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The cubic moment of central *L*-values in the weight aspect

par Hui WANG et Xin WANG

Résumé. Motivés par la liste des problèmes concernant les moments des fonctions L établie par Hamieh et al. [7], nous démontrons une formule asymptotique pour le moment cubique des valeurs centrales des fonctions L associées aux formes modulaires paraboliques holomorphes de niveau un, où les poids k varient dans un court intervalle $K-\Delta<k<K+\Delta$ avec $K^{\varepsilon}\le\Delta\le K^{1-\varepsilon}$, lorsque $K\to\infty$.

Abstract. Motivated by the list of problems for moments of L-functions due to Hamieh et al. [7], we prove an asymptotic formula for the cubic moment of central L-values associated to holomorphic cusp forms of level one, where the weights k lie in a short interval $K-\Delta<k<K+\Delta$ with $K^{\varepsilon}\le\Delta\le K^{1-\varepsilon}$, as $K\to\infty$.

1. Introduction

The theory of moments of L-functions occupies a central position in analytic number theory, serving as a driving force behind the development of transformative techniques with profound implications, particularly in the study of subconvexity bounds for L-functions and the non-vanishing of central L-values. The classical investigation of these moments originates in the pioneering work of Hardy and Littlewood on the Riemann zeta function, later refined and generalized by a series of influential mathematicians (see, e.g., [1, 8, 11, 17, 24]). Building on foundational contributions such as the seminal work of Conrey and Iwaniec [4], recent years have witnessed significant advances in the theory of cubic moments of automorphic L-functions, with applications spanning a variety of settings (e.g., [6, 18, 19, 23]). Inspired by the framework for moments of L-functions introduced by Hamieh et al. [7], this paper focuses on the cubic moment of holomorphic L-functions over GL_2.

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Let S_k, $k \equiv 0 \pmod 2$, denote the space of weight k modular cusp forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, and let H_k denote a basis of eigenfunctions for all Hecke operators, with normalizing first Fourier coefficients 1. The cardinality of H_k is asymptotically $k/12$. Any $f \in H_k$ admits a Fourier expansion

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \quad z = x + \mathrm{i}y \in \mathbb{H}.$$

The Fourier coefficients $\lambda_f(n)$ are real, satisfy the Hecke relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\Big(\frac{mn}{d^2}\Big),$$

and obey the Ramanujan–Petersson bound (see Deligne [5])

$$|\lambda_f(n)| \leq \tau(n),$$

where $\tau(n)$ is the divisor function. For $\mathrm{Re}(s) > 1$, the standard L-function associated to f is defined as

$$L(s,f) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \Big(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\Big)^{-1},$$

which extends analytically to the entire complex plane. The completed L-function

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma\Big(s + \frac{k-1}{2}\Big)L(s,f)$$

satisfies the functional equation

$$\Lambda(s,f) = \mathrm{i}^k \Lambda(1-s,f).$$

The asymptotic behavior of the cubic moment of $L(1/2,f)$ in the weight aspect is conjectured via random matrix theory by Conrey et al. [3]. Specifically, we have the following conjecture.

Conjecture 1.1. *For any $\varepsilon > 0$ and $k \equiv 0 \pmod 4$,*

$$\sum_{f\in H_k}^{h} L(1/2,f)^3 = P_3(\log k) + O\Big(k^{\varepsilon-\frac{1}{2}}\Big), \tag{1.1}$$

where the superscript h denotes the harmonic weight defined in Section 2.2, and

$$P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \tag{1.2}$$

with coefficients

$$\begin{aligned}
a_3 &= \frac{8}{3}, \quad a_2 = 16\gamma - 8\log(2\pi),\\
a_1 &= 8\log^2(2\pi) - 32\gamma\log(2\pi) - 16\gamma_1 + 24\gamma^2,\\
a_0 &= -\frac{8}{3}\log^3(2\pi) + 16\gamma\log^2(2\pi)\\
&\qquad + (16\gamma_1 - 24\gamma^2)\log(2\pi) + 4\gamma_2 - 24\gamma\gamma_1 + 8\gamma^3.
\end{aligned}$$

Here $\gamma := \gamma_0$ *is Euler's constant, and* γ_i $(i \geq 1)$ *are Stieltjes constants defined by the Laurent expansion*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n (s-1)^n.$$

The derivation of (1.1) from [3, Conjecture 4.5.2] is detailed in Frolenkov [6], as the original formulation in [3] pertains to shifted cubic moments. The first progress towards Conjecture 1.1 was made by Peng [18], who established the upper bound

$$\sum_{f\in H_k}^{h} L(1/2,f)^3 \ll k^{\varepsilon} \tag{1.3}$$

and deduced the Weyl-strength subconvexity bound

$$L(1/2,f) \ll k^{1/3+\varepsilon}. \tag{1.4}$$

To refine (1.3) and (1.4), Frolenkov [6] investigated the shifted cubic moment

$$\sum_{f\in H_k}^{h} L(1/2+u+v,f)L(1/2+u-v,f)L(1/2+\rho,f)$$

for $\mathrm{Re}(v) = 0$, $\mathrm{Re}(\rho) > 1 + \mathrm{Re}(u)$, and $0 < \mathrm{Re}(u) < k$. He derived an explicit formula comprising a main term $P_3(\log k)$ and an integral involving the fourth moment of $\zeta(s)$ weighted by a generalized hypergeometric function ${}_3F_2$:

$$\sum_{f\in H_k}^{h} L(1/2,f)^3 = P_3(\log k) + \frac{1}{2\pi \mathrm{i}}\int_{(0)} |\zeta(1/2+w)|^4 \widehat{g}(k,w)\,\mathrm{d}w + O(k^{-1+\varepsilon}),$$

where P_3 is given in (1.2) and

$$\widehat{g}(k,w) = \frac{2\pi^2\sin(\pi w)}{\cos^2(\pi w)}\,{}_3F_2(1-k,k,1/2+w;1,1,1).$$

By combining bounds for $L(1,\mathrm{sym}^2 f)$ (see (2.7) below), approximation for the fourth moment of $\zeta(s)$, and sharp estimates for $\widehat{g}(k,w)$, Frolenkov [6] obtained (with a correction in Humphries and Khan [10, (1.8)])

$$L(1/2,f) \ll k^{1/3}\log^{5/2} k.$$

In this paper, we concentrate on the cubic moment of $L(1/2,f)$ by taking extra averaging over the weight and get an asymptotic formula matching with Conjecture 1.1.

Theorem 1.2. *For large enough K and any $K^\varepsilon \le \Delta \le K^{1-\varepsilon}$, we have*

$$\begin{aligned}\sum_{k\equiv 0\,(\mathrm{mod}\,4)} h_{K,\Delta}(k)\sum_{f\in H_k}^{h} L(1/2,f)^3 \\ = \frac{\Delta\Pi}{4}P_3(\log K)+O\bigg(\frac{\Delta^3\log^2 K}{K^2}+K^\varepsilon\bigg),\end{aligned} \tag{1.5}$$

where P_3 is given in (1.2) *and the test function $h_{K,\Delta}(k)$ and Π are given by* (2.14) *as follows.*

Theorem 1.3. *For any $K^\varepsilon \le H \le K/3$, we have*

$$\sum_{\substack{|k-K|\le H\\ k\equiv 0\,(\mathrm{mod}\,4)}}\sum_{f\in H_k}^{h} L(1/2,f)^3 = \frac{1}{4}\int_{K-H}^{K+H} P_3(\log T)\,\mathrm{d}T + O(K^\varepsilon).$$

Notation. Throughout the paper, ε and A are arbitrarily small and arbitrarily large positive real numbers, which may be different at each occurrence.

As usual, $e(z)=\exp(2\pi\mathrm{i}z)=e^{2\pi\mathrm{i}z}$. The symbol $f\asymp g$ means that $c_1 g\le |f|\le c_2 g$ for some positive constants c_1 and c_2. We write $f=O(g)$ or $f\ll g$ to mean $|f|\le cg$ for some unspecified positive constant c. And $c\sim C$ means $C<c\le 2C$.

2. Preliminaries

In this section, we present some essential facts and tools needed later.

2.1. Approximate functional equations. Firstly, we give the approximate functional equations for $L(1/2,f)$ and $L(1/2,f)^2$.

Lemma 2.1. *Let $G(s)=e^{s^2}$. For any $f\in H_k$, we have*

$$L(1/2,f) = 2\sum_{n\ge 1}\frac{\lambda_f(n)}{\sqrt{n}}W_k(n), \quad \textit{for } k\equiv 0 \pmod 4, \tag{2.1}$$

and

$$L(1/2,f)^2 = 2\sum_{m\ge 1}\frac{\lambda_f(m)\tau(m)}{\sqrt{m}}V_k(m), \tag{2.2}$$

where

$$W_k(x) = \frac{1}{2\pi\mathrm{i}}\int_{(3)}(2\pi x)^{-s}G(s)\frac{\Gamma(s+k/2)}{\Gamma(k/2)}\frac{\mathrm{d}s}{s}, \tag{2.3}$$

and

$$V_k(x) = \frac{1}{2\pi \mathrm{i}} \int_{(3)} (4\pi^2 x)^{-s} G(s) \frac{\Gamma(s+k/2)^2}{\Gamma(k/2)^2} \zeta(1+2s) \frac{\mathrm{d}s}{s}. \tag{2.4}$$

Proof. See Iwaniec and Kowalski [14, Theorem 5.3]. □

The functions W_k and V_k have the following properties, which restrict the sums in (2.1) and (2.2) to $n \ll k^{1+\varepsilon}$ and $m \ll k^{2+\varepsilon}$, respectively, up to a negligible error.

Lemma 2.2. *For any $A > 0$ and integer $j \geq 0$, we have*

$$x^j W_k^{(j)}(x) \ll_{j,A} \left(1 + \frac{x}{k}\right)^{-A}, \quad x^j V_k^{(j)}(x) \ll_{j,A} \left(1 + \frac{x}{k^2}\right)^{-A}. \tag{2.5}$$

Furthermore, for $x \ll k^{2+\varepsilon}$, we have

$$V_k(x) = -\frac{1}{2}\log(4\pi^2 x) + \gamma + \log\frac{k}{2} - \frac{1}{k} + O\left(\frac{x}{k^{2-\varepsilon}}\right). \tag{2.6}$$

Proof. Moving the integrations of $V_k(x)$ and $W_k(x)$ to $\mathrm{Re}(s) = A$ with A sufficiently large, and using Stirling's formula (see Section 2.3 below), we can imply (2.5). To get (2.6), we move the line of the integration of $V_k(x)$ to $\mathrm{Re}(s) = -1 + \varepsilon$, and pick up a double pole at $s = 0$ with residue

$$-\frac{1}{2}\log(4\pi^2 x) + \gamma + \frac{\Gamma'(k/2)}{\Gamma(k/2)}.$$

Using Stirling's formula and (2.9), we complete the proof of this lemma. □

2.2. Petersson trace formula. For any complex number α_f depending on f, let

$$\sum_{f \in H_k}^{h} \alpha_f = \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{\alpha_f}{L(1, \mathrm{sym}^2 f)}$$

denote the "harmonic" average of α_f. In particular, $L(1, \mathrm{sym}^2 f)$ is positive and satisfies the bounds (see [15, Proposition 3.2] and [9, Appendix])

$$(\log k)^{-1} \ll L(1, \mathrm{sym}^2 f) \ll (\log k)^3. \tag{2.7}$$

Note that $\sum_{f \in H_k}^{h} 1 = 1 + O(2^{-k})$ by the following Petersson trace formula.

Lemma 2.3. *For any $m, n \geq 1$, we have*

$$\sum_{f \in H_k}^{h} \lambda_f(m)\lambda_f(n) = \delta_{m,n} + 2\pi \mathrm{i}^k \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where the value of $\delta_{m,n}$ *is* 1 *if* $m = n$ *and* 0 *otherwise,* J_{k-1} *is the Bessel function of the first kind and*

$$S(m,n;c) = \sum_{d \,(\mathrm{mod}\, c)}^{\star} e\bigg(\frac{md + n\bar{d}}{c}\bigg)$$

is the classical Kloosterman sum. Here the notation $\star$ *restricts the summation to the primitive residue classes and* $\bar{d}$ *denotes the multiplicative inverse of* d *modulo* c*.*

We recall here Weil's bound in [22] for the Kloosterman sum as follows:

$$|S(m,n;c)| \leq (m,n,c)^{1/2} c^{1/2} \tau(c).$$

2.3. Stirling's formula. For $s = \sigma + \mathrm{i}\tau$ with σ fixed and $|\tau| \geq 10$, we have Stirling's formula

$$\Gamma(\sigma + \mathrm{i}\tau) = |\tau|^{\sigma - 1/2} e^{-\pi|\tau|/2} \exp\bigg(\mathrm{i}\tau \log \frac{|\tau|}{e}\bigg)\Big(g_{\sigma,J}(\tau) + O_{\sigma,J}(|\tau|^{-J})\Big), \tag{2.8}$$

where

$$\tau^j \frac{\partial^j}{\partial \tau^j} g_{\sigma,J}(\tau) \ll_{j,\sigma,J} 1$$

for all fixed $j \in \mathbb{N}_0$. More precisely, we have

$$\log \Gamma(s) = s \log s + \frac{1}{2} \log \frac{2\pi}{s} + \frac{1}{12s} - \frac{1}{360 s^3} + \frac{1}{1260 s^5} + O(|s|^{-7}).$$

In addition, we have the asymptotic formula of the logarithmic derivative for $\Gamma(z)$,

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O_{\varepsilon}\bigg(\frac{1}{|z|^2}\bigg), \quad \text{for } |\arg z| \leq \pi - \varepsilon. \tag{2.9}$$

2.4. The large sieve inequality.

Lemma 2.4. *Let* a_n *be a sequence of complex numbers. Then for* $T \geq 1$*, we have*

$$\int_{-T}^{T} \bigg|\sum_{n \leq N} a_n n^{\mathrm{i}t}\bigg|^2 \mathrm{d}t \ll (T + N) \sum_{n \leq N} |a_n|^2.$$

Proof. See Young [23, Lemma 9.1]. $\square$

2.5. The Voronoi summation formula. To make our exposition succinct, we introduce the Bessel kernel $B_s(x)$ defined as (see [16, Definition 3.2]),

$$
\begin{aligned}
B_s(x) &= \frac{\pi}{\sin(\pi s)}\big(J_{-2s}(4\pi\sqrt{x}) - J_{2s}(4\pi\sqrt{x})\big)\\
B_s(-x) &= 4\cos(\pi s)K_{2s}(4\pi\sqrt{x}),
\end{aligned}
\tag{2.10}
$$

for $s \in \mathbb{C}$ and real $x > 0$. From (2.10) and Qi [20, (2.2)] we get for $x > 0$,

$$
B_0(x) = -2\pi Y_0(4\pi\sqrt{x}), \quad B_0(-x) = 4K_0(4\pi\sqrt{x}), \tag{2.11}
$$

and for $x > 1$,

$$
\begin{aligned}
B_0(x) &= \sum_{\pm} \frac{e(\pm(2\sqrt{x} + 1/8))}{x^{1/4}} W_0(\pm\sqrt{x}),\\
B_0(-x) &= O\bigg(\frac{\exp(-4\pi\sqrt{x})}{x^{1/4}}\bigg),
\end{aligned}
\tag{2.12}
$$

in which $x^j W_0^{(j)}(x) \ll_j 1$. From the above analysis (2.10)–(2.12), we can write the Voronoi summation formula for the divisor function $\tau(n)$ been from [14, (4.49)] in the following form.

Lemma 2.5. *Let $\omega \in C_c^\infty(0,\infty)$. Let $a, \bar{a}, c$ be integers with $c \geq 1$ and $a\bar{a} \equiv 1 \pmod{c}$. Then*

$$
\begin{aligned}
&c\sum_n \tau(n) e\Big(-\frac{an}{c}\Big)\omega(n)\\
&\qquad = 2(\gamma - \log c)\widetilde{\omega}_0(0) + \widetilde{\omega}_0'(0) + \sum_{n\neq 0} \tau(n) e\Big(\frac{\bar{a}n}{c}\Big)\widetilde{\omega}_0\Big(\frac{n}{c^2}\Big),
\end{aligned}
$$

where

$$
\widetilde{\omega}_0(0) = \int_0^\infty \omega(x)\,\mathrm{d}x, \qquad \widetilde{\omega}_0'(0) = \int_0^\infty \omega(x)\log x\,\mathrm{d}x,
$$

and $\widetilde{\omega}_0$ is the Hankel transform of ω (with kernel B_0) defined by

$$
\widetilde{\omega}_0(y) = \int_0^\infty \omega(x) B_0(xy)\,\mathrm{d}x, \quad \textit{for real } y \neq 0.
$$

In addition, for any $\delta \in \mathbb{R}$, we define

$$
\widetilde{w}_{\pm\delta}(0) = \int_0^\infty w(x) x^{\pm\delta}\,\mathrm{d}x,
$$

then

$$
2(\gamma - \log c)\widetilde{\omega}_0(0) + \widetilde{\omega}_0'(0) = \lim_{\delta\to 0}\sum_{\pm} \frac{\zeta(1 \pm 2\delta)\widetilde{w}_{\pm\delta}(0)}{c^{\pm 2\delta}}.
$$

Our last analytic lemma is on the Hankel transform of a special kind of functions that involve $x^{\mathrm{i}t}$ (see [20, Lemma 5.1 & Corollary 5.2]).

Lemma 2.6. *Let $t, N > K^{\varepsilon}$. For fixed $v(x) \in C_c^{\infty}(0,\infty)$, we have*

$$\sum_{n \asymp N} \frac{\tau(n)}{n^{1/2-\mathrm{i}t}} v(n/N) = \sum_{m \asymp t^2/N} \frac{\tau(m)}{m^{1/2+\mathrm{i}t}} \widetilde{v}_0(Nm;t) + O(T^{-A}),$$

where

$$\widetilde{v}_0(y;t) = y^{1/2+\mathrm{i}t} \int_0^{\infty} v(x) B_0(xy) \frac{\mathrm{d}x}{x^{1/2-\mathrm{i}t}}$$

is bounded but negligibly small unless $y \asymp t^2$.

2.6. Analysis of the integral transforms. Let $w_1(x), w_2(x) \in C_c^{\infty}(0,\infty)$ be such that $w_1^{(j)}(x)$, $w_2^{(j)}(x) \ll_j U^j$ for some large U. Let

$$J(u; y_1, y_2) = \int_0^{\infty} w_1(y) w_2(u/y) e(-yy_1) B_0(uy_2/y) \frac{\mathrm{d}y}{y}. \tag{2.13}$$

Lemma 2.7. *Assume that $|y_2| > k^{\varepsilon}$. Then $J(u;y_1,y_2) = O(k^{-A})$ unless $y_2 > k^{\varepsilon}$ is positive and*

$$|y_1| \asymp \sqrt{y_2},$$

and under these conditions, there is a smooth function $W(\lambda; y_1, y_2)$ with support in $\lambda \asymp \sqrt[3]{y_1 y_2}$ satisfying

$$\lambda^j \frac{\partial^j W(\lambda; y_1, y_2)}{\partial \lambda^j} \ll_j U^j,$$

with the implied constants uniform in y_1 and y_2, such that

$$J(u; y_1, y_2) = \frac{e(-3\sqrt[3]{uy_1y_2})}{\sqrt[3]{|uy_1y_2|}} W(\sqrt[3]{uy_1y_2}; y_1, y_2) + O(k^{-A}).$$

Proof. See [20, Lemma 4.1]. □

Moreover, we have the following estimates for exponential integrals. Let

$$I = \int_{\mathbb{R}} w(y) e^{\mathrm{i}\varrho(y)} \, \mathrm{d}y,$$

we need the evaluation for exponential integrals which are of [2, Lemma 8.1 & Proposition 8.2] in the language of inert functions.

Let $\mathcal{F}$ be an index set, $Y : \mathcal{F} \to \mathbb{R}_{\geq 1}$ and under this map $T \mapsto Y_T$ be a function of $T \in \mathcal{F}$. A family $\{w_T\}_{T \in \mathcal{F}}$ of smooth functions supported on a product of dyadic intervals in $\mathbb{R}_{>0}^d$ is called Y-inert if for each $j = (j_1, \dots, j_d) \in \mathbb{Z}_{\geq 0}^d$ we have

$$C(j_1,\dots,j_d) = \sup_{T \in \mathcal{F}} \sup_{(y_1,\dots,y_d) \in \mathbb{R}_{>0}^d} Y_T^{-j_1-\cdots-j_d} \left| y_1^{j_1} \cdots y_d^{j_d} w_T^{(j_1,\dots,j_d)}(y_1,\dots,y_d) \right| < \infty.$$

Lemma 2.8. *Suppose that $w = w_T(y)$ is a family of Y-inert functions, with compact support on $[Z, 2Z]$, so that $w^{(j)}(y) \ll (Z/Y)^{-j}$. Also suppose that ϱ is smooth and satisfies $\varrho^{(j)}(y) \ll H/Z^j$ for some $H/Y^2 \geq R \geq 1$ and all y in the support of w.*

(1) *If $|\varrho'(y)| \gg H/Z$ for all y in the support of w, then $I \ll_A ZR^{-A}$ for A arbitrarily large.*
(2) *If $\varrho''(y) \gg H/Z^2$ for all y in the support of w, and there exists $y_0 \in \mathbb{R}$ such that $\varrho'(y_0) = 0$ (note y_0 is necessarily unique), then*

$$I = \frac{e^{\mathrm{i}\varrho(y_0)}}{\sqrt{\varrho''(y_0)}} F^\sharp(y_0) + O_A(ZR^{-A}),$$

where $F^\sharp(y_0)$ is an Y-inert function (depending on A) supported on $y_0 \asymp Z$.

2.7. The choice of test function. Let $K^\varepsilon \leq \Delta \leq K^{1-\varepsilon}$ and define the function

$$h_{K,\Delta}(t) := w\left(\frac{|t-K|}{\Delta}\right), \quad \text{with } \Pi := \int_{-\infty}^{\infty} w(|x|)\,\mathrm{d}x. \tag{2.14}$$

where w is an arbitrary smooth function compactly supported on $\mathbb{R}^+$. The function $h_{K,\Delta}(t)$ effectively restricts t to the interval $[K - \Delta \log K, K + \Delta \log K]$. A suitable choice for the weight $w(x)$ is to set it equal to 1 on $[0, 1]$, zero for $x \geq 3/2$, and to have polynomial decay in the transition regions. Alternatively, one could adapt the test function from [16, Section 5]; in this case, it must be multiplied by an additional bump function to constrain its support to a compact subset of $\mathbb{R}^+$, which is essential for the subsequent analysis. Given $3H \leq K$ and $K^\varepsilon \leq \Delta \leq H^{1-\varepsilon}$, we define the averaged function

$$\omega(t) = \omega_{K,\Delta,H}(t) = \frac{1}{\pi^{1/2}\Delta} \int_{K-H}^{K+H} h_{T,\Delta}(t)\,\mathrm{d}T.$$

Let $\chi(t) = \chi_{K,H}(t)$ be the characteristic function for $|t - K| \leq H$. By adapting the arguments in [12, Section 3] or [16, Section 5], it is easy to show that $\omega(t) - 1$ is exponentially small if $|t - K| \leq H - \Delta^{1+\varepsilon}$ and

$$\omega(t) - \chi(t) = O\left(\frac{\Delta^3}{(\Delta + \min\{|t - K \pm H|\})^3}\right), \quad \text{if } |t - K \pm H| \leq \Delta^{1+\varepsilon},$$

and $\omega(t)$ is exponentially small if otherwise.

Combined these, one can prove the following lemma, which is the counterpart of [16, Lemma 5.3] in the weight aspect.

Lemma 2.9. *Let λ be a real constant. Suppose that $a_f \geq 0$ and that*

$$\sum_{k\equiv 0 \pmod 4} h_{K,\Delta}(k) \sum_{f\in H_k}^{h} a_f = O_{\lambda,\varepsilon}(\Delta K^{\lambda})$$

for any Δ with $K^{\varepsilon} \leq \Delta \leq K^{1-\varepsilon}$. Then for $\Delta^{1+\varepsilon} \leq 3H \leq K$ we have

$$\sum_{\substack{k\equiv 0 \pmod 4\\ |k-K|\leq H}} \sum_{f\in H_k}^{h} a_f = \sum_{k\equiv 0 \pmod 4} \omega(k) \sum_{f\in H_k}^{h} a_f + O_{\lambda,\varepsilon}(\Delta K^{\lambda}).$$

As a simple application, we have the following lemma.

Lemma 2.10. *For $K^{\varepsilon} \leq \Delta^{1+\varepsilon} \leq H \leq K/3$, we have*

$$\begin{aligned}(2.15)\qquad &\sum_{\substack{k\equiv 0 \pmod 4\\ |k-K|\leq H}} \sum_{f\in H_k}^{h} L(1/2,f)^3 \\ &= \frac{1}{\sqrt{\pi}\Delta} \int_{K-H}^{K+H} \sum_{k\equiv 0 \pmod 4} h_{T,\Delta}(t) \sum_{f\in H_k}^{h} L(1/2,f)^3 \, \mathrm{d}T + O(\Delta K^{\varepsilon}),\end{aligned}$$

where the error term is derived by (1.3).

2.8. J-Bessel function. Let $J_\nu(z)$ be the Bessel function of the first kind (see Watson [21]). We shall be only concerned with $J_{2k-1}(x)$ for real $x > 0$ and integer $k \geq 1$. First, we have the following standard estimate (see [21, Sections 2.11, 7.1]),

$$J_{k-1}(x) \ll_k \begin{cases} x^{k-1}, & \text{if } x \leq 1,\\ x^{-1/2}, & \text{if } x > 1.\end{cases}$$

Moreover, we have

$$\begin{aligned} &|J_{k-1}(x)| \leq 1, && \text{for all } x \geq 0,\\ &J_{k-1}(x) \sim (1/\Gamma(k))(x/2)^{k-1}, && \text{for } x \leq k^{1/2-\varepsilon},\end{aligned}$$

and

$$J_{k-1}(x) \sim ((\pi/2)x)^{-1/2} \cos(x - (\pi/2)k + \pi/4), \qquad \text{for } x \geq k^{2+\varepsilon}.$$

The Mellin transform is given by

$$(2.16)\qquad \int_0^{\infty} J_{k-1}(x) x^{s-1}\, \mathrm{d}x = 2^{s-1} \frac{\Gamma(\frac{k-1+s}{2})}{\Gamma(\frac{k+1-s}{2})}.$$

Next, we consider the following weighted average of J-Bessel functions over $k \equiv 0 \pmod 4$:

$$(2.17)\qquad B^{\mathrm{hol}}(x) = \sum_{k\equiv 0 \pmod 4} h_{2K+1,\Delta}(k) J_{k-1}(x),$$

where the weight function $h_{K,\Delta}$ is defined in (2.14). By Iwaniec [13, pp. 85–86] or Young [23, p. 1563], we have

$$B^{\mathrm{hol}}(x) = \frac{1}{4}\int_{\mathbb{R}} W(t)c(t)\,\mathrm{d}t,$$

where

$$W(t) = \int_{\mathbb{R}} h_{2K,\Delta}(y)e(ty)\,\mathrm{d}y,$$

and

$$c(t) = -2\mathrm{i}\sin(x\sin(2\pi t)) - 2\sin(x\cos(2\pi t)).$$

Through a direct calculation, there exists a function $g(x)$ satisfying $g^{(j)}(x) \ll_{j,A} (1+|x|)^{-A}$ such that

$$B^{\mathrm{hol}}(x) = \Delta\int_{-\infty}^{\infty} e\Big(\frac{vK}{\pi}\Big)g(\Delta v)(-2\mathrm{i}\sin(x\sin v) - 2\sin(x\cos v))\,\mathrm{d}v.$$

One can easily show that $B^{\mathrm{hol}}(x) \ll x$ since $\sin x \ll x$. Let

$$\phi(v) = \pm\cos(v) \quad \text{or} \quad \pm\sin(v), \tag{2.18}$$

then $B^{\mathrm{hol}}(x)$ can be split into a combination of $B^{\pm}(x)$:

$$B^{\pm}(x) = \Delta\int_{-\infty}^{\infty} e\Big(\frac{vK}{\pi}\Big)g(\Delta v)e\Big(\frac{x}{2\pi}\phi(v)\Big)\,\mathrm{d}v,$$

where $B^{+}(x)$ corresponds to $\phi(v) = \pm\cos(v)$, and $B^{-}(x)$ corresponds to $\phi(v) = \pm\sin(v)$.

For the estimates of $B^{\pm}(x)$, we invoke the following lemmas from [23, Lemmas 7.1 & 7.2].

Lemma 2.11. *The following two statements hold.*

(1) *We have*

$$B^{+}(x) = \Delta\int_{|v|\le\Delta^{\varepsilon}/\Delta} e\Big(\frac{vK}{\pi}\Big)g(\Delta v)\cos(x\phi(v))\,\mathrm{d}v + O(K^{-A}).$$

Moreover, $B^{+}(x) \ll K^{-A}$ unless $x \gg \Delta K^{1-\varepsilon}$, and if $x \ll K$, one has $B^{+}(x) \ll \Delta x/K$.

(2) *We have*

$$B^{-}(x) = \Delta\int_{|v|\le\Delta^{\varepsilon}/\Delta} e\Big(\frac{vK}{\pi}\Big)g(\Delta v)\cos(x\phi(v))\,\mathrm{d}v + O(K^{-A}).$$

Moreover, $B^{-}(x) \ll (x+K)^{-A}$ unless $x \asymp K$, and for $x \ll K$, one has $B^{-}(x) \ll_{\delta} \Delta K^{\delta-1}x^{1-\delta}$ for any $\delta \in (0,1)$.

Remark 2.12. In defining B^{hol} in (2.17), the center of $h_{K,\Delta}$ at $2K+1$ is a technical choice to streamline intermediate computations. Crucially, B^{hol} arises solely in the treatment of off-diagonal terms, its impact is entirely confined to implied constants in O-terms. Consequently, the main term remains unaltered, and all asymptotic conclusions hold verbatim. For brevity, we will henceforth use B^{hol} without further reference to this localized choice.

3. Proof of Theorem 1.2

In this section, we consider the smooth cubic moment,

$$\sum_{k\equiv 0\,(\mathrm{mod}\,4)} h_{K,\Delta}(k) \sum_{f\in H_k}^{h} L(1/2,f)^3,$$

where $h_{K,\Delta}$ is given by (2.14). For $k\equiv 0 \pmod 4$, by the approximate functional equations in Lemma 2.1 and the Petersson trace formula in Lemma 2.3, we have

$$\begin{aligned}\sum_{f\in H_k}^{h} L(1/2,f)^3 &= 4\sum_{m\geq 1}\sum_{n\geq 1}\frac{\tau(m)}{\sqrt{mn}}\sum_{f\in H_k}^{h}\lambda_f(m)\lambda_f(n)V_k(m)W_k(n)\\ &:= \mathcal{D}+\mathcal{ND},\end{aligned}$$

where the diagonal term is

$$\mathcal{D} = 4\sum_{m\geq 1}\frac{\tau(m)}{m}V_k(m)W_k(m),$$

and the non-diagonal term is

$$\mathcal{ND} = 8\pi\sum_{m,n\geq 1}\frac{\tau(m)}{\sqrt{mn}}\sum_{c\geq 1}\frac{S(n,m;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)V_k(m)W_k(n).$$

3.1. The off-diagonal term $\mathcal{ND}$. Using the definitions (2.3) and (2.4) of $W_k(n)$ and $V_k(m)$, we have

$$\begin{aligned}\mathcal{ND} = \frac{8\pi}{(2\pi \mathrm{i})^2}\int_{(3)}\int_{(3)}\sum_{c\geq 1}\sum_{m,n\geq 1}&\frac{\tau(m)}{n^{1/2+v_1}m^{1/2+v_2}}\\ &\times (2\pi)^{-v_1-2v_2}\frac{S(n,m;c)}{c}\frac{\Gamma(k/2+v_1)}{\Gamma(k/2)}\frac{\Gamma(k/2+v_2)^2}{\Gamma(k/2)^2}\\ &\qquad\times J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\zeta(1+2v_2)G(v_1)G(v_2)\frac{\mathrm{d}v_1}{v_1}\frac{\mathrm{d}v_2}{v_2}.\end{aligned}$$

Inserting the smooth dyadic partitions for m and n-sums, respectively, up to a negligible error term, we can transform $\mathcal{ND}$ into

$$\begin{aligned}\frac{8\pi}{(2\pi\mathrm{i})^2}\int_{(3)}\int_{(3)}\frac{1}{N^{1/2+v_1}M^{1/2+v_2}}\sum_{c\geq 1}\sum_{m,n\geq 1}\tau(m)w\Big(\frac{n}{N};v_1\Big)w\Big(\frac{m}{M};v_2\Big)\\ \times(2\pi)^{-v_1-2v_2}\frac{S(n,m;c)}{c}\frac{\Gamma(k/2+v_1)}{\Gamma(k/2)}\frac{\Gamma(k/2+v_2)^2}{\Gamma(k/2)^2}\\ \times J_{k-1}\bigg(\frac{4\pi\sqrt{mn}}{c}\bigg)\zeta(1+2v_2)G(v_1)G(v_2)\frac{\mathrm{d}v_1}{v_1}\frac{\mathrm{d}v_2}{v_2},\end{aligned}$$

for dyadic $1/2 < M \leq k^{2+\varepsilon}$ and $1/2 < N \leq k^{1+\varepsilon}$, where

$$w(x;v_i)=\frac{\nu(x)}{x^{1/2+v_i}},\quad i=1,2,$$

for some suitable function $\nu(x)\in C_c^\infty[1,2]$ satisfying $x^j\nu^{(j)}(x)\ll_j 1$ for any $j\geq 0$.

For the m-sum, we open the Kloosterman sum and apply the Voronoi summation formula in Lemma 2.5. Denote $h(x;n)=w\big(\frac{x}{M};v_2\big)J_{k-1}\Big(\frac{4\pi\sqrt{xn}}{c}\Big)$, we have

$$\begin{aligned}m\text{-sum}&=\sum_m\tau(m)e\Big(\frac{m\alpha}{c}\Big)h(m;n)\\ &=\frac{1}{c}\Big(2(\gamma-\log c)\widetilde{h}_0(0)+\widetilde{h}_0'(0)\Big)+\frac{1}{c}\sum_{m\neq 0}\tau(m)e\Big(-\frac{\bar{\alpha}m}{c}\Big)\widetilde{h}_0\Big(\frac{m}{c^2};n\Big).\end{aligned}$$

For the entire zero-frequency, up to a negligible error, we will reverse the procedures of truncation and partition of sums. More precisely, we deal with the terms

$$\begin{aligned}\frac{8\pi}{(2\pi\mathrm{i})^2}\int_{(3)}\int_{(3)}\sum_{c\geq 1}\sum_{n\geq 1}\frac{1}{n^{1/2+v_1}}\frac{S(n,0;c)}{c^2}\\ \times\big(2(\gamma-\log c)\widetilde{\omega}_0(0)+\widetilde{\omega}_0'(0)\big)(2\pi)^{-v_1-2v_2}\frac{\Gamma(k/2+v_1)}{\Gamma(k/2)}\\ \times\frac{\Gamma(k/2+v_2)^2}{\Gamma(k/2)^2}\zeta(1+2v_2)G(v_1)G(v_2)\frac{\mathrm{d}v_1}{v_1}\frac{\mathrm{d}v_2}{v_2},\end{aligned}\tag{3.1}$$

where $\omega(x;n)=x^{-1/2-v_2}J_{k-1}\Big(\frac{4\pi\sqrt{xn}}{c}\Big)$. Let

$$\sum_{c\geq 1}\frac{S(n,0;c)}{c^2}\big(2(\gamma-\log c)\widetilde{\omega}_0(0)+\widetilde{\omega}_0'(0)\big):=\widetilde{Z}(n,v_2),$$

where

$$\widetilde{Z}(n,v_2)=\lim_{\delta\to 0}\sum_{\pm}\sum_{c\geq 1}\frac{\zeta(1\pm 2\delta)}{c^{2\pm 2\delta}}S(n,0;c)\,\widetilde{\omega}_{\pm\delta}\Big(\frac{n}{c^2},\frac{1}{2}-v_2\pm\delta\Big), \tag{3.2}$$

with

$$\widetilde{\omega}_{\pm\delta}(y,s)=\int_0^\infty J_{k-1}(4\pi\sqrt{xy})x^{s-1}\mathrm{d}x=(2\pi\sqrt{y})^{-2s}\frac{\Gamma\Big(\frac{k-1}{2}+s\Big)}{\Gamma\Big(\frac{k+1}{2}-s\Big)},$$

by making changes of variable $x\to x'/y$ and $4\pi\sqrt{x'}\to x''$ and using (2.16). Using the Ramanujan sum $S(n,0;c)=\sum_{d|(n,c)}d\mu(\frac{c}{d})$ and the definition $\tau_s(n)=\sum_{ab=n}\big(\frac{a}{b}\big)^s$, we obtain that the c-sum in (3.2) is equal to

$$\sum_c\frac{S(n,0;c)}{c^{2\pm 2\delta-1+2v_2\mp 2\delta}}=\sum_c\frac{S(n,0;c)}{c^{1+2v_2}}=\frac{\tau_{v_2}(n)}{n^{v_2}\zeta(1+2v_2)}.$$

Thus, by means of the above equation and the definition (2.3) of $W_k(n)$, we get the zero-frequency term (3.1) becomes

$$\mathcal{D}':=\frac{2}{\pi\mathrm{i}}\int_{(3)}\sum_n\frac{\tau_{v_2}(n)}{n}W_k(n)G(v_2)\frac{\Gamma(k/2+v_2)^2}{\Gamma(k/2)^2}Z(n,v_2)\frac{\mathrm{d}v_2}{v_2}, \tag{3.3}$$

where

$$Z(n,v)=\lim_{\delta\to 0}\sum_{\pm}\zeta(1\pm 2\delta)\frac{1}{(4\pi^2n)^{\pm\delta}}\frac{\Gamma(-v\pm\delta+k/2)}{\Gamma(v\mp\delta+k/2)}.$$

By analyzing the Laurent expansion of $\zeta(1\pm 2\delta)$ at its simple pole (see (3.5) below) and the Taylor expansions of the remaining factors in $Z(n,v)$ at $\delta=0$, and then taking the limit $\delta\to 0$, we arrive at

$$Z(n,v)=\frac{\Gamma(k/2-v)}{\Gamma(k/2+v)}\Big(2\gamma+\psi(k/2-v)+\psi(k/2+v)-\log(4\pi^2n)\Big),$$

where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Since $\tau_{v_2}(n)$ and $G(v_2)$ are even functions by definition, it follows that $\tau_{v_2}(n)\frac{\Gamma(k/2+v_2)^2}{\Gamma(k/2)^2}G(v_2)Z(n,v_2)$ is even in the v_2-variable. Therefore the v_2-integral in (3.3) is equal to exactly its value at $v_2=0$ (to see this, apply $v\to -v$ to half of the integral), i.e., $\tau(n)\Big(-\frac{1}{2}\log(4\pi^2n)+\gamma+\frac{\Gamma'(k/2)}{\Gamma(k/2)}\Big)$. Thus we conclude that

$$\mathcal{D}'=4\sum_{n\geq 1}\frac{\tau(n)}{n}\Big(-\frac{1}{2}\log(4\pi^2n)+\gamma+\frac{\Gamma'(k/2)}{\Gamma(k/2)}\Big)W_k(n).$$

Now, with the help of (2.6), $\mathcal{D}$ and $\mathcal{D}'$ are equal to each other up to an error term $O(k^{-1+\varepsilon})$.

3.2. The main term $\mathcal{D}+\mathcal{D}'$. Using the definition (2.3) of $W_k(m)$, the asymptotic formula (2.6) of $V_k(m)$, and changing the order of the integral and summation, we have that $\mathcal{D}+\mathcal{D}'$ is given by

$$\begin{aligned}(3.4)\quad \frac{4}{\pi \mathrm{i}}\int_{(3)}\frac{G(s)}{(2\pi)^s}\frac{\Gamma(s+k/2)}{\Gamma(k/2)}\Big(\zeta(1+s)\zeta'(1+s)\\ +\zeta^2(1+s)\big(\gamma+\log k-\log 4\pi\big)\Big)\frac{\mathrm{d}s}{s}+O(k^{-1+\varepsilon}),\end{aligned}$$

where we used the Dirichlet series of $\zeta^2(s)$ and $\zeta(s)\zeta'(s)$,

$$\zeta^2(s)=\sum_{m=1}^{\infty}\frac{\tau(m)}{m^s}\quad\text{and}\quad \zeta(s)\zeta'(s)=-\frac{1}{2}\sum_{m=1}^{\infty}\frac{\tau(m)\log m}{m^s}.$$

For $s\to 1$, the following Laurent series holds for $\zeta(s)$,

$$(3.5)\qquad \zeta(s)=\frac{1}{s-1}+\gamma-\gamma_1(s-1)+\frac{\gamma_2}{2}(s-1)^2+O(|s|^3).$$

Then, for $s\to 0$, we have

$$\zeta^2(1+s)=\frac{1}{s^2}+\frac{2\gamma}{s}+\gamma^2-2\gamma_1+O(|s|),$$

and

$$\zeta(1+s)\zeta'(1+s)=-\frac{1}{s^3}-\frac{\gamma}{s^2}-\gamma\gamma_1+\frac{\gamma_2}{2}+O(|s|).$$

Moreover, as $s\to 0$, we have the following Taylor expansion

$$\frac{\Gamma(s+k/2)}{\Gamma(k/2)}=1+\psi_1(k)s+\frac{\psi_2(k)}{2}s^2+\frac{\psi_3(k)}{6}s^3+O\Big((\log k)^4|s|^4\Big),$$

where $\psi_j(k)=\frac{\Gamma^{(j)}(k/2)}{\Gamma(k/2)}=\Big(\log\frac{k}{2}\Big)^j+O_j\Big(\frac{(\log k)^{j-1}}{k}\Big)$, for $j\geq 1$, by Stirling's formula (2.8) and the logarithmic derivative (2.9) for $\Gamma(z)$. Shifting the integral contour in (3.4) to $\mathrm{Re}(s)=-1+\varepsilon$ and then calculating the residues at $s=0$, it follows that (3.4) turns into

$$P_3(\log k)+O\Big(k^{-1+\varepsilon}\Big),$$

with P_3 given in (1.2). Then the contribution from the error term to the left of (1.5) is $O(\Delta K^{-1+\varepsilon})$. Now we are lead to deal with the contribution from the main term. Define

$$g(m)=w\bigg(\frac{|4m-K|}{\Delta}\bigg)P_3(\log(4m)).$$

Then by the Poisson summation formula, we obtain

$$\sum_{k\equiv 0\,(\mathrm{mod}\,4)}h_{K,\Delta}(k)P_3(\log k)=\sum_{m\in\mathbb{Z}}g(m)=\widehat{g}(0)+\sum_{m\neq 0}\widehat{g}(m),$$

where

$$\begin{aligned}\widehat{g}(0) &= \int_{-\infty}^{\infty} w\biggl(\frac{|4x-K|}{\Delta}\biggr) P_3(\log(4x))\,\mathrm{d}x\\ &= \frac{1}{4}\int_{-\infty}^{\infty} w\biggl(\frac{|u|}{\Delta}\biggr) P_3(\log(u+K))\,\mathrm{d}u,\end{aligned}$$

by performing the change of variables $u = 4x - K$. Since $w(|u|/\Delta)$ is even, we expand $P_3(\log(u+K))$ about $u = 0$ and arrive at

$$\widehat{g}(0) = \frac{1}{4}\int_{-\infty}^{\infty} w\biggl(\frac{|u|}{\Delta}\biggr)\biggl(P_3(\log K) + \frac{P_3''(\log K) - P_3'(\log K)}{2K^2}u^2 + \cdots\biggr)\,\mathrm{d}u,$$

where the odd terms vanish due to symmetry. The higher-order terms contribute an error of $O(\Delta^3 \log^2 K/K^2)$. Hence,

$$\widehat{g}(0) = \frac{\Delta\Pi}{4}P_3(\log K) + O\biggl(\frac{\Delta^3}{K^2}\log^2 K\biggr).$$

For $m \neq 0$, we have

$$\widehat{g}(m) = \frac{1}{4}\int_{-\infty}^{\infty} w\biggl(\frac{|u|}{\Delta}\biggr) P_3(\log(u+K)) e\biggl(-\frac{u+K}{4}m\biggr)\,\mathrm{d}u.$$

Integrating by parts twice yields

$$\widehat{g}(m) \ll \frac{1}{m^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}u^2}\biggl(w\biggl(\frac{|u|}{\Delta}\biggr) P_3(\log(u+K))\biggr)\,\mathrm{d}u,$$

where the implied constant is independent of K and Δ. Therefore,

$$\sum_{m\neq 0} \widehat{g}(m) \ll \sum_{m\neq 0} \frac{1}{m^2}\cdot\frac{\log^3 K}{\Delta} \ll K^{\varepsilon}.$$

Under the condition $K^{\varepsilon} \leq \Delta \leq K^{1-\varepsilon}$, the error term $O\bigl(\frac{\Delta^3}{K^2}\log^2 K\bigr)$ may dominate K^{ε}, while other error terms are bounded by K^{ε}. Thus we conclude that

$$\sum_{k\equiv 0\,(\mathrm{mod}\,4)} h_{K,\Delta}(k) P_3(\log k) = \frac{\Delta\Pi}{4}P_3(\log K) + O\biggl(\frac{\Delta^3\log^2 K}{K^2} + K^{\varepsilon}\biggr). \tag{3.6}$$

3.3. The error term $\boldsymbol{\mathcal{S}(M,N)}$. Next, we denote the remainder part by $\mathcal{S}(M,N)$, which is given

$$\begin{aligned}
(3.7)\quad & \mathcal{S}(M,N) \\
&= \frac{8\pi}{(2\pi \mathrm{i})^2}\int_{(3)}\int_{(3)} \frac{G(v_1)G(v_2)}{N^{1/2+v_1}M^{1/2+v_2}} \\
&\quad\times \sum_{c\geq 1}\sum_{m\neq 0}\sum_{n\geq 1}\tau(m)w\Big(\frac{n}{N};v_1\Big)\frac{S(n-m,0;c)}{c^2}\widetilde{h}_0\Big(\frac{m}{c^2};n\Big) \\
&\quad\times (2\pi)^{-v_1-2v_2}\frac{\Gamma(k/2+v_1)}{\Gamma(k/2)}\frac{\Gamma(k/2+v_2)^2}{\Gamma(k/2)^2}\zeta(1+2v_2)\frac{\mathrm{d}v_1}{v_1}\frac{\mathrm{d}v_2}{v_2}.
\end{aligned}$$

For n-sum, we open the Ramanujan sum $S(n-m,0;c)$ and apply the Poisson summation formula,

$$\begin{aligned}
n\text{-sum} &= \sum_{n\geq 1} w\Big(\frac{n}{N};v_1\Big)e\Big(\frac{\alpha n}{c}\Big)\widetilde{h}_0\Big(\frac{m}{c^2};n\Big) \\
&= \sum_{n\equiv -\alpha\,(\mathrm{mod}\,c)}\int_{\mathbb{R}} w\Big(\frac{y}{N};v_1\Big)\widetilde{h}_0\Big(\frac{m}{c^2};y\Big)e\Big(-\frac{yn}{c}\Big)\mathrm{d}y,
\end{aligned}$$

where

$$\widetilde{h}_0\Big(\frac{m}{c^2};n\Big) = M\int_0^\infty w(x;v_2)J_{k-1}\bigg(\frac{4\pi\sqrt{xnM}}{c}\bigg)B_0\Big(\frac{mM}{c^2}x\Big)\mathrm{d}x.$$

Now the dual exponential sum

$$\sideset{}{^\star}\sum_{\substack{\alpha\,(\mathrm{mod}\,c)\\ n\equiv -\alpha\,(\mathrm{mod}\,c)}} e\Big(-\frac{m\alpha}{c}\Big)$$

reduces to $e(\frac{mn}{c})$ along with the condition $(n,c)=1$. Note that the zero-frequency in the n-sum exists only when $c=1$, the m,n,c-sums of (3.7) become

$$\begin{aligned}
M^{1/2-v_2}N^{1/2-v_1}\sum_{m\neq 0}\tau(m)\int_{\mathbb{R}}\int_0^\infty & w(y;v_1)w(x;v_2) \\
&\times J_{k-1}\Big(4\pi\sqrt{xyMN}\Big)B_0(mMx)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By a change of variable $xy\to u$, the integral above becomes

$$\begin{aligned}
\int_0^\infty\int_0^\infty w(y;v_1)w(u/y;v_2)J_{k-1}\Big(4\pi\sqrt{uMN}\Big)B_0\Big(\frac{umM}{y}\Big)\frac{\mathrm{d}y}{y}\mathrm{d}u& \\
= \int_0^\infty J_{k-1}\Big(4\pi\sqrt{uMN}\Big)J(u;0,mM)\,\mathrm{d}u&,
\end{aligned}$$

where $J(u;y_1,y_2)$ is defined by (2.13). For $mM \gg k^{\varepsilon}$, the contribution from this part is negligibly small by Lemma 2.7. For the complementary part $mM \ll k^{\varepsilon}$, it is evident that $\sqrt{MN} \ll k^{1-\varepsilon}$. Using the trivial estimate and the fact $J_{k-1}(x) \ll e^{-k}$ for $x < k/10$, the contribution from this part is $O(k^{-1+\varepsilon})$. We are left to consider the non-zero frequency

$$M^{1/2-v_2}N^{1/2-v_1}\sum_{c\geq 1}\sum_{\substack{m,n\neq 0\\(n,c)=1}}\frac{\tau(m)}{c^2}e\Big(\frac{mn}{c}\Big)\int_{\mathbb{R}}\int_0^{\infty} w(y;v_1)w(x;v_2) \\ \times e\Big(-\frac{nNy}{c}\Big)J_{k-1}\bigg(\frac{4\pi\sqrt{xyMN}}{c}\bigg)B_0\Big(\frac{mMx}{c^2}\Big)\,\mathrm{d}x\mathrm{d}y.$$

Next, we need to apply an additional average over weight k since the individual J-Bessel function lacks the satisfactory asymptotic formula at this stage, and deal with

$$M^{1/2-v_2}N^{1/2-v_1}\sum_{c\geq 1}\sum_{\substack{m,n\neq 0\\(n,c)=1}}\frac{\tau(m)}{c^2}e\Big(\frac{mn}{c}\Big) \\ \times \sum_{k\equiv 0\,(\mathrm{mod}\,4)} h_{K,\Delta}(k)\int_{\mathbb{R}}\int_0^{\infty} w(y;v_1)w(x;v_2) \\ \times e\Big(-\frac{nNy}{c}\Big)J_{k-1}\bigg(\frac{4\pi\sqrt{xyMN}}{c}\bigg)B_0\Big(\frac{mMx}{c^2}\Big)\,\mathrm{d}x\mathrm{d}y,$$

which can be rewritten as

$$\mathcal{S}^{\star}(M,N) := M^{1/2-v_2}N^{1/2-v_1} \\ \times\sum_{c\geq 1}\sum_{\substack{m,n\neq 0\\(n,c)=1}}\frac{\tau(m)}{c^2}e\Big(\frac{mn}{c}\Big)\int_{\mathbb{R}}\int_0^{\infty} w(y;v_1)w(x;v_2) \\ \times e\Big(-\frac{nNy}{c}\Big)B^{\mathrm{hol}}\bigg(\frac{4\pi\sqrt{xyMN}}{c}\bigg)B_0\Big(\frac{mMx}{c^2}\Big)\,\mathrm{d}x\mathrm{d}y,$$

where B^{hol} is given by (2.17). By the discussion of Section 2.8, we are lead to estimate

$$\mathcal{S}^{\pm}(M,N) = M^{1/2-v_2}N^{1/2-v_1} \\ \times\sum_{c\geq 1}\sum_{\substack{m,n\neq 0\\(n,c)=1}}\frac{\tau(m)}{c^2}e\Big(\frac{mn}{c}\Big)\int_{\mathbb{R}}\int_0^{\infty} w(y;v_1)w(x;v_2) \\ \times e\Big(-\frac{nNy}{c}\Big)B^{\pm}\bigg(\frac{4\pi\sqrt{xyMN}}{c}\bigg)B_0\Big(\frac{mMx}{c^2}\Big)\,\mathrm{d}x\mathrm{d}y. \tag{3.8}$$

By Lemma 2.11, we have $c \ll \frac{\sqrt{MN}}{\Delta K^{1-\varepsilon}}$ in the positive case and $c \asymp \frac{\sqrt{MN}}{K}$ in the negative case, up to a negligible term. Then we have

$$\frac{mM}{c^2} \geq \frac{M}{c^2} \gg \frac{K^{2-\varepsilon}}{N} \gg K^{1-\varepsilon}.$$

Changing of variable $xy \to u$, the double integrals in (3.8) become

$$\begin{aligned}\int_0^\infty \int_0^\infty & w(y;v_1)w(u/y;v_2)e\Big(-\frac{nNy}{c}\Big) \\ & \times B^{\pm}\Big(\frac{4\pi\sqrt{uMN}}{c}\Big)B_0\Big(\frac{mMu}{c^2y}\Big)\frac{\mathrm{d}y}{y}\mathrm{d}u \\ &= \int_0^\infty B^{\pm}\Big(\frac{4\pi\sqrt{uMN}}{c}\Big)J\Big(u;\frac{nN}{c},\frac{mM}{c^2}\Big)\,\mathrm{d}u.\end{aligned}$$

With the help of Lemma 2.7, we have $\mathcal{S}^{\pm}(M,N)$ equals to

$$\begin{aligned}M^{1/6-v_2}N^{1/6-v_1}&\sum_{c\geq 1}\sum_{\substack{m,n\neq 0\\(n,c)=1}}\frac{\tau(m)}{c|mn|^{1/3}}e\Big(\frac{mn}{c}\Big) \\ &\times \int_0^\infty W^{\natural}(u;v_1,v_2)B^{\pm}\Big(\frac{4\pi\sqrt{uMN}}{c}\Big)e\Big(\frac{-3(umnMN)^{1/3}}{c}\Big)\frac{\mathrm{d}u}{u^{1/3}} \\ &+ O(K^{-A}).\end{aligned}\tag{3.9}$$

Here we redefine the smooth compactly supported function as $W^{\natural}$ depending on v_1 and v_2 from the function w; the support of $W^{\natural}$ is $u \asymp 1$. Moreover, $W^{\natural}$ vanishes unless

$$nN \asymp \sqrt{mM}.\tag{3.10}$$

Then by Lemma 2.11, the u-integral in (3.9) becomes

$$\begin{aligned}\Delta\int_0^\infty\int_{|v|\leq \Delta^{\varepsilon}/\Delta} & e\Big(\frac{vK}{\pi}\Big)g(\Delta v) \\ &\times e\Big(\frac{2\sqrt{uMN}}{c}\phi(v) - \frac{3(umnMN)^{1/3}}{c}\Big)W^{\natural}(u;v_1,v_2)\,\mathrm{d}v\frac{\mathrm{d}u}{u^{1/3}},\end{aligned}\tag{3.11}$$

with ϕ given by (2.18). By the stationary phase method, the u-integral is negligibly small unless

$$\phi(v) \asymp |mn|^{1/3}(MN)^{-1/6}.\tag{3.12}$$

For $\phi(v) = \pm\cos(v)$, then $|\phi(v)| \asymp 1$, so this implies $|mn| \asymp \sqrt{MN}$, which by (3.10) leads to the assumption $m \asymp N$ and $n \asymp \sqrt{M/N}$. For $\phi(v) = \pm\sin(v)$, then $|v| \asymp |mn|^{1/3}(MN)^{-1/6}$, we conclude that $|mn| \ll \frac{\sqrt{MN}}{\Delta^{3-\varepsilon}}$.

Assuming (3.12), the stationary point $u_0 = \frac{(mn)^2}{\phi(v)^6 MN}$ potentially lies inside the support of $W^\natural(u; v_1, v_2)$, and taking the parameters $Z = Y = 1$, $H = R = \frac{(mnMN)^{1/3}}{c} \gg K^\varepsilon$ in Lemma 2.8, so the u-integral in (3.11) is equal to

$$\frac{c^{1/2}}{(mnMN)^{1/6}} e\Big(-\frac{mn}{c\phi(v)^2}\Big) W^\flat(v; v_1, v_2) + O(K^{-A}),$$

where $W^\flat$ is inert in terms of v as well as the m, n and c, and it has support on (3.12). Inserting this into (3.11) and then (3.9) we derive that, up to a negligible error,

$$\mathcal{S}^\pm(M,N) = \frac{\Delta}{N^{v_1} M^{v_2}} \sum_{c \geq 1} \frac{1}{\sqrt{c}} \sum_{\substack{m, |n| > 0 \\ (n,c)=1}} \frac{\tau(m)}{\sqrt{|mn|}} \Phi^\pm\Big(\frac{mn}{c}\Big), \tag{3.13}$$

with $\mathrm{Re}(v_1) = \mathrm{Re}(v_2) = \varepsilon$ and

$$\Phi^\pm(x) = \int_{|v| \leq \Delta^\varepsilon/\Delta} e\Big(\frac{vK}{\pi}\Big) g(\Delta v) e\Big(x(1 - \phi(v)^{-2})\Big) w(v) \, \mathrm{d}v, \tag{3.14}$$

where w is an inert function having the same properties as $W^\flat$, and the parameters v_1 and v_2 have been suppressed in the notation for brevity. Then we can gather all conditions,

$$nN \asymp \sqrt{mM},$$

and

$$c \ll \frac{\sqrt{MN}}{\Delta K^{1-\varepsilon}}, \quad |mn| \asymp \sqrt{MN}, \quad m \asymp N, \quad |n| \asymp \sqrt{M/N},$$

in the positive case, or

$$c \asymp \frac{\sqrt{MN}}{K}, \quad |mn| \ll \frac{\sqrt{MN}}{\Delta^{3-\varepsilon}}, \quad m \ll \frac{N}{\Delta^{2-\varepsilon}},$$

in the negative case. In particular, it follows that

$$\frac{mn}{c} \ll |mn| \ll \sqrt{MN} \ll K^{3/2+\varepsilon}$$

by $M \ll K^{2+\varepsilon}$ and $N \ll K^{1+\varepsilon}$. For $\Phi^\pm(x)$, we have the following asymptotic formula (see [23, Lemma 8.2]).

Lemma 3.1. *Suppose that g is a function satisfying $g^{(j)}(x) \ll_{j,A} (1+|x|)^{-A}$, $\phi(v)$ is given via* (2.18)*, and Φ is defined by* (3.14) *for some inert function w. Then for $x \asymp X \gg 1$, we have*

$$\Phi^\pm(x) = \frac{1}{K} \int_{|t| \ll U^\pm} \lambda^\pm(t) x^{\mathrm{i}t} \, \mathrm{d}t + O(K^{-A}),$$

where $\lambda^{\pm}$ *and* $U^{\pm}$ *depend on* X, K *and the choice of* $\phi(v)$, *and satisfy* $\lambda^{\pm}(t) \ll 1$ *and*

$$\begin{cases} U^{+} = K^2/X, & \phi(v) = \pm\cos(v), \\ U^{-} = X^{1/3}K^{2/3}, & \phi(v) = \pm\sin(v). \end{cases}$$

Therefore, we have

$$\mathcal{S}^{\pm}(M,N) \ll \frac{\Delta}{K^{1-\varepsilon}} \sup_{C^{\pm},L_1^{\pm},L_2^{\pm}} \left| \int_{|t|\ll U^{\pm}} \lambda^{\pm}(t)\overline{S_{\mathrm{it}}}(C^{\pm})S_{\mathrm{it}}(L_1^{\pm})S_{\mathrm{it}}^{\flat}(L_2^{\pm})\,\mathrm{d}t \right|,$$

where we use the Möbius function to relax the condition $(n,c) = 1$ and then introduce dyadic portions to the variables c, m, n and

$$\overline{S_{\mathrm{it}}}(C^{\pm}) = \sum_{c\sim C^{\pm}} \frac{1}{c^{1/2+\mathrm{it}}}, \qquad S_{\mathrm{it}}(L_1^{\pm}) = \sum_{n\sim L_1^{\pm}} \frac{1}{n^{1/2-\mathrm{it}}},$$

$$S_{\mathrm{it}}^{\flat}(L_2^{\pm}) = \sum_{m\sim L_2^{\pm}} \frac{\tau(m)v(m/L_2^{\pm})}{m^{1/2-\mathrm{it}}}$$

for suitable $v \in C_c^{\infty}[1,2]$. Here the dyadic parameters $C^{\pm}, L_1^{\pm}$ and $L_2^{\pm}$ are in ranges

$$C^{+} \ll \frac{\sqrt{MN}}{\Delta K^{1-\varepsilon}}, \quad L_1^{+} \asymp \frac{\sqrt{L_2^{+}M}}{N}, \quad L_2^{+} \asymp N, \tag{3.15}$$

and

$$C^{-} \asymp \frac{\sqrt{MN}}{K}, \quad L_1^{-} \ll \frac{\sqrt{L_2^{-}M}}{N}, \quad L_2^{-} \leq \frac{N}{\Delta^{2-\varepsilon}}. \tag{3.16}$$

In addition, by Lemma 3.1 we have

$$U^{+} = \frac{C^{+}K^2}{L_1^{+}L_2^{+}}, \quad U^{-} = \frac{\sqrt[3]{L_1^{-}L_2^{-}K^2}}{\sqrt[3]{C^{-}}}, \tag{3.17}$$

and

$$K^{\varepsilon} \ll U^{\pm} \ll \frac{K}{\Delta^{1-\varepsilon}}. \tag{3.18}$$

In view of Lemma 2.6, the lengths of the m-sum and its dual sum are respectively $L_2^{\pm}$ and ${U^{\pm}}^2/L_2^{\pm}$, so we can always ensure that the length of summation does not exceed $U^{\pm}$. For the c-sum and the n-sum, we take the complex conjugate of the former and group them together so that the new sum is over cn and of length $C^{\pm}L_1^{\pm}$. Furthermore, it follows from (3.15)–(3.17), along with $M \leq K^{2+\varepsilon}$, that

$$C^{+}L_1^{+} \ll \frac{C^{+}L_2^{+}M}{L_1^{+}N^2} \ll \frac{C^{+}M}{L_1^{+}L_2^{+}} \ll U^{+}K^{\varepsilon},$$

and

$$C^- L_1^- \ll \frac{C^- \sqrt[3]{L_1^- L_2^- M}}{\sqrt[3]{N^2}} \ll \frac{\sqrt[3]{L_1^- L_2^- M}}{\sqrt[3]{C^- K^4}} \ll U^- K^\varepsilon.$$

We conclude by the Cauchy–Schwarz inequality and Lemma 2.4 that

$$\mathcal{S}^\pm(M,N) \ll \frac{\Delta}{K^{1-\varepsilon}} U^\pm, \tag{3.19}$$

and hence by (3.18) that

$$\mathcal{S}^\pm(M,N) \ll K^\varepsilon. \tag{3.20}$$

3.4. Conclusion. Now by (3.6) and (3.20) we conclude that

$$\sum_{k\equiv 0 \,(\mathrm{mod}\, 4)} h_{K,\Delta}(k) \sum_{f\in H_k}^h L(1/2,f)^3 = \frac{\Delta\Pi}{4} P_3(\log K) + O\biggl(\frac{\Delta^3 \log^2 K}{K^2} + K^\varepsilon\biggr).$$

Remark 3.2. This work inherits an explicit t-dependence in $\widetilde{\nu}_0(y;t)$ from Lemma 2.6, a nuance resolved via the strategy in [20, p. 19231]; details are omitted for simplicity.

4. Proof of Theorem 1.3

To prove Theorem 1.3, we need an average over T in the interval $K-H, K+H$. We first write $\Phi^\pm(x)$ in (3.14) as $\Phi^\pm_{K,\Delta}(x)$ to indicate its dependence on K and Δ. We need to point that the quantity conditions in Section 3.3 are kept when we average $T^v \Phi^\pm_{T,\Delta}(x)$ over T, since $T \asymp K$.

Lemma 4.1. *Let notations be as Section 3.3. Then,*

$$\begin{aligned}\int_{K-H}^{K+H} & T^v \Phi^\pm_{T,\Delta}(x)\,\mathrm{d}T \\ &= \frac{T}{U^\pm}\Bigl((K+H)^v \Phi^{1\pm}_{K+H,\Delta}(x) - (K-H)^v \Phi^{1\pm}_{K-H,\Delta}(x)\Bigr) \\ &\qquad\qquad - \frac{vK}{U^\pm}\int_{K-H}^{K+H} T^v \Phi^{1\pm}_{T,\Delta}(x)\,\mathrm{d}T,\end{aligned}$$

for $\Phi^{1\pm}_{K,\Delta}(x)$ of the same shape as $\Phi^\pm_{K+H,\Delta}(x)$.

It is clear that (3.6) yields the main term in Theorem 1.3 along with an error $O(\Delta^2 H \log^2 K/K^2) = O(\Delta K^\varepsilon)$. As for the error term in (3.13), after the dyadic partitions, we use Lemma 4.1 for

$$\int_{K-H}^{K+H} T^v \Phi^\pm_{T,\Delta}\Bigl(\frac{mn}{c}\Bigr)\,\mathrm{d}T.$$

It follows from (3.19) and (3.20) that the contributions from the terms with $\Phi^{1\pm}(\frac{mn}{c})$ are bounded by $O(K^\varepsilon)$. We conclude that

$$\frac{1}{\Pi\Delta}\int_{K-H}^{K+H}\sum_{k\equiv 0\,(\mathrm{mod}\,4)} h_{T,\Delta}(k)\sum_{f\in H_k}^{h} L(1/2,f)^3\,\mathrm{d}T = \int_{K-H}^{K+H} P_3(\log T)\,\mathrm{d}T + O(K^\varepsilon).$$

Note that there is an error term $O(\Delta K^\varepsilon)$ in (2.15). Consequently, we obtain Theorem 1.3 on choosing $\Delta = K^\varepsilon$.

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