(k)) &= \frac{1}{x_k} \sum_{n < x_k} \int_{\Gamma} \widehat{F}_V(g) \bar{g}(f(n) - A(k)) dm(g) \\ &= \int_{\Gamma} \left(\frac{1}{x_k} \sum_{n < x_k} \widehat{F}_V(g) \bar{g}(f(n) - A(k)) \right) dm(g). \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \frac{1}{x_k} \sum_{n \leq x_k - 1} F(f(n) - A(k)) \\
&= \lim_{k \rightarrow +\infty} \int_{\Gamma} \left(\frac{1}{x_k} \sum_{n \leq x_k - 1} \hat{F}_V(g) \bar{g}(f(n) - A(k)) \right) dm(g) \\
&= \int_{\Gamma} \hat{F}_V(g) \left(\lim_{k \rightarrow +\infty} \frac{1}{x_k} \sum_{n \leq x_k - 1} \bar{g}(f(n) - A(k)) \right) dm(g) \\
&= \nu(F_V) \mu\text{-a.e.}
\end{aligned}$$

Now, since ν is a probability measure and $|F - F_V| \leq \varepsilon$, the sequence

$$\left\{ \frac{1}{x_k} \sum_{n \leq x_k - 1} F(f(n) - A(k)) - \nu(F) \right\}_k$$

is bounded in modulus by 2ε ; this implies

$$\lim_{k \rightarrow +\infty} \frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} F(f(n) - A(k)) = \nu(F) \mu\text{-a.e.}$$

Therefore, the sequence $\left\{ \frac{1}{x_k(\cdot)} \sum_{n < x_k(\cdot)} \delta_{(f(n) - A(k))} \right\}_k$ converges vaguely μ -a.e. to ν .

3. PROOF OF THEOREM 1

Notation and conventions

Given an arbitrary arithmetical function f , we set

$$S_N(f) = \sum_{0 \leq n \leq N} f(n), \quad M_{N-1}(f) = \sum_{0 \leq n < Q_N} f(n), \quad \widetilde{M}_N(f) = Q^{-1} M_N(f).$$

Notice that we have the identity $M_{N-1}(f) = S_{Q_{N-1}}(f)$ and for any Q -multiplicative function f ,

$$M_{N-1}(f) = \prod_{0 \leq k < N} m_k(f).$$

By convention, the result of a summation (resp. a product) on an empty set will be 0 (resp. 1).

A - Toolbox.

Proposition 1. *Let g be a Q -multiplicative function of modulus 1 and assume that the sequence $\{\widetilde{M}_k(g)\}_k$ does not tend to 0. Then, there exists a*

sequence $\{\alpha_k\}_{k \geq 0}$ of complex numbers of modulus 1 such that

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{a=0}^{q_k-1} |1 - g(aQ_k)\alpha_k|^2 < +\infty.$$

Proof. By our assumption, all the complex numbers $m_j(g)$ are different from zero. Put $\alpha_j = m_j(\bar{g}(\cdot))|m_j(g(\cdot))|^{-1}$ where $\bar{g}(\cdot)$ is the complex conjugate of $g(\cdot)$. The product $\alpha_j m_j(g)$ is equal to $|m_j(g)|$ and the sequence $\{|\widetilde{M}_{k+1}|\}_k$ is convergent. Therefore,

$$\sum_{k=0}^{+\infty} (1 - \alpha_k m_k(g)) < +\infty.$$

From

$$\sum_{k=0}^{+\infty} (1 - \alpha_k m_k(g)) = \sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{0 \leq a < q_k} (1 - g(aQ_k)\alpha_k)$$

we get a fortiori that the series $\sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{0 \leq a < q_k} \operatorname{Re}(1 - g(aQ_k)\alpha_k)$ converges and since $|g(aQ_k)\alpha_k| = 1$, we deduce

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{0 \leq a < q_k} |1 - g(aQ_k)\alpha_k|^2 < +\infty.$$

□

According to Proposition 1, we introduce the sequence of arithmetical functions $g_k^*(n)$ defined by

$$g_k^*(n) = \prod_{0 \leq j \leq k} g(a_j Q_j)\alpha_j$$

where n is written in base Q as $n = \sum_{j=0}^k a_j Q_j$. This means that if $k(n)$ is the index of the last digit of n which is different from zero, $g_k^*(n)$ is equal to

$$\left(\prod_{0 \leq j \leq k(n)} g(a_j Q_j)\alpha_j \right) \cdot \left(\prod_{k(n) < j \leq k} \alpha_j \right).$$

We extend g_k^* by $g_k^*(x) = g_k^* \circ x_k$ which we also denote g_k^* . Moreover, for simplification, we shall use the notation $g^*(aQ_j) = g(aQ_j)\alpha_j$.

Proposition 2. *If the sequence $\{\widetilde{M}_{k+1}(f)\}_{k \geq 0}$ does not converge to 0, there exists a subset E_∞ of \mathbf{Z}_Q such that $\mu(E_\infty) = 1$ and for every $a = (a_0, a_1, \dots)$ in E_∞ , the sequence $k \mapsto g_k^*(a)$ converges.*

Proof. The sequence of finite groups $\mathbf{Z}/Q_k\mathbf{Z}$, $k \geq 0$, induces a filtration on the μ -measured space \mathbf{Z}_Q , and the complex-valued sequence of adapted functions for this filtration defined by

$$g_k^*(\cdot) \left(\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g^*(aQ_j) \right)^{-1}$$

is a martingale. Since we have

$$\prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g^*(aQ_j) = \left| \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g^*(aQ_j) \right|$$

and

$$\left| \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g^*(aQ_j) \right|^{-1}$$

is bounded, this martingale is bounded and so, it converges μ -a.e. But the sequence

$$\left\{ \prod_{0 \leq j \leq k} \frac{1}{q_j} \sum_{0 \leq a < q_j} g^*(aQ_j) \right\}_k$$

is convergent. Hence we obtain that the sequence $\{g_k^*(\cdot)\}$ converges μ -a.e. \square

Proposition 3. *If the sequence $\{\widetilde{M}_{k+1}(f)\}_k$ does not tend to 0, there exists a subset F_∞ of \mathbf{Z}_Q such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), \dots)$ in F_∞ , one has*

$$\lim_{\substack{k \rightarrow +\infty \\ a_k(x) \neq 0}} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} |1 - g^*(aQ_k)|^2 = 0.$$

Proof. Assume that the sequence $\{\widetilde{M}_{k+1}(f)\}_k$ does not tend to 0. Using the same notations as in Proposition 2, we have by Proposition 1

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{a=0}^{q_k-1} |1 - g(aQ_k)\alpha_k|^2 < +\infty.$$

Let σ_k be defined by $\sigma_k = \frac{1}{q_k} \sum_{a=0}^{q_k-1} |1 - g(aQ_k)\alpha_k|^2$. For x in \mathbf{Z}_Q , we write $x = (a_0(x), a_1(x), \dots)$, $0 \leq a_k(x) \leq q_k - 1$, $0 \leq k$ and we remark that, on the sequence of the $a_k(x)$ different from 0, one has

$$\begin{aligned} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} |1 - g^*(aQ_k)|^2 &\leq \frac{1}{a_k(x)} \sum_{0 \leq a < q_k} |1 - g^*(aQ_k)|^2 \\ &\leq \frac{q_k}{a_k(x)} \left(\frac{1}{q_k} \sum_{0 \leq a < q_k} |1 - g^*(aQ_k)|^2 \right) \end{aligned}$$

$$\leq \frac{q_k}{a_k(x)} \sigma_k.$$

Since $\sum_k \sigma_k < +\infty$, it is known that there exists an increasing positive function h tending to infinity when k tends to infinity such that $\sum_k \sigma_k h(k) < +\infty$ and $\prod_{k=0}^{+\infty} (1 - \sigma_k h(k)) > 0$. We consider the set $F(h)$ of points x in \mathbf{Z}_Q such that for all k , the inequality

$$[q_k \sigma_k h(k)] \leq a_k(x) \leq q_k - 1$$

holds, where $[\cdot]$ denotes the integral part function. This set $F(h)$ is closed, and its measure $\mu(F(h))$ is equal to

$$\prod_{k=0}^{+\infty} \frac{1}{q_k} (q_k - [q_k \sigma_k h(k)]),$$

and we have

$$\mu F(h) \geq \prod_{k=0}^{+\infty} \frac{1}{q_k} (q_k - q_k \sigma_k h(k)).$$

Now, we remark that this last product can be written $\prod_{k=0}^{+\infty} (1 - \sigma_k h(k))$ and so, $\mu F(h) \neq 0$. For an x in $F(h)$, we consider the condition $[q_k \sigma_k h(k)] \leq a_k(x) \leq q_k - 1$, when $a_k(x) \neq 0$. If $[q_k \sigma_k h(k)]$ is not 0, then we have

$$\begin{aligned} \frac{q_k}{a_k(x)} \sigma_k &\leq \frac{q_k}{[q_k \sigma_k h(k)]} \sigma_k \leq \frac{q_k \sigma_k h(k)}{[q_k \sigma_k h(k)]} \cdot \frac{q_k}{q_k \sigma_k h(k)} \sigma_k \\ &\leq \frac{q_k \sigma_k h(k)}{[q_k \sigma_k h(k)]} \frac{1}{h(k)} \leq \frac{2}{h(k)} \end{aligned}$$

and in this case, we get $\lim_{k \rightarrow +\infty} \frac{q_k}{a_k(x)} \sigma_k = 0$. The case where $[q_k \sigma_k h(k)] = 0$ remains. We have $0 \leq q_k \sigma_k h(k) < 1$, i.e. $q_k \sigma_k < 1/h(k)$. Hence

$$\frac{q_k}{a_k(x)} \sigma_k \leq \frac{q_k}{1} \sigma_k \leq q_k \sigma_k \leq \frac{1}{h(k)} = o(1), \quad k \rightarrow +\infty.$$

To obtain our result, we remark that the sequence of functions h_r indexed by positive integers r and defined by $h_r(n) = h(n)$ if $n > r$ and $h(n)r^{-1}$ otherwise, satisfies the same requirements as h . Now, the sequence of closed sets $F(h_r)$ is increasing with r and $\lim_{r \rightarrow +\infty} \mu(F(h_r)) = 1$. This gives immediately that F_∞ , the union of the $F(h_r)$, is a measurable set of measure 1. Now, if x belongs to F_∞ , it belongs to some $F(h_r)$ and as a consequence, along the sequence k such that $a_k(x) \neq 0$, we have

$$\begin{aligned} \frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} |1 - g^*(aQ_k)|^2 &\leq \frac{q_k}{a_k(x)} \sigma_k \leq q_k \sigma_k \\ &\leq \frac{2}{h_r(k)} = o(1), \quad k \rightarrow +\infty. \end{aligned}$$

□

Proposition 4. *If the sequence $\{\widetilde{M}_{k+1}(f)\}_{k \geq 0}$ converges to zero, then*

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{0 \leq n < N} g(n) = 0.$$

Proof. This Proposition is due to J.Coquet [2]. □

B- End of the proof

1- First case: the sequence $\{\widetilde{M}_{k+1}(f)\}_k$ tends to zero.

From Proposition 4, $\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{0 \leq n < N} f(n) = 0$, and $(x_k)_k$ tends to infinity μ -a.e. due to the fact that $x_k(a)$ is bounded if and only if a has only a finite number of nonzero digits. This means exactly that a is an integer; but $\mu(\mathbf{N}) = 0$.

2- Second case: the sequence $\{\widetilde{M}_{k+1}(f)\}_k$ does not tend to zero.

We consider the intersection of the sets E_∞ and F_∞ given in Proposition 2 and Proposition 3 respectively. Notice that $\mu(E_\infty \cap F_\infty) = 1$. Our aim is to prove that for every ξ in $E_\infty \cap F_\infty$

$$\lim_{k \rightarrow +\infty} \left(\frac{1}{x_k(\xi)} \sum_{n < x_k(\xi)} g(n) - \widetilde{M}_{k+1}(g) \right) = 0.$$

The sequence of functions $k \mapsto g_k^*(n)$ and the constants α_j are defined as in Proposition 2. Let ξ be an element of $E_\infty \cap F_\infty$ and denote $x_k(\xi)$ by x_k for short. We have:

$$\begin{aligned} S_{x_k}(g_k^*) &= \left(\sum_{0 \leq a < a_k} g(aQ_k)\alpha_k \right) M_{k-1}(g_{k-1}^*) + (g(a_k Q_k)\alpha_k) S_{x_{k-1}}(g_{k-1}^*) \\ &= \left(\sum_{0 \leq a < a_k} g(aQ_k)\alpha_k \right) M_{k-1}(g_{k-1}^*) + (g_k^*(\xi))(\overline{g_{k-1}^*(\xi)}) S_{x_{k-1}}(g_{k-1}^*), \end{aligned}$$

and by iteration

$$\begin{aligned} S_{x_k}(g_k^*) &= \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} \left(\sum_{0 \leq a < a_j(\xi)} g(aQ_j)\alpha_j \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} g_{j-1}^*(aQ_r) \right) \\ &= \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} \left(\sum_{0 \leq a < a_j(\xi)} g(aQ_j)\alpha_j \right) (M_{j-1}(g_{j-1}^*)) \end{aligned}$$

If $a_j(\xi) \neq 0$, the choice of ξ in F_∞ implies

$$\sum_{0 \leq a < a_j(\xi)} g^*(aQ_j) = a_j(\xi)(1 + \varepsilon_j),$$

with $\varepsilon_j = o(1)$ when j tends to infinity. Since g is of modulus 1 and $Q_j^{-1}M_{j-1}(g_{j-1}^*)$ is bounded by 1,

$$\begin{aligned} & \left| S_{x_k}(g_k^*) - \sum_{j=0}^k (g_k^*(\xi) \overline{g_j^*(\xi)})(a_j(\xi)) (M_{j-1}(g_{j-1}^*)) \right| \\ &= \left| S_{x_k}(g_k^*) - \sum_{j=0}^k (g_k^*(\xi) \overline{g_j^*(\xi)})(a_j(\xi)) \cdot ((Q_j^{-1}M_{j-1}(g_{j-1}^*))Q_j) \right| \\ &\leq \sum_{j=0}^k \varepsilon_j a_j(\xi) Q_j. \end{aligned}$$

Consequently

$$\left| S_{x_k}(g_k^*) - \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} a_j(\xi) M_{j-1}(g_{j-1}^*) \right| = o(x_k), \quad (k \rightarrow +\infty).$$

Since ξ belongs to E_∞ , $\{g_k^*(\xi)\}_k$ converges, and as a consequence, the sequence $\eta_{j,k} = |g_k^*(\xi) \overline{g_j^*(\xi)} - 1|$ tends to 0 when k and j , $j \leq k$, tend to infinity independently.

This implies

$$\left| \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} a_j(\xi) M_{j-1}(g_{j-1}^*) - \sum_{j=0}^k a_j(\xi) M_{j-1}(g_{j-1}^*) \right| \leq \sum_{j=0}^k \eta_{j,k} a_j(\xi) Q_j,$$

and so, when $k \rightarrow +\infty$,

$$\left| \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} a_j(\xi) M_{j-1}(g_{j-1}^*) - \sum_{j=0}^k a_j(\xi) M_{j-1}(g_{j-1}^*) \right| = o(x_k).$$

Moreover, $Q_j^{-1}M_{j-1}(g_{j-1}^*)$ tends to a limit, say $\widetilde{M}_\infty(g_\infty^*)$. Hence

$$\begin{aligned} & \left| \sum_{j=0}^k a_j(\xi) \cdot M_{j-1}(g_{j-1}^*) - \widetilde{M}_\infty(g_\infty^*) \sum_{j=0}^k a_j(\xi) Q_j \right| \\ &\leq \sum_{j=0}^k |Q_j^{-1}M_{j-1}(g_{j-1}^*) - \widetilde{M}_\infty(g_\infty^*)| \cdot a_j(\xi) Q_j = o(x_k), \quad (k \rightarrow +\infty). \end{aligned}$$

Finally

$$\begin{aligned}
& \left| S_{x_k}(g_k^*) - \widetilde{M}_\infty(g_\infty^*) \sum_{j=0}^k a_j(\xi) Q_j \right| \\
& \leq \left| S_{x_k}(g_k^*) - \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} \cdot a_j(\xi) \cdot M_{j-1}(g_{j-1}^*) \right| \\
& + \left| \sum_{j=0}^k g_k^*(\xi) \overline{g_j^*(\xi)} \cdot a_j(\xi) \cdot M_{j-1}(g_{j-1}^*) - \sum_{j=0}^k a_j(\xi) \cdot M_{j-1}(g_{j-1}^*) \right| \\
& + \left| \sum_{j=0}^k a_j(\xi) \cdot M_{j-1}(g_{j-1}^*) - \widetilde{M}_\infty(g_\infty^*) \sum_{j=0}^k a_j(\xi) Q_j \right|.
\end{aligned}$$

It then follows that

$$S_{x_k}(g_k^*) = \widetilde{M}_\infty(g_\infty^*) \cdot x_k + o(x_k), (k \rightarrow +\infty).$$

To obtain the result, it is enough to notice that from $Q_{k+1}^{-1} M_k(g_k^*) - \widetilde{M}_\infty(g_\infty^*) = o(1)$ we obtain $S_{x_k}(g_k^*) = Q_{k+1}^{-1} M_k(g_k^*) \cdot x_k + o(x_k)$, and replacing g_k^* by its value, we get

$$S_{x_k}(g_k^*) = S_{x_k}(g) \prod_{j=0}^k \alpha_j, \quad M_k(g_k^*) = M_k(g) \prod_{j=0}^k \alpha_j.$$

and this leads to $S_{x_k}(g) - (M_k(g) Q_{k+1}^{-1}) \cdot x_k = o(x_k)$.

REFERENCES

- [1] G. Barat, *Echelles de numération et fonctions arithmétiques associées*. Thèse de doctorat, Université de Provence, Marseille, 1995.
- [2] J. Coquet, *Sur les fonctions S-multiplicatives et S-additives*. Thèse de doctorat de Troisième Cycle, Université Paris-Sud, Orsay, 1975.
- [3] H. Delange, *Sur les fonctions q-additives ou q-multiplicatives*. Acta Arithmetica **21** (1972), 285–298.
- [4] A.O. Gelfond, *Sur les nombres qui ont des propriétés additives ou multiplicatives données*. Acta Arithmetica **13** (1968), 259–265.
- [5] E. Hewitt, K.A. Ross, *Abstract harmonic analysis*. Springer-Verlag, 1963.
- [6] E. Manstavičius, *Probabilistic theory of additive functions related to systems of numeration*. New trends in Probability and Statistics Vol.4 (1997), VSP BV & TEV, 412–429.
- [7] J.-L. Maucclair, *Sur la répartition des fonctions q-additives*. J. Théorie des Nombres de Bordeaux **5** (1993), 79–91.

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