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Makoto KAWASHIMA

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Hermite's approach to Abelian integrals revisited

par Makoto KAWASHIMA

Dedicated to the Memory of Professor Marc Huttner

Résumé. Dans cet article, nous établissons un nouveau critère d'indépendance linéaire pour les valeurs de certaines *séries hypergéométriques de Lauricella F_D* à paramètres rationnels, aussi bien dans le cadre complexe que p-adique, sur un corps de nombres algébriques. Ce résultat généralise un théorème de C. Hermite [14] sur l'indépendance linéaire de certaines intégrales abéliennes. Notre démonstration repose sur des approximations de type Padé explicites pour les solutions d'une équation différentielle de Jordan–Pochhammer réductible, ce qui étend les approximations de Padé pour certaines intégrales abéliennes de [14]. La principale nouveauté de notre approche réside dans la preuve de la non-annulation des déterminants associés à ces approximants de type Padé.

Abstract. In this article, we establish a new linear independence criterion for the values of certain *Lauricella hypergeometric series F_D* with rational parameters, in both the complex and p-adic settings, over an algebraic number field. This result generalizes a theorem of C. Hermite [14] on the linear independence of certain Abelian integrals. Our proof relies on explicit Padé-type approximations to solutions of a reducible Jordan–Pochhammer differential equation, which extends the Padé approximations for certain Abelian integrals in [14]. The main novelty of our approach lies in the proof of the non-vanishing of the determinants associated with these Padé-type approximants.

1. Introduction

In this paper we extend Hermite's construction of Padé approximants for Abelian integrals to a broad class of Lauricella hypergeometric series, and derive an explicit criterion (with an effective measure) for the linear independence of their values over algebraic number fields - simultaneously in the complex and p-adic contexts.

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Padé approximation, originating in the works of C. Hermite in his study of the transcendence of e [13] and H. Padé [22, 23], has long played a central role in Diophantine approximation and transcendental number theory. In arithmetic applications, one typically constructs explicit systems of Padé approximations to certain functions, often by linear algebraic arguments combined with bounds derived from Siegel's lemma via Dirichlet's box principle. However, these general constructions are not always sufficient for arithmetic purposes, such as establishing linear independence results. In such cases, it becomes essential to construct explicit Padé approximations providing analytic estimates sharp enough for the intended application–a task that can usually be carried out only for special classes of functions.

Over the complex field, Hermite [16] established a criterion for the $\mathbb{Q}$-linear independence of the values of certain Abelian integrals associated with the Jordan–Pochhammer differential equation, by explicitly constructing Padé approximations of such integrals. In his work, all parameters s_i determining the exponents of the Jordan–Pochhammer equation (see equation (1.2)) are assumed to be of the form $1/k$ for positive integers k. Following Hermite's approach, G. Rhin and P. Toffin [26] proved a linear independence criterion for distinct logarithmic values over imaginary quadratic fields by constructing explicit Padé-type approximants of logarithmic functions, corresponding to the case $s_i = 0$. Later, M. Huttner [16] refined Hermite's method and, through a detailed analysis of Hermite's approximants, obtained sharper measures of linear independence than those of Hermite's original results (see also [15]). In [18], the present author established a linear independence criterion for the values of Gauss' hypergeometric functions with varying parameters, both in the complex and in the p-adic settings, for a special case where all $s_i = -1/m$ with m the number of parameters. Related Diophantine properties of the values of solutions to Jordan–Pochhammer equations were investigated by G. V. and D. V. Chudnovsky in [7, Section 2].

In the present work, we extend Hermite's criterion to the case where the parameters s_i are arbitrary rational numbers whose sum does not lie in $\mathbb{Z}_{<-1}$. This leads to a new linear independence criterion for the corresponding Abelian integrals over algebraic number fields, valid simultaneously in the complex and p-adic settings. Our approach relies on a detailed study of Padé-type approximants (beyond the classical Padé approximants) and their algebraic and analytic structures, following the framework developed in [18].

To investigate the arithmetic nature of the values of holonomic Laurent series, it is crucial to construct their Padé-type approximants and to elucidate their analytic and algebraic properties. In [18], explicit constructions were given for holonomic series on which a first-order differential operator with polynomial coefficients acts so that the resulting function becomes a polynomial. In this paper, we establish more general sufficient conditions ensuring the non-vanishing of determinants associated with such Padé-type approximants. This analysis is motivated by the goal of extending the method to broader classes of holonomic Laurent series and by the desire to understand the arithmetic implications of non-vanishing–a property that plays a central role in Siegel's method [27].

A key technical ingredient is the introduction of the *formal f-integration map* φ_f (see equation (2.1)), which enables the explicit–yet purely formal–construction of Padé-type approximants and the proof of their principal properties. This concept appears in various forms in earlier works of S. David, N. Hirata-Kohno, and the author [8, 9, 10, 11], as well as in [19, 20] by A. Poëls and the author.

To apply Siegel's method [27] in proving our main theorem, it is essential to establish the non-vanishing of the determinant formed by these Padé-type approximants. In previous works such as [8, 9, 10, 18, 19], this step required explicit computation of the determinant, which is often a intricate and technically demanding task. Here, we develop a new approach based on the study of the kernel of the formal f-integration map, as introduced in [20] and [11]. This method allows us to prove the non-vanishing property in a simpler and more conceptual way, without computing the explicit determinant values. We emphasize that the non-vanishing condition is governed entirely by the first-order differential operators involved–specifically, by the coefficients of operator (see Theorem 3.1).

1.1. Main result. Our main result concerns a linear independence criterion for the values of the Lauricella hypergeometric series over an algebraic number field, in both the complex and p-adic settings. We begin by recalling the definition of the m-variable Lauricella hypergeometric series with parameters $\alpha, \beta_1, \dots, \beta_m, \gamma$, defined by

$$F_D^{(m)}(\alpha, \beta_1, \dots, \beta_m, \gamma; z_1, \dots, z_m) = \sum_{k_1,\dots,k_m=0}^{\infty} \frac{(\alpha)_{k_1+\dots+k_m} (\beta_1)_{k_1} \dots (\beta_m)_{k_m}}{(\gamma)_{k_1+\dots+k_m}\, k_1! \dots k_m!}\, z_1^{k_1} \dots z_m^{k_m},$$

where $(a)_k$ denotes the Pochhammer symbol, given by $(a)_0 = 1$ and $(a)_k = a(a+1)\dots(a+k-1)$ for $k \geq 1$.

To state our main result, we first fix notation. Let K be an algebraic number field and denote by $\mathfrak{M}_K$ the set of places of K, and for $v \in \mathfrak{M}_K$, let K_v denote the completion of K at v. We normalize the absolute value $|\cdot|_v$ by

$$|p|_v = p^{-\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} \quad \text{if } v \mid p, \qquad |x|_v = |\iota_v(x)|^{\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}} \quad \text{if } v \mid \infty,$$

where p is a rational prime and $\iota_v : K \hookrightarrow \mathbb{C}$ the embedding associated to v. On K_v^n, the norm $|\cdot|_v$ is taken to be the supremum norm.

For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in K^m$, we define the logarithmic v-adic height and the global logarithmic height by

$$\begin{aligned} \mathrm{h}_v(\boldsymbol{\beta}) &= \log \max\{1, |\beta_1|_v, \ldots, |\beta_m|_v\}, \\ \mathrm{h}(\boldsymbol{\beta}) &= \sum_{v \in \mathfrak{M}_K} \mathrm{h}_v(\boldsymbol{\beta}). \end{aligned}$$

We also define the denominator of $\boldsymbol{\beta}$ as

$$\mathrm{den}(\boldsymbol{\beta}) = \min\left\{ n \in \mathbb{Z} \,\middle|\, \begin{array}{l} n > 0 \text{ such that } n\beta_i \text{ are} \\ \text{algebraic integers for all } 1 \le i \le m \end{array} \right\}.$$

For a rational number α, we put

$$\mu(\alpha) = \mathrm{den}(\alpha) \prod_{\substack{q:\,\text{prime} \\ q \mid \mathrm{den}(\alpha)}} q^{\frac{1}{q-1}}.$$

Now we are ready to state our main result. Let $m \ge 2$ be a fixed positive integer. Fix an algebraic number field K. Consider polynomials $a(z), b(z) \in K[z]$, where $a(z)$ is assumed to be monic of degree m and $b(z)$ satisfies $\deg b \le m-1$. We denote the derivative of $a(z)$ by $a'(z)$, and by b_{m-1} the coefficient of z^{m-1} in $b(z)$. Note that b_{m-1} can be 0. Assume that $a(z)$ decomposes over K, and let $\alpha_1, \ldots, \alpha_m \in K$ denote its roots, counted with multiplicity. We impose the following conditions:

$$\alpha_1, \ldots, \alpha_m \text{ are pairwise distinct,} \tag{1.1}$$

$$s_i := \frac{b(\alpha_i)}{a'(\alpha_i)} \in \mathbb{Q} \setminus \mathbb{Z}_{\le -1} \quad \text{for all } 1 \le i \le m, \tag{1.2}$$

$$b_{m-1} \notin \mathbb{Z}_{<-1}. \tag{1.3}$$

For $v \in \mathfrak{M}_K$ and $\beta \in K \setminus \{0\}$, we introduce the quantity

$$\begin{aligned} V_v(\beta) = m\,\mathrm{h}_v(\beta) - (m-1)\,\mathrm{h}(\beta) - m\left(\mathrm{h}(\boldsymbol{\alpha}) + \sum_{i=1}^m \log \mu(s_i) + \log 4\right) \\ - \sum_{i=1}^m \mathrm{h}(\alpha_i) - (m-1)\frac{\mathrm{den}(b_{m-1})}{\varphi(\mathrm{den}(b_{m-1}))} \sum_{\substack{1 \le j \le \mathrm{den}(b_{m-1}) \\ (j, \mathrm{den}(b_{m-1}))=1}} \frac{1}{j}, \end{aligned} \tag{1.4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in K^m$ and φ denotes Euler's totient function.

Theorem 1.1. *Retain the above notation and assumptions* (1.1), (1.2) *and* (1.3). *For $0 \leq j \leq m-2$, we define formal Laurent series $f_j(z)$ by*

$$\prod_{i=1}^{m}\left(1-\frac{\alpha_i}{z}\right)^{s_i}\cdot\frac{1}{z^{j+1}}\cdot F_D^{(m)}\Bigg(b_{m-1}+j+1,\,1+s_1,\dots,1+s_m,\\ b_{m-1}+j+2;\,\frac{\alpha_1}{z},\dots,\frac{\alpha_m}{z}\Bigg).$$

Let $v_0 \in \mathfrak{M}_K$. We assume $V_{v_0}(\beta) > 0$[1]*. Then each series $f_j(z)$ converges at $z=\beta$ in K_{v_0} and the m elements of K_{v_0} :*

$$1, f_0(\beta), f_1(\beta), \dots, f_{m-2}(\beta),$$

are linearly independent over K.

We will give a proof of Theorem 1.1 together with a linear independence measure (see Theorem 5.1).

Remark 1.2. We denote L by the differential operator of order 1 with polynomial coefficients $-a(z)\frac{\mathrm{d}}{\mathrm{d}z}+b(z)$. A result due to S. Fischler and T. Rivoal [12, Proposition 3 (ii)] shows that L is a G-operator (see the definition of G-operator [1, IV]) under the assumptions (1.1) and (1.2). We observe that the Laurent series f_j satisfies $L\cdot f_j \in K[z]$ with degree $m-j-2$ (see Lemma 4.1). In particular, f_j is a solution of a reducible Jordan–Pochhammer equation $\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{m-1}L$ (see Example A.1). Combining these results, by a well-known theorem of Y. André, G. V. & D. V. Chudnovsky and N. Katz (refer [2, Théorème 3.5]), we conclude f_j is a G-function in the sense of C. F. Siegel [27].

Outline of the article. Section 2 is based on the results given in [18]. We begin by introducing the Padé-type approximants of Laurent series. In Subsection 2.1, we introduce the *formal f-integration map* associated with a Laurent series f, which plays a central role throughout this paper, and we describe its fundamental properties in case of f is holonomic. Subsection 2.3 provides an overview of Padé-type approximants and Padé-type approximation for Laurent series that become polynomials under the action of a first-order differential operator with polynomial coefficients. Section 3 formulates, in terms of the coefficients of the differential operator, sufficient conditions ensuring the non-vanishing of the determinant formed by the Padé-type approximants constructed in Section 2 (see Theorem 3.1). This part constitutes the main novel contribution of the present

[1] The positivity condition $V_{v_0}(\beta) > 0$ roughly means that β is arithmetically large enough at v_0 compared with the heights of the parameters. This guarantees both the convergence of the involved Laurent series at $z=\beta$ and the arithmetic control required in the application of Siegel's method.

work. In Section 4, we treat the case where the differential operator considered in Section 3 is a G-operator. We give explicit expressions for the corresponding Laurent series and establish estimates for their Padé-type approximants and Padé approximations with respect to both Archimedean and non-Archimedean valuations. Section 5 is devoted to the proof of our main theorem, together with a quantitative measure of linear independence. Finally, Section A serves as an appendix, summarizing some basic facts on the Jordan–Pochhammer equation.

2. Padé-type approximants of Laurent series

Throughout this section, we fix a field K of characteristic 0. We denote the formal power series ring in the variable $1/z$ with coefficients K by $K[\![1/z]\!]$ and the field of fractions by $K((1/z))$. We say that an element of $K((1/z))$ is a formal Laurent series. We define the order function at $z=\infty$ by

$$\mathrm{ord}_\infty : K((1/z)) \longrightarrow \mathbb{Z}\cup\{\infty\};\quad \sum_k \frac{a_k}{z^k} \longmapsto \min\{k\in\mathbb{Z}\cup\{\infty\} \mid a_k\neq 0\}.$$

Note that, for $f\in K((1/z))$, $\mathrm{ord}_\infty f=\infty$ if and only if $f=0$. We recall without proof the following elementary fact:

Lemma 2.1. *Let m be a nonnegative integer, $f_1(z),\dots,f_m(z)\in(1/z)\cdot K[\![1/z]\!]$ and $\boldsymbol{n}=(n_1,\dots,n_m)\in\mathbb{N}^m$. Put $N=\sum_{j=1}^m n_j$. For a nonnegative integer M with $M\geq N$, there exist polynomials $(P,Q_1,\dots,Q_m)\in K[z]^{m+1}\setminus\{\mathbf{0}\}$ satisfying the following conditions:*

(i) $\deg P\leq M$,
(ii) $\mathrm{ord}_\infty(P(z)f_j(z)-Q_j(z))\geq n_j+1$ *for* $1\leq j\leq m$.

Definition 2.2. We say that a vector of polynomials $(P,Q_1,\dots,Q_m)\in K[z]^{m+1}$ satisfying properties (i) and (ii) is a weight $\boldsymbol{n}$, degree M Padé-type approximant of $(f_1,\dots,f_m)$. For such approximants $(P,Q_1,\dots,Q_m)$ of $(f_1,\dots,f_m)$, we call the formal Laurent series $(P(z)f_j(z)-Q_j(z))_{1\leq j\leq m}$, that is to say remainders, as weight $\boldsymbol{n}$ degree M Padé-type approximations of $(f_1,\dots,f_m)$.

2.1. Formal $\boldsymbol{f}$-integration map. Let $f(z)=\sum_{k=0}^\infty f_k/z^{k+1}\in(1/z)\cdot K[\![1/z]\!]$. We define a K-linear map $\varphi_f\in\mathrm{Hom}_K(K[t],K)$ by

$$\varphi_f : K[t]\longrightarrow K;\quad t^k\longmapsto f_k\quad (k\geq 0). \tag{2.1}$$

The above linear map extends naturally to a $K[z]$-linear map $\varphi_f : K[z,t]\to K[z]$, and then to a $K[z][\![1/z]\!]$-linear map $\varphi_f : K[z,t][\![1/z]\!]\to K[z][\![1/z]\!]$. With this notation, the formal Laurent series $f(z)$ satisfies the following crucial identities (see [21, (6.2) p. 60 and (5.7) p. 52]):

$$f(z) = \varphi_f\left(\frac{1}{z-t}\right),$$

$$P(z)f(z) - \varphi_f\left(\frac{P(z)-P(t)}{z-t}\right) \in (1/z)\cdot K[\![1/z]\!] \quad \text{for any } P(z)\in K[z].$$

We recall a criterion, based on the morphism φ_f, for determining when the given polynomials form Padé approximants.

Lemma 2.3 (cf. [18, Lemma 2.3])**.** *Let m, M be positive integers, $f_1(z),\dots,f_m(z) \in (1/z)\cdot K[\![1/z]\!]$. Let $\boldsymbol{n} = (n_1,\dots,n_m) \in \mathbb{N}^m$ with $\sum_{j=1}^m n_j \le M$. Let $P(z)\in K[z]$ be a non-zero polynomial with $\deg P \le M$, and put $Q_j(z) = \varphi_{f_j}((P(z)-P(t))/(z-t)) \in K[z]$ for $1\le j\le m$. The following assertions are equivalent.*

(i) *The vector of polynomials $(P, Q_1,\dots,Q_m)$ is a weight $\boldsymbol{n}$ Padé-type approximant of $(f_1,\dots,f_m)$.*
(ii) *We have $t^kP(t)\in\ker\varphi_{f_j}$ for any pair of integers (j,k) with $1\le j\le m$ and $0\le k\le n_j-1$.*

2.2. Holonomic series and the kernel of $\boldsymbol{f}$-integration map. Lemma 2.3 implies that the study of the kernel of the formal f-integration map is essential for constructing Padé approximants of Laurent series. This aspect has been studied in [18] for holonomic Laurent series. In this subsection we recall a result from [18, Corollary 2.6].

We denote the action of a differential operator L on a function f, such as a polynomial or a Laurent series, by $L\cdot f$. Consider the map

$$\iota : K(z)[\tfrac{\mathrm{d}}{\mathrm{d}z}] \longrightarrow K(t)[\tfrac{\mathrm{d}}{\mathrm{d}t}];\quad \sum_j P_j(z)\tfrac{\mathrm{d}^j}{\mathrm{d}z^j} \longmapsto \sum_j (-1)^j \tfrac{\mathrm{d}^j}{\mathrm{d}t^j}P_j(t). \tag{2.2}$$

Note that for $L\in K(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$, $\iota(L)$ is called the *formal adjoint* of L and relates to the dual of differential module $K(z)[\frac{\mathrm{d}}{\mathrm{d}z}]/K(z)[\frac{\mathrm{d}}{\mathrm{d}z}]L$ (see [1, III Exercises 3)]). For $L\in K(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$, we denote $\iota(L)$ by L^*. Notice that we have $(L_1L_2)^* = L_2^*L_1^*$ for any $L_1, L_2\in K(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$.

Proposition 2.4 ([18, Corollary 2.6])**.** *Let $f(z)\in(1/z)\cdot K[\![1/z]\!]$ and $L\in K[z,\frac{\mathrm{d}}{\mathrm{d}z}]$. Then the following are equivalent*:

(i) *$L\cdot f(z)\in K[z]$.*
(ii) *$L^*\cdot K[t]\subseteq \ker\varphi_f$.*

2.3. Order 1 differential operator and Padé-type approximants. Let K be a field of characteristic 0 and $a(z), b(z)\in K[z]$. Put $\deg a = u, \deg b = v$, $w=\max\{u-2, v-1\}$ and

$$a(z)=\sum_{i=0}^u a_iz^i,\quad b(z)=\sum_{j=0}^v b_jz^j. \tag{2.3}$$

We now assume

$$w \geq 0, \tag{2.4}$$

$$a_u(k+u)+b_v \neq 0 \quad \text{for all } k \geq 0 \text{ if } u-1=v. \tag{2.5}$$

Define the order 1 differential operator with polynomial coefficients

$$L = -a(z)\frac{\mathrm{d}}{\mathrm{d}z} + b(z) \in K\Big[z, \tfrac{\mathrm{d}}{\mathrm{d}z}\Big].$$

We consider the Laurent series $f(z) \in (1/z)\cdot K[\![1/z]\!]$ such that $L\cdot f(z) \in K[z]$ and construct the Padé approximation of $f(z)$. The following lemma is proven in [18, Lemma 4.1]. For the sake of completion, we recall the proof in the present article.

Lemma 2.5. *We keep the notation above. Then there exist* $f_0(z),\dots,f_w(z) \in (1/z)\cdot K[\![1/z]\!]$ *that are linearly independent over* K *and satisfy* $L\cdot f_j(z) \in K[z]$ *with at most degree* w *for* $0 \leq j \leq w$.

Proof. Let $f(z) = \sum_{k=0}^{\infty} f_k/z^{k+1} \in (1/z)\cdot K[\![1/z]\!]$ be a Laurent series. There exists a polynomial $A(z) \in K[z]$ that depends on the operator L and f with $\deg A \leq w$ and satisfying

$$L\cdot f(z) = A(z) + \sum_{k=0}^{\infty} \frac{\sum_{i=0}^{u} a_i(k+i)f_{k+i-1} + \sum_{j=0}^{v} b_j f_{k+j}}{z^{k+1}}. \tag{2.6}$$

Put

$$\sum_{i=0}^{u} a_i(k+i)f_{k+i-1} + \sum_{j=0}^{v} b_j f_{k+j} = c_{k,0}f_{k-1} + \cdots + c_{k,w}f_{k+w} + c_{k,w+1}f_{k+w+1}$$

for $k \geq 0$ with $c_{0,0} = 0$. We note that $c_{k,l}$ depends only on $a(z), b(z)$. Notice that $c_{k,w+1}$ is $a_u(k+u)$ if $u-2 > v-1$, b_v if $u-2 < v-1$ and $a_u(k+u)+b_v$ if $u-2 = v-1$. Then the assumption (2.5) ensures that $\min\{k \geq 0 \mid c_{k',w+1} \neq 0 \text{ for all } k' \geq k\} = 0$ and thus the K-linear map:

$$\begin{aligned} K^{w+1} &\longrightarrow \{f \in (1/z)\cdot K[\![1/z]\!] \mid L\cdot f \in K[z]\}; \\ (f_0,\dots,f_w) &\longmapsto \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}}, \end{aligned} \tag{2.7}$$

where, for $k \geq w+1$, f_k is determined inductively by

$$\sum_{i=0}^{u} a_i(k+i)f_{k+i-1} + \sum_{j=0}^{v} b_j f_{k+j} = 0 \quad \text{for} \quad k \geq 0 \tag{2.8}$$

is an isomorphism. This completes the proof of Lemma 2.5. $\square$

Let us fix Laurent series $f_0(z),\dots,f_w(z)\in(1/z)\cdot K[\![1/z]\!]$ such that these series are linearly independent over K and $L\cdot f_j(z)\in K[z]$ for $0\le j\le w$. We denote the formal integration associated to f_j by $\varphi_{f_j}=\varphi_j$. For a nonnegative integer n, we denote the n-th Rodriges operator associated with L (cf. [4, Equation (2.5)] and [18, Definition 3.1]) by

$$R_n=\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}+\frac{b(z)}{a(z)}\right)^n a(z)^n.$$

The differential operator R_n can be decomposed as follows:

Lemma 2.6 ([18, Lemma 4.3])**.** *Let $w(z)$ be a non-zero solution of the differential operator L in a differential extension $\mathcal{K}$ of $K(z)$. In the ring $\mathcal{K}[\frac{\mathrm{d}}{\mathrm{d}z}]$, we have the equalities:*

$$R_n=\frac{1}{n!}w(z)^{-1}\frac{\mathrm{d}^n}{\mathrm{d}z^n}w(z)a(z)^n=\frac{1}{n!}R_1(R_1+a'(z))\dots(R_1+(n-1)a'(z)).$$

Making use of the Rodrigues operator associated with L, we construct explicit Padé approximants of the family $(f_j)_j$. The following theorem is a particular case of [18, Theorem 4.2, Lemma 5.1].

Theorem 2.7. *For a nonnegative integer ℓ, define polynomials*

$$P_{n,\ell}(z)=R_n\cdot z^\ell,\quad Q_{n,j,\ell}(z)=\varphi_j\left(\frac{P_{n,\ell}(z)-P_{n,\ell}(t)}{z-t}\right)\ \textit{for}\ 0\le j\le w. \tag{2.9}$$

Then the following properties hold.

(i) *The vector of polynomials $(P_{n,\ell}(z),Q_{n,j,\ell}(z))$ forms a weight n Padé-type approximant for $f_0(z),\dots,f_w(z)$.*

(ii) *Put the Padé-type approximation of $(f_j)_j$ by*

$$\mathfrak{R}_{n,j,\ell}(z)=P_{n,\ell}(z)f_j(z)-Q_{n,j,\ell}(z)\quad\textit{for}\quad 0\le j\le w. \tag{2.10}$$

We have

$$\mathfrak{R}_{n,j,\ell}(z)=(-1)^n\sum_{k=n}^{\infty}\binom{k}{n}\frac{\varphi_j(t^{k+\ell-n}a(t)^n)}{z^{k+1}}.$$

Proof.

(i). The statement is derived by applying [18, Theorem 4.2] in the special case where $d=1$ and $D_1=L$.

(ii). By definition of $\mathfrak{R}_{n,j,\ell}$, we see $\mathfrak{R}_{n,j,\ell}(z)=\varphi_j(P_{n,\ell}(t)/(z-t))$. Substituting the expansion $1/(z-t)=\sum_{k=0}^{\infty}t^k/z^{k+1}$ into this expression, we obtain

$$\mathfrak{R}_{n,j,\ell}(z)=\sum_{k=n}^{\infty}\frac{\varphi_j(t^kP_{n,\ell}(t))}{z^{k+1}}.$$

The above equality implies it is sufficient to prove

$$\varphi_j(t^k P_{n,\ell}(t)) = (-1)^n \binom{k}{n} \varphi_j(t^{\ell+k-n} a(t)^n) \quad \text{for} \quad k \geq n. \tag{2.11}$$

Now we fix an integer $k \geq n$. To prove equation (2.11), we define the differential operator $\mathcal{E}_{a,b} = \frac{\mathrm{d}}{\mathrm{d}t} + b(t)/a(t) \in K(t)[\frac{\mathrm{d}}{\mathrm{d}t}]$. Notice $L^* = \mathcal{E}_{a,b} a(t)$. By [18, Proposition 3.2 (i)], there exists a set of integers $\{c_{n,k,i};\ 0 \leq i \leq n\}$ such that $c_{n,k,n} = (-1)^n k(k-1)\dots(k-n+1)$ and

$$t^k \mathcal{E}_{a,b}^n = \sum_{i=0}^{n} c_{n,k,i} \mathcal{E}_{a,b}^{n-i} t^{k-i}.$$

This implies

$$t^k P_{n,\ell}(t) = \frac{t^k}{n!} \mathcal{E}_{a,b}^n a(t)^n \cdot t^\ell = \sum_{i=0}^{n} \frac{c_{n,k,i}}{n!} \mathcal{E}_{a,b}^{n-i} t^{k-i} a(t)^n \cdot t^\ell. \tag{2.12}$$

From [18, Proposition 3.2 (ii)], we know

$$\mathcal{E}_{a,b}^{n-i} t^{k-i} a(t)^n \cdot t^\ell \in L^* \cdot K[t] \quad \text{for} \quad 0 \leq i \leq n-1. \tag{2.13}$$

Combining Equations (2.12), (2.13) together with $L^* \cdot K[t] \subseteq \ker \varphi_j$ (refer Proposition 2.4) deduces

$$\varphi_j(t^k P_{n,\ell}(t)) = \varphi_j\left(\frac{c_{n,k,n}}{n!} t^{k+\ell-n} a(t)^n\right) = (-1)^n \binom{k}{n} \varphi_j(t^{k+\ell-n} a(t)^n).$$

This establishes equation (2.11) and completes the proof of (ii). □

Let n be a nonnegative integer. Denote the determinant of the matrix formed by the Padé approximants of $(f_j)_j$ in Theorem 2.7 by

$$\Delta_n(z) = \det \begin{pmatrix} P_{n,0}(z) & P_{n,1}(z) & \dots & P_{n,w+1}(z) \\ Q_{n,0,0}(z) & Q_{n,0,1}(z) & \dots & Q_{n,0,w+1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n,w,0}(z) & Q_{n,w,1}(z) & \dots & Q_{n,w,w+1}(z) \end{pmatrix}.$$

In the next section, we consider the non-vanishing of $\Delta_n(z)$.

3. Linear independence of Padé-type approximants

In this section, we keep the notation in Subsection 2.3. Recall the polynomials $a(z)$, $b(z) \in K[z]$ defined in equation (2.3) which satisfy (2.4) and (2.5). Recall $w = \max\{\deg a - 2, \deg b - 1\}$. This section is devoted to prove the next theorem.

Theorem 3.1. *For the polynomials $a(z), b(z)$, we assume*

$$na'(z) + b(z) \text{ and } a(z) \text{ are coprime for any positive integer } n, \tag{3.1}$$

$$a_u((k+1)u + n) + b_v \neq 0 \quad \text{for any} \quad k, n \geq 0 \quad \text{if} \quad u - 1 = v. \tag{3.2}$$

Then $\Delta_n(z) \in K \setminus \{0\}$ for any $n \geq 0$.

Our strategy of the proof of Theorem 3.1 is the following. By definition $\Delta_n(z)$ is a polynomial. We show in Proposition 3.3, which is essentially an application of [9, Lemma 4.2 (ii)], that this polynomial is a constant, and we reduce the problem to showing that another determinant $\det \mathcal{M}_n$ is non-zero. This last property, established in Subsection 3.2, will imply Theorem 3.1. In order to prove above results, we prepare the following lemma.

Lemma 3.2. *Denote the differential operator $-a(z)\frac{\mathrm{d}}{\mathrm{d}z} + b(z)$ by L. Let $k \in \mathbb{N}$ and $P(t) \in K[t] \setminus \{0\}$. For the polynomials $a(z), b(z)$, we assume equation* (3.2) *holds. Then we have*

$$\deg(L^* + ka'(t)) \cdot P(t) = \deg P + w + 1.$$

Proof. We may assume $P(t) = t^N$ for a nonnegative integer N. Put

$$\mathcal{P}(t) = \big(L^* + ka'(t)\big) \cdot t^N.$$

Then a direct computation yields that $\deg \mathcal{P} \leq N + w + 1$ and the coefficient of t^{N+w+1} of $\mathcal{P}$ is $a_u(u(k+1) + N) + b_v$ if $u - 1 = v$, b_v if $v > u - 1$ and $a_u((k+1)u + N)$ if $u - 1 > v$. The assumption (3.2) ensures the assertion. This completes the proof of the statement. $\square$

Let n be a nonnegative integer. Recall the polynomials $P_{n,\ell}(z), Q_{n,j,\ell}(z)$ defined in equation (2.9). Combining Lemmas 2.6 and 3.2 yields

$$\deg P_{n,\ell} = \ell + n(w+1). \tag{3.3}$$

Put the following $w+1$ by $w+1$ matrix by

$$\mathcal{M}_n = \begin{pmatrix} \varphi_0(a(t)^n) & \dots & \varphi_0(t^w a(t)^n) \\ \vdots & \ddots & \vdots \\ \varphi_w(a(t)^n) & \dots & \varphi_w(t^w a(t)^n) \end{pmatrix} \in \mathrm{Mat}_{w+1}(K).$$

We now prove $\Delta_n(z)$ is a constant.

Proposition 3.3. *There exists a constant $c \in K$ such that c is non-zero under the assumption* (3.2) *and*

$$\Delta_n(z) = c \cdot \det \mathcal{M}_n.$$

In particular, we have $\Delta_n(z) \in K$.

Proof. Note that this proposition is a particular case of [18, Proposition 5.2]. For the matrix in the definition of $\Delta_n(z)$, adding $-f_j(z)$ times first row to $j+1$ th row for each $0 \leq j \leq w$,

$$\Delta_n(z) = (-1)^w \det \begin{pmatrix} P_{n,0}(z) & \dots & P_{n,w+1}(z) \\ \mathfrak{R}_{n,0,0}(z) & \dots & \mathfrak{R}_{n,0,w+1}(z) \\ \vdots & \ddots & \vdots \\ \mathfrak{R}_{n,w,0}(z) & \dots & \mathfrak{R}_{n,w,w+1}(z) \end{pmatrix}.$$

We denote the (s,t)th cofactor of the matrix in the right hand side of above equality by $\Delta_{s,t}(z)$. Then we have, developing along the first row

$$\Delta_n(z) = (-1)^w \left(\sum_{l=0}^{w} P_{n,\ell}(z) \Delta_{1,\ell+1}(z) \right). \tag{3.4}$$

The property of the Padé approximation $\mathrm{ord}_\infty \, \mathfrak{R}_{n,j,\ell}(z) \geq n+1$ for $0 \leq j \leq w$, $0 \leq \ell \leq w+1$ implies

$$\mathrm{ord}_\infty \, \Delta_{1,\ell+1}(z) \geq (n+1)(w+1) \quad \text{for} \quad 0 \leq \ell \leq w.$$

Combining equation (3.3) and above inequality yields

$$P_{n,\ell}(z)\Delta_{1,\ell+1}(z) \in (1/z) \cdot K[\![1/z]\!] \quad \text{for} \quad 0 \leq \ell \leq w,$$

and

$$P_{n,w+1}(z)\Delta_{1,w+1}(z) \in K[\![1/z]\!].$$

Note that in above relation, the constant term of $P_w(z)\Delta_{1,w+1}(z)$ is

$$\begin{aligned} &\text{“Coefficient of } z^{(n+1)(w+1)} \text{ of } P_{n,w+1}(z)\text{”} \\ &\qquad \cdot \text{“Coefficient of } 1/z^{(n+1)(w+1)} \text{ of } \Delta_{1,w+1}(z)\text{”}. \end{aligned} \tag{3.5}$$

Notice that the coefficient of $z^{(n+1)(w+1)}$ of $P_{n,w+1}(z)$ is non-zero under the assumption (3.2) by Lemma 3.2. equation (3.4) implies $\Delta_n(z)$ is a polynomial in z with non-positive valuation with respect to ord_∞. Thus, it has to be a constant. Finally, by Theorem 2.7 (ii), the coefficient of $1/z^{(n+1)(w+1)}$ of $\Delta_{1,w+1}(z)$ is

$$\det \begin{pmatrix} (-1)^n \varphi_0(a(t)^n) & \dots & (-1)^n \varphi_0(t^w a(t)^n) \\ \vdots & \ddots & \vdots \\ (-1)^n \varphi_w(a(t)^n) & \dots & (-1)^n \varphi_w(t^w a(t)^n) \end{pmatrix} = (-1)^{wn} \cdot \det \mathcal{M}_n.$$

Combining Equations (3.4), (3.5) and above equality yields the assertion. This completes the proof of Proposition 3.3. $\square$

3.1. Study of kernels of φ_j. Let n be a nonnegative integer. Denote $K[t]_{\leq n} \subset K[t]$ the K-vector space of polynomials of degree at most n. In this subsection we consider the kernel of φ_j and prove the following crucial lemma.

Lemma 3.4. *We have*

$$\bigcap_{j=0}^{w} \ker \varphi_j = L^* \cdot K[t].$$

Proof. Denote the K-vector space $\bigcap_{j=0}^{w} \ker \varphi_j$ by W. Since $L \cdot f_j(z) \in K[z]$, Proposition 2.4 yields $L^* \cdot K[t] \subseteq W$. Let us show the opposite inclusion. Let $P(t) \in W$. Applying Lemma 3.2 for $k = 0$, we see that there exists a polynomial $\widetilde{P}(t) \in L^* \cdot K[t]$ such that $P(t) - \widetilde{P}(t) \in K[t]_{\leq w}$. This implies that it is sufficient to prove the following equality:

$$W \cap K[t]_{\leq w} = \{0\}. \tag{3.6}$$

Put $f_j(z) = \sum_{k=0}^{\infty} f_{j,k}/z^{k+1}$. By the definition of φ_j,

$$\mathcal{M}_0 = \begin{pmatrix} f_{0,0} & \cdots & f_{0,w} \\ \vdots & \ddots & \vdots \\ f_{w,0} & \cdots & f_{w,w} \end{pmatrix} \in \mathrm{Mat}_{w+1}(K).$$

Since the Laurent series f_j is determined by $(f_{j,k})_{0\leq k\leq w}$ (see the K-isomorphism in equation (2.7)) and the Laurent series $\{f_j(z)\}_{0\leq j\leq w}$ are linearly independent over K, we have $\det \mathcal{M}_0 \neq 0$. For $P(t) = \sum_{k=0}^{w} p_j t^j \in W \cap K[t]_{\leq w}$, we put $\boldsymbol{p} = {}^t(p_0, \ldots, p_w)$. By the definition of W, we have

$$\mathcal{M}_0 \cdot \boldsymbol{p} = \boldsymbol{0},$$

and thus $\boldsymbol{p} = \boldsymbol{0}$. This implies $P = 0$ and equation (3.6) holds. □

Remark 3.5. The non-vanishing of the determinant of $\mathcal{M}_0$ yields the non-vanishing of $\Delta_0(z)$.

3.2. Proof of Theorem 3.1. Let n be a positive integer. By Proposition 3.3, it is sufficient to show the non-vanishing of $\det \mathcal{M}_n$ to prove Theorem 3.1. Let $\boldsymbol{q} = {}^t(q_0, \ldots, q_w) \in K^w$ such that

$$\mathcal{M}_n \cdot \boldsymbol{q} = \boldsymbol{0}.$$

Put $Q(t) = \sum_{k=0}^{w} q_k t^k \in K[t]$. Then the linearity of φ_j derives $\varphi_j(Q(t)a(t)^n) = 0$ for $0 \leq j \leq w$ and thus, using Lemma 3.4,

$$Q(t)a(t)^n \in \bigcap_{j=0}^{w} \ker \varphi_j = L^* \cdot K[t]. \tag{3.7}$$

We complete the proof of Theorem 3.1 by showing $Q(t) = 0$. In order to prove $Q(t) = 0$, we prepare the following key lemma.

Lemma 3.6. *Let $a(t), b(t) \in K[t]$. Assume $Na'(t)+b(t)$ and $a(t)$ are coprime for any positive integer N. Let n be a positive integer and $P(t), Q(t) \in K[t]$ such that*

$$Q(t)a(t)^n = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{b(t)}{a(t)}\right) a(t) \cdot P(t).$$

Then $P(t)$ is divisible by $a(t)^n$.

Proof. Let us prove the statement by induction on n. Let $n=1$. Then

$$Q(t)a(t) = (a'(t)+b(t))P(t) + a(t)P'(t).$$

Since $a'(t)+b(t)$ is coprime with $a(t)$, the polynomial $P(t)$ is divisible by $a(t)$. Assume the claim holds for $n \geq 1$. Let us consider the case $n+1$. The induction hypothesis yields $P(t)$ is divisible by $a(t)^n$. Put $P(t) = a(t)^n \widetilde{P}(t)$. Then

$$Q(t)a(t)^{n+1} = \big((n+1)a'(t)+b(t)\big)\widetilde{P}(t)a(t)^n + a(t)^{n+1}\widetilde{P}'(t).$$

Combining the hypothesis of $a(t), b(t)$ and this equality yields $\widetilde{P}(t)$ is divisible by $a(t)$ and thus $P(t)$ is divisible by $a(t)^{n+1}$. $\square$

We now finish the proof of Theorem 3.1.

Proof of Theorem 3.1. Recall $a(z), b(z)$ be polynomials defined in equation (2.3) satisfying the assumptions (2.4), (3.1) and (3.2). For a vector $\boldsymbol{q} = {}^t(q_0, \ldots, q_w)$ satisfying $\mathcal{M}_n \cdot \boldsymbol{q} = \boldsymbol{0}$, we put $Q(t) = \sum_{k=0}^{w} q_k t^k$. Assume $Q(t) \neq 0$. From equation (3.7), there exists a non-zero polynomial $P(t) \in K[t]$ such that

$$Q(t)a(t)^n = L^* \cdot P(t).$$

Using the assumption (3.1), by Lemma 3.6, the polynomial $P(t)$ must be divisible by $a(t)^n$ and thus we have $\deg P \geq nu$. Using the assumption (3.2) together with Lemma 3.2 for $k=0$, we have

$$\deg L^* \cdot P(t) \geq nu + w + 1.$$

However, by the definition of $Q(t)$, we know

$$\deg Q(t)a(t)^n \leq nu + w.$$

This leads to a contradiction, as the degrees cannot simultaneously satisfy both conditions. Thus we conclude $Q(t)=0$, completing the proof of Theorem 3.1. $\square$

Remark 3.7. Let K be an algebraic number field, and let $a(z), b(z) \in K[z]$ satisfy the assumptions (2.3), (2.4) and (2.5). Consider the differential operator $L = -a(z)\frac{\mathrm{d}}{\mathrm{d}z} + b(z)$. In Theorem 1.1, we apply Theorem 3.1 to the case where $\deg a = m$ and $\deg b \leq m-1$, under the assumptions (1.1), (1.2) and (1.3), and to Laurent series $f_j(z) \in (1/z) \cdot K[[1/z]]$ that are linearly independent over K and satisfy $L \cdot f_j \in K[z]$. In this setting, as mentioned in Remark 1.2, the Laurent series f_j are all G-functions.

In [2], André introduced the notion of *arithmetic Gevrey series* as a general framework encompassing the theories of E-functions and G-functions originally developed by Siegel [27] in his study of transcendental number theory (refer [3]). It would be interesting to explore possible applications of Theorem 3.1 to the study of the arithmetic properties of other classes of arithmetic Gevrey series $f(z) \in (1/z) \cdot K[\![1/z]\!]$ satisfying $L \cdot f \in K[z]$.

4. Estimates

Recall notation in Subsection 1.1. We fix a positive integer $m \geq 2$. Let K be an algebraic number field and $a(z), b(z) \in K[z]$ where $a(z)$ is a monic polynomial satisfying

$$\deg a = m \quad \text{and} \quad \deg b \leq m-1.$$

Assume that $a(z)$ decomposes over K, and let $\alpha_1, \ldots, \alpha_m \in K$ denote its roots, counted with multiplicity. Set

$$a(z) = \sum_{i=0}^{m} a_i z^i, \quad b(z) = \sum_{j=0}^{m-1} b_j z^j.$$

Assume (1.1), (1.2), and (1.3). Denote the differential operator of order 1

$$L = -a(z)\frac{\mathrm{d}}{\mathrm{d}z} + b(z).$$

The assumption (1.3), together with Lemma 2.5, ensures that there exist Laurent series $f_0(z), \ldots, f_{m-2}(z) \in (1/z) \cdot K[\![1/z]\!]$ such that f_j are linearly independent over K and

$$L \cdot f_j(z) \in K[z] \quad \text{for} \quad 0 \leq j \leq m-2. \tag{4.1}$$

We now compute the exact form of $f_j(z)$. Recall $s_i := b(\alpha_i)/a'(\alpha_i)$ for $1 \leq i \leq m$ and the quantity $b_{m-1} = s_1 + \cdots + s_m$ is not in $\mathbb{Z}_{<-1}$.

Lemma 4.1. *For $0 \leq j \leq m-2$, the formal Laurent series $f_j(z)$ defined by*

$$\prod_{i=1}^{m}\left(1 - \frac{\alpha_i}{z}\right)^{s_i} \cdot \frac{1}{z^{j+1}} F_D^{(m)}\left(b_{m-1}+j+1,\, 1+s_1, \ldots, 1+s_m,\, b_{m-1}+j+2,\, \frac{\alpha_1}{z}, \ldots, \frac{\alpha_m}{z}\right),^{2}$$

are linearly independent over K and satisfy relation (4.1).

[2] In case of $b_{m-1} = -1$, we note that $f_0(z) = \frac{1}{z}\prod_{i=1}^{m}\left(1 - \frac{\alpha_i}{z}\right)^{s_i}$.

Proof. Since $\mathrm{ord}_\infty\, f_j = j+1$, it is easy to see that f_j are linearly independent over K. Let us show f_j satisfy (4.1). Set

$$\Phi(z) = \prod_{i=1}^{m}\left(1-\frac{\alpha_i}{z}\right)^{s_i},$$

$$g_j(z) = \frac{1}{z^{j+1}} F_D^{(m)}\Bigl(b_{m-1}+j+1,\, 1+s_1,\dots,1+s_m,\, b_{m-1}+j+2; \frac{\alpha_1}{z},\dots,\frac{\alpha_m}{z}\Bigr).$$

It is sufficient to prove that each $f_j(z) = \Phi(z)g_j(z)$ satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{b(z)}{a(z)}\right)\Phi(z)g_j(z) = \frac{(b_{m-1}+j+1)z^{m-j-2}}{a(z)}. \tag{4.2}$$

A direct computation yields the following:

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = \frac{\Phi(z)}{z}\sum_{j=1}^{m}\frac{\alpha_j s_j}{z-\alpha_j},$$

$$\begin{aligned}\left(-z\frac{\mathrm{d}}{\mathrm{d}z} + b_{m-1}\right)g_j(z) &= -\frac{b_{m-1}+j+1}{z^{j+1}}\prod_{i=1}^{m}\left(1-\frac{\alpha_i}{z}\right)^{-1-s_i} \\ &= -\frac{(b_{m-1}+j+1)z^{m-j-1}}{a(z)\Phi(z)}.\end{aligned} \tag{4.3}$$

From the second equality we obtain

$$g_j'(z) = \frac{b_{m-1}}{z}\, g_j(z) + \frac{(b_{m-1}+j+1)z^{m-j-2}}{a(z)\Phi(z)}. \tag{4.4}$$

Now, applying (4.3) and (4.4), we compute

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{b(z)}{a(z)}\right)\Phi(z)g_j(z) &= \Phi'(z)g_j(z) + \Phi(z)g_j'(z) - \frac{b(z)}{a(z)}\Phi(z)g_j(z) \\ &= \left[\frac{1}{z}\left(\sum_{j=1}^{m}\frac{\alpha_j s_j}{z-\alpha_j} + b_{m-1}\right) - \frac{b(z)}{a(z)}\right]\Phi(z)g_j(z) \\ &\quad + \frac{(b_{m-1}+j+1)z^{m-j-2}}{a(z)}.\end{aligned}$$

Finally, observe the identity

$$\frac{zb(z)}{a(z)} = z\sum_{j=1}^{m}\frac{s_j}{z-\alpha_j} = \sum_{j=1}^{m}\left(\frac{\alpha_j s_j}{z-\alpha_j} + s_j\right) = \sum_{j=1}^{m}\frac{\alpha_j s_j}{z-\alpha_j} + b_{m-1},$$

which implies

$$\frac{1}{z}\left(\sum_{j=1}^{m}\frac{\alpha_j s_j}{z-\alpha_j}+b_{m-1}\right)-\frac{b(z)}{a(z)}=0.$$

Therefore, (4.2) holds. This completes the proof of Lemma 4.1. $\square$

Now we fix the Laurent series $f_j(z)$ in Lemma 4.1 and denote φ_{f_j} by φ_j. We define the polynomials

$$\begin{aligned} P_{n,\ell}(z) &= \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}+\frac{b(z)}{a(z)}\right)^n a(z)^n\cdot z^\ell, \\ Q_{n,j,\ell}(z) &= \varphi_j\left(\frac{P_{n,\ell}(z)-P_{n,\ell}(t)}{z-t}\right), \end{aligned} \tag{4.5}$$

for $0\le j\le m-2$. By Theorem 2.7 (i), the vector of polynomials

$$(P_{n,\ell}(z),Q_{n,0,\ell}(z),\dots,Q_{n,m-2,\ell}(z))$$

is a weight n Padé-type approximant of $(f_j)_j$. Denote the Padé-type approximations of $(f_j)_j$ by

$$\mathfrak{R}_{n,j,\ell}(z)=P_{n,\ell}(z)f_j(z)-Q_{n,j,\ell}(z)\quad\text{for } 0\le j\le m-2. \tag{4.6}$$

A direct computation shows the algebraic function

$$w(z)=\prod_{i=1}^{m}(z-\alpha_i)^{s_i}$$

is a non-zero solution of the differential operator L (cf. [12, Prop. 3 (i)]). Notice that Lemma 2.6 implies

$$P_{n,\ell}(z)=\frac{w(z)^{-1}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}w(z)\cdot a(z)^n z^\ell \tag{4.7}$$

and, applying the Leibniz formula to equation (4.7) along with the expression $a(z)=\prod_{i=1}^{m}(z-\alpha_i)$ and the definition of $w(z)$, the following identity holds.

Lemma 4.2. *We have*

$$\begin{aligned} P_{n,\ell}(z)=\sum_{k=0}^{n}(-1)^k \sum_{\substack{0\le k_1,\dots,k_m\le k\\ \sum k_i=k}}\ \sum_{\substack{0\le j_1,\dots,j_{m+1}\le n-k\\ \sum j_i=n-k}}\binom{\ell}{j_{m+1}} \\ \cdot\left[\prod_{i=1}^{m}\frac{(-s_i)_{k_i}}{k_i!}\binom{n}{j_i}(z-\alpha_i)^{n-j_i-k_i}\right]z^{\ell-j_{m+1}}, \end{aligned}$$

with the convention $\binom{\ell}{j_{m+1}}=0$ *if* $j_{m+1}>\ell$.

Proof. The Leibniz formula yields

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\cdot w(z)a(z)^n z^\ell = \sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\cdot w(z)\times\frac{\mathrm{d}^{n-k}}{\mathrm{d}z^{n-k}}\cdot a(z)^n z^\ell. \tag{4.8}$$

From the definition of $w(z)$ and the equality $a(z) = \prod_{i=1}^m (z-\alpha_i)$, we compute

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\cdot w(z) = w(z)\sum_{\substack{0\le k_1,\dots,k_m\le k\\ \sum k_i=k}}(-1)^k\frac{k!}{k_1!\dots k_m!}\prod_{i=1}^{m}\frac{(-s_i)_{k_i}}{(z-\alpha_i)^{k_i}},$$
$$\frac{\mathrm{d}^{n-k}}{\mathrm{d}z^{n-k}}\cdot a(z)^n z^\ell = (n-k)!\sum_{\substack{0\le j_i\le n-k\\ \sum j_i=n-k}}\binom{\ell}{j_{m+1}}z^{\ell-j_{m+1}}\prod_{i=1}^{m}\binom{n}{j_i}(z-\alpha_i)^{n-j_i}.$$

Substituting (4.8) into equation (4.7) together with the above equalities, we obtain the expansion of $P_{n,\ell}(z)$. □

4.1. Absolute values of the Padé approximants. Let v be a place of K. In this subsection we describe the asymptotic behavior, as n tends to infinity, of the v-adic absolute values of the polynomials $P_{n,\ell}(z)$ and $Q_{n,j,\ell}(z)$ evaluated at a fixed $\beta\in K$.

Now we use the following notations. Denote the v-adic Weil height of the m-tuple of algebraic number $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$ by $\mathrm{H}_v(\boldsymbol{\alpha}) = \exp(\mathrm{h}_v(\boldsymbol{\alpha}))$. For a polynomial $P = \sum_{k=0}^n p_k z^k \in K[z]$ and a valuation v of K, we denote the maximum v-adic absolute value of its coefficients of P by

$$\|P\|_v := \max_{0\le k\le n}\{|p_k|_v\}.$$

The floor function is denoted by $\lfloor\cdot\rfloor$. For a nonnegative integer n, $s\in\mathbb{Q}$ and $b\in\mathbb{Q}\setminus\mathbb{Z}_{\le -1}$, we denote

$$\mu_n(s) = \mathrm{den}(s)^n\prod_{\substack{q:\text{prime}\\ q\mid\mathrm{den}(s)}} q^{\lfloor n/q-1\rfloor},$$
$$d_n(b) = \mathrm{den}\left(\frac{1}{b+1},\dots,\frac{1}{b+n+1}\right).$$

The aim of this section is to prove the following proposition.

Proposition 4.3. *Let v be a place of K and $\beta \in K$.*

(i) *Assume that v is non-Archimedean. Then*

$$\begin{aligned}\log\max\{|P_{n,\ell}(\beta)|_v, |Q_{n,j,\ell}(\beta)|_v\} \\ \le n\left(\sum_{i=1}^{m} \mathrm{h}_v(\alpha_i) + (m-1)(\mathrm{h}_v(\boldsymbol{\alpha}) + \mathrm{h}_v(\beta))\right) \\ + m\sum_{i=1}^{m} \log|\mu_n(s_i)|_v^{-1} + \log|d_{(m-1)(n+1)}(b_{m-1})|_v + o(n),\end{aligned}$$

where $o(n) = 0$ for all but finitely many non-Archimedean places v.

(ii) *Assume that v is Archimedean. Then*

$$\begin{aligned}\log\max\{|P_{n,\ell}(\beta)|_v, |Q_{n,j,\ell}(\beta)|_v\} \\ \le n\left(\sum_{i=1}^{m} \mathrm{h}_v(\alpha_i) + (m-1)(\mathrm{h}_v(\boldsymbol{\alpha}) + \mathrm{h}_v(\beta)) + m\log|4|_v + o(1)\right).\end{aligned}$$

4.1.1. *Proof of Proposition 4.3 (i).* We will use the following classical lemma to control the denominator of $(\alpha)_n/n!$ and the growth of $d_n(b)$.

Lemma 4.4. *Let $\alpha \in \mathbb{Q}$ and $b \in \mathbb{Q} \setminus \mathbb{Z}_{\le -1}$.*

(i) *Let n be a nonnegative integer. For $k = 0, \dots, n$, we have*

$$\mu_n(\alpha)\frac{(\alpha)_k}{k!} \in \mathbb{Z}.$$

(ii) *Let k be an integer and n_1, n_2 be positive integers. Then*

$$\mu_n(\alpha + k) = \mu_n(\alpha) \quad \textit{and} \quad \mu_{n_1+n_2}(\alpha) \textit{ is divisible by } \mu_{n_1}(\alpha)\mu_{n_2}(\alpha).$$

(iii) *We have*

$$\limsup_{n\to\infty} \frac{1}{n}\log d_n(b) = \frac{\mathrm{den}(b)}{\varphi(\mathrm{den}(b))} \sum_{\substack{1\le j\le \mathrm{den}(b_{m-1}) \\ (j,\mathrm{den}(b_{m-1}))=1}} \frac{1}{j},$$

where φ is Euler's totient function.

(iv) *Let p be a prime. We have*

$$\lim_{n\to\infty} \frac{|d_n(b)|_p^{-1}}{n} = 0.$$

Proof.

(i). This property was proved in [6, Lemma 2.2].

(ii). These properties follow directly from the definition of $\mu_n(\alpha)$.

(iii), (iv). These statements were established in [5]. □

Lemma 4.5. *Let v be a non-Archimedean place of K. Let $f(z) = \sum_{k=0}^{\infty} f_k/z^{k+1}$, belonging to $K[\![1/z]\!]$, and $P(z) \in K[z]$ with $\deg P = N$. Put $Q(z) = \varphi_f\Big(\frac{P(z)-P(t)}{z-t}\Big)$. Then we have*

$$\|Q\|_v \leq \|P\|_v \cdot \max_{0 \leq k \leq N-1} \{|f_k|_v\}.$$

Proof. Set $P(z) = \sum_{i=0}^{N} p_i z^i$. Then, using the identity

$$z^i - t^i = (z-t) \sum_{k=0}^{i-1} t^{i-1-k} z^k \text{ for } 1 \leq i,$$

we get

$$Q(z) = \varphi_f\left(\sum_{k=0}^{N-1}\left(\sum_{i=k+1}^{N} p_k t^{i-1-k}\right) z^k\right) = \sum_{k=0}^{N-1}\left(\sum_{i=k+1}^{N} p_k f_{i-1-k}\right) z^k. \tag{4.9}$$

Combining above equality with the strong triangle inequality leads us to get

$$\|Q\|_v = \max_{0 \leq k \leq N-1}\left\{\left|\sum_{i=k+1}^{N} p_k f_{i-1-k}\right|_v\right\} \leq \|P\|_v \cdot \max_{0 \leq k \leq N-1} \{|f_k|_v\}. \qquad \square$$

Lemma 4.6. *Let v be a non-Archimedean place of K. Let $f(z) = \sum_{k=0}^{\infty} f_k/z^{k+1} \in K[\![1/z]\!]$ be a Laurent series satisfying $L \cdot f(z) \in K[z]$. For any positive integer n, there exists a positive integer D which depends only on f such that*

$$\max_{0 \leq k \leq n} \{|f_k|_v\} \leq |D|_v^{-1} \, \mathrm{H}_v(\boldsymbol{\alpha})^n \cdot \prod_{i=1}^{m} |\mu_n(s_i)|_v^{-1} \cdot |d_n(b_{m-1})|_v^{-1}. \tag{4.10}$$

Proof. Put

$$f_j(z) = \sum_{k=0}^{\infty} \frac{f_{j,k}}{z^{k+1}}.$$

Since the Laurent series $f(z)$ can be expressed as a K-linear combination of $f_j(z)$ for $0 \leq j \leq m-2$ (cf. isomorphism (2.7)), it suffices to prove that

$$\max_{0 \leq k \leq n} \{|f_{j,k}|_v\} \leq |b_{m-1}|_v^{-1} \, \mathrm{H}_v(\boldsymbol{\alpha})^n \cdot \prod_{i=1}^{m} |\mu_n(s_i)|_v^{-1} \cdot |d_n(b_{m-1})|_v^{-1}. \tag{4.11}$$

We have the following expansions:

$$\prod_{i=1}^{m}\Big(1-\frac{\alpha_i}{z}\Big)^{s_i}=\sum_{k=0}^{\infty}\left[\sum_{\substack{0\le k_i\\ \sum k_i=k}}\prod_{i=1}^{m}\frac{(-s_i)_{k_i}}{k_i!}\alpha_i^{k_i}\right]\frac{1}{z^k},$$

$$F_D^{(m)}\Big(b_{m-1}+j+1,\,1+s_1,\dots,1+s_m;\,b_{m-1}+j+2;\,\frac{\alpha_1}{z},\dots,\frac{\alpha_m}{z}\Big)$$
$$=\sum_{k=0}^{\infty}\left[\sum_{\substack{0\le k_i\\ \sum k_i=k}}\prod_{i=1}^{m}\frac{(1+s_i)_{k_i}}{k_i!}\alpha_i^{k_i}\right]\frac{b_{m-1}+j+1}{b_{m-1}+j+k+1}\frac{1}{z^k}.$$

From these we conclude that, for $0\le j\le m-2$ and $k\ge 0$,

$$f_{j,k+j}=\sum_{w=0}^{k}\left[\sum_{\substack{0\le k_i\le w\\ \sum k_i=w}}\prod_{i=1}^{m}\frac{(-s_i)_{k_i}}{k_i!}\alpha_i^{k_i}\right]\cdot\left[\sum_{\substack{0\le l_i\le k-w\\ \sum l_i=k-w}}\prod_{i=1}^{m}\frac{(1+s_i)_{l_i}}{l_i!}\alpha_i^{l_i}\right]\cdot\frac{b_{m-1}+j+1}{b_{m-1}+j+k-w+1}. \tag{4.12}$$

Now let $0\le w\le k\le n$. For any choice of positive integers k_i,l_i with $\sum k_i=w$ and $\sum l_i=k-w$, Lemma 4.4 (i) and (ii) implies

$$\left|\prod_{i=1}^{m}\frac{(-s_i)_{k_i}}{k_i!}\frac{(1+s_i)_{l_i}}{l_i!}\alpha_i^{k_i+l_i}\right|_v\le\prod_{i=1}^{m}|\mu_n(s_i)|_v^{-1}\ \mathrm{H}_v(\boldsymbol{\alpha})^n.$$

Therefore, by (4.12) and the strong triangle inequality, we get

$$\max_{0\le k\le n}\{|f_{j,k}|_v\}\le|b_{m-1}|_v\prod_{i=1}^{m}|\mu_n(s_i)|_v\ \mathrm{H}_v(\boldsymbol{\alpha})^n\cdot\max_{0\le k\le n}\left\{\left|\frac{1}{b_{m-1}+k+1}\right|_v\right\}$$
$$=|b_{m-1}|_v\prod_{i=1}^{m}|\mu_n(s_i)|_v^{-1}\ \mathrm{H}_v(\boldsymbol{\alpha})^n\cdot|d_n(b_{m-1})|_v^{-1}.$$

This completes the proof of the assertion. □

Lemma 4.7. *Let v be a non-Archimedean place and n be a positive integer. Then*

$$\log\max_{0\le\ell\le m}\{\|P_{n,\ell}\|_v\}\le\sum_{i=1}^{m}\Big(\log|\mu_n(s_i)|_v^{-1}+n\,\mathrm{h}_v(\alpha_i)\Big).$$

Proof. Put the equality

$$(z-\alpha_i)^{n-j_i-k_i} = \sum_{w_i=0}^{n-j_i-k_i} \binom{n-j_i-k_i}{w_i} (-\alpha_i)^{n-j_i-k_i-w_i} z^{w_i},$$

for $0 \le k_i \le k \le n, 0 \le j_i \le n-k_i$, into the expression of $P_{n,\ell}$ obtained in Lemma 4.2. We then have

$$\begin{aligned}(4.13)\quad P_{n,\ell}(z) = \sum_k (-1)^k \sum_{k_i} \sum_{j_i} & \binom{\ell}{j_{m+1}} \\ \cdot & \left[\prod_{i=1}^m \frac{(-s_i)_{k_i}}{k_i!} \binom{n}{j_i} \sum_{w_i} \binom{n-j_i-k_i}{w_i} (-\alpha_i)^{n-j_i-k_i-w_i} z^{w_i}\right] z^{\ell-j_{m+1}}.\end{aligned}$$

Making use of Lemma 4.4 (i) together with the strong triangle inequality for the above identity, we obtain the desire estimate. □

Proof of Proposition 4.3 (i). We apply Lemma 4.5 for $P = P_{n,\ell}$. Since $\deg P_{n,\ell} = \ell + (m-1)n$, using Lemma 4.6, the value $\log \max_{j,\ell}\{\|Q_{n,j,\ell}\|_v\}$ is bounded by

$$\begin{aligned}&\|P_{n,\ell}\|_v + (m-1)n\,\mathrm{h}_v(\boldsymbol{\alpha}) \\ &\qquad + \sum_{i=1}^m \log|\mu_{(m-1)(n+1)}(s_i)|_v^{-1} + \log|d_{(m-1)(n+1)}(b_{m-1})|_v + |b_{m-1}|_v.\end{aligned}$$

Finally, Lemma 4.7 together with the strong triangle inequality yields

$$\begin{aligned}&\log\max\{|P_{n,\ell}(\beta)|_v, |Q_{n,j,\ell}(\beta)|_v\} \\ &\qquad \le \sum_{i=1}^m \Big(m\log|\mu_n(s_i)|_v^{-1} + n\,\mathrm{h}_v(\alpha_i)\Big) + (m-1)n(\mathrm{h}_v(\boldsymbol{\alpha}) \\ &\qquad\qquad + \mathrm{h}_v(\beta)) + \log|d_{(m-1)(n+1)}(b_{m-1})|_v + o(n),\end{aligned}$$

where $o(n) = 0$ if v does not divide $\mathrm{den}(b_m - 1, s_1, \dots, s_m)$. This completes the proof of Proposition 4.3 (i). □

4.1.2. *Poincaré–Perron type recurrence.* We now turn attention to the proof of Proposition 4.3 (ii). To this end, in this subsection, let us consider the following Poincaré-type recurrence of some order $m > 0$.

$$(4.14)\qquad a_m(n)u(n+m) + a_{m-1}(n)u(n+m-1) + \cdots + a_0(n)u(n) = 0$$

for large enough n, where the coefficients $a_i(t) \in \mathbb{C}[t]$ are polynomials and $a_m(t) \ne 0$. Then, we can apply Perron's Second Theorem below (see [24] and [25, Theorem C]) to estimate precisely the growth of a solution of the above recurrence.

Theorem 4.8 (Perron's Second Theorem). *Let m be a positive integer. Assume that for $i = 0, \dots, m$ there exist a function $a_i : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ and $a_i \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} a_i(n) = a_i \in \mathbb{C},$$

with $a_m \neq 0$. Denote by $\lambda_1, \dots, \lambda_m$ the (not necessarily distinct) roots of the characteristic polynomial

$$\chi(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0.$$

Then, there exist m linearly independent solutions $u_1, \dots, u_m$ of (4.14)*, such that, for each $i = 1, \dots, m$,*

$$\limsup_{n\to\infty} |u_i(n)|^{1/n} = |\lambda_i|.$$

In particular, any solution u of (4.14) *satisfies* $\limsup_{n\to\infty} |u(n)|^{1/n} \leq \max_{1\leq i\leq m}\{|\lambda_i|\}$.

Remark 4.9. In the above theorem, there are no restriction on the roots of $\chi(z)$, whereas in Poincaré's Theorem and Perron's First Theorem, we ask that

$$|\lambda_i| \neq |\lambda_j| \text{ for } i \neq j, \tag{4.15}$$

see [25, Theorem A and B].

Corollary 4.10. *Let us fix an embedding K into $\mathbb{C}$. Let $f(z) = \sum_{k=0}^{\infty} f_k/z^{k+1} \in \mathbb{C}[\![1/z]\!]$ be a Laurent series satisfying $L \cdot f(z) \in \mathbb{C}[z]$. Then we have*

$$\limsup_{n\to\infty} |f_n|^{1/n} \leq \max_{1\leq i\leq m}\{|\alpha_i|\}.$$

Proof. Equation (2.8) in Lemma 2.5 yields that the vector $(f_n)_{n\geq -1}$ with $f_{-1} = 0$ is a solution of the recurrence equation:

$$\sum_{i=0}^{m} a_i(k+i) f_{k+i-1} + \sum_{j=0}^{m-1} b_j f_{k+j} = 0 \quad \text{for} \quad k \geq 0.$$

A straightforward computation yields that this recurrence equation is of Poincaré type and the characteristic polynomial $\chi(z)$ is $a(z)$. Thus Perron's Second Theorem ensures

$$\limsup_{n\to\infty} |f_n|^{1/n} \leq \max_{1\leq i\leq m}\{|\alpha_i|\}.$$

This completes the proof of Corollary 4.10. ☐

4.1.3. *Proof of Proposition 4.3 (ii).*

Lemma 4.11. *Let v be an Archimedean valuation of K. Define a constant $c_v(\boldsymbol{\alpha})$ which depends on $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ and v by*

$$c_v(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \mathrm{h}_v(\alpha_i) + m \log |4|_v. \tag{4.16}$$

Then we have

$$\log \max_{0 \le \ell \le m-1} \{\|P_{n,\ell}(z)\|_v\} \le n c_v(\boldsymbol{\alpha}) + o(n).$$

Proof. We apply the estimate

$$\frac{(s)_k}{k!} \le e^{o(n)}, \quad \binom{n}{k} \le 2^n \quad \text{for} \quad s \in \mathbb{Q} \quad \text{and} \quad 0 \le k \le n,$$

together with the triangular inequality for equation (4.13), we conclude the desire estimate. □

Lemma 4.12. *Let n be a positive integer and $P(z) \in K[z]$ with $\deg P = n$. Let v be an Archimedean valuation of K and $\beta \in K$. Let $f(z) = \sum_{k=0}^{\infty} f_k / z^{k+1} \in K[[1/z]]$ be a Laurent series satisfying $L \cdot f(z) \in K[z]$. Put*

$$Q(z) = \varphi_f\left(\frac{P(z) - P(t)}{z - t}\right).$$

Then the following inequalities hold.

(i) $|P(\beta)|_v \le (n+1)\|P\|_v \,\mathrm{H}_v(\beta)^n$.
(ii) $\|Q\|_v \le n\|P\|_v \,\mathrm{H}_v(\boldsymbol{\alpha})^{n+o(n)}$.
(iii) $|Q(\beta)|_v \le n^2\|P\|_v \,\mathrm{H}_v(\boldsymbol{\alpha})^{n+o(n)}\,\mathrm{H}_v(\beta)^{n-1}$.

Proof.

(i). This is an easy consequence of the triangle inequality.

(ii). Equation (4.9) together with the triangle inequality and the estimate

$$\max_{0 \le k \le n-1} \{|f_k|_v\} \le \mathrm{H}_v(\boldsymbol{\alpha})^{n+o(n)}$$

deduced from Corollary 4.10 allows us to obtain the desire inequality.

(iii). Combining (i) and (ii) yields

$$|Q(\beta)|_v \le n\|Q\|_v \,\mathrm{H}_v(\beta)^{n-1} \le n^2\|P\|_v \,\mathrm{H}_v(\boldsymbol{\alpha})^{n+o(n)}\,\mathrm{H}_v(\beta)^{n-1}.$$

This completes the proof of Lemma 4.12. □

Proof of Proposition 4.3 (ii). Let $c_v(\boldsymbol{\alpha})$ be the constant defined in (4.16). We apply Lemma 4.12 (i) with $P = P_{n,\ell}$. Using Lemma 4.11 and the equality $\deg P_{n,\ell} = \ell + (m-1)n$ yields

$$\log |P_{n,\ell}(\beta)|_v \leq n(c_v(\boldsymbol{\alpha}) + (m-1)\,\mathrm{h}_v(\beta) + o(1)). \tag{4.17}$$

For the estimating of $Q_{n,j,\ell}$, we invoke Lemma 4.12 (iii), which gives

$$\log |Q_{n,j,\ell}(\beta)|_v \leq n(c_v(\boldsymbol{\alpha}) + (m-1)(\mathrm{h}_v(\boldsymbol{\alpha}) + \mathrm{h}_v(\beta)) + o(1)). \tag{4.18}$$

Combining inequalities (4.17) and (4.18), we arrive at the desired estimate. $\square$

4.2. Absolute values of the Padé approximations. For a rational number s and a place v of K, we define

$$\mu_v(s) = \begin{cases} 1 & \text{if } v \in \mathfrak{M}_K^\infty \text{ or } v \in \mathfrak{M}_K^f \ \&\ |s|_v \leq 1, \\ |\mathrm{den}(s)|_v |p|_v^{\frac{1}{p-1}} & \text{otherwise,} \end{cases}$$

where p is the prime below v.

Proposition 4.13. *Let v be a place of K and $\beta \in K$.*

(i) *Assume that v is non-Archimedean and*

$$|\beta|_v > \prod_{i=1}^m \mu_v(s_i)^{-1} \cdot \mathrm{H}_v(\boldsymbol{\alpha}). \tag{4.19}$$

Then the series $\mathfrak{R}_{n,j,\ell}(z)$ converges to an element of K_v at $z = \beta$ and

$$\log \max_{\substack{0\leq j\leq m-2 \\ 0\leq \ell\leq m-1}} \{|\mathfrak{R}_{n,j,\ell}(\beta)|_v\} \leq n\left(-\log|\beta|_v + \sum_{i=1}^m \mathrm{h}_v(\alpha_i) + m\,\mathrm{h}_v(\boldsymbol{\alpha}) - m\sum_{i=1}^m \log\mu_v(s_i) + o(1)\right).$$

(ii) *Assume that v is Archimedean and $|\beta|_v > \max\{|\alpha_i|_v\}$. Then the series $\mathfrak{R}_{n,j,\ell}(z)$ converges to an element of K_v at $z = \beta$ and*

$$\log \max_{\substack{0\leq j\leq m-2 \\ 0\leq \ell\leq m-1}} \{|\mathfrak{R}_{n,j,\ell}(\beta)|_v\} \leq n\left(-\log|\beta|_v + \sum_{i=1}^m \mathrm{h}_v(\alpha_i) + m\,\mathrm{h}_v(\boldsymbol{\alpha}) + m\log|2|_v + o(1)\right).$$

Proof. Before starting the proof, we give an expansion of $\mathfrak{R}_{n,j,\ell}(z)$. Combining the expansion

$$t^{k+\ell}a(t)^n = \sum_{w=0}^{mn} \sum_{\substack{k_i \le n \\ \sum k_i = w}} \prod_{i=1}^{m} \binom{n}{k_i} (-\alpha_i)^{n-k_i} t^{w+k+\ell},$$

and Theorem 2.7 (ii) implies

$$(4.20)\quad \mathfrak{R}_{n,j,\ell}(z) = \frac{(-1)^n}{z^{n+1}} \sum_{k=0}^{\infty} \binom{k+n}{n} \left(\sum_{w=0}^{mn} \sum_{\substack{k_i \le n \\ \sum k_i = w}} \prod_{i=1}^{m} \binom{n}{k_i} (-\alpha_i)^{n-k_i} f_{w+k+\ell} \right) z^{-k}.$$

(i). Let v be a non-Archimedean valuation. By the strong triangle inequality together with Lemma 4.6, we obtain

$$(4.21)\quad \left| \binom{k+n}{n} \sum_{w=0}^{mn} \sum_{\substack{k_i \le n \\ \sum k_i = w}} \prod_{i=1}^{m} \binom{n}{k_i} (-\alpha_i)^{n-k_i} f_{w+k+\ell}\, \beta^{-k} \right|_v$$
$$\le \prod_{i=1}^{m} \mathrm{H}_v(\alpha_i)^n \cdot \mathrm{H}_v(\boldsymbol{\alpha})^{mn+k+m} \cdot \prod_{i=1}^{m} |\mu_{mn+k+m}(s_i)|_v^{-1} \cdot |d_{mn+k+m}(b_{m-1})|_v^{-1} \cdot |b_{m-1}|_v^{-1} |\beta|_v^{-k},$$

for every $k \ge 0$.

By Lemma 4.4 (ii) and the divisibility relation

$$d_{mn+k+m}(b_{m-1}) \mid d_{m(n+1)}(b_{m-1})\, d_k(b_{m-1} + m(n+1)),$$

we have

$$\prod_{i=1}^{m} |\mu_{mn+k+m}(s_i)|_v^{-1}\, |d_{mn+k+m}(b_{m-1})|_v^{-1} \le \prod_{i=1}^{m} |\mu_{m(n+1)}(s_i)|_v^{-1}\, |d_{(m+1)n}(b_{m-1})|_v^{-1} \cdot \prod_{i=1}^{m} |\mu_k(s_i)|_v^{-1}\, |d_k(b_{m-1} + m(n+1))|_v^{-1}.$$

Combining this with (4.21), we obtain

$$\left|\binom{k+n}{n}\sum_{w=0}^{mn}\sum_{\substack{k_i\le n\\ \sum k_i=w}}\prod_{i=1}^{m}\binom{n}{k_i}(-\alpha_i)^{n-k_i}f_{w+k+\ell}\,\beta^{-k}\right|_v$$

$$\le \prod_{i=1}^{m}|\mu_{m(n+1)}(s_i)|_v^{-1}|d_{(m+1)n}(b_{m-1})|_v^{-1}\prod_{i=1}^{m}\mathrm{H}_v(\alpha_i)^n\,\mathrm{H}_v(\boldsymbol{\alpha})^{m(n+1)}$$

$$\cdot|b_{m-1}|_v^{-1}|d_k(b_{m-1}+m(n+1))|_v^{-1}\,\mathrm{H}_v(\boldsymbol{\alpha})^k\prod_{i=1}^{m}|\mu_k(s_i)|_v^{-1}|\beta|_v^{-k}.$$

Since Lemma 4.4 (iv) implies that

$$|d_k(b_{m-1}+m(n+1))|_v^{-1}=o(k)\qquad(k\to\infty),$$

and by assumption (4.19), we deduce that

$$\limsup_{k\to\infty}|b_{m-1}|_v^{-1}|d_k(b_{m-1}+m(n+1))|_v^{-1}\,\mathrm{H}_v(\boldsymbol{\alpha})^k\prod_{i=1}^{m}|\mu_k(s_i)|_v^{-1}|\beta|_v^{-k}=0.$$

Hence $\mathfrak{R}_{n,j,\ell}(z)$ converges to an element of K_v at $z=\beta$, and

$$\begin{aligned}
&\log|\mathfrak{R}_{n,j,\ell}(\beta)|_v\\
&\le \log|\beta|_v^{-n-1}\\
&\qquad+\log\max_{k\ge0}\left\{\left|\binom{k+n}{n}\sum_{w=0}^{mn}\sum_{\substack{k_i\le n\\ \sum k_i=w}}\prod_{i=1}^{m}\binom{n}{k_i}(-\alpha_i)^{n-k_i}f_{w+k+\ell}\,\beta^{-k}\right|_v\right\}\\
&\le n\left(-\log|\beta|_v+\sum_{i=1}^{m}\mathrm{h}_v(\alpha_i)+m\,\mathrm{h}_v(\boldsymbol{\alpha})-m\sum_{i=1}^{m}\log\mu_v(s_i)+o(1)\right).
\end{aligned}$$

(ii). Let v be an Archimedean valuation. The triangle inequality and Corollary 4.10 lead us to

$$\left|\sum_{w=0}^{mn}\sum_{\substack{k_i\le n\\ \sum k_i=w}}\prod_{i=1}^{m}\binom{n}{k_i}(-\alpha_i)^{n-k_i}f_{w+k+\ell}\right|_v$$

$$\le e^{o(n)}|2|_v^{mn}\prod_{i=1}^{m}\mathrm{H}_v(\alpha_i)^n\max\{|\alpha_i|_v\}^{mn+k}$$

for any $k \geq 0$. Therefore combining (4.20) and the assumption $|\beta|_v > \max\{|\alpha_i|_v\}$ yields that $\mathfrak{R}_{n,j,\ell}(z)$ converges at β in K_v and

$$\begin{aligned}|\mathfrak{R}_{n,j,\ell}(\beta)|_v &\leq e^{o(n)}|\beta|_v^{-n}|2|_v^{mn}\prod_{i=1}^{m}\mathrm{H}_v(\alpha_i)^n\max\{|\alpha_i|_v\}^{mn}\\ &\quad\cdot\sum_{k=0}^{\infty}\binom{k+n}{n}\left(\frac{\max\{|\alpha_i|_v\}}{|\beta|_v}\right)^k\\ &\leq e^{o(n)}|\beta|_v^{-n}|2|_v^{mn}\prod_{i=1}^{m}\mathrm{H}_v(\alpha_i)^n\,\mathrm{H}_v(\boldsymbol{\alpha})^{mn}.\end{aligned}$$

Taking the logarithm, we obtain the conclusion of (ii). $\square$

5. Proof of Theorem 1.1

We keep the notation in Section 4. For a positive integer n, we recall that the polynomials $P_{n,\ell}(z)$ and $Q_{n,j,\ell}(z)$ are defined in equation (4.5). Let us fix a place v_0 of K and let $\beta \in K$. Define the following m by m matrix M_n as

$$M_n = \begin{pmatrix} P_{n,0}(\beta) & P_{n,1}(\beta) & \dots & P_{n,m-1}(\beta)\\ Q_{n,0,0}(\beta) & Q_{n,0,1}(\beta) & \dots & Q_{n,0,m-1}(\beta)\\ \vdots & \vdots & \ddots & \vdots\\ Q_{n,m-2,0}(\beta) & Q_{n,m-2,1}(\beta) & \dots & Q_{n,m-2,m-1}(\beta)\end{pmatrix} \in \mathrm{Mat}_m(K).$$

Our proof relies on a qualitative linear independence criterion [8, Proposition 5.6] which is based on the method of C. F. Siegel (cf. [27]). Define real numbers:

$$\mathbb{A}_{v_0}(\beta) = \begin{cases} \log|\beta|_{v_0} - \displaystyle\sum_{i=1}^{m}\mathrm{h}_{v_0}(\alpha_i) - m\,\mathrm{h}_{v_0}(\boldsymbol{\alpha}) + m\sum_{i=1}^{m}\log\mu_{v_0}(s_i) & \text{if } v\nmid\infty,\\ \log|\beta|_{v_0} - \displaystyle\sum_{i=1}^{m}\mathrm{h}_{v_0}(\alpha_i) - m\,\mathrm{h}_{v_0}(\boldsymbol{\alpha}) - m\log|2|_{v_0} & \text{if } v\mid\infty,\end{cases}$$

$$U_{v_0}(\beta) = \left(\sum_{i=1}^{m}\mathrm{h}_v(\alpha_i) + (m-1)(\mathrm{h}_v(\boldsymbol{\alpha}) + \mathrm{h}_v(\beta))\right) + m\sum_{i=1}^{m}\log\mu_{v_0}(s_i).$$

We now restate Theorem 1.1 together with a linear independence measure.

Theorem 5.1. *We use the same notations in Theorem 1.1. Let $v_0 \in \mathfrak{M}_K$ such that $V_{v_0}(\beta) > 0$. Then the series $f_j(z)$ for $0 \leq j \leq m-2$ converge around β in K_{v_0} and for any positive number ε with $\varepsilon < V_{v_0}(\beta)$, there exists an effectively computable positive number H_0 depending on ε and the given* *data such that the following property holds. For any* $\boldsymbol{\lambda} = (\lambda, \lambda_j)_{0\le j\le m-2} \in K^m \setminus \{\mathbf{0}\}$ *satisfying* $H_0 \le \mathrm{H}(\boldsymbol{\lambda})$*, then*

$$\left|\lambda + \sum_{j=0}^{m-2} \lambda_j f_j(\beta)\right|_{v_0} > C(\beta,\varepsilon)\, \mathrm{H}_{v_0}(\boldsymbol{\lambda})\, \mathrm{H}(\boldsymbol{\lambda})^{-\mu(\beta,\varepsilon)} \ ,$$

where

$$\mu(\beta,\varepsilon) = \frac{\mathbb{A}_{v_0}(\beta) + U_{v_0}(\beta)}{V_{v_0}(\beta) - \varepsilon},$$
$$C(\beta,\varepsilon) = \exp\left(-\left(\frac{\log(2)}{V_{v_0}(\beta)-\varepsilon} + 1\right)(\mathbb{A}_{v_0}(\beta) + U_{v_0}(\beta))\right).$$

Proof. Firstly, we claim that M_n is invertible by applying Theorem 3.1. To this end, we show that equations (3.1) and (3.2) in Theorem 3.1 hold for our $a(z), b(z)$.

First, we claim that the assumptions (1.1) and (1.2) imply (3.1). Suppose not. Then there exist a positive integer n and a root α_i of $a(t)$ such that

$$na'(\alpha_i) + b(\alpha_i) = 0.$$

The assumption (1.1) implies $a'(\alpha_i) \neq 0$, and the above equality yields

$$\frac{b(\alpha_i)}{a'(\alpha_i)} = -n \in \mathbb{Z}_{\le -1}.$$

This contradicts the assumption (1.2). Moreover, the assumption (1.3) implies that equation (3.2) holds. Therefore, Theorem 3.1 ensures that $\det M_n \in K \setminus \{0\}$.

For $v \in \mathfrak{M}_K$, we define functions $F_v : \mathbb{N} \to \mathbb{R}_{\ge 0}$ by

$$F_v(n) = n\left(\sum_{i=1}^{m} \mathrm{h}_v(\alpha_i) + (m-1)(\mathrm{h}_v(\boldsymbol{\alpha}) + \mathrm{h}_v(\beta))\right)$$
$$+\begin{cases} m\displaystyle\sum_{i=1}^{m} \log|\mu_n(s_i)|_v^{-1} + \log|d_{(m-1)(n+1)}(b_{m-1})|_v + o(n) & \text{if } v \nmid \infty, \\ m\log|4|_v + o(n) & \text{if } v \mid \infty, \end{cases}$$

where $o(n) = 0$ for all but finitely many finite places non-Archimedean places v. Notice that

$$\limsup_{n\to\infty} \frac{1}{n} F_{v_0}(n) = U_{v_0}(\beta).$$

Proposition 4.3 allows us to get

$$\max_{\substack{0\le j\le m-2\\ 0\le \ell\le m-1}} \log\max\{|P_{n,j}(\beta)|_v, |Q_{n,j,\ell}(\beta)|_v\} \le F_v(n).$$

Then Proposition 4.13 yields

$$\max_{\substack{0\le j\le m-2\\ 0\le \ell\le m-1}} \log |\mathfrak{R}_{n,j,\ell}(\beta)|_{v_0} \le -\mathbb{A}_{v_0}(\beta)n + o(n).$$

Lemma 4.4 (iii) yields

$$\limsup_{n\to\infty} \frac{1}{n}\log d_{(m-1)(n+1)}(b_{m-1}) = (m-1)\frac{\mathrm{den}(b_{m-1})}{\varphi(\mathrm{den}(b_{m-1}))}\sum_{\substack{1\le j\le \mathrm{den}(b_{m-1})\\ (j,\mathrm{den}(b_{m-1}))=1}} \frac{1}{j},$$

and we obtain

$$\mathbb{A}_{v_0}(\beta) - \limsup_{n} \frac{1}{n}\sum_{v\ne v_0} F_v(n) \le V_{v_0}(\beta),$$

where $V_{v_0}(\beta)$ is the real number defined in (1.4). Using a qualitative linear independence criterion in [8, Proposition 5.6] for

$$\vartheta_j = f_j(\beta) \quad \text{for} \quad 0\le j\le m-2,$$

and the family of invertible matrices $(M_n)_n$, and applying above estimates, we obtain Theorem 1.1. □

Appendix A. Jordan–Pochhammer equation

The equation of the following form is called the Jordan–Pochhammer equation (cf. [17, 18.4]):

$$\text{(A.1)}\qquad \sum_{i=0}^{m} \binom{-\mu}{i} Q^{(i)}(z)\frac{\mathrm{d}^{m-i}}{\mathrm{d}z^{m-i}} - \sum_{j=0}^{m-1}\binom{-\mu-1}{j} R^{(j)}(z)\frac{\mathrm{d}^{m-1-j}}{\mathrm{d}z^{m-1-j}}$$

where μ is a complex number, $\binom{\mu}{0}=1$ and

$$\binom{\mu}{i} = \frac{\mu(\mu-1)\dots(\mu-i+1)}{i!} \quad \text{for} \quad i\ge 1,$$
$$Q(z) = (z-\alpha_1)\dots(z-\alpha_m)\in\mathbb{C}[z] \quad \text{with disinct roots},$$
$$R(z)\in\mathbb{C}[z] \text{ of degree at most } m-1.$$

We observe that the equation (A.1) is of Fuchsian type with singularities $\alpha_1,\dots,\alpha_m,\infty$, and has the following Riemann scheme:

$$\left\{\begin{matrix} \alpha_1 & \dots & \alpha_m & \infty \\ 0 & \dots & 0 & -\mu-1 \\ 1 & \dots & 1 & -\mu-2 \\ \vdots & \ddots & \vdots & \vdots \\ m-2 & \dots & m-2 & -\mu-m+1 \\ m+\mu+s_1-1 & \dots & m+\mu+s_m-1 & -\mu-\gamma \end{matrix}\right\},$$

where $s_i := R(\alpha_i)/Q'(\alpha_i)$ for $1\le i\le m$ and $\gamma = s_1+\dots+s_m$.

Example A.1. Let $m \geq 2$ be an integer and $a(z), b(z) \in \mathbb{C}[z]$, where $a(z)$ is a monic polynomial of degree m with distinct roots, and $b(z)$ is a polynomial of degree at most $m-1$. Denote the differential operator

$$L = \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left(a(z)\frac{\mathrm{d}}{\mathrm{d}z} - b(z)\right).$$

Then, the following identity holds:

$$L = \sum_{i=0}^{m} \binom{m}{i} a^{(i)}(z) \frac{\mathrm{d}^{m-i}}{\mathrm{d}z^{m-i}} - \sum_{j=0}^{m-1} \binom{m-1}{j} (a'(z) + b(z))^{(j)} \frac{\mathrm{d}^{m-1-j}}{\mathrm{d}z^{m-1-j}}.$$

Thus, L takes a form of a Jordan–Pochhammer equation with $Q(z) = a(z), R(z) = a'(z) + b(z)$ and $\mu = -m$.

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Makoto KAWASHIMA
Totsuka, Yokohama, Kanagawa, 224-8539, Japan
E-mail: kawasima@mi.meijigakuin.ac.jp
URL: https://sites.google.com/view/makotokawashimashp/home